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# Three dimensional contact metric manifolds with Cotton solitons

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**Abstract.** In this article we study a three dimensional contact metric manifold $M^3$ with Cotton solitons. We mainly consider two classes of contact metric manifolds admitting Cotton solitons. Firstly, we study a contact metric manifold with $Q\xi = \rho\xi$, where $\rho$ is a smooth function on $M$ constant along Reeb vector field $\xi$ and prove that it is Sasakian or has constant sectional curvature 0 or 1 if the potential vector field of Cotton soliton is collinear with $\xi$ or is a gradient vector field. Moreover, if $\rho$ is constant we prove that such a contact metric manifold is Sasakian, flat or locally isometric to one of the following Lie groups: $SU(2)$ or $SO(3)$ if it admits a Cotton soliton with the potential vector field being orthogonal to Reeb vector field $\xi$. Secondly, it is proved that a $(\kappa, \mu, \nu)$-contact metric manifold admitting a Cotton soliton with the potential vector field being Reeb vector field is Sasakian. Furthermore, if the potential vector field is a gradient vector field, we prove that $M$ is Sasakian, flat, a contact metric $(0, -4)$-space or a contact metric $(\kappa, 0)$-space with $\kappa < 1$ and $\kappa \neq 0$. For the potential vector field being orthogonal to $\xi$, if $\nu$ is constant we prove that $M$ is either Sasakian, or a $(\kappa, \mu)$-contact metric space.

## 1. Introduction

A *Cotton soliton* is a metric defined on a three dimensional smooth manifold $M$ such that the following equation

$$\mathscr{L}_V g + C - \sigma g = 0 \tag{1}$$

holds for a constant $\sigma$ and one vector field $V$, called *potential vector field*, where $C$ is the $(0,2)$-Cotton tensor defined by

$$C_{ij} = \frac{1}{2\sqrt{g}} C_{nmi} \varepsilon^{nm\ell} g_{\ell j} \tag{2}$$

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in a local frame of $M$. Here $\varepsilon$ is a tensor density, in an orthonormal frame $\varepsilon^{123} = 1$ and $C_{ijk}$ is Cotton tensor. As the Ricci soliton being fixed point of Ricci flow, Cotton solitons are fixed points of the Cotton flow up to diffeomorphisms and rescaling:

$$\frac{\partial}{\partial t} g(t) = C_{g(t)},$$

introduced in [14], where $C_{g(t)}$ is the $(0,2)$-Cotton tensor of $(M, g(t))$. Cotton soliton is said to be *trivial* if $C = 0$ (i.e. locally conformally flat). Using the terminology of Ricci solitons, we call a Cotton soliton *shrinking, steady* and *expanding* according as $\sigma$ is positive, zero and negative respectively. If the potential vector field $V$ is a gradient field for some function, then $g$ is called a *gradient Cotton soliton*, i.e. the following equation

$$2 \operatorname{Hess} f + C = \sigma g \tag{3}$$

is satisfied for a smooth function $f$ on $M$.

For a Riemannian case, in [18] it proved that a compact Riemannian Cotton soliton is locally conformally flat, and in the noncompact case the existence of a nontrivial shrinking Cotton soliton on Heisenberg group $\mathscr{H}$ is given. Meanwhile, for a non-Riemannian case, they gave the existence of Lorentzian Cotton solitons. Furthermore, E. Calviño-Louzao et al. studied left-invariant Cotton solitons on homogeneous manifolds, see [17].

In fact, Cotton solitons are closely related to Ricci and Yamabe solitons, which are defined respectively by

$$\mathscr{L}_V g + Ric = \sigma g \qquad \text{and} \qquad \mathscr{L}_V g = (r - \sigma) g,$$

where $Ric$ and $r$ are denoted by the Ricci tensor and scalar curvature, respectively (see the examples [16, 7]). We notice that many authors studied Ricci solitons and Yamabe solitons on contact metric manifolds, for instance, Cho and Sharma in [5, 6] studied a contact metric manifold with a Ricci soliton such that potential vector field $V$ being collinear with $\xi$, and Venkatesha-Naik [21] proved that a contact metric manifold with a Yamabe soliton is flat or it has constant scalar curvature under the assumption that $\phi Q = Q\phi$. More results can refer to [10, 11, 19, 20].

