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Asymptotic expansion of the null distribution of the likelihood ratio statistic for testing the equality of variances in a nonnormal one-way ANOVA model

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Abstract. This paper is concerned with the null distribution of the likelihood ratio statistic for testing the equality of variances of q nonnormal populations. It is known that the null distribution of this statistic converges to χ^2_{q-1} under normality. We extend this result by obtaining an asymptotic expansion under general conditions. Numerical accuracies are studied for some approximations of the percentage points and actual test sizes of this statistic based on the limiting distribution and the asymptotic expansion.

1. Introduction

The one-way ANOVA test is a familiar procedure for comparing several populations. Let X_{ij} be the j-th sample observation $(j = 1, \dots, n_i)$ from the i-th population Π_i $(i = 1, \dots, q)$ with mean μ_i and common variance σ^2, where μ_i's and σ^2 are unknown. The null hypothesis which is considered in this test is $H_0 : \mu_1 = \cdots = \mu_q$. Let $n = n_1 + \cdots + n_q$, $\overline{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$ and $\overline{X} = n^{-1} \sum_{i=1}^{q} \sum_{j=1}^{n_i} X_{ij}$. A commonly used statistic is $T = (n-q)S_h/S_e$, which is the likelihood ratio statistic for the normal case, where $S_h = \sum_{i=1}^{q} n_i(\overline{X}_i - \overline{X})^2$, $S_e = \sum_{i=1}^{q}(n_i - 1)s_i^2$ and $s_i^2 = (n_i - 1)^{-1} \sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2$. Under normality, i.e., $\Pi_i : \mathrm{N}(\mu_i, \sigma^2)$, it is well known that the null distribution of $(q-1)^{-1}T$ is distributed as F^{q-1}_{n-q}. Under nonnormality, it is known that the null distribution of this statistic converges to χ^2_{q-1} and an asymptotic expansion of the null distribution was obtained by Fujikoshi, Ohmae and Yanagihara (1999). Under normality, it is known that this test is robust against heteroscedasticy of the variances and under nonnormality an asymptotic expansion of the null distribution of the test statistic, proposed by James (1951), was obtained by Yanagihara (2000). As thses tests depend on the assumption of variances, it is important to test the equality of variances as a preliminary to one-way ANOVA test.

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In this paper we consider testing the null hypothesis

$$H_0 : \sigma_1^2 = \cdots = \sigma_q^2. \tag{1.1}$$

The treated test statistic is

$$T = (n - q) \log \frac{S_e}{n - q} - \sum_{i=1}^{q} (n_i - 1) \log s_i^2,$$

which is the likelihood ratio statistic for the normal case. Under normality it is well known that the null distribution of T converges to χ_{q-1}^2, as the sample sizes n_i $(i = 1, \dots, q)$ tend to infinity, and an asymptotic expansion was obtained by Hartley (1940). Sugiura and Nagao (1969) compared Bartlett's test and Lehmann's test by deriving asymptotic expansion of the non-null distributions under normality. Under nonnormality Boos and Brownie (1989) have proposed a bootstrap approach for the hypothesis (1.1) and the corresponding multivariate results were obtained by Zhang and Boos (1992). The main purpose of this paper is to obtain an asymptotic expansion of the null distribution of T up to the order n^{-1} under general conditions. In the multivariate case, we will be able to obtain an asymptotic expansion formula by using similar calculation methods in this paper. However, it needs enormous calculations, and we consider that it has some difficulty to use for the approximation.

The present paper is organized in the following way. In section 2 we prepare Edgeworth expansions for the density function of the sample variance. In section 3 we derive an asymptotic expansion of the null distribution of T, by expanding the characteristic function of T. In section 4 numerical accuracies are studied for some approximations of the percentage points and actual test sizes of T based on the limiting distribution and the asymptotic expansion.

2. Preliminary result

Let $Y, Y_1, \dots, Y_n$ be independently and identically distributed with $\mathrm{E}(Y) = 0$ and $\mathrm{E}(Y^2) = 1$. Let the j-th cumulant of Y be denoted by κ_j. Consider the sample mean, sample squared mean and sample variance defined by

$$\overline{Y} = \frac{1}{n} \sum_{j=1}^{n} Y_j, \qquad \tilde{S}^2 = \frac{1}{n} \sum_{j=1}^{n} Y_j^2, \qquad s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (Y_j - \overline{Y})^2$$

and their standardized statistics defined by

$$Z = \sqrt{n}\,\overline{Y}, \qquad \tilde{V} = \sqrt{n}(\tilde{S}^2 - 1), \qquad V = \sqrt{n}(s^2 - 1)$$

From Barndorff-Nielsen and Cox (1989) and Hall (1992) etc, we can write the joint characteristic function $(Z, \tilde{V})$ in the following lemma.

