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Bessel-type functions of matrix variables

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Abstract. In the present work we compute explicitly a certain type of hypergeometric functions of matrix variables given as an integral of a Gaussian-type kernel. In the case of one variable, these functions are related to the modified Bessel function of the third kind.

1. Introduction

This paper deals with explicit computations of certain type of hypergeometric functions related to the linear groups $U(p,q)$ and $Sp(2n,\mathbf{R})$. In doing this, some integral formulas over the group of unitary matrices are given. To be more precise, let us take the case of $U(p,q)$.

For $p,q \in \mathbf{N}$ and $n = p+q$, let $I_{p,q} := \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ be the diagonal matrix with p copies of $(+1)$ and q copies of (-1) along the diagonal. Define $U(p,q)$ as the set of invertible matrices $g \in M(n,\mathbf{C})$ such that $gI_{p,q}g^* = I_{p,q}$, where $g^* := \bar{g}^t$.

For diagonal matrices $\boldsymbol{\alpha} := \operatorname{diag}(\alpha_1,\dots,\alpha_p)$ and $\boldsymbol{\beta} := \operatorname{diag}(\beta_1,\dots,\beta_q)$, such that $\alpha_i + \beta_j \neq 0$, we define

$$\zeta_{p,q}(\boldsymbol{\alpha},\boldsymbol{\beta}) := \int_{U(p,q)} e^{-\operatorname{tr}\left\{\left[\begin{smallmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\beta}\end{smallmatrix}\right](gg^*)^{-1}\right\}}\,dg.$$

Here "tr" means the usual trace of a matrix. If $p=q=1$, we can easily show that

$$\zeta_{1,1}(\alpha,\beta) = c_0(\alpha+\beta)^{-1/2}K_{1/2}(\alpha+\beta),$$

where $K_\nu(z)$ is the modified Bessel function of the third kind

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\,\frac{e^{-z/2}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+\frac{t}{z}\right)^{\nu-1/2}dt$$

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for $\operatorname{Re}(\nu + \frac{1}{2}) > 0$ and $|\arg z| < \pi$. As we can see, the function $\zeta_{p,q}$ is a multivariate analogue of the modified Bessel function K_ν. To compute $\zeta_{p,q}$, the main idea is to write $\zeta_{p,q}$ as an integral over the unit ball $\mathfrak{D}_{p,q} := \{z \in M(p,q;\mathbf{C}) \mid \det(I_p - zz^*) > 0\}$, and to use the polar decomposition of $\mathfrak{D}_{p,q}$. In doing this, we also obtain the explicit formula of

$${}_0F_0(S,T) := \int_{U(m)} e^{\operatorname{tr}(uSu^*T)}\,du$$

for $S = \operatorname{diag}(s_1,\ldots,s_{m_1},0,\ldots,0)$ and $T = \operatorname{diag}(t_1,\ldots,t_{m_2},0,\ldots,0)$. Here $U(m)$ denotes the set of unitary matrices $u \in M(m,\mathbf{C})$. It turns out that ${}_0F_0(S,T)$ was introduced by A. T. James in [James, 1964] as a generalization of the usual hypergeometric function ${}_0F_0(S) = e^{\operatorname{tr}(S)}$.

The Bessel-type functions under investigation play a crucial role in the theory of random matrices, mainly when one needs to derive explicit formulas for the correlation functions of the random variables (see for instance [Brézin-Hikami, 2001, Brézin-Hikami, 2003]).

The following notations will be used through out the paper. For a matrix x we write $x^* = \bar{x}^t$ where x^t is the transpose of x. If $x_1, x_2, \ldots, x_r$ are complex numbers, $\operatorname{diag}\underbrace{(x_1,x_2,\ldots,x_r)}_{r\times r}$ denotes the diagonal matrix of size $r\times r$. If x and y are two square matrices of size $r\times r$ and $s\times s$, respectively, $\exp[\operatorname{tr}(x+y)]$ stands for $\exp[\operatorname{tr}(x)]\exp[\operatorname{tr}(y)]$ where "exp" is the exponential function. For $r,s\in\mathbf{N}$, the element $I_{r,s}$ is the diagonal matrix $\operatorname{diag}[I_r;-I_s]$, where I_N is the $N\times N$ identity matrix. For $r\in\mathbf{N}$, S_r denotes the group of permutations.

2. The $U(p,q)$-case

Let $p,q\in\mathbf{N}$, and assume that $q\geq p$. We define

$$U(p,q) = \{g\in GL(n,\mathbf{C}) \mid gI_{p,q}g^* = I_{p,q}\} \qquad (n = p+q),$$

where $GL(n,\mathbf{C})$ denotes the set of $n\times n$-invertible matrices. For $g = \begin{bmatrix} A & B \\ C & D\end{bmatrix} \in U(p,q)$, the defining condition of $U(p,q)$ implies the following relations

$$\begin{aligned}
&\text{(a)}\quad AA^* - BB^* = I_p && \text{(e)}\quad C = DB^*A^{*-1},\\
&\text{(b)}\quad CC^* - DD^* = -I_q && \text{(f)}\quad B = AC^*D^{*-1},\\
&\text{(c)}\quad A^*A - C^*C = I_p && \text{(g)}\quad C = D^{*-1}B^*A,\\
&\text{(d)}\quad B^*B - D^*D = -I_q && \text{(h)}\quad B = A^{*-1}C^*D.
\end{aligned}$$

