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Trigonometric series involving logarithmic function

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Abstract. Extensions of the closed-form expressions for the series represented as a product of sine and logarithmic functions mentioned in reference [7] have been reformulated utilizing the Hurwitz zeta function and its first derivative. Furthermore, analogous closed-form expressions have been derived for these series that involve the cosine function instead.

1. Introduction and preliminaries

In [7, p. 748], the entries 24–26 are closed-form formulas for the series over the product of the sine and logarithmic functions. Here, we give generalizations of these formulas. In addition, we consider the same series, but instead of the sine, we include the cosine function.

For $\alpha > 0$, by setting $a = 1$, $b = 0$, $s = 1$, $f = \cos$ in the general summation formula for the trigonometric series we derived in [9], one obtains the formula

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha) \sin \frac{\pi}{2}\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(\alpha - 2k - 1)}{(2k+1)!} x^{2k+1}, \tag{1.1}$$

where ζ is Riemann's zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \operatorname{Re} s > 1, \tag{1.2}$$

which can be analytically extended to the whole complex plane except for $s = 1$, where it has a pole [6]. Similarly, for $a = 1$, $b = 0$, $s = 1$, $f = \cos$ in [9], we have

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha) \cos \frac{\pi}{2}\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(\alpha - 2k)}{(2k)!} x^{2k}, \tag{1.3}$$

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where $0 < x < 2\pi$. If we place $\alpha = 2m - 1$ (m is a positive integer) in (1.1), we get

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} = \frac{(-1)^{m-1}\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \zeta(2m-2k-2)}{(2k+1)!} x^{2k+1}, \tag{1.4}$$

and taking $\alpha = 2m$ in (1.3) yields

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} = \frac{(-1)^m \pi x^{2m-1}}{2(2m-1)!} + \sum_{k=0}^{m} \frac{(-1)^k \zeta(2m-2k)}{(2k)!} x^{2k}. \tag{1.5}$$

Formulas (1.4) and (1.5) arise because the ζ function takes the value zero at negative even integers, which results from setting $s = 2m + 1$ (m is a positive integer) in the functional equation for Riemann's zeta function [1]

$$(2\pi)^s \zeta(1-s) = 2\zeta(s)\Gamma(s) \cos \frac{\pi s}{2}.$$

In the sequel, the set of positive integers will be denoted by $\mathbb{N}$.

However, we cannot replace α immediately with $2m$ in (1.1) nor with $2m - 1$ in (1.3), since we encounter singularities both in denominators of the first terms and in the term for $k = m - 1$ of the formulas (1.1) and (1.3), but dealing with this problem, relying on the Hurwitz zeta function initially defined by

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \qquad \operatorname{Re} s > 1,\ 0 < a \leqslant 1, \tag{1.6}$$

we deduced in [8] the following closed-form formulas

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m}} = \frac{(-1)^m (2\pi)^{2m-1}}{(2m-1)!} \left(\zeta'\left(1-2m, 1-\frac{x}{2\pi}\right) - \zeta'\left(1-2m, \frac{x}{2\pi}\right) \right) \tag{1.7}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m-1}} = \frac{(-1)^{m-1} (2\pi)^{2m-2}}{(2m-2)!} \left(\zeta'\left(2-2m, 1-\frac{x}{2\pi}\right) + \zeta'\left(2-2m, \frac{x}{2\pi}\right) \right). \tag{1.8}$$

Note that $\zeta'(s, a)$ in (1.7) and (1.8) and further on in the text denotes the first derivative of Hurwitz's zeta function with respect to s.

The functional equation of $\zeta(s, a)$ [1] written in the equivalent form

$$\zeta(1-s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \cos\left(\frac{\pi}{2}s - 2n\pi a\right) \tag{1.9}$$

provides the analytic continuation of $\zeta(s,a)$ in the whole complex plane except for a simple pole at $s=1$, because the values $\Gamma(1-s)$ are defined for $\operatorname{Re} s<1$, where the Γ function is introduced by Euler as the integral [3]

$$\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x, \qquad \operatorname{Re} s>0. \tag{1.10}$$

Hence, we deduce the equality

$$\begin{aligned}\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,\mathrm{d}x&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\int_0^1 x^{s+n-1}\,\mathrm{d}x+\int_1^\infty x^{s-1}e^{-x}\,\mathrm{d}x\\&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\frac{1}{s+n}+\int_1^\infty x^{s-1}e^{-x}\,\mathrm{d}x,\end{aligned}$$

from which the Γ function is analytically extended to all the complex values of s except for negative integers and zero, where it has simple poles.