Lemma 2.1. *Suppose that Y has the eighth moment, then the joint characteristic function of* $(Z, \tilde{V})$ *can be expanded as*

$$C_{(Z,\tilde{V})}(t_1,t_2) = \exp\left(\frac{i^2}{2}w_2\right)\left[1+\frac{i^3}{6\sqrt{n}}w_3+\frac{i^4}{24n}w_4+\frac{i^6}{72n}w_3^2\right]+\mathrm{o}(n^{-1}), \quad (2.2)$$

where w_h *is h-th cumulant of* $W = t_1Y + t_2(Y^2-1)$,

$$w_2 = t_1^2 + 2\kappa_3 t_1 t_2 + (\kappa_4+2)t_2^2,$$

$$w_3 = \kappa_3 t_1^3 + 3(\kappa_4+2)t_1^2 t_2 + 3(\kappa_5+8\kappa_3)t_1 t_2^2, + (\kappa_6+12\kappa_4+8\kappa_3^2+8)t_2^3$$

$$w_4 = \kappa_4 t_1^4 + 4(\kappa_5+6\kappa_3)t_1^3 t_2 + 6(\kappa_6+12\kappa_4+8\kappa_3^2+8)t_1^2 t_2^2,$$
$$+4(\kappa_7+18\kappa_5+32\kappa_3\kappa_4+72\kappa_3)t_1 t_2^3$$
$$+(\kappa_8+24\kappa_6+56\kappa_3\kappa_5+32\kappa_4^2+144\kappa_4+240\kappa_3^2+48)t_2^4,$$

Futher, noting that $V = \tilde{V} + n^{-1/2}(1-Z^2) + n^{-1}\tilde{V} + \mathrm{O}_p(n^{-3/2})$, we can obtain an expansion of the characteristic function of V as follow

$$C_V(t) = \mathrm{E}\left[\exp(it\tilde{V})\left\{\left(1+\frac{it}{\sqrt{n}}-\frac{t^2}{2n}\right)\right.\right.$$
$$\left.\left.+Z^2\left(-\frac{it}{\sqrt{n}}+\frac{t^2}{n}\right)-\frac{t^2}{2n}Z^4+\frac{it}{n}\tilde{V}\right\}\right]+\mathrm{o}(n^{-1}).$$

In order to compute $\mathrm{E}(e^{it\tilde{V}}Z^2)$, $\mathrm{E}(e^{it\tilde{V}}Z^4)$ and $\mathrm{E}(e^{it\tilde{V}}\tilde{V})$, we use differentiation of (2.2) in Lemma 2.1. Note that

$$\mathrm{E}[\tilde{V}\exp(it\tilde{V})] = \frac{1}{i}\frac{\partial}{\partial t_2}C_{(Z,\tilde{V})}(t_1,t_2)|_{t_1=0},$$

$$\mathrm{E}[Z^k\exp(it\tilde{V})] = \frac{1}{i^k}\frac{\partial^k}{\partial t_1^k}C_{(Z,\tilde{V})}(t_1,t_2)|_{t_1=0}.$$

Using the result we obtain the following lemma.

Lemma 2.2. *Suppose that Y has the eighth moment, then the characteristic function of V can be expanded as*

$$C_V(t) = \exp\left\{\frac{1}{2}m_0(it)^2\right\}\left[1+\frac{(it)^3}{6\sqrt{n}}m_1\right.$$
$$\left.+\frac{1}{n}\left\{(it)^2+\frac{1}{24}m_2(it)^4+\frac{1}{72}m_3(it)^6\right\}\right]+\mathrm{o}(n^{-1}),$$

where

$$m_0 = 2 + \kappa_4,$$

$$m_1 = 8 + 4\kappa_3^2 + 12\kappa_4 + \kappa_6,$$

$$m_2 = 48 + 96\kappa_3^2 + 144\kappa_4 + 32\kappa_4^2 + 32\kappa_3\kappa_5 + 24\kappa_6 + \kappa_8,$$

$$m_3 = (8 + 4\kappa_3^2 + 12\kappa_4 + \kappa_6)^2 = m_1^2. \tag{2.3}$$

In order to obtain the Edgeworth expansion for the density function of V, we assume that the characteristic function of Y and Y^2 satisfies

$$C : \int\int |C_{(Y,\,Y^2)}(t_1, t_2)|^r dt_1 dt_2 < \infty,$$

for some $r \geq 1$.

LEMMA 2.3. *Under the same condition as in Lemma* 2.2 *and the assumption* C, *it holds that*

$$f(v) = \phi(v; 0, m_0)\left[1 + \frac{1}{\sqrt{n}} q_1(v) + \frac{1}{n} q_2(v)\right] + \mathrm{o}(n^{-1}),$$

where

$$q_1(v) = -\frac{m_1}{6} H_3(m_0^{-1/2} v) m_0^{-3/2},$$

$$q_2(v) = H_2(m_0^{-1/2} v) m_0^{-1} + \frac{m_2}{24} H_4(m_0^{-1/2} v) m_0^{-2} + \frac{m_3}{72} H_6(m_0^{-1/2} v) m_0^{-3},$$

and $\phi(v; 0, m_0)$ *is the probability density function of* $\mathrm{N}(0, m_0)$, $H_j(v)$ *is the Hermite polynomial of order* j, *for example*, $H_2(v) = v^2 - 1$, $H_3(v) = v^3 - 3v$, $H_4(v) = v^4 - 6v^2 + 3$, $H_6(v) = v^6 - 15v^4 + 45v^2 - 15$.