The previous works motivate us to study Cotton solitons on a three dimensional contact metric manifold. In this article, we study two classes of contact metric 3-manifolds admitting a Cotton soliton including a contact metric 3-manifolds with $Q\xi = \rho\xi$ and a $(\kappa, \mu, \nu)$-contact metric 3-manifold. In Section 3, for a contact metric 3-manifolds with $Q\xi = \rho\xi$, we first assume that the function $\rho$ is constant along Reeb vector field $\xi$. Such a class of contact metric manifolds was studied in [2] under the hypothesis of pseudosymmetric.

We classify such a class of contact metric manifold admitting a Cotton soliton with potential vector field $V$ being collinear with $\xi$ or a gradient vector field. For $V$ being orthogonal to Reeb vector field, we need to assume that $\rho$ is a constant function. For a $(\kappa, \mu, \nu)$-contact metric manifold, in Section 4 we also consider the potential vector field of a Cotton soliton being Reeb vector field, a gradient vector field and orthogonal to $\xi$, respectively. In order to state and prove our conclusions, we need to give some preliminaries of contact manifolds, which are presented in Section 2.

## 2. Preliminaries

A contact metric manifold is a smooth manifold $M^{2n+1}$ with a global one form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. The one form $\eta$ induces an almost contact structure $(\phi, \xi, \eta)$ on $M$, which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi \circ \xi = 0.$$

Here $\xi$ is a unique vector field (called *Reeb* or *characteristic vector field*) dual to $\eta$ and satisfying $d\eta(\xi, X) = 0$ for all $X$. It is well-known that there exists a Riemannian metric $g$ such that

$$d\eta(X, Y) = g(X, \phi Y), \qquad g(X, \xi) = \eta(X)$$

for any $X, Y \in \mathfrak{X}(M)$. We refer to $(M^{2n+1}, \phi, \xi, \eta, g)$ as a *contact metric manifold*. A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ for which Reeb vector field $\xi$ is Killing, i.e. $\mathscr{L}_\xi g = 0$, is called a $K$-contact manifold.

On a contact metric manifold, we recall a operator $h = \frac{1}{2}\mathscr{L}_\xi \phi$, which is a self-dual operator, and $\ell = R(\cdot, \xi)\xi$. Concerning the operators the following identities, which were given in [3], are satisfied:

$$\begin{cases} h\xi = 0, \quad \phi h = -h\phi, \quad \nabla_X \xi = -\phi X - \phi h X, \quad g(hX, Y) = g(X, hY), \\ \text{trace}(h) = \text{trace}(\phi h) = 0, \quad \eta \circ h = 0, \\ \text{trace}(\ell) = g(Q\xi, \xi) = 2n - \text{trace}(h^2). \end{cases} \tag{4}$$

If $h = 0$ then we have $\mathscr{L}_\xi g = 0$, that means that $M^{2n+1}$ is a $K$-contact manifold.

One can define a complex structure $J$ on $M \times \mathbb{R}$ by $J\big(X, f\frac{d}{dt}\big) = \big(\phi X - f\xi, \eta(X)\frac{d}{dt}\big)$ for any $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M \times \mathbb{R})$. A contact metric structure $(\phi, \xi, \eta, g)$ is said to be *normal* and $M$ is called *Sasakian* if the corresponding complex structure $J$ on $M \times \mathbb{R}$ is integrable. A Sasakian manifold is a $K$-contact manifold and the converse does not hold, but if $\dim M = 3$ then a $K$-contact manifold is Sasakian.

In the following we assume that $M$ is a 3-dimensional contact metric manifold. Let $U$ be the open subset where the tensor $h \neq 0$ and $U'$ be the open subset such that $h$ is identically zero. $U \cup U'$ is open dense in $M$ because $h$ is a smooth function on $M$, thus a property that is satisfied in $U' \cup U$ is also satisfied in $M$. For any $p \in U' \cup U$, there exists a local orthonormal frame field $\mathscr{E} = \{e_1 = e, e_2 = \phi e, e_3 = \xi\}$ such that $he = \lambda e$ and $h\phi e = -\lambda\phi e$ on $U$, where $\lambda$ is a positive non-vanishing smooth function of $M$.