Integration by parts of the integral (1.10) gives rise to the basic relation

$$\Gamma(s+1)=s\Gamma(s),$$

and it can be easily shown that, for arbitrary $n\in\mathbb{N}$, there holds

$$\Gamma(s+n)=(s+n-1)(s+n-2)\cdots s\Gamma(s),$$

and that formula we use to express the Pochhammer symbol $(s)_n$, i.e.

$$(s)_n=s(s+1)(s+2)\cdots(s+n-1)=\frac{\Gamma(s+n)}{\Gamma(s)}. \tag{1.11}$$

2. Main results

Initially, the following lemma establishes a connection between the two zeta functions, Hurwitz's and Riemann's.

Lemma 2.1. *The Hurwitz zeta function $\zeta(s,a)$ for $s=2-2m$, and $a=x/2\pi$ $(m\in\mathbb{N})$ presents a polynomial in x of the degree $2m-1$ with coefficients containing values of Riemann zeta at positive even integers, i.e. there holds*

$$\begin{aligned}&\zeta\left(2-2m,\frac{x}{2\pi}\right)\\&=\frac{1}{2}\left(\frac{x}{2\pi}\right)^{2m-2}+\frac{(-1)^{m-1}(2m-2)!}{2^{2m-2}\pi^{2m-1}}\sum_{k=0}^{m-1}\frac{(-1)^k\zeta(2m-2k-2)}{(2k+1)!}x^{2k+1}.\end{aligned} \tag{2.1}$$

PROOF. Replacing s and a with $2m-1$ and $x/2\pi$, respectively, in (1.9), we obtain

$$\zeta\left(2-2m,\frac{x}{2\pi}\right)=\frac{2\Gamma(2m-1)}{(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{2}(2m-1)-nx\right)}{n^{2m-1}}$$
$$=\frac{2(-1)^{m-1}(2m-2)!}{(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}}. \tag{2.2}$$

Employing the formula (1.4), we find (2.1). □

LEMMA 2.2. *The Hurwitz zeta function $\zeta(s,a)$ for $s=1-2m$ and $a=x/2\pi$ $(m\in\mathbb{N})$ presents a polynomial in x of the degree $2m$ with coefficients containing values of Riemann zeta at positive even integers, i.e. there holds*

$$\zeta\left(1-2m,\frac{x}{2\pi}\right)$$
$$=\frac{1}{2}\left(\frac{x}{2\pi}\right)^{2m-1}+\frac{(-1)^m(2m-1)!}{2^{2m-1}\pi^{2m}}\sum_{k=0}^{m-1}\frac{(-1)^k\zeta(2m-2k)}{(2k)!}x^{2k}. \tag{2.3}$$

PROOF. Replacing s and a with $2m$ and $x/2\pi$, respectively, in (1.9), we obtain

$$\zeta\left(1-2m,\frac{x}{2\pi}\right)=\frac{2\Gamma(2m)}{(2\pi)^{2m}}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{2}(2m)-nx\right)}{n^{2m}}$$
$$=\frac{(-1)^m(2m-1)!}{2^{2m-1}\pi^{2m}}\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m}}. \tag{2.4}$$

Employing the formula (1.5), we find (2.3). □

Relying on them here we shall deduce closed-form formulas.

THEOREM 2.3. *For $0<x<2\pi$ and $m\in\mathbb{N}$, the closed form of the sine series involving logarithmic function is*

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}}\log n=\frac{(-1)^m(2\pi)^{2m-1}}{4(2m-2)!}\left(2(\log 2\pi-\psi(2m-1))\zeta\left(2-2m,\frac{x}{2\pi}\right)\right.$$
$$\left.+\zeta'\left(2-2m,1-\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{x}{2\pi}\right)\right). \tag{2.5}$$

PROOF. Putting $a=x/2\pi$ in (1.9), then taking the first derivative on both sides with respect to s yields

$$-\zeta'\left(1-s,\frac{x}{2\pi}\right)=\frac{2\Gamma(s)}{(2\pi)^s}(\psi(s)-\log 2\pi)\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi s}{2}-nx\right)}{n^s}$$

$$-\frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi s}{2}-nx\right)}{n^s}\log n-\frac{\pi\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi s}{2}-nx\right)}{n^s}.$$

After replacing then s with $2m-1$, we find

$$\zeta'\left(2-2m,\frac{x}{2\pi}\right)=\frac{(-1)^{m-1}(2m-2)!}{(2\pi)^{2m-1}}\left(2(\log 2\pi-\psi(2m-1))\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}}\right.$$

$$\left.+2\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}}\log n+\pi\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m-1}}\right).$$

