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Integer group determinants for abelian groups of order 16

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ABSTRACT. For any positive integer n, let C_n be the cyclic group of order n. We determine all possible values of the integer group determinant of $C_4 \times C_2^2$, which is the only unsolved abelian group of order 16.

1. Introduction

A circulant determinant is the determinant of a square matrix in which each row is obtained by a cyclic shift of the previous row one step to the right. At the meeting of the American Mathematical Society in Hayward, California, in April 1977, Olga Taussky-Todd [16] suggested a problem that is to determine all the possible values of an $n \times n$ circulant determinant when all the entries are integers (see e.g., [5, 7]). The solution for the case $n = 2$ is well known. In the cases of $n = p$ and $2p$, where p is an odd prime, the problem was solved [3, 7]. Also, the problem was solved for the cases $n = 9$ [6, Theorem 4], $n = 4$ and 8 [2, Theorem 1.1], $n = 12$ [13, Theorem 5.3], $n = 15$ [8, Theorem 1.3], $n = 16$ [20], and $n = 25$ and 27 [5, Theorems 1.2 and 1.3].

For a finite group G, let x_g be a variable for each $g \in G$. The group determinant of G is defined as $\det(x_{gh^{-1}})_{g,h \in G}$. Let C_n be the cyclic group of order n. Note that the group determinant of C_n becomes an $n \times n$ circulant determinant. The group determinant of G is called an integer group determinant of G when the variables x_g are all integers. Let $S(G)$ denote the set of all possible values of the integer group determinant of G:

$$S(G) := \{\det(x_{gh^{-1}})_{g,h \in G} \mid x_g \in \mathbb{Z}\}.$$

The problem suggested by Taussky-Todd is extended to the problem that is to determine $S(G)$ for any finite group G. For some groups, the problem was solved in [1, 4, 8, 9, 12, 13, 17, 22, 21]. As a result, for every group G of order at most 15, $S(G)$ was determined. Also, $C_4 \times C_2^2$ is left as the only unsolved abelian group of order 16 (the integer group determinants for the

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non-abelian groups of order 16 have also been characterized recently in [10, Theorems 3.1 and 4.1], [11], [23], [14], [15] and [19]).

In this paper, we determine $S(\mathrm{C}_4 \times \mathrm{C}_2^2)$. For any $r \in \mathbb{Z}$, let

$$P_r := \{p \mid p \equiv r \pmod 8 \text{ is a prime number}\},$$

$$P' := \{p \mid p = a^2 + b^2 \equiv 1 \pmod 8 \text{ is a prime number satisfying } a + b \equiv \pm 3 \pmod 8\},$$

$$A := \{(8k - 3)(8l + 3) \mid k, l \in \mathbb{Z}\} \subsetneq \{8m - 1 \mid m \in \mathbb{Z}\},$$

$$B := \{p(8m - 1) \mid p \in P', m \in \mathbb{Z}\} \subsetneq \{8m - 1 \mid m \in \mathbb{Z}\}.$$

Theorem 1.1. *We have*

$$S(\mathrm{C}_4 \times \mathrm{C}_2^2) = \{16m + 1, 2^{16}(4m + 1), 2^{16}(8m + 3), 2^{17}p(2m + 1), 2^{18}m \mid m \in \mathbb{Z}, p \in P_5\} \cup \{2^{16}m \mid m \in A \cup B\}.$$

Let D_n denote the dihedral group of order n and let

$$C := \{(8k - 3)(8l - 3)(8m - 3)(8n - 3) \mid k \in \mathbb{Z}, 8l - 3, 8m - 3, 8n - 3 \in P_5, k + l \not\equiv m + n \pmod 2\} \subsetneq \{16m - 7 \mid m \in \mathbb{Z}\},$$

$$D := \{(8k - 3)(8l - 3) \mid k, l \in \mathbb{Z}, k \equiv l \pmod 2\} \subsetneq \{16m - 7 \mid m \in \mathbb{Z}\}.$$

Remark that $C \subset D$ holds. In [21], [22], [17, Theorem 1.5], [1, Theorem 5.3] and [20], the following are obtained respectively:

$$S(\mathrm{C}_2^4) = \{16m + 1, 2^{16}(4m + 1), 2^{24}(4m + 1), 2^{24}(8m + 3), 2^{24}m', 2^{26}m \mid m \in \mathbb{Z}, m' \in A\},$$

$$S(\mathrm{C}_4^2) = \{16m + 1, m', 2^{15}p(2m + 1), 2^{16}m \mid m \in \mathbb{Z}, m' \in C, p \in P_5\},$$

$$S(\mathrm{C}_8 \times \mathrm{C}_2) = \{16m + 1, m', 2^{10}(2m + 1), 2^{11}p(2m + 1), 2^{11}q^2(2m + 1), 2^{12}m \mid m \in \mathbb{Z}, m' \in D, p \in P' \cup P_5, q \in P_3\},$$

$$S(\mathrm{D}_{16}) = \{4m + 1, 2^{10}m \mid m \in \mathbb{Z}\},$$

$$S(\mathrm{C}_{16}) = \{2m + 1, 2^6p(2m + 1), 2^6q^2(2m + 1), 2^7m \mid m \in \mathbb{Z}, p \in P' \cup P_5, q \in P_3\}.$$