For all $\boldsymbol{\alpha} = \operatorname{diag}(\alpha_1, \dots, \alpha_p)$ with $\alpha_i > 0$, and $\boldsymbol{\beta} = \operatorname{diag}(\beta_1, \dots, \beta_q)$ with $\beta_i > 0$, let

$$\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{U(p,q)} \exp[-\operatorname{tr}(\operatorname{diag}[\boldsymbol{\alpha}, \boldsymbol{\beta}](gg^*)^{-1})] dg.$$

For $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$,

$$\operatorname{diag}[\boldsymbol{\alpha}, \boldsymbol{\beta}](gg^*)^{-1} = \begin{bmatrix} \boldsymbol{\alpha}(AA^* + BB^*) & \boldsymbol{\alpha}(-AC^* - BD^*) \\ \boldsymbol{\beta}(-CA^* - DB^*) & \boldsymbol{\beta}(CC^* + DD^*) \end{bmatrix}.$$

Therefore, by the relations (a) and (b) we have

$$\begin{aligned} \operatorname{tr}(\operatorname{diag}[\boldsymbol{\alpha}, \boldsymbol{\beta}](gg^*)^{-1}) &= \operatorname{tr}(\boldsymbol{\alpha}(AA^* + BB^*) + \boldsymbol{\beta}(CC^* + DD^*)) \\ &= \operatorname{tr}(\boldsymbol{\alpha}(2AA^* - I_p) + \boldsymbol{\beta}(2DD^* - I_q)). \end{aligned}$$

Let $\mathfrak{D}_{p,q}$ be the domain defined by

$$\mathfrak{D}_{p,q} = \{T \in M(p,q,\mathbf{C}) \mid \det(I_p - TT^*) > 0\}.$$

The measure $d\mu(T) = \det(I_p - TT^*)^{-p-q} dT$ is the $U(p,q)$-invariant measure on $\mathfrak{D}_{p,q}$ where dT is the Lebesgue measure on $\mathfrak{D}_{p,q}$.

The map $U(p,q) \to \mathfrak{D}_{p,q}$ defined by

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto T = BD^{-1}$$

is an isomorphism. Using the relations (a), ..., (g) and (h), we can write $AA^* = (I_p - TT^*)^{-1}$ and $DD^* = (I_q - T^*T)^{-1}$.

Next, we write $U(N) \equiv U(N,0)$. It is well known that for all functions F defined on $U(p,q)$, such that $F(gk) = F(g)$ for all $k \in \begin{bmatrix} U(p) & \mathbf{0} \\ \mathbf{0} & U(q) \end{bmatrix}$, there exists a function $F^{\#} : \mathfrak{D}_{p,q} \to \mathbf{C}$ defined by $F^{\#}(T) = F(g)$ such that

$$\int_{U(p,q)} F(g) dg = \int_{\mathfrak{D}_{p,q}} F^{\#}(T) d\mu(T).$$

Therefore, if $F(g) = \exp[-\operatorname{tr}(\operatorname{diag}[\boldsymbol{\alpha}, \boldsymbol{\beta}](gg^*)^{-1})]$, there exists a complex valued function $F^{\#}$ on $\mathfrak{D}_{p,q}$ such that

$$\begin{aligned} F^{\#}(T) &= \exp[-\operatorname{tr}(\boldsymbol{\alpha}[2(I_p - TT^*)^{-1} - I_p] + \boldsymbol{\beta}[2(I_q - T^*T)^{-1} - I_q])] \\ &= \exp[-\operatorname{tr}(\boldsymbol{\alpha}(I_p - TT^*)^{-1}(I_p + TT^*) + \boldsymbol{\beta}(I_q - T^*T)^{-1}(I_q + T^*T))] \\ &= \exp[-\operatorname{tr}(\boldsymbol{\alpha} + \boldsymbol{\beta} + 2\boldsymbol{\alpha}(I_p - TT^*)^{-1}TT^* + 2\boldsymbol{\beta}(I_q - T^*T)^{-1}T^*T)]. \end{aligned}$$

By [Hua, 1963], for $T \in \mathfrak{D}_{p,q}$, there exists $u \in U(p)$ and $v \in U(q)$ such that $T = u\Lambda v$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & \lambda_p & 0 & \cdots & 0 \end{bmatrix} \in M(p,q;\mathbf{R})$$

and $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$. Hence

$$TT^* = u \operatorname{diag} \underbrace{[\lambda_1^2, \ldots, \lambda_p^2]}_{p \times p} u^*, \qquad T^*T = v^* \operatorname{diag} \underbrace{[\lambda_1^2, \ldots, \lambda_p^2, 0, \ldots, 0]}_{q \times q} v.$$

Therefore $F^{\#}$ can be written in terms of u, v and Λ as

$$F^{\#}(T) = \exp[-\operatorname{tr}(\boldsymbol{\alpha} + \boldsymbol{\beta})] \exp(-2 \operatorname{tr}(u^{-1}\boldsymbol{\alpha} u(I_p - \Lambda\Lambda^*)^{-1}\Lambda\Lambda^*))$$
$$\times \exp(-2 \operatorname{tr}(v^{-1}\boldsymbol{\beta} v(I_q - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda)).$$