Based on the formulas (1.8) and (2.2), we have

$$\frac{1}{2}\zeta'\left(2-2m,\frac{x}{2\pi}\right)=(\log 2\pi-\psi(2m-1))\zeta\left(2-2m,\frac{x}{2\pi}\right)$$

$$+2\frac{(-1)^{m-1}(2m-2)!}{(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}}\log n$$

$$+\frac{1}{2}\zeta'\left(2-2m,1-\frac{x}{2\pi}\right).$$

After a rearrangement, we arrive at (2.5). □

Remark 2.4. *The formula* (2.5) *comprises the entry* 24 *in* [7, p. 748].

To show that, we put $m=1$ in (2.5), knowing that for $n=1$ from

$$H_{n-1}=\psi(n)+\gamma,\qquad n\in\mathbb{N},\tag{2.6}$$

there implies $\psi(1)=-\gamma$, where γ is Euler-Mascheroni's constant, H_n is the nth harmonic number given by

$$H_n=\sum_{k=1}^{n}\frac{1}{k},\qquad H_0=0,\tag{2.7}$$

and ψ is the function (also known as the digamma function) defined as the logarithmic derivative of the gamma function [2]

$$\psi(s)=\frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s)=\frac{\Gamma'(s)}{\Gamma(s)}.\tag{2.8}$$

Thus, we have

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} \log n = -\frac{\pi}{2}\left(2(\log 2\pi + \gamma)\zeta\left(0, \frac{x}{2\pi}\right) + \zeta'\left(0, 1 - \frac{x}{2\pi}\right) - \zeta'\left(0, \frac{x}{2\pi}\right)\right).$$

For $s = 0$ in (1.9), we evaluate

$$\begin{aligned}\zeta(0, a) &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} = \frac{1}{2\pi i}\left(\sum_{n=1}^{\infty} \frac{e^{2n\pi i a}}{n} - \sum_{n=1}^{\infty} \frac{e^{-2n\pi i a}}{n}\right) \\ &= \sum_{n=1}^{\infty} \int e^{2n\pi i a}\, \mathrm{d}a + \sum_{n=1}^{\infty} \int e^{-2n\pi i a}\, \mathrm{d}a = \frac{i}{2\pi} \log \frac{1 - e^{2\pi i a}}{1 - e^{-2\pi i a}} = \frac{1}{2} - a.\end{aligned}$$

Using this and the identity [2]

$$\zeta'(0, a) = \log \Gamma(a) - \frac{1}{2} \log 2\pi,$$

we obtain

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} \log n = \frac{\pi}{2}\left(2(\log 2\pi + \gamma)\left(\frac{x}{2\pi} - \frac{1}{2}\right) - \log \Gamma\left(1 - \frac{x}{2\pi}\right) + \log \Gamma\left(\frac{x}{2\pi}\right)\right).$$

Applying Euler's reflection formula [5]

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \qquad z \notin \mathbb{Z}, \tag{2.9}$$

the preceding left-hand side series becomes

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin nx}{n} \log n &= (\log 2\pi + \gamma)\left(\frac{x}{2} - \frac{\pi}{2}\right) + \frac{\pi}{2} \log \frac{\pi}{\Gamma\left(\frac{x}{2\pi}\right) \sin \frac{x}{2}} - \frac{\pi}{2} \log \Gamma\left(\frac{x}{2\pi}\right) \\ &= \frac{1}{2}(\log 2\pi + \gamma)(x - \pi) + \frac{\pi}{2} \log \frac{1}{\pi} \sin \frac{x}{2} + \pi \log \Gamma\left(\frac{x}{2\pi}\right),\end{aligned}$$

which is exactly the entry 24 in [7, p. 748] for $a = 1$.

We note that it is easy to deduce the formula for $a \neq 1$ (entry 24 in [7, p. 748]) by rewriting its left-hand side as follows

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} \log na = \log a \sum_{n=1}^{\infty} \frac{\sin nx}{n} + \sum_{n=1}^{\infty} \frac{\sin nx}{n} \log n,$$

then we apply formulas (1.4) and (2.5) for $m = 1$.

Employing a similar method, we can deduce a summation formula for the cosine function, which is missing in [7].