From the Lemma 2.3 the probability density function of $\boldsymbol{V} = (V_1, \ldots, V_q)'$ can be expanded as

$$f(\boldsymbol{v}) = \phi_q(\boldsymbol{v}; \boldsymbol{0}, m_0 I)\left[1 + \frac{1}{\sqrt{n}} Q_1(\boldsymbol{v}) + \frac{1}{n} Q_2(\boldsymbol{v})\right] + \mathrm{o}(n^{-1}), \tag{2.4}$$

where $\phi_q(\boldsymbol{v}; \boldsymbol{0}, m_0 I)$ is the probability density function of $\mathrm{N}(\boldsymbol{0}, m_0 I)$ and

$$Q_1(\boldsymbol{v}) = \sum_{i=1}^{q} \rho_i^{-1} q_1(v_i), \tag{2.5}$$

$$Q_2(\boldsymbol{v}) = \sum_{i=1}^{q} \rho_i^{-2} q_2(v_i) + \frac{1}{2}\sum_{i \neq j}^{q} \rho_i^{-1}\rho_j^{-1} q_1(v_i) q_1(v_j), \tag{2.6}$$

where $\rho_i = \sqrt{n_i/n}$ and $q_i(v)$'s are given in Lemma 2.3.

3. Asymptotic expansion of T

In this section we derive an asymptotic expansion of the null distribution of T up to the order n^{-1}. We consider the null distribution. Let $Y_{ij} = (X_{ij} - \mu_i)/\sigma$. Then under the null hypothesis, $\mathrm{E}(Y_{ij}) = 0$ and $\mathrm{Var}(Y_{ij}) = 1$. Let Y be independent with Y_{ij} $(i = 1, \dots, q; j = 1, \dots, n_i)$ and have the same distribution as Y_{ij}. Then Y, Y_{ij} $(i = 1, \dots, q; j = 1, \dots, n_i)$ are independently and identically distributed with $\mathrm{E}(Y) = 0$ and $\mathrm{Var}(Y) = 1$ without loss of generality. Let the jth cumulant of Y be denoted by κ_j. For $i = 1, \dots, q$, let $V_i = \sqrt{n_i}(s_i^2 - 1)$ and

$$\boldsymbol{V} = (V_1, \dots, V_q)', \qquad \boldsymbol{\rho} = (\rho_1, \dots, \rho_q)',$$

where ρ_i's are defined in the previous section. Suppose that Y and n_i $(i = 1, \dots, q)$ satisfy the following assumptions.

ASSUMPTIONS: A1. (Y, Y^2) satisfies Cramér condition,
A2. Y has the eighth moment,
A3. $\rho_i^{-1} = \mathrm{O}(1)$ as $n \to \infty$.

Cramér condition is stated as

$$\limsup_{\|\boldsymbol{t}\| \to \infty} |\mathrm{E}[\exp(it_1 Y + it_2 Y^2)]| < 1,$$

with $\boldsymbol{t} = (t_1, t_2)'$ and $\|\boldsymbol{t}\| = (t_1^2 + t_2^2)^{1/2}$.

Note that T is a smooth function of $V_1, \dots, V_q$. So, from the results of Chandra and Ghosh (1979) it can be shown that T has a valid expansion up to the order n^{-1} under the assumptions A1, A2 and A3. In the following we will find an asymptotic expansion of the characteristic function of T up to the order n^{-1}, which may be inverted formally. We can expand T as

$$T = T_0 + \frac{1}{\sqrt{n}} T_1 + \frac{1}{n} T_2 + \mathrm{O}_p(n^{-3/2}) \tag{3.7}$$

where

$$T_0 = \frac{1}{2} \boldsymbol{V}'(I - \boldsymbol{\rho}\boldsymbol{\rho}')\boldsymbol{V}, \qquad T_1 = \frac{1}{3}\{(\boldsymbol{\rho}'\boldsymbol{V})^3 - (\boldsymbol{\rho}^{-1})'\boldsymbol{V}^3\},$$

$$T_2 = \frac{1}{4}\{4\boldsymbol{\rho}'(\boldsymbol{V}\boldsymbol{V}')\boldsymbol{\rho}^{-1} - (\boldsymbol{\rho}'\boldsymbol{V})^4 - 2q(\boldsymbol{\rho}'\boldsymbol{V})^2 + (\boldsymbol{\rho}^{-2})'\boldsymbol{V}^4 - 2(\boldsymbol{\rho}^{-2})'\boldsymbol{V}^2\}.$$