Consider the map $\psi : \mathfrak{D}_{p,q} \to \Upsilon$ taking each $T \in \mathfrak{D}_{p,q}$ to the collection of the eigenvalues of $\sqrt{TT^*}$. The image of the Lebesgue measure dT on $\mathfrak{D}_{p,q}$ with respect to the map ψ is the measure on Υ given by

$$c \prod_{1 \leq i < j \leq p} (\lambda_i^2 - \lambda_j^2)^2 \prod_{i=1}^{p} \lambda_i^{2(q-p)+1} \, d\lambda_i,$$

for some constant c. Thus, the image of the measure $d\mu(T) = \det(I_p - TT^*)^{-p-q} dT$ is

$$c \prod_{1 \leq i < j \leq p} (\lambda_i^2 - \lambda_j^2)^2 \prod_{i=1}^{p} \lambda_i^{2(q-p)+1} (1 - \lambda_i^2)^{-p-q} \, d\lambda_i. \tag{2.1}$$

Hence, the function $\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is given by

$$\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = c \exp[-\operatorname{tr}(\boldsymbol{\alpha} + \boldsymbol{\beta})] \int_{U(p)} \int_{U(q)} \int_{\Upsilon} \exp(-2 \operatorname{tr}(u^{-1}\boldsymbol{\alpha} u(I_p - \Lambda\Lambda^*)^{-1}\Lambda\Lambda^*))$$
$$\times \exp(-2 \operatorname{tr}(v^{-1}\boldsymbol{\beta} v(I_q - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda))$$
$$\times \prod_{1 \leq i < j \leq p} (\lambda_i^2 - \lambda_j^2)^2 \prod_{i=1}^{p} \lambda_i^{2(q-p)+1} (1 - \lambda_i^2)^{-p-q} \prod_{i=1}^{p} d\lambda_i du dv.$$

Let

$$A := 2(I_p - \Lambda\Lambda^*)^{-1}\Lambda\Lambda^* = \operatorname{diag} \underbrace{\left[\frac{2\lambda_1^2}{1 - \lambda_1^2}, \ldots, \frac{2\lambda_p^2}{1 - \lambda_p^2}\right]}_{p \times p}, \tag{2.2}$$

and

$$B := 2(I_q - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda = \operatorname{diag}\underbrace{\left[\frac{2\lambda_1^2}{1-\lambda_1^2}, \dots, \frac{2\lambda_p^2}{1-\lambda_p^2}, 0, \dots, 0\right]}_{q\times q}. \tag{2.3}$$

It will be convenient for us to define new coordinates $x_i = \dfrac{2\lambda_i^2}{(1-\lambda_i^2)}$. Then the set $\Upsilon = \{\Lambda \mid 1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0\}$ becomes the set

$$\mathfrak{X} := \{\operatorname{diag}(x_1, x_2, \dots, x_p) \mid x_1 \geq x_2 \geq \cdots \geq x_p \geq 0\}.$$

The measure (2.1) in the coordinates x_i has the form

$$c \prod_{1\leq i<j\leq j} (x_i - x_j)^2 \prod_{i=1}^{p} x_i^{q-p}\, dx_i,$$

and the function $\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be written as

$$\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = c \exp[-\operatorname{tr}(\boldsymbol{\alpha}+\boldsymbol{\beta})] \int_{U(p)} \int_{U(q)} \int_{\mathfrak{X}} \exp(-\operatorname{tr}(u^{-1}\boldsymbol{\alpha} uA)) \exp(-\operatorname{tr}(v^{-1}\boldsymbol{\beta} vB))$$

$$\times \prod_{1\leq i<j\leq p} (x_i - x_j)^2 \prod_{i=1}^{p} x_i^{q-p}\, dx_i du dv,$$

where A and B are given by (2.2) and (2.3).

Now we turn our attention to the integral formula over $U(p)$ and $U(q)$. For this we need to introduce some terminology.

For a multi-parameter $t = (t_1, t_2, \dots, t_N)$, the Vandermonde polynomial is defined by $D(t) = \prod_{1\leq i<j\leq N} (t_i - t_j)$. Let $\ell = (\ell_1, \dots, \ell_N) \in \mathbf{N}^N$. The Schur polynomial $S_\ell(t_1, \dots, t_N)$ is defined by

$$S_\ell(t_1, \dots, t_N) = \frac{\det(t_i^{\ell_j+N-j})_{1\leq i,j\leq N}}{D(t)}.$$

For more details on Schur polynomials, we refer to [Macdonald, 1979, Chapter I]. We also need the following lemma.

Lemma 2.1 (cf. [Hua, 1963], Theorem 1.2.1). *Let the power series*

$$f_i(y) = \sum_{\kappa=0}^{\infty} a_\kappa^{(i)} y^\kappa.$$

Then for all $(y_1, y_2, \dots, y_N)$ *the following identity holds*

$$\det(f_i(y_j))_{1\le i,j\le N} = \sum_{\ell_1>\ell_2>\cdots>\ell_N\ge 0} \det(a^{(i)}_{\ell_j})_{1\le i,j\le N} \det(y_i^{\ell_j})_{1\le i,j\le N},$$

where $\ell_1, \ell_2, \dots, \ell_N$ *are integers.*

Now we are in position to compute the integral over the compact groups $U(p)$ and $U(q)$.