THEOREM 2.5. *For $0 < x < 2\pi$ and $m \in \mathbb{N}$, the closed form of the cosine series involving logarithmic function is*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} \log n = \frac{(-1)^{m-1}(2\pi)^{2m}}{4(2m-1)!} \left(2(\log 2\pi - \psi(2m))\zeta\left(1-2m, \frac{x}{2\pi}\right) - \zeta'\left(1-2m, 1-\frac{x}{2\pi}\right) - \zeta'\left(1-2m, \frac{x}{2\pi}\right) \right). \tag{2.10}$$

PROOF. Applying again the first derivative with respective to s to (1.9), where $a = x/2\pi$, we have

$$-\zeta'\left(1-s, \frac{x}{2\pi}\right) = \frac{2\Gamma(s)}{(2\pi)^s}(\psi(s) - \log 2\pi) \sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi s}{2} - nx\right)}{n^s} - \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi s}{2} - nx\right)}{n^s} \log n - \frac{\pi\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi s}{2} - nx\right)}{n^s}.$$

After replacing then s with $2m$, we find

$$\zeta'\left(1-2m, \frac{x}{2\pi}\right) = \frac{(-1)^m(2m-1)!}{(2\pi)^{2m}} \left(2(\log 2\pi - \psi(2m)) \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} \log n - \pi \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m}} \right).$$

Based on the formulas (1.7) and (2.4), we have

$$\frac{1}{2}\zeta'\left(1-2m, \frac{x}{2\pi}\right) = (\log 2\pi - \psi(2m))\zeta\left(1-2m, \frac{x}{2\pi}\right) + 2\frac{(-1)^{m-1}(2m-1)!}{(2\pi)^{2m}} \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m-1}} \log n - \frac{1}{2}\zeta'\left(1-2m, 1-\frac{x}{2\pi}\right).$$

After a rearrangement, we arrive at (2.10). □

We can find the summation formulas and closed forms of alternating series by relying on prior results.

Theorem 2.6. *For* $-\pi < x < \pi$ *and* $m \in \mathbb{N}$, *the closed form of the alternating sine series involving logarithmic function is*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^{2m-1}} \log n \\
&\quad = \frac{(-1)^{m-1} \pi^{2m-1}}{(2m-2)!} \left(\zeta\left(2-2m, \frac{x}{\pi}\right) \log 2 \right. \\
&\quad + (\log 2\pi - \psi(2m-1)) \left(2^{2m-2} \zeta\left(2-2m, \frac{x}{2\pi}\right) - \zeta\left(2-2m, \frac{x}{\pi}\right) \right) \\
&\quad + \frac{1}{2} \left(2^{2m-2} \zeta'\left(2-2m, 1-\frac{x}{2\pi}\right) - 2^{2m-2} \zeta'\left(2-2m, \frac{x}{2\pi}\right) \right. \\
&\quad \left.\left. - \zeta'\left(2-2m, 1-\frac{x}{\pi}\right) + \zeta'\left(2-2m, \frac{x}{\pi}\right) \right) \right). \qquad (2.11)
\end{aligned}$$

Proof. Using representation as follows

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^{\alpha}} \log n \\
&\quad = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}} \log n - 2 \sum_{n=1}^{\infty} \frac{\sin 2nx}{(2n)^{\alpha}} \log 2n \\
&\quad = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}} \log n - \frac{\log 2}{2^{\alpha-1}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{\alpha}} - \frac{1}{2^{\alpha-1}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{\alpha}} \log n,
\end{aligned}$$

where the second row suggests taking $2x$ instead of x in the corresponding formulas, then replacing here α with $2m-1$, according to Theorem 2.3 and (2.2), after a rearrangement, we arrive at (2.11). □

Remark 2.7. *By an immediate check, one ascertains misprints in* the entry 25 *of* [7, p. 748]*, and, in keeping with the notation,* p. 792*, it should go like this*

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k} \sin kx}{k} \log ak = \left(\frac{\pi}{2} - x\right) \log 2 + \frac{x}{2}\left(\gamma - \log \frac{a}{2\pi}\right) \\
+ \frac{\pi}{2} \log \Gamma \left[\frac{x/\pi, 1 - x/2\pi}{x/2\pi, 1 - x/\pi} \right],
\end{aligned}$$

which one obtains for $m = 1$ *from* (2.11).