Pinner and Smyth [13, p. 427] noted the following inclusion relations for every groups of order 8: $S(\mathrm{C}_2^3) \subsetneq S(\mathrm{C}_4 \times \mathrm{C}_2) \subsetneq S(\mathrm{Q}_8) \subsetneq S(\mathrm{D}_8) \subsetneq S(\mathrm{C}_8)$, where Q_8

denotes the generalized quaternion group of order 8. From the above results, we have $S(C_2^4) \subsetneq S(C_4 \times C_2^2) \subsetneq S(C_4^2) \subsetneq S(C_8 \times C_2) \subsetneq S(D_{16}) \subsetneq S(C_{16})$.

2. Preliminaries

For any $\bar{r} \in C_n$ with $r \in \{0, 1, \ldots, n-1\}$, we denote the variable $x_{\bar{r}}$ by x_r, and let $D_n(x_0, x_1, \ldots, x_{n-1}) := \det(x_{gh^{-1}})_{g,h \in C_n}$. For any $(\bar{r}, \bar{s}) \in C_4 \times C_2$ with $r \in \{0, 1, 2, 3\}$ and $s \in \{0, 1\}$, we denote the variable $y_{(\bar{r}, \bar{s})}$ by y_j, where $j := r + 4s$, and let $D_{4\times 2}(y_0, y_1, \ldots, y_7) := \det(y_{gh^{-1}})_{g,h \in C_4 \times C_2}$. For any $(\bar{r}, \bar{s}, \bar{t}) \in C_4 \times C_2^2$ with $r \in \{0, 1, 2, 3\}$ and $s, t \in \{0, 1\}$, we denote the variable $z_{(\bar{r}, \bar{s}, \bar{t})}$ by z_j, where $j := r + 4s + 8t$, and let $D_{4\times 2\times 2}(z_0, z_1, \ldots, z_{15}) := \det(z_{gh^{-1}})_{g,h \in C_4 \times C_2^2}$. From the $G = C_4$ and $H = \{\bar{0}, \bar{2}\}$ case of [17, Theorem 1.1], we have the following corollary.

Corollary 2.1. *We have*

$$D_4(x_0, x_1, x_2, x_3) = D_2(x_0 + x_2, x_1 + x_3) D_2(x_0 - x_2, \sqrt{-1}(x_1 - x_3))$$
$$= \{(x_0 + x_2)^2 - (x_1 + x_3)^2\}\{(x_0 - x_2)^2 + (x_1 - x_3)^2\}.$$

Remark 2.2. *From Corollary* 2.1, *we have*

$$D_4(x_0, x_1, x_2, x_3) = -D_4(x_1, x_2, x_3, x_0).$$

Lemma 2.3. *The following hold:*

(1) $D_{4\times 2}(y_0, \ldots, y_7) = D_4(y_0 + y_4,\ y_1 + y_5,\ y_2 + y_6,\ y_3 + y_7) D_4(y_0 - y_4,\ y_1 - y_5, y_2 - y_6, y_3 - y_7)$;

(2) $D_{4\times 2\times 2}(z_0, \ldots, z_{15}) = D_{4\times 2}(z_0 + z_8,\ z_1 + z_9, \ldots, z_7 + z_{15}) D_{4\times 2}(z_0 - z_8,\ z_1 - z_9, \ldots, z_7 - z_{15})$.

Proof. Theorem 1.1 of [18] describes a formula for $S(H \times K)$ when H and K are finite abelian groups. Part (1) follows from this by taking $H = C_4$ and $K = C_2$, and (2) by choosing $H = C_4 \times C_2$ and $K = C_2$. □

Throughout this paper, we assume that $a_0, a_1, \ldots, a_{15} \in \mathbb{Z}$, and for any $0 \le i \le 3$, let

$$b_i := (a_i + a_{i+8}) + (a_{i+4} + a_{i+12}), \qquad c_i := (a_i + a_{i+8}) - (a_{i+4} + a_{i+12}),$$
$$d_i := (a_i - a_{i+8}) + (a_{i+4} - a_{i+12}), \qquad e_i := (a_i - a_{i+8}) - (a_{i+4} - a_{i+12}).$$

Also, let $\boldsymbol{a} := (a_0, a_1, \ldots, a_{15})$ and let

$$\boldsymbol{b} := (b_0, b_1, b_2, b_3), \qquad \boldsymbol{c} := (c_0, c_1, c_2, c_3),$$
$$\boldsymbol{d} := (d_0, d_1, d_2, d_3), \qquad \boldsymbol{e} := (e_0, e_1, e_2, e_3).$$

Then, from Lemma 2.3, we have

$$D_{4\times2\times2}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e}). \tag{$*$}$$

REMARK 2.4. *For any* $0 \le i \le 3$, *the following hold:*
(1) $b_i \equiv c_i \equiv d_i \equiv e_i \pmod 2$*;*
(2) $b_i + c_i + d_i + e_i \equiv 0 \pmod 4$.