Proposition 2.2. (i) *For* $A = \mathrm{diag}(x_1, \dots, x_p)$, *and for* $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1, \dots, \alpha_p)$, *we have*

$$\int_{U(p)} \exp[-\mathrm{tr}(u^{-1}\boldsymbol{\alpha}uA)]du = (-1)^{p(p-1)/2}\prod_{i=1}^{p}(i-1)!\frac{\det(e^{-x_i\alpha_j})_{1\le i,j\le p}}{\prod_{1\le i<j\le p}(x_i-x_j)(\alpha_i-\alpha_j)}.$$

(ii) *For* $B = \mathrm{diag}\underbrace{(x_1, x_2, \dots, x_p; 0, \dots, 0)}_{q\times q}$, *and for* $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_q)$, *we have*

$$\int_{U(q)} \exp[-\mathrm{tr}(v^{-1}\boldsymbol{\beta}vB)]dv$$

$$= \frac{(-1)^{q(q-1)/2}\prod_{i=1}^{q}(i-1)!}{\prod_{1\le i<j\le p}(x_i-x_j)\prod_{1\le i<j\le q}(\beta_i-\beta_j)}$$

$$\times \sum_{\ell_1>\ell_2>\cdots>\ell_p\ge 0}\frac{\det(x_i^{\ell_j})_{1\le i,j\le p}}{\prod_{j=1}^{p}(\ell_j+q-p)!}\begin{vmatrix} (-\beta_1)^{\ell_1+q-p} & \cdots & (-\beta_q)^{\ell_1+q-p} \\ \vdots & \cdots & \vdots \\ (-\beta_1)^{\ell_p+q-p} & \cdots & (-\beta_q)^{\ell_p+q-p} \\ (-\beta_1)^{q-p-1} & \cdots & (-\beta_q)^{q-p-1} \\ \vdots & \cdots & \vdots \\ (-\beta_1) & \cdots & (-\beta_q) \\ 1 & \cdots & 1 \end{vmatrix}$$

where $\ell_1, \dots, \ell_p$ *are integers.*

Proof. (i) First, let us write the Taylor series of $\exp[-\mathrm{tr}(u^{-1}\boldsymbol{\alpha}uA)]$ as a series of Schur polynomials S_ℓ, in the form

$$\exp[-\mathrm{tr}(u^{-1}\boldsymbol{\alpha}uA)] = \sum_{\ell_1\ge\cdots\ge\ell_p\ge 0} d_\ell\frac{\delta!}{(\ell+\delta)!}S_\ell(\wp(-u^{-1}\boldsymbol{\alpha}uA)),$$

where $\ell = (\ell_1, \ldots, \ell_p)$, $\delta = (p-1, p-2, \ldots, 0)$, $d_\ell = \dfrac{D(\ell+\delta)}{D(\delta)}$, and $\wp(g)$ stands for the collection $(z_1, \ldots, z_p)$ of the eigenvalues of g. Therefore,

$$\begin{aligned} I(\boldsymbol{a}, A) &\equiv \int_{U(p)} \exp[-\mathrm{tr}(u^{-1}\boldsymbol{a}uA)]du \\ &= \sum_{\ell_1 \geq \cdots \geq \ell_p \geq 0} d_\ell \frac{\delta!}{(\ell+\delta)!} \int_{U(p)} S_\ell(\wp(-u^{-1}\boldsymbol{a}uA))du \\ &= \sum_{\ell_1 \geq \cdots \geq \ell_p \geq 0} \frac{\delta!}{(\ell+\delta)!} S_\ell(\boldsymbol{a}) S_\ell(-A). \end{aligned}$$

To obtain the latter equality, we used the following well known functional equation

$$\int_{U(p)} \chi_\ell(xuyu^{-1})du = \frac{1}{d_\ell}\chi_\ell(x)\chi_\ell(y)$$

where χ_ℓ is the central function on $U(p)$ whose restriction to the set of diagonal matrices in $U(p)$ is equal to S_ℓ (see for instance [Macdonald, 1979, Chapter I]).

Using the determinant formula of S_ℓ, we deduce

$$\begin{aligned} I(\boldsymbol{a}, A) &= \frac{\delta!}{D(\boldsymbol{a})D(-A)} \sum_{\ell_1 \geq \cdots \geq \ell_p \geq 0} \frac{\det(\alpha_i^{\ell_j+p-j})_{1\leq i,j\leq p} \det(-x_i^{\ell_j+p-j})_{1\leq i,j\leq p}}{(\ell_1+p-1)!(\ell_2+p-2)!\ldots\ell_p!} \\ &= \frac{\delta!}{D(\boldsymbol{a})D(-A)} \sum_{\ell_1 > \cdots > \ell_p \geq 0} \frac{\det(\alpha_i^{\ell_j})_{1\leq i,j\leq p} \det(-x_i^{\ell_j})_{1\leq i,j\leq p}}{\ell_1!\ell_2!\ldots\ell_p!}. \end{aligned}$$

Let

$$f_i(\alpha) = e^{-x_i\alpha} = \sum_{\kappa=0}^{\infty} \frac{(-x_i)^\kappa}{\kappa!}\alpha^\kappa.$$

Using Lemma 2.1 where $a_\kappa^{(i)} = \dfrac{(-x_i)^\kappa}{\kappa!}$, we obtain

$$\begin{aligned} \det(e^{-x_i\alpha_j})_{1\leq i,j\leq p} &= \det(f_i(\alpha_j))_{1\leq i,j\leq p} \\ &= \sum_{\ell_1 > \cdots > \ell_p \geq 0} \det\left(\frac{(-x_i)^{\ell_j}}{\ell_j!}\right)_{1\leq i,j\leq p} \det(\alpha_i^{\ell_j})_{1\leq i,j\leq p}. \end{aligned}$$

Therefore, statement (i) holds.