Theorem 2.8. *For $-\pi < x < \pi$ and $m \in \mathbb{N}$, the closed form of the alternating cosine series involving logarithmic function is*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^{2m}} \log n$$
$$= \frac{(-1)^{m-1}\pi^{2m}}{(2m-1)!} \left(\zeta\left(1-2m, \frac{x}{\pi}\right) \log 2 \right.$$
$$+ (\log 2\pi - \psi(2m)) \left(2^{2m-1} \zeta\left(1-2m, \frac{x}{2\pi}\right) - \zeta\left(1-2m, \frac{x}{\pi}\right)\right)$$
$$- \frac{1}{2}\left(2^{2m-1}\zeta'\left(1-2m, 1-\frac{x}{2\pi}\right) + 2^{2m-1}\zeta'\left(1-2m, \frac{x}{2\pi}\right)\right.$$
$$\left.\left. - \zeta'\left(1-2m, 1-\frac{x}{\pi}\right) - \zeta'\left(1-2m, \frac{x}{\pi}\right)\right)\right). \tag{2.12}$$

Proof. Similarly, as in the case of the preceding proof, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^{\alpha}} \log n$$
$$= \sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}} \log n - 2\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^{\alpha}} \log 2n,$$
$$= \sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}} \log n - \frac{\log 2}{2^{\alpha-1}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{\alpha}} - \frac{1}{2^{\alpha-1}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{\alpha}} \log n.$$

Replacing here α with $2m$, according to (2.4) and Theorem 2.5, after a rearrangement, we arrive at (2.12). □

Relying on the proofs of the last two theorem, we easily find the representations

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{\alpha}} \log(2n-1)$$
$$= \sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}} \log n - \frac{\log 2}{2^{\alpha}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{\alpha}} - \frac{1}{2^{\alpha}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{\alpha}} \log n, \tag{2.13}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{\alpha}} \log(2n-1)$$
$$= \sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}} \log n - \frac{\log 2}{2^{\alpha}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{\alpha}} - \frac{1}{2^{\alpha}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{\alpha}} \log n. \tag{2.14}$$

Theorem 2.9. *For $0 < x < \pi$ and $m \in \mathbb{N}$, the closed form of the sine series involving logarithmic function over odd positive integers is*

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{2m-1}} \log(2n-1)$$
$$= \frac{(-1)^m \pi^{2m-1}}{4(2m-2)!} \left(2 \log 2\zeta\left(2-2m, \frac{x}{\pi}\right) \right.$$
$$+ 2(\log 2\pi - \psi(2m-1)) \left(2^{2m-1} \zeta\left(2-2m, \frac{x}{2\pi}\right) - \zeta\left(2-2m, \frac{x}{\pi}\right) \right)$$
$$+ 2^{2m-1} \zeta'\left(2-2m, 1-\frac{x}{2\pi}\right) - 2^{2m-1} \zeta'\left(2-2m, \frac{x}{2\pi}\right)$$
$$\left. - \zeta'\left(2-2m, 1-\frac{x}{\pi}\right) + \zeta'\left(2-2m, \frac{x}{\pi}\right) \right). \tag{2.15}$$

Proof. Using (2.13), replacing α with $2m-1$, we have

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{2m-1}} \log(2n-1)$$
$$= \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} \log n - \frac{\log 2}{2^{2m-1}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{2m-1}} - \frac{1}{2^{2m-1}} \sum_{n=1}^{\infty} \frac{\sin 2nx}{n^{2m-1}} \log n.$$

Applying (2.2) and (2.5), after a rearrangement, we arrive at (2.15). □

Remark 2.10. *There is a misprint in the entry* 26, [7, p. 748], *and in keeping with the notation there on page* 792, *for $a = 1$, it should go like this*

$$\frac{\pi}{4} \left(\log \frac{1}{2\pi} - \gamma + \log \operatorname{tg} \frac{x}{2} - 2 \log \Gamma \left[\frac{(\pi + x)/2\pi}{x/2\pi} \right] \right).$$

It can be obtained from the formula (2.15) *for $m = 1$.*

Theorem 2.11. *For $0 < x < \pi$ and $m \in \mathbb{N}$, the closed form of the cosine series involving logarithmic function over odd positive integers is*

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2m}} \log(2n-1)$$
$$= \frac{(-1)^m \pi^{2m}}{4(2m-1)!}\Bigg(2 \log 2\zeta\left(1-2m, \frac{x}{\pi}\right)$$
$$+ 2(\log 2\pi - \psi(2m))\left(2^{2m}\zeta\left(1-2m, \frac{x}{2\pi}\right) - \zeta\left(1-2m, \frac{x}{\pi}\right)\right)$$
$$+ 2^{2m}\zeta'\left(1-2m, 1-\frac{x}{2\pi}\right) - 2^{2m}\zeta'\left(1-2m, \frac{x}{2\pi}\right)$$
$$- \zeta'\left(1-2m, 1-\frac{x}{\pi}\right) + \zeta'\left(1-2m, \frac{x}{\pi}\right)\Bigg). \tag{2.16}$$

