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Retractions of H-spaces

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Abstract. Stasheff showed that if a map between H-spaces is an H-map, then the suspension of the map is extendable to a map between projective planes of the H-spaces. Stahseff also proved the converse under the assumption that the multiplication of the target space of the map is homotopy associative. We show by giving an example that the assumption of homotopy associativity of the multiplication of the target space is necessary to show the converse. We also show an analogous fact for maps between A_n-spaces.

1. Introduction

Let X and Y be H-spaces, and $f : X \to Y$ a map. Stasheff [4] showed that if f is an H-map, then it's suspension $\Sigma f : \Sigma X \to \Sigma Y$ is extendable to a map $P_2 f : P_2 X \to P_2 Y$ between projective planes $P_2 X$ and $P_2 Y$ of X and Y, respectively. He also showed the converse under the assumption that the multiplication μ_Y of Y is homotopy associative. It has not been known whether the converse holds without the assumption of the homotopy associativity of μ_Y. In this paper we show by giving an example that the assumption of homotopy associativity of μ_Y is necessary to show the converse.

Our example is the retraction $r : J(X) \to X$ for an H-space X. Here, $J(X)$ is the reduced power space of X introduced by James [2], which has the homotopy type of $\Omega\Sigma X$. By definition $J(X)$ is an identification space of $\bigcup_{i \geq 1} X^i$. Then the map r is defined by

$$r([x_1, \dots, x_i]) = (\cdots((x_1 \cdot x_2) \cdot x_3) \cdots) \cdot x_i,$$

where $[x_1, \dots, x_i]$ is the class of $(x_1, \dots, x_i) \in X^i$ and $x \cdot y$ denotes the multiplication of x and y. Our result is stated as follows.

Theorem 1.1. *For any H-space X, there is an extension $P_2 r : P_2 J(X) \to P_2 X$ of $\Sigma r : \Sigma J(X) \to \Sigma X$.*

Stasheff showed the following

Theorem 1.2 ([4]). *The retraction r is an H-map if and only if the multiplication of X is homotopy associative.*

Thus in particular, if the multiplication of X is not homotopy associative, then r is not an H-map even though there exists a map between projective planes extending the suspension of r.

Now, the above result is a special case of the main theorem of this paper, which deals with the case that the H-space X is an A_n-space. An A_n-space is an H-space such that the multiplication satisfies higher homotopy associativity of order n. For example, an A_2-space is just an H-space, an A_3-space is a homotopy associative H-space, and an A_∞-space is a space with the homotopy type of a loop space.

Any A_n-space X has an associated space P_iX for each i with $1 \le i \le n$ which is called the projective i-space of X. By definition, P_1X is the suspension ΣX, P_2X is the projective plane, and $P_\infty X$ is the classifying space of X.

Maps preserving A_n-space structures are called A_n-maps. An A_2-map is an H-map, and an A_∞-map is a map homotopic to a loop map. See [1] for the definition. By definition, if $f : X \to Y$ is an A_n-map, then there are maps $P_if : P_iX \to P_iY$ $(1 \le i \le n)$ such that

$$P_1f = \Sigma f, \qquad P_{i+1}f \,|\, P_iX \simeq P_if \qquad (1 \le i \le n-1). \tag{1.1}$$

Then the problem becomes whether the converse of the above fact holds. To state our main theorem we call a map $f : X \to Y$ between A_n-spaces a *quasi* A_n*-map* if there are maps $P_if : P_iX \to P_iY$ for $(1 \le i \le n)$ with (1.1). Then we shall prove the following

Theorem 1.3. *Let X be an A_n-space for some $n \ge 2$. Then the retraction $r : J(X) \to X$ is a quasi A_n-map.*

We notice that the above theorem for $n = 2$ is just Theorem 1.1.

We can show a fact analogous to Theorem 1.2 for A_n-spaces. Thus the existence of an A_{n+1}-space structure for X is essential for the quasi A_n-map $r : J(X) \to X$ to be an A_n-map. We discuss it in §3.

2. Proof of the main theorem

First we recall some facts on the reduced product space given by James [2]. Let $f : Z \times J(X) \to Y$ be a map. Put $f_n = f \circ (\mathrm{id}_Z \times \nu_n) : Z \times X^n \to Y$ for $n \ge 1$, where $\nu_n : X^n \to J(X)$ $(n \ge 1)$ is the canonical map. Then we have

$$f_n \,|\, Z \times X^{i-1} \times * \times X^{n-i} = f_{n-1} \qquad \text{for } 1 \le i \le n,$$

where $X^{i-1} \times * \times X^{n-i}$ is identified with X^{n-1} by the obvious way.

