Hiroshima Math. J.
33 (2003), 371–377

Stable extendibility of vector bundles over real projective spaces and bounds for the Schwarzenberger numbers $\beta(k)$

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(Received April 8, 2003)
(Revised July 7, 2003)

Abstract. For a non-negative integer k, R. L. E. Schwarzenberger defined in [**7**] an integer $\beta(k) \geq 0$ which we call the Schwarzenberger number of k. Let ζ be a k-dimensional F-vector bundle over the real projective n-space RP^n, where F is either the real number field R or the complex number field C. Then $\beta(k)$ is closely related to the problem to find the dimension m with $m \geq n$ which has the property that ζ is stably equivalent to a sum of k F-line bundles if ζ is stably extendible to RP^m. The problem for $F = R$ has been studied in [**7**], [**5**] and [**4**], and that for $F = C$ has been studied in [**6**] and [**4**]. In this note we obtain further results on the problem and determine bounds for the Schwarzenberger numbers $\beta(k)$.

1. Introduction

Throughout this note, F denotes either the real number field R or the complex number field C, and N is the set of all non-negative integers. Let X be a space and A its subspace. A k-dimensional F-vector bundle ζ over A is said to be *extendible* (respectively *stably extendible*) to X, if there is a k-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ, that is, if ζ is equivalent (respectively stably equivalent) to the induced bundle $i^*\eta$ of a k-dimensional F-vector bundle η over X under the inclusion map $i : A \to X$ (cf. [**7**, p. 20], [**8**, p. 191] and [**3**, p. 273]).

For a positive integer i, write $i = (2a+1)2^{\nu(i)}$, where $a \in N$, and for $k \in N$ define an integer $\beta(k) \in N$ by

$$\beta(k) = \min\{i - \nu(i) - 1 \mid k < i\}$$

which we call the Schwarzenberger number of k.

Let ζ be a k-dimensional F-vector bundle over the real projective n-space RP^n where $k > 0$. We study the problem to find the dimension m with $m \geq n$ which has the property that ζ is stably equivalent to a sum of k F-line bundles if ζ is stably extendible to RP^m.

2000 *Mathematics Subject Classification*. Primary 55R50; Secondary 55N15.
Key words and phrases. vector bundle, extendible, stably extendible, KO-theory, K-theory, real projective space, Schwarzenberger number.

Let $\phi(n)$ denote the number of integers s with $0 < s \leq n$ and $s \equiv 0, 1, 2$ or 4 mod 8. For $F = R$, we have

Theorem 1. *Let ζ be a k-dimensional R-vector bundle over RP^n, where $k > 0$, and consider the following four conditions.*

(1) *ζ is stably extendible to RP^m for every $m \geq n$.*

(2) *ζ is stably extendible to RP^m, where $m \geq n$, $m \geq 2k - 1$ and $\phi(m) \geq \phi(n) + \beta(k)$.*

(3) *ζ is stably extendible to RP^m, where $m = 2^{\phi(n)} - 1$.*

(4) *ζ is stably equivalent to a sum of k R-line bundles.*

Then all the four conditions are equivalent. Moreover, when $k = 1$ or $n = 1, 3$ or 7, the conditions always hold.

Let $[x]$ denote the largest integer q with $q \leq x$. For $F = C$, we have

Theorem 2. *Let ζ be a k-dimensional C-vector bundle over RP^n, where $k > 0$, and consider the following four conditions.*

(1) *ζ is stably extendible to RP^m for every $m \geq n$.*

(2) *ζ is stably extendible to RP^m, where $m \geq n$, $m \geq 4k - 1$ and $\phi(m) \geq [n/2] + \beta(2k) + 1$.*

(3) *ζ is stably extendible to RP^m, where $m = 2^{[n/2]+1} - 1$.*

(4) *ζ is stably equivalent to a sum of k C-line bundles.*

Then all the four conditions are equivalent. Moreover, when $k = 1$ or $n = 1, 2$ or 3, the conditions always hold.