PROOF. Using (2.14), replacing α with $2m$, we have

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2m}} \log(2n-1)$$
$$= \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} \log n - \frac{\log 2}{2^{2m-1}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{2m}} - \frac{1}{2^{2m}} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{2m}} \log n.$$

Applying (2.4) and (2.10), after a rearrangement, we arrive at (2.16). □

Alternating series related to the series over odd positive integers can be expressed as a power series involving Dirichlet's beta function. Their summation formulas deduced in [9] are

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k)}{(2k)!} x^{2k}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \tag{2.17}$$

where $\beta(s)$ is Dirichlet's beta function (also known as Catalan's beta function), defined by

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \qquad \operatorname{Re} s > 0,$$

and it is related to Hurwitz's zeta function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{1}{(4n+1)^s} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^s} = 4^{-s}\left(\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right). \tag{2.18}$$

The function β is analytical functions for $\operatorname{Re} s > 0$, but by means of their functional equations

$$\beta(1-s) = \left(\frac{2}{\pi}\right)^{s} \sin\frac{\pi s}{2} \Gamma(s)\beta(s) \tag{2.19}$$

β is extended to the whole complex plane. By virtue of Euler's reflection formula (2.9), the identity (2.19) can be rewritten as follows

$$\beta(s) = \left(\frac{\pi}{2}\right)^{s-1} \beta(1-s)\Gamma(1-s) \cos\frac{\pi s}{2}, \tag{2.20}$$

whence we find $\beta(-2n+1) = 0$, $n \in \mathbb{N}$.

By placing $\alpha = 2m$ in the first and $\alpha = 2m-1$ in the second one, the following closed-form formulas are easily obtained because of vanishing Dirichlet's beta function for negative odd integers

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n-1)x}{(2n-1)^{2m}} &= \sum_{k=0}^{m-1} \frac{(-1)^k \beta(2m-2k-1)}{(2k+1)!} x^{2k+1}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2n-1)x}{(2n-1)^{2m-1}} &= \sum_{k=0}^{m} \frac{(-1)^k \beta(2m-2k-1)}{(2k)!} x^{2k}. \end{aligned} \tag{2.21}$$

However, apart from these, there are other representations of theirs that we shall use in the sequel.

Lemma 2.12. *Both series in* (2.21) *can be represented over values of the Hurwitz zeta function.*

Proof. We replace a with $\frac{1}{4} - \frac{x}{2\pi}$ and $\frac{3}{4} - \frac{x}{2\pi}$ successively in (1.9), then making subtraction, obtain

$$\begin{aligned} &\zeta\left(1-s, \frac{1}{4} - \frac{x}{2\pi}\right) - \zeta\left(1-s, \frac{3}{4} - \frac{x}{2\pi}\right) \\ &\qquad = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{2}s - \frac{n\pi}{2} + nx\right) - \cos\left(\frac{\pi}{2}s - \frac{3n\pi}{2} + nx\right)}{n^s}. \end{aligned}$$

We transform the right-hand side further by applying the cosine sum-to-product identity

$$\zeta\left(1-s, \frac{1}{4} - \frac{x}{2\pi}\right) - \zeta\left(1-s, \frac{3}{4} - \frac{x}{2\pi}\right) = \frac{4\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin\frac{n\pi}{2} \sin\left(\frac{\pi s}{2} + nx\right)}{n^s}.$$

Since for even running indices terms are zero, we pick only odd running indices. Thus, we have

$$\zeta\left(1-s,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta\left(1-s,\frac{3}{4}-\frac{x}{2\pi}\right)$$
$$=\frac{4\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left(\frac{\pi s}{2}+(2n-1)x\right)}{(2n-1)^s}. \tag{2.22}$$

Hence, by choosing $s=2m$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}}$$
$$=\frac{(-1)^m(2\pi)^{2m}}{4(2m-1)!}\left(\zeta\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right), \tag{2.23}$$

or by setting $s=2m-1$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}}$$
$$=\frac{(-1)^{m-1}(2\pi)^{2m-1}}{4(2m-2)!}\left(\zeta\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right), \tag{2.24}$$

whereby we complete the proof. □

Except for (2.21), there exist other types of closed-form formulas for the trigonometric series in (2.17).