(ii) Let $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \dots, \beta_q)$ and let $X = \mathrm{diag}\underbrace{(x_1, \dots, x_p; x_{p+1}, \dots, x_q)}_{q \times q}$. Using statement (i), we have

$$\int_{U(q)} \exp[-\mathrm{tr}(v^{-1}\boldsymbol{\beta} v X)]dv = c_q \frac{\det(e^{-x_i\beta_j})_{1\le i,j\le q}}{\prod_{1\le i<j\le q}(x_i - x_j)\prod_{1\le i<j\le q}(\beta_i - \beta_j)},$$

where $c_q = (-1)^{q(q-1)/2}\prod_{i=1}^{q}(i-1)!$. Also, from the proof of statement (i), we have

(2.4)
$$\frac{\det(e^{-x_i\beta_j})_{1\le i,j\le q}}{\prod_{1\le i<j\le q}(x_i - x_j)} = \sum_{\ell_1>\cdots>\ell_{q-1}>\ell_q\ge 0}\prod_{j=1}^{q}\frac{1}{\ell_j!}\frac{\det(x_i^{\ell_j})_{1\le i,j\le q}}{\prod_{1\le i<j\le q}(x_i - x_j)}\det((-\beta_i)^{\ell_j})_{1\le i,j\le q}.$$

Now we set $x_q = 0$ in (2.4). Then all terms with $\ell_q > 0$ vanish, and we get

$(2.4)_{|x_q=0}$

$$= \sum_{\ell_1>\cdots>\ell_{q-1}>0}\prod_{j=1}^{q-1}\frac{1}{\ell_j!}\frac{\det(x_i^{\ell_j})_{1\le i,j\le q-1}}{\prod_{1\le i<j\le q-1}(x_i - x_j)\prod_{i=1}^{q-1}x_i}\begin{vmatrix}(-\beta_1)^{\ell_1} & \cdots & (-\beta_q)^{\ell_1}\\ & \cdots & \\ (-\beta_1)^{\ell_{q-1}} & \cdots & (-\beta_q)^{\ell_{q-1}}\\ 1 & \cdots & 1\end{vmatrix}$$

$$= \sum_{\ell_1>\cdots>\ell_{q-1}>0}\prod_{j=1}^{q-1}\frac{1}{\ell_j!}\frac{\det(x_i^{\ell_j-1})_{1\le i,j\le q-1}}{\prod_{1\le i<j\le q-1}(x_i - x_j)}\begin{vmatrix}(-\beta_1)^{\ell_1} & \cdots & (-\beta_q)^{\ell_1}\\ & \cdots & \\ (-\beta_1)^{\ell_{q-1}} & \cdots & (-\beta_q)^{\ell_{q-1}}\\ 1 & \cdots & 1\end{vmatrix}.$$

After substituting ℓ_i by $\ell_i + 1$, we obtain

$(2.4)_{|x_q=0}$

$$= \sum_{\ell_1>\cdots>\ell_{q-1}\ge 0}\prod_{j=1}^{q-1}\frac{1}{(\ell_j+1)!}\frac{\det(x_i^{\ell_j})_{1\le i,j\le q-1}}{\prod_{1\le i<j\le q-1}(x_i - x_j)}\begin{vmatrix}(-\beta_1)^{\ell_1+1} & \cdots & \beta_q^{\ell_1+1}\\ & \cdots & \\ (-\beta_1)^{\ell_{q-1}+1} & \cdots & \beta_q^{\ell_{q-1}+1}\\ 1 & \cdots & 1\end{vmatrix}.$$

Setting now $x_{q-1} = 0$ and repeating this process $(q-p-1)$-times, we arrive at the following sum: if $x_q = 0, x_{q-1} = 0, \dots, x_{p+1} = 0$, then

$$\int_{U(q)} \exp[-\mathrm{tr}(v^{-1}\boldsymbol{\beta} vX)]dv_{|x_q=\cdots=x_{p+1}=0}$$

$$= \frac{c_q}{\prod_{1\le i<j\le q}(\beta_i-\beta_j)} \sum_{\ell_1>\ell_2>\cdots>\ell_p\ge 0} \prod_{j=1}^{p} \frac{1}{(\ell_j+q-p)!}$$

$$\times \frac{\det(x_i^{\ell_j})_{1\le i,j\le p}}{\prod_{1\le i<j\le p}(x_i-x_j)} \begin{vmatrix} (-\beta_1)^{\ell_1+q-p} & \cdots & (-\beta_q)^{\ell_1+q-p} \\ & \cdots & \\ (-\beta_1)^{\ell_p+q-p} & \cdots & (-\beta_q)^{\ell_p+q-p} \\ (-\beta_1)^{q-p-1} & \cdots & (-\beta_q)^{q-p-1} \\ & \cdots & \\ (-\beta_1) & \cdots & (-\beta_q) \\ 1 & \cdots & 1 \end{vmatrix}. \qquad \blacksquare$$

(After the work on this paper was completed, we learned that the argument presented above for statement (i) was used earlier by G. Olshanski and A. M. Vershik in [Olshanski-Vershik, 1996].)