Here $\boldsymbol{a}^m$ denotes $(a_1^m, \dots, a_q^m)'$ for $\boldsymbol{a} = (a_1, \dots, a_q)'$. From (3.7) we can write the characteristic function of T as

$$C_T(t) = C_0(t) + \frac{1}{\sqrt{n}} C_1(t) + \frac{1}{n} C_2(t) + \mathrm{o}(n^{-1}) \tag{3.8}$$

where

$$C_0(t) = \mathrm{E}[e^{itT_0}], \qquad C_1(t) = \mathrm{E}[itT_1 e^{itT_0}], \qquad C_2(t) = \mathrm{E}\left[\left\{itT_2 + \frac{1}{2}(itT_1)^2\right\}e^{itT_0}\right].$$

For evaluation of each term in (3.8), we will use an asymptotic expansion of the density function of $\boldsymbol{V}$ given by (2.4).

For computing the $C_0(t)$, using (2.4) we obtain

$$C_0(t) = \frac{1}{(2\pi m_0)^{q/2}} \int \exp\left(-\frac{1}{2m_0}\boldsymbol{v}'\boldsymbol{v}\right) \exp\left(\frac{it}{2}\boldsymbol{v}'(I - \boldsymbol{\rho}\boldsymbol{\rho}')\boldsymbol{v}\right) \times \left[1 + \frac{1}{\sqrt{n}}Q_1(\boldsymbol{v}) + \frac{1}{n}Q_2(\boldsymbol{v})\right] + \mathrm{o}(n^{-1})$$

where $Q_1(\boldsymbol{v})$ and $Q_2(\boldsymbol{v})$ are defined by (2.5) and (2.6). Let $\varphi = (1 - m_0 it)^{-1}$ and $\Gamma = \varphi(I_q - \boldsymbol{\rho}\boldsymbol{\rho}') + \boldsymbol{\rho}\boldsymbol{\rho}'$, then we see that

$$\exp\left(-\frac{1}{2m_0}\boldsymbol{V}'\boldsymbol{V}\right) \exp\left\{\frac{it}{2}\boldsymbol{V}'(I - \boldsymbol{\rho}\boldsymbol{\rho}')\boldsymbol{V}\right\} = \exp\left\{-\frac{1}{2m_0}(\Gamma^{-1/2}\boldsymbol{V})'(\Gamma^{-1/2}\boldsymbol{V})\right\}.$$

Considering the transformation $\boldsymbol{V}$ to $\boldsymbol{X} = \Gamma^{-1/2}\boldsymbol{V}$, $C_0(t)$ is expressed as the expectation on $\boldsymbol{X}$ which is distributed as $\mathrm{N}_q(\boldsymbol{0}, m_0 I)$. Then we have

$$C_0(t) = \varphi^{(1/2)(q-1)}\mathrm{E}_{\boldsymbol{X}}\left[1 + \frac{1}{\sqrt{n}}Q_1(\Gamma^{1/2}\boldsymbol{X}) + \frac{1}{n}Q_2(\Gamma^{1/2}\boldsymbol{X})\right] + \mathrm{o}(n^{-1}).$$

Note that $\boldsymbol{U} = \Gamma^{1/2}\boldsymbol{X}$ is distributed as $\mathrm{N}_q(\boldsymbol{0}, m_0\Gamma)$. Therefore, we can write

$$C_0(t) = \varphi^{(1/2)(q-1)}\mathrm{E}_{\boldsymbol{U}}\left[1 + \frac{1}{\sqrt{n}}Q_1(\boldsymbol{U}) + \frac{1}{n}Q_2(\boldsymbol{U})\right] + \mathrm{o}(n^{-1}). \tag{3.9}$$

Applying similar method to $C_1(t)$ and $C_2(t)$, we obtain

$$C_1(t) = \frac{1}{3m_0\sqrt{n}}(1 - \varphi^{-1})\varphi^{(1/2)(q-1)}\mathrm{E}_{\boldsymbol{U}}[Q_1(\boldsymbol{U})] + \mathrm{o}(n^{-1/2}), \tag{3.10}$$

$$\begin{aligned} C_2(t) = {} & \frac{1}{4m_0}(1 - \varphi)\varphi^{(1/2)(q-1)}\mathrm{E}_{\boldsymbol{U}}[4\boldsymbol{\rho}'(\boldsymbol{U}\boldsymbol{U}')\boldsymbol{\rho}^{-1} - (\boldsymbol{\rho}'\boldsymbol{U})^4 \\ & - 2q(\boldsymbol{\rho}'\boldsymbol{U})^2(\boldsymbol{\rho}^{-2})'\boldsymbol{U}^4 - 2(\boldsymbol{\rho}^{-2})'\boldsymbol{U}^2] \\ & + \frac{1}{18m_0^2}(1 - \varphi)^2\varphi^{(1/2)(q-1)}\mathrm{E}_{\boldsymbol{U}}[\{(\boldsymbol{\rho}'\boldsymbol{U})^3 - (\boldsymbol{\rho}^{-1})'\boldsymbol{U}^3\}^2] + \mathrm{o}(1). \end{aligned} \tag{3.11}$$