Theorem 2.13. *The closed form of the second series in* (2.17) *ensuing by setting* $\alpha=2m$ *is*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m}}=\frac{(-1)^{m-1}(2\pi)^{2m-1}}{2(2m-1)!}\left(\zeta'\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)\right.$$
$$-\zeta'\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)+\zeta'\left(1-2m,\frac{1}{4}+\frac{x}{2\pi}\right)$$
$$\left.-\zeta'\left(1-2m,\frac{3}{4}+\frac{x}{2\pi}\right)\right). \tag{2.25}$$

Proof. Upon placing $\alpha=2m$ in the second right-hand side series of (2.17), and using (2.20) for $s=0$, i.e.

$$\beta(0)=\frac{2}{\pi}\sin\frac{\pi}{2}\Gamma(1)\beta(1)=\frac{2}{\pi}\cdot\frac{\pi}{4}=\frac{1}{2},$$

the first step on a path to deduce (2.25) consists of splitting that series in three as follows

$$\sum_{k=0}^{m-1}\frac{(-1)^k\beta(2m-2k)}{(2k)!}x^{2k}+\frac{(-1)^m x^{2m}}{2(2m)!}+\sum_{k=m+1}^{\infty}\frac{(-1)^k\beta(2m-2k)}{(2k)!}x^{2k}. \tag{2.26}$$

By shifting the summation index, $k=m+j$, and making use of (2.20) again, but for $s=-2j$, then applying the connection of the Gamma function Pochhammer's symbol (1.11), after reverting to the letter k instead of j, the last series becomes

$$\sum_{k=m+1}^{\infty}\frac{(-1)^k\beta(2m-2k)}{(2k)!}x^{2k}=(-1)^m x^{2m-1}\sum_{k=1}^{\infty}\frac{\beta(2k+1)}{(2k+1)_{2m}}\left(\frac{2x}{\pi}\right)^{2k+1}.$$

Relying on (2.18), we further change the right-hand side series to

$$(-1)^m x^{2m-1}\sum_{k=1}^{\infty}\frac{\zeta\left(2k+1,\frac{1}{4}\right)-\zeta\left(2k+1,\frac{3}{4}\right)}{(2k+1)_{2m}}\left(\frac{x}{2\pi}\right)^{2k+1}. \tag{2.27}$$

That means we are dealing with two series over the Hurwitz zeta functions. To obtain the first one we set $n=2m-1$, $t=\frac{x}{2\pi}$ and $a=\frac{1}{4}$ in the formula (1.10) of [4, p. 419], considering, apart from t, the same formula for $-t$ as well, and make addition of these cases. We repeat this procedure for the second series but with $a=\frac{3}{4}$ instead.

Afterwards, we subtract these equalities. As a result, on the left-hand side, we obtain (2.27). The right-hand side consists of three parts. In the first one, four derivatives of the Hurwitz zeta function are grouped

$$\frac{(-1)^{m-1}(2\pi)^{2m-1}}{2(2m-1)!}\left(\zeta'\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)+\zeta'\left(1-2m,\frac{1}{4}+\frac{x}{2\pi}\right)\right.$$
$$\left.-\zeta'\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)-\zeta'\left(1-2m,\frac{3}{4}+\frac{x}{2\pi}\right)\right),$$

in the sequel, there are two sums

$$\frac{(-1)^m(2\pi)^{2m-1}}{(2m-1)!}\sum_{k=1}^{m-1}\left(\frac{x}{2\pi}\right)^{2k}\binom{2m-1}{2k}(H_{2m-2}-H_{2m-2k-1})\beta(2k-2m+1)$$
$$+\frac{(-1)^m(2\pi)^{2m-1}}{(2m-1)!}\sum_{k=1}^{m-1}\left(\frac{x}{2\pi}\right)^{2k}\binom{2m-1}{2k}4^{2k-2m+1}\beta'(2k-2m+1), \tag{2.28}$$

and finally

$$\left(\psi\left(\frac{1}{4}\right)-\psi\left(\frac{3}{4}\right)\right)\frac{(-1)^m(2\pi)^{2m-1}}{(2m)!}\left(\frac{x}{2\pi}\right)^{2m}=-4\beta(1)\frac{(-1)^m x^{2m}}{2\pi(2m)!}=-\frac{(-1)^m x^{2m}}{2(2m)!}.$$

The last equality, we obtained by setting $n=1$ in the relation

$$\beta(n)=\frac{(-1)^n}{2^{2n}(2n-1)!}\left(\psi^{(n-1)}\left(\frac{1}{4}\right)-\psi^{(n-1)}\left(\frac{3}{4}\right)\right),\qquad n\in\mathbb{N},$$

which in turn we get form (2.18) by relying on the expression of the nth derivative of the ψ function [2]

$$\psi^{(n)}(a)=(-1)^{n-1}n!\zeta(n+1,a).$$

For $k=1,\dots,m-1$ the expressions $2k-2m+1$ present negative odd integers, which means $\beta(2k-2m+1)=0$, so the first sum in (2.28) equals zero.