Concerning bounds for the Schwarzenberger numbers $\beta(k)$, we obtain

Theorem 3. *Let k be a positive integer, let $\alpha(k)$ denote the number of the non-zero terms of the 2-adic expansion of k, and let $\beta(k)$ denote the Schwarzenberger number of k. Then the inequalities $k - \alpha(k) \leq \beta(k) \leq k$ hold.*

This note is arranged as follows. We study some properties of $\beta(k)$ in Section 2. We prove Theorems 1 and 2 in Section 3, and prove Theorem 3 in Section 4.

2. Some properties of $\beta(k)$

Lemma 2.1. *Let k be a positive integer and t be any integer with $k < 2^t$. Then $\beta(k) = \min\{i - \nu(i) - 1 \mid k < i \leq 2^t\}$.*

Proof. Clearly it suffices to prove that

$$\min\{i - \nu(i) - 1 \mid 2^t < i\} \geq \min\{i - \nu(i) - 1 \mid k < i \leq 2^t\}.$$

Comparing $2^t - \nu(2^t) - 1$ with $i - \nu(i) - 1$ for $i = a2^t + b$, where $a \geq 1$ and $0 < b < 2^t$, and with $i - \nu(i) - 1$ for $i = a2^t$, where $a > 1$, we have

$$a2^t + b - v(a2^t + b) - 1 - \{2^t - v(2^t) - 1\}$$
$$= (a-1)2^t + b - v(b) - 1 + t + 1 > 0,$$

and

$$a2^t - v(a2^t) - 1 - \{2^t - v(2^t) - 1\} = (a-1)2^t - v(a) > a - 1 - v(a) \geq 0,$$

since

$$j - v(j) - 1 = (2x+1)2^y - v((2x+1)2^y) - 1 \geq 2^y - y - 1 \geq 0,$$

where $j = (2x+1)2^y$ $(x, y \in N)$. We therefore obtain the desired inequality. □

Remark. It seems to us that in line 11 of [**7**, p. 21], the last inequality $i < 2^t$ should be replaced by the inequality $i \leq 2^t$. In fact, $\min\{i - v(i) - 1 \mid k < i < 2^t\}$ is not necessarily equal to $\beta(k)$ (for example, if $(k, t) = (6, 3)$, $\min\{i - v(i) - 1 \mid k < i < 2^t\} = 6$ and $\beta(k) = 4$), the first inequality in line 11 of [**7**, p. 21] holds also for $i = 2^t$ and $\min\{i - v(i) - 1 \mid k < i \leq 2^t\}$ is equal to $\beta(k)$ by Lemma 2.1.

Lemma 2.2. *$\beta(2^r) = 2^r$ for $r \geq 2$, $\beta(2) = 1$ and $\beta(1) = 0$.*

Proof. We prove the first equality. Suppose $r \geq 2$. Since $\beta(2^r) \leq 2^r$ (cf. [**7**, Examples]), it suffices to prove that $\min\{i - v(i) - 1 \mid 2^r < i \leq 2^{r+1}\} \geq 2^r$ by Lemma 2.1. If $i = 2^r + b$, where $0 < b < 2^r$, we have $i - v(i) - 1 = 2^r + b - v(b) - 1 \geq 2^r$, and if $i = 2^{r+1}$, we have $i - v(i) - 1 = 2^{r+1} - (r+1) - 1 \geq 2^r$ for $r \geq 2$. Hence we obtain the desired inequality. The other equalities are easily verified. □

Corollary 2.3. *$\beta(0) = 0$, and $\beta(k) > 0$ for $k > 1$.*

Proof. By definition, $j \geq k$ implies $\beta(j) \geq \beta(k)$. Hence the results follow from Lemma 2.2. □

Lemma 2.4. *Let $j = 1, 2, 3$ or 4 and let $k = 2^r - j$, where $r \geq 0$ for $j = 1$, $r \geq 1$ for $j = 2$, and $r \geq 3$ for $j = 3$ or 4. Then*

$$\beta(k) = 2^r - r - 1.$$

Moreover,

$$\beta(2^r - 5) = 2^r - r - 1 \quad \text{for } r \geq 6, \qquad \beta(2^r - 5) = 2^r - 7 \quad \text{for } 3 \leq r \leq 5.$$