For calculating (3.9), (3.10) and (3.11), we use the expectations of the Hermite polynomials. Let the (α, β) element of Γ be denoted by $\gamma_{\alpha\beta}$. Then

$$\mathrm{E}_{\boldsymbol{U}}[H_2(U_\alpha)] = \gamma_{\alpha\alpha} - 1,$$

$$\mathrm{E}_{\boldsymbol{U}}[H_3(U_\alpha)] = 0,$$

$$\mathrm{E}_{\boldsymbol{U}}[H_4(U_\alpha)] = 3(\gamma_{\alpha\alpha} - 1)^2,$$

$$\mathrm{E}_{\boldsymbol{U}}[H_6(U_\alpha)] = 15(\gamma_{\alpha\alpha} - 1)^3,$$

$$\mathrm{E}_{\boldsymbol{U}}[H_3(U_\alpha)H_3(U_\beta)] = 3\gamma_{\alpha\beta}, \{3(\gamma_{\alpha\alpha} - 1)(\gamma_{\beta\beta} - 1) + 2\gamma_{\alpha\beta}^2\}, \qquad (\alpha \neq \beta).$$

Note that $\gamma_{\alpha\beta} = \varphi\delta_{\alpha\beta} + (1-\varphi)\rho_\alpha\rho_\beta$, where $\delta_{\alpha\beta}$ is the Kronecker delta, i.e., $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ otherwise. Substituting these into (3.9), (3.10) and (3.11) yields

$$C_T(t) = \varphi^{(1/2)(q-1)}\left[1 + \frac{1}{n}\sum_{j=0}^{3} b_j\varphi^j\right] + \mathrm{o}(n^{-1}), \tag{3.12}$$

where

$$b_0 = -a_1 + a_2 - a_5,$$

$$b_1 = 3a_1 - 2a_2 - a_4 + a_5,$$

$$b_2 = -3a_1 + a_2 - a_3 + a_4,$$

$$b_3 = a_1 + a_3, \tag{3.13}$$

and

$$a_1 = \{4m_1^2 - 6qm_1^2 - 3q^2(-2m_0^2 + m_1)^2 + 5m_1^2\|\boldsymbol{\rho}^{-1}\|^2\}/24m_0^3,$$

$$a_2 = m_2(1 - 2q + \|\boldsymbol{\rho}^{-1}\|^2)/8m_0^2,$$

$$a_3 = -(m_0^2 - m_1)(-4 + 6q - 5\|\boldsymbol{\rho}^{-1}\|^2)/6m_0,$$

$$a_4 = -\{(5 + 6q)m_0^2 - 6m_1 + (m_0^2 - 6m_1)\|\boldsymbol{\rho}^{-1}\|^2\}/12m_0,$$

$$a_5 = \{-2m_0^2 + q(-2 + m_0 + 3m_0^2 - 2m_1) + m_1 - (-2 + m_0)\|\boldsymbol{\rho}^{-1}\|^2\}/2m_0,$$

where $\|\boldsymbol{\rho}^{-1}\|^2 = \sum_{i=1}^q \rho_i^{-2} = \sum_{i=1}^q n/n_i$ and m_j's are given by (2.3).

Note that the leading term of (3.12) is $\varphi^{(q-1)/2} = (1 - m_0it)^{-(q-1)/2}$, the null distribution of T converges to $(m_0/2)\chi_{q-1}^2$ under nonnormality. Therefore, this test is not robust against nonnormality, because the limiting distribution of the null distribution of T varies according to the value of $m_0/2 = 1 + \kappa_4/2$ under nonnormality (Box (1953)). So, we consider the statistic $2T/m_0$ whose null distribution converges to χ_{q-1}^2.

Finally, by inverting the characteristic function of $2T/m_0$ which is computed from (3.12), we have the following Theorem 3.1.

THEOREM 3.1. *Under the Assumptions* A1, A2 *and* A3, *the null distribution of* $2T/m_0$ *can be expanded as*

$$\mathrm{P}\left(\frac{2T}{m_0} \le x\right) = G_{q-1}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{q-1+2j}(x) + \mathrm{o}(n^{-1}), \tag{3.14}$$

where G_f *is the distribution function of a central chi-squared distribution with* f *degrees of freedom and the coefficient* b_j*'s are given by* (3.13).

Especially, when X_{ij} is normal, we can write

$$\mathrm{P}(T \le x) = G_{q-1}(x) + \frac{1}{6n}(\|\boldsymbol{\rho}^{-1}\|^2 - 1)\{G_{q+1}(x) - G_{q-1}(x)\} + \mathrm{o}(n^{-1}).$$

This formula is same one as in Hartley (1940).