Differentiating the relation (2.20) at $z=2k-2m+1$, for $1\leqslant k\leqslant m-1$, yields

$$\beta'(2k-2m+1)=-\left(\frac{\pi}{2}\right)^{2k-2m+1}(-1)^{k-m}\Gamma(2m-2k)\beta(2m-2k),$$

whereby the second sum in (2.28) becomes

$$-\sum_{k=0}^{m-1}\frac{(-1)^{k-1}\beta(2m-2k)}{(2k)!}.$$

So, summing up these three parts, taking account of (2.26) gives rise to (2.25). □

THEOREM 2.14. *If α is replaced with $2m-1$ in the first formula of* (2.17), *the following closed form is obtained*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m-1}}=\frac{(-1)^{m-1}(2\pi)^{2m-2}}{2(2m-2)!}\left(\zeta'\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{1}{4}+\frac{x}{2\pi}\right)+\zeta'\left(2-2m,\frac{3}{4}+\frac{x}{2\pi}\right)\right).\tag{2.29}$$

PROOF. Acting in the same manner as in deducing the formula (2.25), we come to the formula (2.29). □

Theorem 2.15. *The closed-form formula for the alternating sine series involving logarithmic function over positive odd integers is*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}}\log(2n-1)$$
$$=\frac{(-1)^{m}(2\pi)^{2m}}{8(2m-1)!}\left(\zeta'\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right.$$
$$-\zeta'\left(1-2m,\frac{1}{4}+\frac{x}{2\pi}\right)+\zeta'\left(1-2m,\frac{3}{4}+\frac{x}{2\pi}\right)$$
$$+2(\psi(2m)-\log 2\pi)\left(\zeta\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)\right.$$
$$\left.\left.-\zeta\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)\right). \tag{2.30}$$

Proof. Differentiating (2.22) with respect to s, then taking $s=2m$, we obtain

$$\frac{(-1)^{m-1}(2\pi)^{2m}}{4(2m-1)!}\left(\zeta'\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)$$
$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m}}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}}\log(2n-1)$$
$$+(\psi(2m)-\log 2\pi)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}},$$

whence we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}}\log(2n-1)$$
$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m}}$$
$$+\frac{(-1)^{m}(2\pi)^{2m}}{4(2m-1)!}\left(\zeta'\left(1-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(1-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)$$
$$+(\psi(2m)-\log 2\pi)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}}.$$

Substituting (2.25) and (2.23) for the first and second right-hand series, respectively, after rearrangement, we obtain (2.30). □

Theorem 2.16. *The closed-form formula for the alternating cosine series involving logarithmic function over positive odd integers is*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}}\log(2n-1)$$
$$=\frac{(-1)^{m-1}(2\pi)^{2m-1}}{8(2m-2)!}\left(\zeta'\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)\right.$$
$$-\zeta'\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)+\zeta'\left(2-2m,\frac{1}{4}+\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{3}{4}+\frac{x}{2\pi}\right)$$
$$-2(\log 2\pi-\psi(2m-1))\left(\zeta\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)\right.$$
$$\left.\left.-\zeta\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)\right). \tag{2.31}$$

Proof. Differentiating (2.22) with respect to s, then taking $s=2m-1$, we obtain

$$-\frac{(-1)^{m-1}(2\pi)^{2m-1}}{4(2m-1)!}\left(\zeta'\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)$$
$$=-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m-1}}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}}\log(2n-1)$$
$$+(\psi(2m-1)-\log 2\pi)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}},$$

whence we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}}\log(2n-1)$$
$$=-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m-1}}$$
$$+\frac{(-1)^{m-1}(2\pi)^{2m-1}}{4(2m-1)!}\left(\zeta'\left(2-2m,\frac{1}{4}-\frac{x}{2\pi}\right)-\zeta'\left(2-2m,\frac{3}{4}-\frac{x}{2\pi}\right)\right)$$
$$-(\log 2\pi-\psi(2m-1))\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}}.$$

Substituting (2.29) and (2.24) for the first and second right-hand series, respectively, after rearrangement, we obtain (2.31). □

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