Remark 2.3. The first statement of Proposition 2.2 can be proved in a number of different ways. For instance, it can be obtained by using the Harish-Chandra integral formula (some times also called HIZ integral) [Harish-Chandra, 1957], [Gross-Richards, 1989]. Another interesting way is to obtain the integral formula over $U(p)$ from the spherical function on $GL(p, \mathbf{C})$ by a passage to the limit. For more about the latest way described above, and in a general setting of compact Lie groups, we refer to [Ben Saïd-Ørsted, 2003].

Next we turn to the computation of $\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The proof of the following lemma is obvious.

Lemma 2.4. *Let μ be a measure on $\mathbf{R}$. Then*

$$\int_{\mathbf{R}^N} \det\{f_k(x_\ell)\}_{k,\ell} \det\{g_k(x_\ell)\}_{k,\ell} d\mu(x_1)\dots d\mu(x_N)$$
$$= N! \det\left\{\int_{\mathbf{R}} f_k(x) g_m(x) d\mu(x)\right\}_{k,m},$$

whenever the right-hand side of the equation makes sense.

Using Proposition 2.2, the function $\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is given by

$$\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{c \exp[-\mathrm{tr}(\boldsymbol{\alpha}+\boldsymbol{\beta})]}{\prod_{1\le i<j\le p}(\alpha_i-\alpha_j) \prod_{1\le i<j\le q}(\beta_i-\beta_j)}$$

$$\times \sum_{\ell_1 > \cdots > \ell_p \geq 0} \prod_{j=1}^{p} \frac{1}{(\ell_j + q - p)!} \begin{vmatrix} (-\beta_1)^{\ell_1+q-p} & \cdots & (-\beta_q)^{\ell_1+q-p} \\ & \cdots & \\ (-\beta_1)^{\ell_p+q-p} & \cdots & (-\beta_q)^{\ell_p+q-p} \\ (-\beta_1)^{q-p-1} & \cdots & (-\beta_q)^{q-p-1} \\ & \cdots & \\ (-\beta_1) & \cdots & (-\beta_q) \\ 1 & \cdots & 1 \end{vmatrix}$$

$$\times \int_{\mathfrak{x}} \frac{\det(e^{-x_i \alpha_j})_{1 \leq i,j \leq p} \det(x_i^{\ell_j})_{1 \leq i,j \leq p}}{\prod_{1 \leq i < j \leq p} (x_i - x_j)^2} \prod_{1 \leq i < j \leq p} (x_i - x_j)^2 \prod_{i=1}^{p} x_i^{q-p} \, dx_i.$$

Using Lemma 2.4, we deduce that

$$\int_{\mathfrak{x}} \frac{\det(e^{-x_i \alpha_j})_{1 \leq i,j \leq p} \det(x_i^{\ell_j})_{1 \leq i,j \leq p}}{\prod_{1 \leq i < j \leq p} (x_i - x_j)^2} \prod_{1 \leq i < j \leq p} (x_i - x_j)^2 \prod_{i=1}^{p} x_i^{q-p} \, dx_i$$

$$= c \det\left(\int_0^\infty e^{-x\alpha_i} x^{\ell_j+q-p} \, dx \right)_{1 \leq i,j \leq p}$$

$$= c \det\left(\frac{\Gamma(\ell_j + q - p + 1)}{\alpha_i^{\ell_j+q-p+1}} \right)_{1 \leq i,j \leq p}$$

$$= c \det\left(\frac{(\ell_j + q - p)!}{\alpha_i^{\ell_j+q-p+1}} \right)_{1 \leq i,j \leq p},$$

where c is a constant. Therefore

$$\zeta_{p,q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{c \exp[-\mathrm{tr}(\boldsymbol{\alpha} + \boldsymbol{\beta})]}{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j) \prod_{1 \leq i < j \leq q} (\beta_i - \beta_j)} \sum_{\ell_1 > \cdots > \ell_p \geq 0} \prod_{j=1}^{p} \frac{1}{(\ell_j + q - p)!}$$

$$\times \begin{vmatrix} (-\beta_1)^{\ell_1+q-p} & \cdots & (-\beta_q)^{\ell_1+q-p} \\ & \cdots & \\ (-\beta_1)^{\ell_p+q-p} & \cdots & (-\beta_q)^{\ell_p+q-p} \\ (-\beta_1)^{q-p-1} & \cdots & (-\beta_q)^{q-p+1} \\ & \cdots & \\ (-\beta_1) & \cdots & (-\beta_q) \\ 1 & \cdots & 1 \end{vmatrix} \det\left(\frac{(\ell_j + q - p)!}{\alpha_i^{\ell_j+q-p+1}} \right)_{1 \leq i,j \leq p}$$