The asymptotic expansion (3.14) can be written as

$$\begin{aligned}\mathrm{P}\left(\frac{2T}{m_0} \le x\right) = G_{q-1}(x) - \frac{2x}{n(q-1)} g_{q-1}(x)\Bigg\{ & b_1 + b_2 + b_3 \\ & + \frac{(b_1+b_2)x}{q+1} + \frac{b_3 x^2}{(q+1)(q+3)}\Bigg\} + \mathrm{o}(n^{-1}),\end{aligned} \tag{3.15}$$

where $g_q(x)$ is the density function of a χ^2_q-variate with q degrees of freedom.

Let

$$\mathrm{P}(2T/m_0 \le t(u)) = P(\chi^2_{q-1} \le u).$$

Then, from (3.15) we can expand $t(u)$ as

$$\begin{aligned} t(u) &= u + \frac{2u}{n(q-1)}\Bigg\{b_1 + b_2 + b_3 \\ &\qquad + \frac{(b_1+b_2)u}{q+1} + \frac{b_3 u^2}{(q+1)(q+3)}\Bigg\} + \mathrm{o}(n^{-1}). \\ &= t_E(u) + \mathrm{o}(n^{-1}) \end{aligned} \tag{3.16}$$

4. Numerical accuracies

Numerical accuracies are studied for approximations of the percentage points and actual test sizes of T. The approximations considered are based on the limiting distribution and the asymptotic expansion (3.14). We consider the following five nonnormal models and the normal model with $q = 3$ and 5.

Table I. Cumulants of six mosels.

	κ_3	κ_4	κ_5	κ_6	κ_8
M1	0	1.5	0	15	315
M2	0	−1.2	0	6.86	−86.4
M3	0	3	0	30	630
M4	1.63	4	13.06	53.33	1493.3
M5	1	1.5	3	7.5	78.75
M6	0	0	0	0	0

M1. $X + YZ$, where X, Y, Z are independent normal distribution $\mathrm{N}(0, 1)$,
M2. symmetric uniform distribution $\mathrm{U}(-5, 5)$,
M3. double exponential distribution $\mathrm{DE}(0, 1)$,
M4. χ^2 distribution with 3 degrees of freedom,
M5. χ^2 distribution with 8 degrees of freedom,
M6. normal distribution.

The first three models are symmetric. In M4 and M5 we choose χ^2 distributions with different degrees of freedom which are asymmetric. The cumulants of each model are given in Table I, because we need the cumulants up to eighth for computing the coefficients b_j's given in (3.13).

Table II gives the upper 5% and 1% percentage points of the null distribution in the case $q = 3$. The first row $t(u)$ is the true percentage points which were obtained simulation experiments. The second row is the approximate percentage points $t_E(u)$ given in (3.16) based on the asymptotic expansion. Table III gives the results in the case of $q = 5$.

Table IV gives the actual test sizes for nominal test size 5% and 1% in the case $q = 3$. The first row α_0 is the actual test size based on limiting distribution under normality. The second row α_1 is one based on limiting distribution, χ^2_{q-1}, under nonnormality. The third row α_2 is one based on the asymptotic expansion. The α_j's are defined as follows,

$$\alpha_0 = \mathrm{P}(T \geq u), \qquad \alpha_1 = \mathrm{P}(2T/m_0 \geq u), \qquad \alpha_2 = \mathrm{P}(2T/m_0 \geq t_E(u)).$$

Note that $\alpha_0 = \alpha_1$ in M6. Table V gives the results in the case of $q = 5$.

5. Conclusion

From Table II to V, we can see that the approximation $t_E(u)$ improves the approximation based on the limiting distribution. The α_0 based on normal theory has bad behavior. In the case $\kappa_4 < 0$ (M2), this test becomes very conservative and in the case $\kappa_4 > 0$ (M1, M3, M4, M5), it becomes very liberal. On the other hand, the approximation, based on the asymptotic expansion,

Table II. The percenrage points in the case $q = 3$.

	Sample sizes n_1	n_2	n_3	Upper 5% points $\chi^2_{0.05} = 5.991$ $t(u)$	$t_E(u)$	Upper 1% points $\chi^2_{0.01} = 9.210$ $t(u)$	$t_E(u)$
	10	10	10	4.929	4.143	7.480	7.898
	20	20	20	5.257	5.067	8.076	8.554
M1	30	30	30	5.447	5.375	8.398	8.773
	15	20	25	5.200	5.025	7.996	8.632
	10	20	30	5.131	4.855	8.168	8.943
	10	10	10	8.501	7.787	14.39	12.66
	20	20	20	7.001	6.889	11.35	10.93
M2	30	30	30	6.610	6.590	10.43	10.36
	15	20	25	7.088	6.940	11.32	11.04
	10	20	30	7.499	7.142	12.59	11.48
	10	10	10	4.738	3.840	7.038	6.439
	20	20	20	5.252	4.916	7.917	7.824
M3	30	30	30	5.383	5.274	8.198	8.286
	15	20	25	5.175	4.862	7.762	7.828
	10	20	30	5.046	4.647	7.634	7.844
	10	10	10	4.205	3.377	6.248	7.943
	20	20	20	4.827	4.684	7.332	8.577
M4	30	30	30	5.153	5.120	7.878	8.788
	15	20	25	4.796	4.632	7.242	8.733
	10	20	30	4.701	4.424	7.299	9.361
	10	10	10	4.972	4.336	7.453	7.198
	20	20	20	5.374	5.164	8.343	8.204
M5	30	30	30	5.470	5.440	8.559	8.540
	15	20	25	5.321	5.122	8.187	8.215
	10	20	30	5.192	4.956	7.921	8.258
	10	10	10	6.260	6.258	9.606	9.620
	20	20	20	6.170	6.125	9.351	9.415
M6	30	30	30	6.178	6.080	9.456	9.347
	15	20	25	6.159	6.131	9.427	9.425
	10	20	30	6.094	6.158	9.565	9.466