Proof. By definition and by Lemma 2.2, we have

$$\beta(2^r - 1) = \min\{2^r - v(2^r) - 1, \beta(2^r)\} = \min\{2^r - r - 1, 2^r - \delta\}$$
$$= 2^r - r - 1,$$

where $\delta = 0$ if $r \geq 2$ and $\delta = 1$ if $r = 0$ or 1. By the result above, we have, for $r \geq 1$,

$$\begin{aligned}\beta(2^r - 2) &= \min\{2^r - 1 - v(2^r - 1) - 1, \beta(2^r - 1)\} \\ &= \min\{2^r - 2, 2^r - r - 1\} = 2^r - r - 1.\end{aligned}$$

Similarly, we have, for $r \geq 3$,

$$\begin{aligned}\beta(2^r - 3) &= \min\{2^r - 2 - v(2^r - 2) - 1, \beta(2^r - 2)\} \\ &= \min\{2^r - 4, 2^r - r - 1\} = 2^r - r - 1, \\ \beta(2^r - 4) &= \min\{2^r - 3 - v(2^r - 3) - 1, \beta(2^r - 3)\} \\ &= \min\{2^r - 4, 2^r - r - 1\} = 2^r - r - 1.\end{aligned}$$

Moreover,

$$\begin{aligned}\beta(2^r - 5) &= \min\{2^r - 4 - v(2^r - 4) - 1, \beta(2^r - 4)\} \\ &= \min\{2^r - 7, 2^r - r - 1\} \\ &= 2^r - r - 1 \quad \text{for } r \geq 6, \qquad = 2^r - 7 \quad \text{for } 3 \leq r \leq 5.\end{aligned}$$

□

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Clearly (1) implies (2) and (3). In [**7**, Theorem 3] R. L. E. Schwarzenberger proved that (2) implies (4) (cf. Remark in Section 2). In the original result of Schwarzenberger, the R-vector bundle ζ is assumed to be extendible, but his result is also valid if we only assume that ζ is stably extendible instead of extendible (cf. [**3**, Section 1]). We proved in [**4**, Theorem 3.1(i)] that (3) implies (4) for $n \neq 1, 3, 7$, and in [**4**, Theorem 3.2] that (4) is equivalent to (1). We therefore proved the theorem for the case $n \neq 1, 3, 7$.

When $n = 1, 3$ or 7, it suffices to prove (4). In fact, (4) for $n = 1, 3$ or 7 follows from [**4**, Theorem 3.1 (ii)]. The latter part for $k = 1$ is clear (cf. [**4**, Theorem 3.2]). □

Remark. If $k > 1$, then $\beta(k) > 0$ by Corollary 2.3 and so the inequality $\phi(m) \geq \phi(n) + \beta(k)$ implies the inequality $m > n$.

Proof of Theorem 2. Clearly (1) implies (2) and (3). We proved in [**6**, Theorem 2.2 and Remark] that (3) implies (4) for $n > 3$, and in [**4**, Theorem 3.2] that (4) is equivalent to (1). Hence for the proof for the case $n > 3$, it suffices to prove that (2) implies (4). Though the proof is parallel to that of [**7**, Theorem 3], for completeness we prove that (2) implies (4) below.