On the other hand, if we have a sequence of maps $(f_n : Z \times X^n \to Y)_{n=1,2,\dots}$ with the above property, then there is a map $f : Z \times J(X) \to Y$ such that $f \circ (\mathrm{id}_Z \times \nu_n) = f_n$. Such a sequence $(f_n)_{n=1,2,\dots}$ is called a compatible sequence of invariant maps.

The space $J(X)$ has the homotopy type of $\Omega\Sigma X$. A homotopy equivalence $s : J(X) \to \Omega\Sigma X$ is defined by means of a compatible sequence of invariant maps $(s_n : X^n \to \Omega\Sigma X)_{n=1,2,\dots}$, where $s_1 : X \to \Omega\Sigma X$ is the adjoint of $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$, and s_n $(n \geq 2)$ is defined by using the loop multiplication of $\Omega\Sigma X$ as

$$s_n(x_1, \dots, x_n) = (\cdots(s_1(x_1) \cdot s_1(x_2)) \cdots) \cdot s_1(x_n).$$

Note that to make (s_n) a compatible sequence of invariant maps we need to modify the loop multiplication so that the constant loop is the strict unit of the loop multiplication.

Let $e : \Sigma\Omega\Sigma X \to \Sigma X$ be the evaluation map, that is, the adjoint of the $\mathrm{id}_{\Omega\Sigma X} : \Omega\Sigma X \to \Omega\Sigma X$. Then we prove the following

Lemma 2.1. *Let X be an H-space and $\varepsilon : \Sigma X \to P_2X$ the inclusion. Then $\varepsilon \circ \Sigma r \simeq \varepsilon \circ e \circ \Sigma s$.*

Proof. The projective plane P_2X is the mapping cone of the Dold-Lashoff construction $q : X \cup_\mu X \times CX \to \Sigma X$, where $\mu : X \times X \to X$ is the multiplication of X. Morisugi [3, (1.3)] showed that there exists a homotopy equivalence $X \cup_\mu X \times CX \to \Sigma(X \wedge X)$ such that if we identify $X \cup_\mu X \times CX$ with $\Sigma(X \wedge X)$ by this homotopy equivalence, then q is identified with a map $q' : \Sigma(X \wedge X) \to \Sigma X$ with

$$q' \circ \Sigma\pi \simeq \Sigma p_1 + \Sigma p_2 - \Sigma\mu : \Sigma(X \times X) \to \Sigma X,$$

where $\pi : X \times X \to X \wedge X$ is the quotient map and p_i is the projection to the i-th factor. Thus,

$$\varepsilon \circ \Sigma\mu \simeq \varepsilon \circ (\Sigma p_1 + \Sigma p_2).$$

Put $\mu_n = r \circ v_n : X^n \to X$. Then $\mu_2 = \mu$ and $\mu_n = \mu \circ (\mu_{n-1} \times \mathrm{id}_X)$. We show that there are homotopies $H_n : I \times \Sigma X^n \to P_2X$ $(n \geq 1)$ between $\varepsilon \circ \Sigma\mu_n$ and $\varepsilon \circ e \circ \Sigma s_n$ such that $H_1 = \varepsilon \circ p_2$ and

$$H_n \,|\, I \times \Sigma(X^{j-1} \times * \times X^{n-j}) = H_{n-1} \qquad \text{for any } 1 \leq j \leq n. \tag{2.1}$$

Then $(H_n)_{n=1,2,\dots}$ defines a homotopy between $\varepsilon \circ \Sigma r$ and $\varepsilon \circ e \circ \Sigma s$.

Now $e \circ \Sigma s_2 = \Sigma p_1 + \Sigma p_2$ since the adjoint of the both maps are the same s_2. Thus,

$$\varepsilon \circ \Sigma\mu_2 \simeq \varepsilon \circ (\Sigma p_1 + \Sigma p_2) = \varepsilon \circ e \circ \Sigma s_2.$$

We notice that the above homotopy $H_2 : I \times \Sigma X^2 \to P_2X$ can be chosen to be constant on $I \times \Sigma(X \vee X)$.