LEMMA 2.5. *We have*

$$D_{4\times2\times2}(\boldsymbol{a}) \equiv D_4(\boldsymbol{b}) \equiv D_4(\boldsymbol{c}) \equiv D_4(\boldsymbol{d}) \equiv D_4(\boldsymbol{e}) \pmod 2.$$

PROOF. From Corollary 2.1, for any $x_0, x_1, x_2, x_3 \in \mathbb{Z}$,

$$D_4(x_0, x_1, x_2, x_3) \equiv x_0 + x_1 + x_2 + x_3 \pmod 2.$$

Therefore, we have

$$D_4(\boldsymbol{b}) \equiv D_4(\boldsymbol{c}) \equiv D_4(\boldsymbol{d}) \equiv D_4(\boldsymbol{e}) \pmod 2$$

from Remark 2.4 (1). □

3. Impossible odd numbers

In this section, we consider impossible odd numbers. Let $\mathbb{Z}_{\text{odd}}$ be the set of all odd numbers.

LEMMA 3.1. *We have* $S(\mathrm{C}_4 \times \mathrm{C}_2^2) \cap \mathbb{Z}_{\text{odd}} \subset \{16m + 1 \mid m \in \mathbb{Z}\}$.

To prove Lemma 3.1, we use the following lemma.

LEMMA 3.2 ([17, Lemmas 4.6 and 4.7]). *For any* $k, l, m, n \in \mathbb{Z}$, *the following hold:*
(1) $D_4(2k+1, 2l, 2m, 2n) \equiv 8m + 1 \pmod{16}$*;*
(2) $D_4(2k, 2l+1, 2m+1, 2n+1) \equiv 8(k+l+n) - 3 \pmod{16}$.

PROOF (Proof of Lemma 3.1). Let

$$D_{4\times2\times2}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e}) \in \mathbb{Z}_{\text{odd}}.$$

Then $D_4(\boldsymbol{b}) \in \mathbb{Z}_{\text{odd}}$. From this and Corollary 2.1, we have $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$. Therefore, one of the following cases holds:
(i) exactly three of b_0, b_1, b_2, b_3 are even;
(ii) exactly one of b_0, b_1, b_2, b_3 is even.
First, we consider the case (i). From Remarks 2.2 and 2.4 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv \boldsymbol{d} \equiv \boldsymbol{e} \equiv (1, 0, 0, 0) \pmod 2$. From Remark 2.4, there exist $m_i \in \mathbb{Z}$ satisfying $b_2 = 2m_0$, $c_2 = 2m_1$, $d_2 = 2m_2$,

$e_2 = 2m_3$ and $\sum_{i=0}^{3} m_i \equiv 0 \pmod 2$. Therefore, from Lemma 3.2 (1),

$$D_{4\times 2\times 2}(\boldsymbol{a}) \equiv \prod_{i=0}^{3}(8m_i + 1) \equiv 1 + 8\sum_{i=0}^{3} m_i \equiv 1 \pmod{16}.$$

Next, we consider the case (ii). From Remarks 2.2 and 2.4 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv \boldsymbol{d} \equiv \boldsymbol{e} \equiv (0, 1, 1, 1) \pmod 2$. From Remark 2.4, there exist $k_i, l_i, n_i \in \mathbb{Z}$ satisfying

$$(b_0, b_1, b_2) = (2k_0, 2l_0 + 1, 2n_0 + 1), \qquad (c_0, c_1, c_3) = (2k_1, 2l_1 + 1, 2n_1 + 1),$$
$$(d_0, d_1, d_3) = (2k_2, 2l_2 + 1, 2n_2 + 1), \qquad (e_0, e_1, e_3) = (2k_3, 2l_3 + 1, 2n_3 + 1)$$

and $\sum_{i=0}^{3} k_i \equiv \sum_{i=0}^{3} l_i \equiv \sum_{i=0}^{3} n_i \equiv 0 \pmod 2$. Therefore, from Lemma 3.2 (2), we have

$$\begin{aligned} D_{4\times 2\times 2}(\boldsymbol{a}) &\equiv \prod_{i=0}^{3}\{8(k_i + l_i + n_i) - 3\} \\ &\equiv 1 + 8\sum_{i=0}^{3}(k_i + l_i + n_i) \equiv 1 \pmod{16}. \end{aligned}$$ □

4. Impossible even numbers

With P_5, P', A, B as in Section 1, in this section we aim to establish three statements regarding necessary conditions for even members of $S(\mathrm{C}_4 \times \mathrm{C}_2^2)$.

Lemma 4.1. *We have* $S(\mathrm{C}_4 \times \mathrm{C}_2^2) \cap 2\mathbb{Z} \subset 2^{16}\mathbb{Z}$.