$$= \frac{c \exp[-\mathrm{tr}(\boldsymbol{\alpha}+\boldsymbol{\beta})]}{\prod_{1\le i<j\le p}(\alpha_i-\alpha_j)\prod_{1\le i<j\le q}(\beta_i-\beta_j)\prod_{i=1}^{p}\alpha_i^{q-p+1}} \sum_{\ell_1>\ell_2>\cdots>\ell_p\ge 0}$$

$$\times \begin{vmatrix} (-\beta_1)^{\ell_1+q-p} & \cdots & (-\beta_q)^{\ell_1+q-p} \\ & \cdots & \\ (-\beta_1)^{\ell_p+q-p} & \cdots & (-\beta_q)^{\ell_p+q-p} \\ (-\beta_1)^{q-p-1} & \cdots & (-\beta_q)^{q-p-1} \\ & \cdots & \\ (-\beta_1) & \cdots & (-\beta_q) \\ 1 & \cdots & 1 \end{vmatrix} \det\left(\frac{1}{\alpha_i^{\ell_j}}\right)_{1\le i,j\le p}.$$

LEMMA 2.5 (cf. [HUA, 1963], Theorem 1.2.3). *Let $q \ge p > 0$. The following identity holds*

$$\sum_{\ell_1>\cdots>\ell_p\ge 0} \det(x_i^{\ell_j})_{1\le i,j\le p} \begin{vmatrix} y_1^{\ell_1+q-p} & \cdots & y_q^{\ell_1+q-p} \\ \vdots & \cdots & \vdots \\ y_1^{\ell_p+q-p} & \cdots & y_q^{\ell_p+q-p} \\ y_1^{q-p-1} & \cdots & y_q^{q-p-1} \\ \vdots & \cdots & \vdots \\ y_1 & \cdots & y_q \\ 1 & \cdots & 1 \end{vmatrix}$$

$$= \frac{\prod_{1\le i<j\le p}(x_i-x_j)\prod_{1\le i<j\le q}(y_i-y_j)}{\prod_{i=1}^{p}\prod_{i=1}^{q}(1-x_i y_j)}.$$

Using the above lemma, we obtain the following explicit expression for $\zeta_{p,q}$.

THEOREM 2.6. *Let c_0 be a constant. For $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1,\ldots,\alpha_p)$ and $\boldsymbol{\beta} = \mathrm{diag}(\beta_1,\ldots,\beta_q)$ such that $\alpha_i+\beta_j \ne 0$, the Bessel-type function $\zeta_{p,q}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is given by*

$$\zeta_{p,q}(\boldsymbol{\alpha},\boldsymbol{\beta}) = c_0 \frac{\exp[-\mathrm{tr}(\boldsymbol{\alpha}+\boldsymbol{\beta})]}{\prod_{i=1}^{p}\prod_{j=1}^{q}(\alpha_i+\beta_j)}.$$

3. The $Sp(2n, \mathbf{R})$-case

Let

$$Sp(2n, \mathbf{R}) = \left\{ g = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \in M(2n, \mathbf{C}) \mid gI_{n,n}g^* = I_{n,n} \right\},$$

where $A \in GL(n, \mathbf{C})$ and $B \in M(n, \mathbf{C})$.

A simple calculation shows that all elements $\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \in Sp(2n, \mathbf{R})$ satisfy

$$AA^* - BB^* = I_n, \qquad \text{and} \qquad A^*A - B^t\bar{B} = I_n.$$

For a diagonal matrix $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, such that $\alpha_i \neq 0$, we write

$$\zeta_n(\boldsymbol{\alpha}) = \int_{Sp(2n,\mathbf{R})} \exp[-\mathrm{tr}(\mathrm{diag}[\boldsymbol{\alpha}; \boldsymbol{\alpha}](gg^*)^{-1})]dg.$$

Remark 3.1. For $n = 1$ and $\alpha > 0$

$$\zeta_1(\alpha) = c_0(4\alpha)^{-1/2}K_{1/2}(4\alpha),$$

where $K_\nu(z)$ is the modified Bessel function of the third kind.

Let

$$\mathfrak{D}_n = \{T \in Sym(n, \mathbf{C}) \mid \det(I_n - T\overline{T}) > 0\},$$

where $Sym(n, \mathbf{C})$ denotes the set of $n \times n$-symmetric matrices. The $Sp(2n, \mathbf{R})$-invariant measure $d\mu(T)$ on $\mathfrak{D}_n$ is given by $d\mu(t) = \det(I_n - T\overline{T})^{-(n+1)}dT$, where dT is the Lebesgue measure on $\mathfrak{D}_n$.

Using the same method used in section 2, we can deduce that if

$$F(g) = \exp[-\mathrm{tr}(\mathrm{diag}[\boldsymbol{\alpha}, \boldsymbol{\alpha}](gg^*)^{-1})], \qquad g \in Sp(2n, \mathbf{R}),$$

then there exists a function $F^\# : \mathfrak{D}_n \to \mathbf{C}$ such that

$$F^\#(T) = \exp[-2\ \mathrm{tr}(\boldsymbol{\alpha})]\ \exp[-4\ \mathrm{tr}(\boldsymbol{\alpha}(I_n - T\overline{T})^{-1}T\overline{T})].$$

By [Hua, 1944], every symmetric matrix $Z \in Sym(n, \mathbf{C})$ can be written as $Z = u\Lambda u^t$, where $u \in U(n)$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Therefore the function $F^\#$ can be written as

$$F^\#(T) = \exp[-2\ \mathrm{tr}(\boldsymbol{\alpha})]\ \exp[-4\ \mathrm{tr}(\boldsymbol{\alpha}u(I_n - \Lambda^2)^{-1}\Lambda^2u^*)],$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $u \in U(n)$.