shows good behaviors under the nonnormal distributions close to normal distribution. However, if the distribution is not close to normal distribution, it is not useful. The test statistic T is optimized against the normal distribution. Therefore, if the underlying distribution is far from the normal distribution, other test statistics should be considered to use.

Our aims are to find influence factors of nonnormality and to examine a behavior of the studentized statistic $T_1 = 2T/\hat{m}_0$. For them we consider the

Table III. The percenrage points in the case $q = 5$.

	Sample sizes					Upper 5% points $\chi^2_{0.05} = 9.488$		Upper 1% points $\chi^2_{0.01} = 13.28$	
	n_1	n_2	n_3	n_4	n_5	$t(u)$	$t_E(u)$	$t(u)$	$t_E(u)$
M1	10	10	10	10	10	7.928	7.581	11.15	13.47
	20	20	20	20	20	8.532	8.534	12.27	13.37
	30	30	30	30	30	8.788	8.852	12.57	13.34
	10	15	20	25	30	8.399	8.421	11.94	13.77
	10	10	10	10	30	8.124	7.909	11.43	14.17
M2	10	10	10	10	10	13.56	12.32	20.54	18.11
	20	20	20	20	20	11.17	10.90	16.33	15.69
	30	30	30	30	30	10.54	10.43	15.27	14.89
	10	15	20	25	30	11.52	11.16	16.98	16.18
	10	10	10	10	30	13.07	12.00	19.62	17.65
M3	10	10	10	10	10	7.629	6.840	10.49	11.02
	20	20	20	20	20	8.368	8.164	11.58	12.15
	30	30	30	30	30	8.672	8.605	12.15	12.53
	10	15	20	25	30	8.211	7.967	11.40	12.22
	10	10	10	10	30	7.778	7.221	10.88	11.80
M4	10	10	10	10	10	6.890	7.021	9.627	14.74
	20	20	20	20	20	7.827	8.254	11.09	14.01
	30	30	30	30	30	8.283	8.666	11.82	13.77
	10	15	20	25	30	7.770	8.142	10.92	14.75
	10	10	10	10	30	7.129	7.512	10.16	15.74
M5	10	10	10	10	10	7.992	7.489	11.18	11.78
	20	20	20	20	20	8.599	8.489	12.05	12.53
	30	30	30	30	30	8.812	8.822	12.51	12.78
	10	15	20	25	30	8.546	8.342	11.96	12.62
	10	10	10	10	30	8.246	7.780	11.42	12.38
M6	10	10	10	10	10	9.902	9.867	13.78	13.81
	20	20	20	20	20	9.649	9.678	13.53	13.54
	30	30	30	30	30	9.552	9.614	13.17	13.45
	10	15	20	25	30	9.695	9.709	13.52	13.59
	10	10	10	10	30	9.749	9.819	13.71	13.74

following two steps, because the test statistic is slightly different according to whether the assumption that population cumulants are known or not.

As the first step we consider to derive an asymptotic expansion assuming population cumulants are given. This paper is concerned with the first step, and the first aim is achieved by evaluating its coefficients of the asymptotic expansion.

Table IV. The actual test sizes in the case $q = 3$.