Lemma 4.2. *We have*

$$S(\mathrm{C}_4 \times \mathrm{C}_2^2) \cap 2^{16}\mathbb{Z}_{\mathrm{odd}} \subset \{2^{16}(4m+1), 2^{16}(8m+3), 2^{16}m' \mid m \in \mathbb{Z},\ m' \in A \cup B\}.$$

Lemma 4.3. *We have*

$$S(\mathrm{C}_4 \times \mathrm{C}_2^2) \cap 2^{17}\mathbb{Z}_{\mathrm{odd}} \subset \{2^{17}p(2m+1) \mid p \in P_5,\ m \in \mathbb{Z}\}.$$

Lemma 4.1 is immediately obtained from Equation $(*)$, Lemma 2.5 and Kaiblinger's [2, Theorem 1.1] result $S(\mathrm{C}_4) = \mathbb{Z}_{\mathrm{odd}} \cup 2^4\mathbb{Z}$. To prove Lemma 4.2, we use the following six lemmas.

Lemma 4.4 ([22, Lemma 3.2]). *For any* $k, l, m, n \in \mathbb{Z}$, *the following hold:*

(1) $$D_4(2k, 2l, 2m, 2n) \in \begin{cases} 2^4\mathbb{Z}_{\mathrm{odd}}, & k + m \not\equiv l + n \pmod 2, \\ 2^8\mathbb{Z}, & k + m \equiv l + n \pmod 2; \end{cases}$$

(2) $D_4(2k+1, 2l+1, 2m+1, 2n+1)$

$$\in \begin{cases} 2^4\mathbb{Z}_{\mathrm{odd}}, & k+m \not\equiv l+n \pmod{2}, \\ 2^7\mathbb{Z}_{\mathrm{odd}}, & (k+m)(l+n) \equiv -1 \pmod{4}, \\ 2^9\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(3) $D_4(2k, 2l+1, 2m, 2n+1)$

$$\in \begin{cases} 2^5\mathbb{Z}_{\mathrm{odd}}, & k-m \equiv l-n \equiv 1 \pmod{2}, \\ 2^6\mathbb{Z}_{\mathrm{odd}}, & k \equiv m \pmod{2}, \quad (2k+2l+1)(2m+2n+1) \equiv \pm 3 \pmod{8}, \\ 2^7\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(4) $D_4(2k, 2l, 2m+1, 2n+1)$

$$\in \begin{cases} 2^4\mathbb{Z}_{\mathrm{odd}}, & (2k+2m+1)(2l+2n+1) \equiv \pm 3 \pmod{8}, \\ 2^5\mathbb{Z}, & (2k+2m+1)(2l+2n+1) \equiv \pm 1 \pmod{8}. \end{cases}$$

LEMMA 4.5. *Suppose that* $(x_0, x_1, x_2, x_3) \equiv (0, 0, 1, 1) \pmod{2}$ *and* $(x_0+x_2)(x_1+x_3) \equiv \pm 3 \pmod{8}$ *hold. Then the following hold:*
(1) *if* $x_0 \equiv x_1 \pmod{4}$, *then* $(x_0-x_2)^2 + (x_1-x_3)^2 \in \{2(8k-3) \mid k \in \mathbb{Z}\}$;
(2) *if* $x_0 \not\equiv x_1 \pmod{4}$, *then* $(x_0-x_2)^2 + (x_1-x_3)^2 \in \{2(8k+1) \mid k \in \mathbb{Z}\}$.

PROOF. We prove (1). If $x_0 \equiv x_1 \pmod{4}$, then

$$\begin{aligned}(x_0-x_2)(x_1-x_3) &= (x_0+x_2)(x_1+x_3) - 2x_0x_3 - 2x_2x_1 \\ &\equiv (x_0+x_2)(x_1+x_3) - 2x_0(x_3+x_2) \\ &\equiv \pm 3 \pmod{8}.\end{aligned}$$

Thus, $(x_0-x_2)^2 + (x_1-x_3)^2 \equiv -6 \pmod{16}$ holds. We prove (2). If $x_0 \not\equiv x_1 \pmod{4}$, then

$$\begin{aligned}(x_0-x_2)(x_1-x_3) &= (x_0+x_2)(x_1+x_3) - 2x_0x_3 - 2x_2x_1 \\ &\equiv (x_0+x_2)(x_1+x_3) - 2x_0x_3 - 2x_2(x_0+2) \\ &\equiv (x_0+x_2)(x_1+x_3) - 2x_0(x_3+x_2) - 4x_2 \\ &\equiv \pm 1 \pmod{8}.\end{aligned}$$

Thus, $(x_0-x_2)^2 + (x_1-x_3)^2 \equiv 2 \pmod{16}$ holds. □

LEMMA 4.6. *Suppose that* $(x_0+x_2)^2 - (x_1+x_3)^2$ *has no prime factor of the form* $8k \pm 3$. *Then the following hold:*
(1) *if* $x_0+x_2 \equiv \pm 3$, $x_1+x_3 \equiv \pm 1 \pmod{8}$, *then* $(x_0+x_2)^2 - (x_1+x_3)^2 \in \{8(8k+1) \mid k \in \mathbb{Z}\}$;

(2) *if* $x_0 + x_2 \equiv \pm 1$, $x_1 + x_3 \equiv \pm 3 \pmod 8$, *then* $(x_0 + x_2)^2 - (x_1 + x_3)^2 \in \{8(8k - 1) \mid k \in \mathbf{Z}\}$.