Assume that (2) holds for ζ. Then there is a k-dimensional C-vector bundle η over RP^m such that $i^*\eta$ is stably equivalent to ζ, where $i : RP^n \to RP^m$ is the standard inclusion. According to [**1**, Theorem 7.3], we have, for some integer q with $0 \leq q < 2^{[m/2]}$, $\eta - k = qc(\xi_m - 1)$ in the reduced K-group $\tilde{K}(RP^m)$, and so

$$\zeta - k = i^*\eta - k = qc(i^*\xi_m - 1) = qc(\xi_n - 1)$$

in $\tilde{K}(RP^n)$, where ξ_N is the canonical R-line bundle over RP^N and c denotes the complexification. Let r denote the forgetful map. Then

$$r\eta - 2k = rcq(\xi_m - 1) = 2q(\xi_m - 1),$$

since $rc = 2$. In the terminology of [**2**, Section 2], the element $2q(\xi_m - 1)$ has geometrical dimension not exceeding $2k$. If $q \leq k$, then ζ is stably equivalent to a sum of k C-line bundles. If $q > k$, then, by [**1**, Theorem 7.4] and [**2**, Proposition 2.3], for all $i > 2k$,

$$\gamma^i(2q(\xi_m - 1)) = C_{2q,i}(\xi_m - 1)^i = (-1)^{i-1}2^{i-1}C_{2q,i}(\xi_m - 1) = 0,$$

where γ^i is the Grothendiek operator and $C_{j,i}$ is the binomial coefficient $j!/((j-i)!i!)$, and so, by [**1**, Theorem 7.4],

$$i - 1 + v(C_{2q,i}) \geq \phi(m) \qquad \text{for all } i \text{ with } 2k < i \leq 2q.$$

Since $r\eta$ is stably equivalent to $2q\xi_m$, the total Stiefel-Whitney class $w(r\eta)$ of $r\eta$ is $(1 + x_m)^{2q}$, where x_m is the generator of $H^1(RP^m; Z_2)$. On the other hand, by [**7**, Theorem 2],

$$w(r\eta) = w(s\xi_m \oplus (2k - s)) = (1 + x_m)^s \qquad \text{for some } s \text{ with } 0 \leq s \leq 2k,$$

since $m \geq 4k - 1$, where $\oplus$ denotes the Whitney sum. Hence

$$2q = (2a + 1)2^t + s \qquad \text{for some } a \in N \text{ and } t \in N \text{ with } m < 2^t.$$

It follows from the inequalities $3 \leq m < 2^t$ that $t \geq 2$. So s is even and $0 \leq s/2 \leq k$. Now,

$$v(C_{(2a+1)2^t+s,i}) = v(C_{2^t+s,i}) \leq v(C_{2^t,i}) \qquad \text{for all } i \text{ with } s < i \leq 2^t,$$

and $v(C_{2^t,i}) = t - v(i)$. Therefore

$$i - 1 + t - v(i) \geq \phi(m) \qquad \text{for all } i \text{ with } 2k < i \leq 2^t,$$

and so $t - 1 \geq \phi(m) - \beta(2k) - 1 \geq [n/2]$ by Lemma 2.1 and by the assumption. Therefore $\zeta - k = (s/2)c(\xi_n - 1)$, since $c(\xi_n - 1)$ is of order $2^{[n/2]}$ by [**1**, Theorem 7.3]. Hence ζ is stably equivalent to a sum of k C-line bundles and so (4) holds. Thus we have completed the proof of the former part.

To prove the theorem for the case $n = 1, 2$ or 3, it suffices to prove (4) (cf. [**4**, Theorem 3.2]). By [**1**, Theorem 7.3] there is an integer l such that $\zeta - k = (k+l)c(\xi_n - 1)$, where $0 \le k + l < 2^{[n/2]}$. If $l > 0$, $[n/2] < k + l$ by [**6**, Theorem 2.1]. This contradicts the inequality $k + l < 2^{[n/2]}$ if $n = 1, 2$ or 3. Hence $l \le 0$, and so (4) holds. The latter part for $k = 1$ is clear (cf. [**4**, Theorem 3.2]). □

Remark. For $k > 0$, $\beta(2k) > 0$ by Corollary 2.3 and so the inequality $\phi(m) \ge [n/2] + \beta(2k) + 1$ implies the inequality $m > n$.

4. Proof of Theorem 3

Lemma 4.1. *For* $0 < i < 2^r$, $\alpha(2^r - i - 1) = \alpha(2^r - i) + \nu(i) - 1$.