Let $n > 2$. Suppose inductively that we have H_i for $i < n$ with the desired properties. Then H_n is defined as the composition of homotopies as follows.

$$\begin{aligned}\varepsilon\circ\Sigma\mu_n &= \varepsilon\circ\Sigma\mu\circ\Sigma(\mu_{n-1}\times\mathrm{id}_X)\\ &\simeq \varepsilon\circ(\Sigma p_1+\Sigma p_2)\circ\Sigma(\mu_{n-1}\times\mathrm{id}_X)\\ &= \varepsilon\circ\Sigma\mu_{n-1}\circ\Sigma p'+\varepsilon\circ e\circ\Sigma s_1\circ\Sigma p_n\\ &\simeq \varepsilon\circ e\circ\Sigma s_{n-1}\circ\Sigma p'+\varepsilon\circ e\circ\Sigma s_1\circ\Sigma p_n\\ &= \varepsilon\circ e\circ\Sigma s_n,\end{aligned}$$

where $p' : X^n \to X^{n-1}$ is the projection to the first $(n-1)$-factors, and the second homotopy is given by using H_{n-1}. It is clear that we can modify H_n to satisfy (2.1). Thus we have H_n for all n by induction. □

Now we prove Theorem 1.3. Theorem 1.1 is a special case of Theorem 1.3.

Proof of Theorem 1.3. Since $J(X)$ is a topological monoid, we have the projective ∞-space $P_\infty J(X)$. It is known that $P_\infty J(X)$ has the homotopy type of ΣX such that the inclusion $\Sigma J(X) \to P_\infty J(X)$ followed by the homotopy equivalence $P_\infty J(X) \simeq \Sigma X$ is homotopic to $e\circ\Sigma s$ (cf. [5, Proof of Theorem 4.8]).

Define $P_i r : P_i J(X) \to P_i(X)$ for $2 \le i \le n$ by the following composition

$$P_i J(X) \subset P_\infty J(X) \simeq \Sigma X \xrightarrow{\varepsilon} P_2 X \subset P_i X.$$

Then by Lemma 2.1 we have the result. □

3. A_n-form of the retraction

In this section we show the following theorem which is analogous to Theorem 1.2.

Theorem 3.1. *Let X be an A_n-space for some $n \ge 2$. Then the retraction $r : J(X) \to X$ is an A_{n-1}-map. Moreover, if r is an A_n-map then the A_n-space structure of X is extendable to an A_{n+1}-space structure.*

Proof. The idea of the proof is not so hard to understand. But, writing down the explicit proof is very complicated.

Let $\{\mu_i : K_i \times X^i \to X\}_{2\le i\le n}$ be the A_n-form on X, where K_i is an $i-2$ dimensional CW ball called the associahedron. The second part of the theorem is a corollary to Iwase-Mimura [1, p. 196, P10)]. They claim that if $f : X \to Y$ and $g : Y \to X$ are maps between A_n-spaces such that $g\circ f \simeq \mathrm{id}_X$, and if one of f and g is an A_n-map, then the A_n-space structure of X is extendable to an A_{n+1}-space structure. In fact, in our case the extended A_{n+1}-form on X is given as follows.

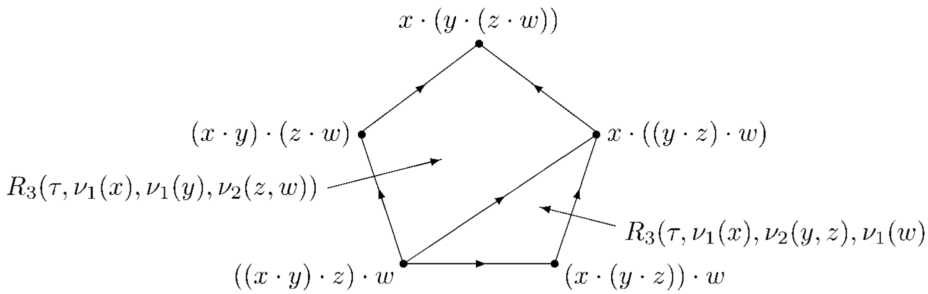

Fig. 1. $\mu_4(\tau, x, y, z, w)$

Let $\{R_i : J_i \times J(X)^i \to X\}_{i \le n}$ be the A_n-form on r, where J_i is an $i-1$ dimensional CW ball called the multiplihedron. We consider $n-1$ higher homotopies

$$R_n \circ (1 \times v_1^s \times v_2 \times v_1^{n-s-1}) : J_n \times X^{n+1} \to X \qquad (1 \le s \le n-1).$$

Then by combining these higher homotopies, we can construct a map $\mu_{n+1} : K_{n+1} \times X^{n+1} \to X$ which extend $\{\mu_i\}_{i \le n}$ to an A_{n+1}-form on X. For example, the associating homotopy $\mu_3 : K_3 \times X^3 \to X$ is given as $\mu_3(t, x, y, z) = R_2(t, [x], [y, z])$ $(t \in J_2 = K_3, x, y, z \in X)$, and the homotopy $\mu_4 : K_4 \times X^4 \to X$ is illustrated in Figure 1.