Proof. We prove (1). First, we consider the case of $(x_0 + x_2, x_1 + x_3) \equiv (3, 1) \pmod 8$. Then $(x_0 + x_2, x_1 + x_3) \equiv (3, 1), (3, -7), (-5, 1)$ or $(-5, -7)$ $\pmod{16}$. From

$$\begin{aligned}(16l + 3)^2 - (16m + 1)^2 &= (16l + 3 + 16m + 1)(16l + 3 - 16m - 1)\\ &= 8(4l + 4m + 1)(8l - 8m + 1),\\ (16l + 3)^2 - (16m - 7)^2 &= (16l + 3 + 16m - 7)(16l + 3 - 16m + 7)\\ &= 8(4l + 4m - 1)(8l - 8m + 5),\\ (16l - 5)^2 - (16m + 1)^2 &= (16l - 5 + 16m + 1)(16l - 5 - 16m - 1)\\ &= 8(4l + 4m - 1)(8l - 8m - 3),\\ (16l - 5)^2 - (16m - 7)^2 &= (16l - 5 + 16m - 7)(16l - 5 - 16m + 7)\\ &= 8(4l + 4m - 3)(8l - 8m + 1),\end{aligned}$$

we find that if $(x_0 + x_2, x_1 + x_3) \equiv (3, -7)$ or $(-5, 1) \pmod{16}$, then $(x_0 + x_2)^2 - (x_1 + x_3)^2$ has at least one prime factor of the form $8k \pm 3$. Also, if $(x_0 + x_2, x_1 + x_3) \equiv (3, 1)$ or $(-5, -7) \pmod{16}$, then $(x_0 + x_2)^2 - (x_1 + x_3)^2$ is of the form $8(8k + 1)$ or has at least one prime factor of the form $8k \pm 3$. In the same way, we can prove for the cases $(x_0 + x_2, x_1 + x_3) \equiv (3, -1), (-3, 1)$ and $(-3, -1) \pmod 8$. Replacing (x_0, x_1, x_2, x_3) with (x_1, x_2, x_3, x_0) in (1), we obtain (2). □

Lemma 4.7. *Suppose that* $(x_0 - x_2)^2 + (x_1 - x_3)^2$ *has no prime factor of the form* $8k \pm 3$ *and* $x_0 - x_2 \equiv \pm 3$, $x_1 - x_3 \equiv \pm 3 \pmod 8$ *hold. Then*

$$\begin{aligned}(x_0 - x_2)^2 + (x_1 - x_3)^2 \in \{2pm \mid m \in \mathbf{Z},\ p = a^2 + b^2 \equiv 1 \pmod 8,\\ a + b \equiv \pm 3 \pmod 8\}.\end{aligned}$$

Proof. From the assumption, there exist primes $p_i \equiv 1$, $q_i \equiv -1 \pmod 8$ and integers $k_i, l_i \geq 0$ satisfying $(x_0 - x_2)^2 + (x_1 - x_3)^2 = 2p_1^{k_1} \cdots p_r^{k_r} q_1^{2l_1} \cdots q_s^{2l_s}$. We prove by contradiction. If $p_i = a_i^2 + b_i^2$ with $a_i + b_i \equiv \pm 1 \pmod 8$ for any $1 \leq i \leq r$, then $x_0 - x_2, x_1 - x_3 \in \{8k \pm 1 \mid k \in \mathbf{Z}\}$ hold from [20, Lemma 4.8]. This is a contradiction. □

Lemma 4.8. *Suppose that* $b_0 + b_2 \equiv b_1 + b_3 \equiv 0 \pmod 2$ *and* $D_{4\times2\times2}(\boldsymbol{a}) \in 2^{16}\mathbf{Z}_{\mathrm{odd}}$. *Then we have* $D_{4\times2\times2}(\boldsymbol{a}) \in \{2^{16}(4m + 1) \mid m \in \mathbf{Z}\}$.

PROOF. From Equation $(*)$, Remarks 2.2 and 2.4 (1) and Lemma 4.4, one of the following cases holds:

(i) $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 0 \pmod 2$ and $D_4(\boldsymbol{b}), D_4(\boldsymbol{c}), D_4(\boldsymbol{d}), D_4(\boldsymbol{e}) \in 2^4\mathbb{Z}_{\mathrm{odd}}$;

(ii) $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 1 \pmod 2$ and $D_4(\boldsymbol{b}), D_4(\boldsymbol{c}), D_4(\boldsymbol{d}), D_4(\boldsymbol{e}) \in 2^4\mathbb{Z}_{\mathrm{odd}}$.