Proof. Let $\nu(i) = s$. Then $\nu(2^r - i) = s$ and $2^r - i = (2a+1)2^s$, for some $a \in N$. Hence $2^r - i - 1 = a2^{s+1} + 2^{s-1} + 2^{s-2} + \cdots + 2 + 1$ and so $\alpha(2^r - i - 1) = \alpha(a2^{s+1} + 2^{s-1}) + s - 1 = \alpha(a2^{s+1} + 2^s) + s - 1 = \alpha(2^r - i) + \nu(i) - 1$. □

Theorem 3 is a consequence of the following result.

Theorem 4.2. *Let k be a positive integer. Then we have the following.*
(i) *If $k \ne 2^r - 1$ and $k \ne 2^r - 2$, $k - \alpha(k) < \beta(k) \le k - \varepsilon$, where $\varepsilon = 0$ for k even and $\varepsilon = 1$ for k odd.*
(ii) *If $k = 2^r - 1$ or $k = 2^r - 2$, $\beta(k) = k - \alpha(k)$.*

Proof. (i) The inequality $\beta(k) \le k - \varepsilon$ follows from [**7**, Examples].

By Lemma 2.2, clearly the inequality $k - \alpha(k) < \beta(k)$ holds for $k = 2^r$ for $r \ge 2$. We prove the inequality $k - \alpha(k) < \beta(k)$ for $k \ne 2^r - 1$ and $k \ne 2^r - 2$ by a downward induction on $k = 2^r - i$, where $r \ge 3$ and $3 \le i < 2^{r-1}$. If $i = 3$, by Lemma 2.4, $\beta(2^r - 3) = 2^r - r - 1$. On the other hand, $\alpha(2^r - 3) = r - 1$, and so $k - \alpha(k) < \beta(k)$ holds for $k = 2^r - 3$.

Suppose that the inequality $k - \alpha(k) < \beta(k)$ holds for $k = 2^r - i$, where $4 \le i + 1 < 2^{r-1}$. Since $\beta(2^r - i - 1) = \min\{2^r - i - \nu(2^r - i) - 1, \beta(2^r - i)\}$, $\beta(2^r - i - 1) = 2^r - i - \nu(2^r - i) - 1$ or $\beta(2^r - i - 1) = \beta(2^r - i)$.

If $\beta(2^r - i - 1) = 2^r - i - \nu(2^r - i) - 1$, we have, by Lemma 4.1,

$$\begin{aligned}\beta(2^r - i - 1) &- \{2^r - i - 1 - \alpha(2^r - i - 1)\} \\ &= 2^r - i - \nu(2^r - i) - 1 - \{2^r - i - 1 - (\alpha(2^r - i) + \nu(i) - 1)\} \\ &= \alpha(2^r - i) - 1 > 0,\end{aligned}$$

since $\alpha(2^r - i) > 1$ for $3 \le i < 2^{r-1}$. If $\beta(2^r - i - 1) = \beta(2^r - i)$, we have, by Lemma 4.1 and by the inductive assumption,

$$\begin{aligned}\beta(2^r - i - 1) &- \{2^r - i - 1 - \alpha(2^r - i - 1)\} \\ &= \beta(2^r - i) - \{2^r - i - 1 - (\alpha(2^r - i) + \nu(i) - 1)\} \\ &= \beta(2^r - i) - \{2^r - i - \alpha(2^r - i)\} + \nu(i) > 0.\end{aligned}$$

So the inequality $k - \alpha(k) < \beta(k)$ holds for $k = 2^r - i - 1$.

(ii) Note that if $k = 2^r - 1$ or $k = 2^r - 2$, $r = \alpha(k)$ or $r = \alpha(k) + 1$, respectively. By Lemma 2.4, if $k = 2^r - 1$, $\beta(k) = 2^r - r - 1 = k - \alpha(k)$, and if $k = 2^r - 2$, $\beta(k) = 2^r - r - 1 = k - \alpha(k)$. □

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