As in section 2, we consider the map $\psi : \mathfrak{D}_n \to \Upsilon$. The image of the Lebesgue measure dT on $\mathfrak{D}_n$ with respect to ψ is the measure on Υ given by

$$c \prod_{1 \le i < j \le n} (\lambda_i^2 - \lambda_j^2) \prod_{i=1}^{n} \lambda_i \, d\lambda_i,$$

for some constant c. Thus the image of $d\mu(T) = \det(I_n - T\overline{T})^{-(n+1)} dT$ is

$$c \prod_{1 \le i < j \le n} (\lambda_i^2 - \lambda_j^2) \prod_{i=1}^{n} \lambda_i (1 - \lambda_i^2)^{-(n+1)} d\lambda_i.$$

Using the above notations and Proposition 2.2(i) for $U(n)$, we obtain

$$\begin{aligned}
\zeta_n(\boldsymbol{a}) &= c \exp[-2 \operatorname{tr}(\boldsymbol{a})] \int_{U(n)} \int_{\Upsilon} \exp[-4 \operatorname{tr}(\boldsymbol{a} u (I_n - \Lambda^2)^{-1} \Lambda^2 u^*)] \\
&\quad \times \prod_{1 \le i < j \le n} (\lambda_i^2 - \lambda_j^2) \prod_{i=1}^{n} \lambda_i (1 - \lambda_i^2)^{-(n+1)} d\lambda_i du \\
&= c \exp[-2 \operatorname{tr}(\boldsymbol{a})] \int_{\mathfrak{X}} \left\{ \int_{U(n)} \exp[-\operatorname{tr}(\boldsymbol{a} u \operatorname{diag}[x_1, \ldots, x_n] u^*)] du \right\} \\
&\quad \prod_{1 \le i < j \le n} (x_i - x_j) dx_1 \ldots dx_n. \\
&= \frac{c \exp[-2 \operatorname{tr}(\boldsymbol{a})]}{\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)} \int_{\mathfrak{X}} \det(e^{-\alpha_i x_j})_{1 \le i, j \le n} dx_1 \ldots dx_n,
\end{aligned}$$

where

$$\mathfrak{X} = \{\operatorname{diag}(x_1, x_2, \ldots, x_n) \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}.$$

To obtain the above second equality, we used the change of variable $x_i = \dfrac{4\lambda_i^2}{1 - \lambda_i^2}$. Since $\det(e^{-\alpha_i x_j})_{1 \le i, j \le n} = \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^{n} e^{-\alpha_{\tau(i)} x_i}$, where S_n is the group of permutations, then

$$\begin{aligned}
\zeta_n(\boldsymbol{a}) &= \frac{c \exp[-2 \operatorname{tr}(\boldsymbol{a})]}{\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)} \int_{0 \le x_1 \le \cdots \le x_n} \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^{n} e^{-\alpha_{\tau(i)} x_i} \, dx_i \\
&= \frac{c \exp[-2 \operatorname{tr}(\boldsymbol{a})]}{\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)} \int_0^1 \cdots \int_0^1 \sum_{\tau \in S_n} \varepsilon(\tau) \xi_1^{\alpha_{\tau(1)} - 1} \ldots \xi_n^{\alpha_{\tau(1)} + \cdots + \alpha_{\tau(n)} - 1} \, d\xi_1 \ldots d\xi_n \\
&= \frac{c \exp[-2 \operatorname{tr}(\boldsymbol{a})]}{\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)} \sum_{\tau \in S_n} \varepsilon(\tau) \frac{1}{\alpha_{\tau(1)} (\alpha_{\tau(1)} + \alpha_{\tau(2)}) \ldots (\alpha_{\tau(1)} + \cdots + \alpha_{\tau(n)})}.
\end{aligned}$$

To finish the computation of $\zeta_n(\boldsymbol{a})$, we need the following lemma.

Lemma 3.2 (cf. [Hua, 1963], Lemma 6.3.1).

$$\sum_{\tau \in S_N} \varepsilon(\tau) \frac{1}{\ell_{\tau(1)}(\ell_{\tau(1)} + \ell_{\tau(2)}) \ldots (\ell_{\tau(1)} + \cdots + \ell_{\tau(N)})}$$
$$= \frac{(-1)^{N(N-1)/N} 2^N \prod_{1 \le i < j \le N} (\ell_i - \ell_j)}{\prod_{1 \le i \le j \le N} (\ell_i + \ell_j)}.$$

Using Lemma 3.2, the following theorem holds.

Theorem 3.3. *Let c_0 be a constant. For $\boldsymbol{a} = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \neq 0$, the Bessel-type function $\zeta_n(\boldsymbol{a})$ is given by*

$$\zeta_n(\boldsymbol{a}) = c_0 \frac{\exp[-2 \operatorname{tr}(\boldsymbol{a})]}{\prod_{1 \le i \le j \le n} (\alpha_i + \alpha_j)}.$$

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