First, we consider the case (i). From Remark 2.4, there exist $k_i, l_i, m_i, n_i \in \mathbb{Z}$ satisfying

$$\boldsymbol{b} = (2k_0, 2l_0, 2m_0, 2n_0), \qquad \boldsymbol{c} = (2k_1, 2l_1, 2m_1, 2n_1),$$
$$\boldsymbol{d} = (2k_2, 2l_2, 2m_2, 2n_2), \qquad \boldsymbol{e} = (2k_3, 2l_3, 2m_3, 2n_3)$$

and $\sum_{i=0}^3 k_i \equiv \sum_{i=0}^3 l_i \equiv \sum_{i=0}^3 m_i \equiv \sum_{i=0}^3 n_i \equiv 0 \pmod 2$. Here, from Lemma 4.4, $k_i + m_i \not\equiv l_i + n_i \pmod 2$ holds for any $0 \le i \le 3$. Thus by Corollary 2.1 we have

$$\begin{aligned} 2^{-4}D_4(2k_i, 2l_i, 2m_i, 2n_i) &= \{(k_i+m_i)^2 - (l_i+n_i)^2\}\{(k_i-m_i)^2 + (l_i-n_i)^2\} \\ &\equiv (-1)^{l_i+n_i} \pmod 4. \end{aligned}$$

Therefore,

$$\begin{aligned} 2^{-16}D_{4\times2\times2}(\boldsymbol{a}) &= 2^{-16}\prod_{i=0}^{3} D_4(2k_i, 2l_i, 2m_i, 2n_i) \\ &\equiv \prod_{i=0}^{3}(-1)^{l_i+n_i} \\ &\equiv (-1)^{l_0+l_1+l_2+l_3}(-1)^{n_0+n_1+n_2+n_3} \\ &\equiv 1 \pmod 4. \end{aligned}$$

In the same way, we can prove for the case (ii). □

LEMMA 4.9. *Let* $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$. *If*

$$D_{4\times2\times2}(\boldsymbol{a}) \in \{2^{16}m \mid m \equiv -1 \pmod 8\},$$

then $D_{4\times2\times2}(\boldsymbol{a}) \in \{2^{16}m \mid m \in A \cup B\}$.

PROOF. Let $D_{4\times2\times2}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e}) = 2^{16}m$ with $m \equiv -1 \pmod 8$. From Remarks 2.2 and 2.4 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv \boldsymbol{d} \equiv \boldsymbol{e} \equiv (0,0,1,1) \pmod 2$. We prove that if $m \notin A$, then $m \in B$. Suppose that $m \notin A$ and let

$$Q := \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid (x_0, x_1, x_2, x_3) \equiv (0, 0, 1, 1) \pmod 2,$$
$$(x_0 + x_2)(x_1 + x_3) \equiv \pm 3 \pmod 8,\ x_0 \not\equiv x_1 \pmod 4\},$$
$$Q_1 := \{(x_0, x_1, x_2, x_3) \in Q \mid x_0 + x_2 \equiv \pm 3,\ x_1 + x_3 \equiv \pm 1 \pmod 8\},$$
$$Q_2 := \{(x_0, x_1, x_2, x_3) \in Q \mid x_0 + x_2 \equiv \pm 1,\ x_1 + x_3 \equiv \pm 3 \pmod 8\},$$
$$Q_1' := \{(x_0, x_1, x_2, x_3) \in Q_1 \mid x_0 \equiv 0,\ x_1 \equiv 2 \pmod 4\},$$
$$Q_2' := \{(x_0, x_1, x_2, x_3) \in Q_2 \mid x_0 \equiv 2,\ x_1 \equiv 0 \pmod 4\}.$$

Since $m \notin A$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e})$ has no prime factor of the form $8k \pm 3$. Thus, from Lemmas 4.4 (4) and 4.5, we have $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e} \in Q$. Moreover, from $m \equiv -1 \pmod 8$ and Lemmas 4.5 and 4.6, either one of the following cases holds:

(i) one of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ is an element of Q_1 and the other three are elements of Q_2;

(ii) one of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ is an element of Q_2 and the other three are elements of Q_1.

Since $b_0 + c_0 + d_0 + e_0 \equiv 0 \pmod 4$ from Remark 2.4 (2), we find that in both cases (i) and (ii), at least one of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ is an element of $Q_1' \cup Q_2'$. On the other hand, for any $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in Q_1'$, we have $-x_0 + x_2 \equiv x_0 + x_2 \equiv \pm 3$, $-x_1 + x_3 \equiv x_1 + x_3 - 4 \equiv \pm 3 \pmod 8$. Thus, it follows from Lemma 4.7 that for any $\boldsymbol{x} \in Q_1'$, if $D_4(\boldsymbol{x})$ has no prime factor of the form $8k \pm 3$, then $D_4(\boldsymbol{x})$ has at least one prime factor of the form $p = a^2 + b^2 \equiv 1 \pmod 8$ with $a + b \equiv \pm 3 \pmod 8$. In the same way, we can obtain the same conclusion for any $\boldsymbol{x} \in Q_2'$. From the above, we have $m \in B$. □

Proof (Proof of Lemma 4.2). Suppose that

$$D_{4\times 2\times 2}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e}) \in 2^{16}\mathbb{Z}_{\mathrm{odd}}.$$

From Corollary 2.1 and Lemma 2.5, we have $b_0 + b_2 \equiv b_1 + b_3 \pmod 2$. Therefore, we have

$$D_{4\times 2\times 2}(\boldsymbol{a}) \in \{2^{16}(4m+1),\ 2^{16}(8m+3),\ 2^{16}m' \mid m \in \mathbb{Z},\ m' \in A \cup B\}$$

from Lemmas 4.8 and 4.9. □

To prove Lemma 4.3, we use the following lemma.