Next we consider the first part of Theorem 3.1. An A_{n-1}-form $\{R_i : J_i \times J(X)^i \to X\}_{i \le n-1}$ is defined by means of compatible sequences of invariant maps $(R_{i,j} : J_i \times J(X)^{i-1} \times X^j \to X)_{j=1,2,\dots}$.

First we define $R_{2,1}$ as the constant homotopy. For $j \ge 2$, $R_{2,j}$ is given as the composition of $\mu_2 \circ (R_{2,j-1} \times \mathrm{id}_X)$ and $\mu_3 \circ (1 \times r \times r \circ v_{j-1} \times \mathrm{id}_X)$.

For $i \ge 3$ the explicit definition for $R_{i,j}$ is very complicated. Unlike with the case of $i = 2$, the homotopy $R_{i,1}$ for $i \ge 3$ is not a constant homotopy. For example, $R_{3,1} : J_3 \times J(X)^2 \times X \to X$ should be a map illustrated in Figure 2, where the double lines mean constant homotopies. By definition, the homotopy $R_2(t, \boldsymbol{x}, \boldsymbol{y} \cdot v_1(z))$ is given as the composition of two homotopies $R_2(t, \boldsymbol{x}, \boldsymbol{y}) \cdot z$ and $\mu_3(t, r(\boldsymbol{x}), r(\boldsymbol{y}), z)$, which means that the homotopy represented by the upper left edge equals to the one represented by the composition of the lower right and the bottom edges. Thus $R_{3,1}$ can be defined by using a suitable degeneracy map $\delta_3 : J_3 \to J_2$ as $R_{3,1}(\tau, \boldsymbol{x}, \boldsymbol{y}, z) = R_2(\delta(\tau), \boldsymbol{x}, \boldsymbol{y} \cdot v_1(z))$.

For $i \ge 4$ the definition of $R_{i,1}$ is similar. It is defined by using a suitable degeneracy map $\delta_i : J_i \to J_{i-1}$ as $R_{i,1}(\tau, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1}, z) = R_{i-1}(\delta_i(\tau), \boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1} \cdot v_1(z))$.

To define $R_{i,j}$ for $j > 1$ we decompose J_i into small polytopes homeomorphic to $K_k \times J_t$ with $k + t = i + 2$. Then we define $R_{i,j}$ by combining

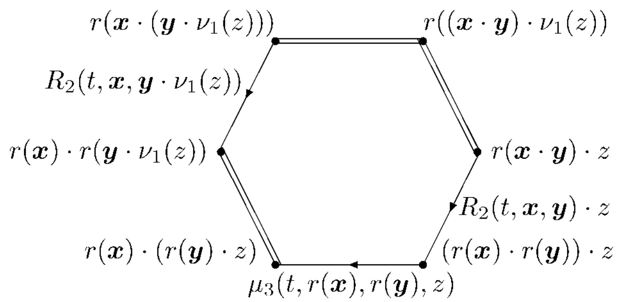

Fig. 2. $R_{3,1}(\tau,\boldsymbol{x},\boldsymbol{y},z)$

higher homotopies $h_{k,s}: K_k\times J_{i+2-k}\times J(X)^{i-1}\times X^j\to X$ $(s+3\le k\le i)$ and $h'_k: K_k\times J_{i+2-k}\times J(X)^{i-1}\times X^j\to X$ $(k\le i+1)$ defined as follows:

$$h_{k,s}(\tau,\rho,\boldsymbol{x}_1,\dots,\boldsymbol{x}_{i-1},(y_1,\dots,y_j))$$
$$=\mu_k(\tau,\boldsymbol{x}_1,\dots,R_{i+2-k}(\rho,\boldsymbol{x}_{s+1},\dots,\boldsymbol{x}_{s+i+2-k}),\dots,\boldsymbol{x}_{i-1},r(v_{j-1}(y_1,\dots,y_{j-1})),y_j)$$
$$h'_k(\tau,\rho,\boldsymbol{x}_1,\dots,\boldsymbol{x}_{i-1},(y_1,\dots,y_j))$$
$$=\mu_k(\tau,\boldsymbol{x}_1,\dots,R_{i+2-k,j-1}(\rho,\boldsymbol{x}_{k-1},\dots,\boldsymbol{x}_{i-1},(y_1,\dots,y_{j-1})),y_j)$$

where we put $R_{1,j-1}(*,y_1,\dots,y_{j-1})=r(v_{j-1}(y_1,\dots,y_{j-1}))$.