	Sample sizes			Nominal 5% test			Nominal 1% test		
	n_1	n_2	n_3	α_0	α_1	α_2	α_0	α_1	α_2
	10	10	10	12.6	2.70	8.20	3.99	0.33	0.75
	20	20	20	14.4	3.26	5.60	5.15	0.54	0.77
M1	30	30	30	15.4	3.72	5.24	5.62	0.67	0.84
	15	20	25	14.1	3.14	5.56	4.81	0.52	0.73
	10	20	30	13.7	3.13	5.86	4.94	0.55	0.63
	10	10	10	0.91	10.8	6.10	0.11	4.07	1.61
	20	20	20	0.30	7.51	5.24	0.02	2.13	1.17
M2	30	30	30	0.16	6.58	5.06	0.01	1.69	1.03
	15	20	25	0.24	7.71	5.28	0.02	2.19	1.11
	10	20	30	0.45	8.69	5.69	0.06	2.87	1.39
	10	10	10	23.0	2.08	9.14	9.86	0.20	1.54
	20	20	20	25.6	3.20	6.06	12.1	0.44	1.06
M3	30	30	30	26.9	3.51	5.31	13.1	0.57	0.96
	15	20	25	26.0	3.08	6.02	12.1	0.42	0.97
	10	20	30	25.0	2.81	6.37	11.8	0.41	0.89
	10	10	10	25.4	1.25	9.32	11.7	0.07	0.22
	20	20	20	29.7	2.35	5.46	15.2	0.30	0.45
M4	30	30	30	31.9	3.09	5.10	16.6	0.48	0.61
	15	20	25	29.7	2.28	5.59	14.9	0.30	0.40
	10	20	30	28.6	2.21	5.95	14.2	0.31	0.28
	10	10	10	13.1	2.63	7.39	4.21	0.33	1.19
	20	20	20	14.7	3.65	5.64	5.24	0.62	1.09
M5	30	30	30	15.3	3.76	5.09	5.75	0.68	1.01
	15	20	25	14.8	3.43	5.62	5.13	0.57	0.99
	10	20	30	14.4	3.10	5.71	4.94	0.48	0.80
	10	10	10	5.78	5.78	5.00	1.21	1.21	1.00
	20	20	20	5.46	5.46	5.11	1.08	1.08	0.98
M6	30	30	30	5.43	5.43	5.24	1.12	1.12	1.05
	15	20	25	5.44	5.44	5.06	1.12	1.12	1.00
	10	20	30	5.26	5.26	4.83	1.16	1.16	1.05

The next step is to derive an asymptotic expansion of T_1 assuming population cumulants are unknown. We have tried similar simulations by using only limiting approximation for the studentized statistic $T_1 = 2T/\hat{m}_0$. Its result shows better performances than that of $T_0 = 2T/m_0$. As a reason of this result, we think that the effect of nonnormality on the null distribution of T_1 becomes smaller than that of T_0. This relation will be shown by comparing coefficients of asymptotic expansions of T_0 and T_1. Therefore, we will derive the asymptotic expansion of T_1 to investigate this.

Table V. The actual test sizes in the case $q = 5$.

	Sample sizes					Nominal 5% test			Nominal 1% test		
	n_1	n_2	n_3	n_4	n_5	α_0	α_1	α_2	α_0	α_1	α_2
	10	10	10	10	10	16.7	2.33	5.94	6.09	0.35	0.31
	20	20	20	20	20	19.2	3.36	5.00	7.62	0.64	0.62
M1	30	30	30	30	30	20.7	3.72	4.86	8.57	0.73	0.71
	10	15	20	25	30	18.7	3.03	4.96	7.22	0.54	0.46
	10	10	10	10	30	17.3	2.65	5.54	6.37	0.42	0.29
	10	10	10	10	10	0.47	13.5	6.69	0.06	5.31	1.72
	20	20	20	20	20	0.09	8.45	5.42	0.01	2.57	1.24
M2	30	30	30	30	30	0.05	7.25	5.22	0.00	1.94	1.12
	10	15	20	25	30	0.16	9.17	5.55	0.02	2.99	1.26
	10	10	10	10	30	0.42	12.3	6.55	0.03	4.79	1.59
	10	10	10	10	10	32.9	1.77	7.52	16.5	0.20	0.75
	20	20	20	20	20	37.0	2.92	5.49	19.8	0.43	0.77
M3	30	30	30	30	30	38.9	3.42	5.15	21.5	0.56	0.81
	10	15	20	25	30	36.3	2.71	5.54	18.9	0.41	0.68
	10	10	10	10	30	33.7	2.11	6.65	17.0	0.28	0.62
	10	10	10	10	10	37.2	1.11	4.67	19.6	0.11	0.04
	20	20	20	20	20	43.1	2.17	4.07	25.2	0.33	0.24
M4	30	30	30	30	30	45.7	2.89	4.16	27.8	0.51	0.39
	10	15	20	25	30	42.4	2.17	4.10	24.6	0.33	0.17
	10	10	10	10	30	38.5	1.42	4.05	20.9	0.20	0.06
	10	10	10	10	10	17.2	2.33	6.41	6.21	0.32	0.73
	20	20	20	20	20	19.7	3.37	5.25	7.65	0.57	0.81
M5	30	30	30	30	30	21.1	3.77	4.99	8.65	0.74	0.92
	10	15	20	25	30	19.5	3.18	5.51	7.53	0.51	0.72
	10	10	10	10	30	18.0	2.63	6.16	6.68	0.38	0.61
	10	10	10	10	10	5.83	5.83	5.06	1.23	1.23	0.99
	20	20	20	20	20	5.33	5.33	4.93	1.10	1.10	0.99
M6	30	30	30	30	30	5.16	5.16	4.87	0.95	0.95	0.85
	10	15	20	25	30	5.44	5.44	4.98	1.13	1.13	0.96
	10	10	10	10	30	5.58	5.58	4.85	1.19	1.19	0.99

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