Lemma 4.10. *Suppose that* $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \equiv (0, 0, 1, 1) \pmod 2$, $(x_0 + x_2)(x_1 + x_3) \equiv \pm 1 \pmod 8$ *and* $x_0 \not\equiv x_1 \pmod 4$. *Then* $D_4(\boldsymbol{x})$ *has at least one prime factor of the form* $8k - 3$.

PROOF. From

$$\begin{aligned}(x_0 - x_2)(x_1 - x_3) &= (x_0 + x_2)(x_1 + x_3) - 2x_0x_3 - 2x_2x_1 \\ &\equiv (x_0 + x_2)(x_1 + x_3) - 2x_0x_3 - 2x_2(x_0 + 2) \\ &\equiv (x_0 + x_2)(x_1 + x_3) - 2x_0(x_3 + x_2) - 4x_2 \\ &\equiv \pm 3 \pmod 8,\end{aligned}$$

we have $(x_0 - x_2)^2 + (x_1 - x_3)^2 \equiv -6 \pmod{16}$. This completes the proof. □

PROOF (Proof of Lemma 4.3). Suppose that

$$D_{4\times2\times2}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})D_4(\boldsymbol{d})D_4(\boldsymbol{e}) \in 2^{17}\mathbb{Z}_{\mathrm{odd}}.$$

Then, from Corollary 2.1 and Lemma 2.5, we have $b_0 + b_2 \equiv b_1 + b_3 \pmod 2$. Therefore, from Remarks 2.2 and 2.4 (1) and Lemma 4.4, we have $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$. We may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv \boldsymbol{d} \equiv \boldsymbol{e} \equiv (0, 0, 1, 1) \pmod 2$. Let

$$\begin{aligned}Q_3 &:= \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid (x_0, x_1, x_2, x_3) \equiv (0, 0, 1, 1) \pmod 2, \\ &\qquad (x_0 + x_2)(x_1 + x_3) \equiv \pm 3 \pmod 8\}, \\ Q_4 &:= \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid (x_0, x_1, x_2, x_3) \equiv (0, 0, 1, 1) \pmod 2, \\ &\qquad (x_0 + x_2)(x_1 + x_3) \equiv \pm 1 \pmod 8\}, \\ Q_3' &:= \{(x_0, x_1, x_2, x_3) \in Q_3 \mid x_0 \equiv x_1 \pmod 4\}, \\ Q_4' &:= \{(x_0, x_1, x_2, x_3) \in Q_4 \mid x_0 \not\equiv x_1 \pmod 4\}.\end{aligned}$$

From Lemma 4.4, three of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ are elements of Q_3 and the other one is an element of Q_4. Moreover, since $(b_0 - b_1) + (c_0 - c_1) + (d_0 - d_1) + (e_0 - e_1) \equiv 0 \pmod 4$ from Remark 2.4 (2), we find that at least one of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ is an element of $Q_3' \cup Q_4'$. On the other hand, it follows from Lemmas 4.5 and 4.10 that for any $\boldsymbol{x} \in Q_3' \cup Q_4'$, $D_4(\boldsymbol{x})$ has at least one prime factor of the form $8k - 3$. From the above, there exist $p \in P_5$ and $m \in \mathbb{Z}$ satisfying $D_{4\times2\times2}(\boldsymbol{a}) = 2^{17}p(2m + 1)$. □

5. Possible numbers

In this section, we determine all possible numbers. Lemmas 3.1 and 4.1–4.3 imply that $S(\mathrm{C}_4 \times \mathrm{C}_2^2)$ does not include every integer that is not mentioned in the following Lemmas 5.1–5.3.

Lemma 5.1. *For any* $m \in \mathbb{Z}$, *the following are elements of* $S(C_4 \times C_2^2)$*:*
(1) $16m + 1$;
(2) $2^{16}(4m + 1)$;
(3) $2^{16}(4m + 1)(8n + 3)$;
(4) $2^{18}(2m + 1)$;
(5) $2^{18}(2m)$.

Lemma 5.2. *For any* $p \in P'$ *and* $m \in \mathbb{Z}$, *we have*

$$2^{16}p(4m - 1) \in S(C_4 \times C_2^2).$$

Lemma 5.3. *For any* $p \in P_5$ *and* $m \in \mathbb{Z}$, *we have*

$$2^{17}p(2m + 1) \in S(C_4 \times C_2^2).$$

Remark 5.4. *From Lemma* 5.1 (3), *we have* $2^{16}m \in S(C_4 \times C_2^2)$ *for any* $m \in A$. *Also, from Lemma* 5.2, *we have* $2^{16}m \in S(C_4 \times C_2^2)$ *for any* $m \in B$.