$R_{3,j}$ is illustrated in Figure 3. Here the points (a)–(k) and the homotopies (A)–(D) are as follows, where $\boldsymbol{z}=v_j(z_1,\dots,z_j)$ and $\boldsymbol{z}'=v_{j-1}(z_1,\dots,z_{j-1})$:

(a): $r(\boldsymbol{x}\cdot(\boldsymbol{y}\cdot\boldsymbol{z}))=r(\boldsymbol{x}\cdot(\boldsymbol{y}\cdot\boldsymbol{z}'))\cdot z_j$
(b): $r((\boldsymbol{x}\cdot\boldsymbol{y})\cdot\boldsymbol{z})=r((\boldsymbol{x}\cdot\boldsymbol{y})\cdot\boldsymbol{z}')\cdot z_j$
(c): $(r(\boldsymbol{x})\cdot r(\boldsymbol{y}\cdot\boldsymbol{z}'))\cdot z_j$
(d): $(r(\boldsymbol{x}\cdot\boldsymbol{y})\cdot r(\boldsymbol{z}'))\cdot z_j$
(e): $r(\boldsymbol{x})\cdot(r(\boldsymbol{y}\cdot\boldsymbol{z}')\cdot z_j)=r(\boldsymbol{x})\cdot r(\boldsymbol{y}\cdot\boldsymbol{z})$
(f): $(r(\boldsymbol{x})\cdot(r(\boldsymbol{y})\cdot r(\boldsymbol{z}')))\cdot z_j$
(g): $((r(\boldsymbol{x})\cdot r(\boldsymbol{y}))\cdot r(\boldsymbol{z}'))\cdot z_j$
(h): $r(\boldsymbol{x}\cdot\boldsymbol{y})\cdot(r(\boldsymbol{z}')\cdot z_j)=r(\boldsymbol{x}\cdot\boldsymbol{y})\cdot r(\boldsymbol{z})$
(i): $r(\boldsymbol{x})\cdot((r(\boldsymbol{y})\cdot r(\boldsymbol{z}'))\cdot z_j)$
(j): $r(\boldsymbol{x})\cdot(r(\boldsymbol{y})\cdot(r(\boldsymbol{z}')\cdot z_j))=r(\boldsymbol{x})\cdot(r(\boldsymbol{y})\cdot r(\boldsymbol{z}))$
(k): $(r(\boldsymbol{x})\cdot r(\boldsymbol{y}))\cdot(r(\boldsymbol{z}')\cdot z_j)=(r(\boldsymbol{x})\cdot r(\boldsymbol{y}))\cdot r(\boldsymbol{z})$
(A): $R_{3,j-1}(\tau,\boldsymbol{x},\boldsymbol{y},(z_1,\dots,z_{j-1}))\cdot z_j$
(B): $\mu_3(t,r(\boldsymbol{x}),R_{2,j-1}(s,\boldsymbol{y},(z_1,\dots,z_{j-1})),z_j)$
(C): $\mu_3(t,R_{2,j-1}(s,\boldsymbol{x},\boldsymbol{y}),r(\boldsymbol{z}'),z_j)$
(D): $\mu_4(\tau,r(\boldsymbol{x}),r(\boldsymbol{y}),r(\boldsymbol{z}'),z_j)$ □

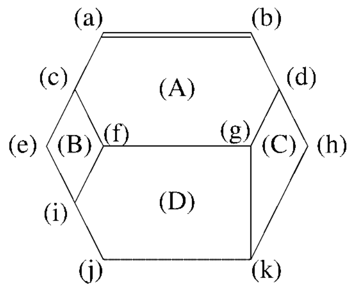

Fig. 3. $R_{3,j}(\tau, \boldsymbol{x}, \boldsymbol{y}, (z_1, \ldots, z_j))$

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