Proof (Proof of Lemma 5.1). We obtain (1) from

$$\begin{aligned} D_{4\times 2\times 2}(m+1, m, \dots, m) &= D_4(4m+1, 4m, 4m, 4m)D_4(1,0,0,0)^3 \\ &= (8m+1)^2 - (8m)^2 \\ &= 16m + 1. \end{aligned}$$

We obtain (2) from

$$\begin{aligned} D_{4\times 2\times 2}(m+1, m+1, m+2, m, \dots, m) &= D_4(4m+1, 4m+1, 4m+2, 4m) \\ &\quad \times D_4(1,1,2,0)^3 \\ &= 2\{(8m+3)^2 - (8m+1)^2\}(2^4)^3 \\ &= 2^{13}(32m+8) \\ &= 2^{16}(4m+1). \end{aligned}$$

We obtain (3) from

$$\begin{aligned} &D_{4\times 2\times 2}(\overbrace{m+n+1, \dots, m+n+1}^{4}, \overbrace{m-n, \dots, m-n}^{4}, \\ &\qquad m+n+1, m+n, m+n+1, m+n-1, m-n-1, m-n, m-n, m-n) \\ &\quad = D_4(4m+1, 4m+1, 4m+2, 4m)D_4(4n+3, 4n+1, 4n+2, 4n) \\ &\qquad \times D_4(1,1,0,2)D_4(-1,1,0,2) \end{aligned}$$

$$= 2\{(8m+3)^2 - (8m+1)^2\}2\{(8n+5)^2 - (8n+1)^2\}(-2^4)(-2^4)$$
$$= 2^{10}(32m+8)(64n+24)$$
$$= 2^{16}(4m+1)(8n+3).$$

We obtain (4) from

$$D_{4\times2\times2}(m+2, m, m+2, m+1, m, m, m, m,$$
$$m+1, m+1, m, m+1, m, m, m, m)$$
$$= D_4(4m+3, 4m+1, 4m+2, 4m+2)D_4(3,1,2,2)D_4(1,-1,2,0)^2$$
$$= 2\{(8m+5)^2 - (8m+3)^2\}2^5(2^4)^2$$
$$= 2^{18}(2m+1).$$

We obtain (5) from

$$D_{4\times2\times2}(m+1, m, m+1, m, m, m-1, m, m,$$
$$m, m+1, m, m+1, m, m-1, m-1, m-1)$$
$$= D_4(4m+1, 4m-1, 4m, 4m)D_4(1,3,2,2)D_4(1,-1,2,0)D_4(1,-1,0,-2)$$
$$= 2\{(8m+1)^2 - (8m-1)^2\}(-2^5)2^4(-2^4)$$
$$= 2^{18}(2m).$$

This completes the proof. □

PROOF (Proof of Lemma 5.2). For any $p \in P'$, there exist $r, s \in \mathbb{Z}$ with $r \not\equiv s \pmod 2$ satisfying $p = (4r)^2 + (4s+1)^2$. Let $k := \frac{r+s+1}{2}$ and $l := \frac{r-s-1}{2}$. Then we have

$$2p = (4r+4s+1)^2 + (4r-4s-1)^2 = (8k-3)^2 + (8l+3)^2.$$

Therefore, from

$$D_{4\times2\times2}(k-m, l-m+1, -k-m+1, -l-m, k+m, l+m+1,$$
$$-k+m+1, -l+m, k-m, l-m+1, -k-m+1, -l-m,$$
$$k+m-1, l+m, -k+m-1, -l+m)$$
$$= D_4(4k-1, 4l+3, 2-4k, -4l)D_4(1-4m, 1-4m, 2-4m, -4m)$$
$$\times D_4(1,1,2,0)D_4(-1,-1,-2,0)$$

$$= -2^3\{(8k-3)^2 + (8l+3)^2\}2\{(3-8m)^2 - (1-8m)^2\}2^4 2^4$$
$$= -2^{12}\{(8k-3)^2 + (8l+3)^2\}(-32m+8)$$
$$= 2^{15}\{(8k-3)^2 + (8l+3)^2\}(4m-1),$$

we have $2^{16}p(4m-1) \in S(C_4 \times C_2^2)$. □

Proof (Proof of Lemma 5.3). For any $p \in P_5$, there exist $r, s \in \mathbb{Z}$ with $r \equiv s \pmod 2$ satisfying $p = (4r+2)^2 + (4s+1)^2$. Let $k := \frac{r+s}{2}$ and $l := \frac{r-s}{2}$. Then we have

$$2p = (4r+4s+3)^2 + (4r-4s+1)^2 = (8k+3)^2 + (8l+1)^2.$$

Therefore, from

$$D_{4\times2\times2}(m+l+1, m+k+1, m-l+1, m-k, m+l+1, m+k+1,$$
$$m-l+1, m-k, m+l+1, m+k+1, m-l+1, m-k,$$
$$m+l, m+k, m-l-1, m-k)$$
$$= D_4(4m+4l+3, 4m+4k+3, 4m-4l+2, 4m-4k)$$
$$\times D_4(1,1,2,0)^2 D_4(-1,-1,-2,0)$$
$$= \{(8m+5)^2 - (8m+3)^2\}\{(8l+1)^2 + (8k+3)^2\}(2^4)^2 2^4$$
$$= 2^{12}(32m+16)\{(8k+3)^2 + (8l+1)^2\}$$
$$= 2^{16}\{(8k+3)^2 + (8l+1)^2\}(2m+1),$$

we have $2^{17}p(2m+1) \in S(C_4 \times C_2^2)$. □

From Lemmas 3.1, 4.1–4.3 and 5.1–5.3, Theorem 1.1 is proved.

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