First of all, we have the following lemma:

LEMMA 1 ([9]). *In the open subset $U$, the Levi-Civita connection $\nabla$ is given by*

$$\begin{aligned} &\nabla_\xi e = a\phi e, &&\nabla_\xi \phi e = -ae, &&\nabla_\xi \xi = 0,\\ &\nabla_e \xi = -(1+\lambda)\phi e, &&\nabla_e e = b\phi e, &&\nabla_e \phi e = -be + (1+\lambda)\xi,\\ &\nabla_{\phi e}\xi = (1-\lambda)e, &&\nabla_{\phi e}\phi e = ce, &&\nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi, \end{aligned}$$

*where $a$ is a smooth function,*

$$b = \frac{1}{2\lambda}[\phi e(\lambda) + A] \qquad \text{with } A = Ric(e,\xi), \tag{5}$$

$$c = \frac{1}{2\lambda}[e(\lambda) + B] \qquad \text{with } B = Ric(\phi e,\xi). \tag{6}$$

The components of Ricci operator $Q$ are given by

$$\begin{cases} Qe = \left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right)e + Z\phi e + A\xi,\\ Q\phi e = Ze + \left(\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda\right)\phi e + B\xi,\\ Q\xi = Ae + B\phi e + 2(1-\lambda^2)\xi, \end{cases} \tag{7}$$

where $Z = \xi(\lambda)$ and the scalar curvature

$$r = \operatorname{trace}(Q) = 2(1 - \lambda^2 - b^2 - c^2 + 2a + e(c) + \phi e(b)). \tag{8}$$

Moreover, it follows from Lemma 1 that

$$\begin{cases} [e,\phi e] = \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi,\\ [e,\xi] = \nabla_e \xi - \nabla_\xi e = -(a+\lambda+1)\phi e,\\ [\phi e,\xi] = \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\lambda+1)e. \end{cases} \tag{9}$$

Putting $X = e$, $Y = \phi e$ and $Z = \xi$ in the Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ and using (9), we conclude

$$\begin{cases} b(a+\lambda+1) - \xi(c) - \phi e(\lambda) - \phi e(a) = 0,\\ c(a-\lambda+1) + \xi(b) + e(\lambda) - e(a) = 0. \end{cases} \tag{10}$$

PROPOSITION 1. *If the Reeb vector field $\xi$ is an eigenvector of $Q$, in the open subset $U$ the components of $(0,2)$-Cotton tensor $C$ can be expressed* *as follows:*

$$C_{11} = C(e,e) = -(1-\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 - 2a\lambda\right) - \xi(Z) + 4a^2\lambda, \tag{11}$$

$$C_{12} = C(e,\phi e) = -2\lambda\xi(a) - 4aZ - (1-\lambda)Z + \frac{1}{4}\xi(r), \tag{12}$$

$$C_{13} = C(e,\xi) = e(Z) - 4ab\lambda - \phi e(\lambda^2 - 2a\lambda) - 2cZ - \frac{1}{4}\phi e(r), \tag{13}$$

$$C_{22} = C(\phi e,\phi e) = \xi(Z) - 4a^2\lambda - (1+\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 + 2a\lambda\right), \tag{14}$$

$$C_{23} = C(\phi e,\xi) = e(\lambda^2 + 2a\lambda) + 2bZ - \phi e(Z) - 4ac\lambda + \frac{1}{4}e(r), \tag{15}$$

$$C_{33} = C(\xi,\xi) = r + 4a\lambda^2 - 6(1-\lambda^2). \tag{16}$$

Proof. It is well-known that the Cotton tensor is defined by

$$C(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \tag{17}$$

for all $X$, $Y$, $Z$, where

$$S(X,Y) = Ric(X,Y) - \frac{r}{4}g(X,Y)$$

is the Schouten tensor. In the frame field $\mathscr{E}$, by (2) the $(0,2)$-Cotton tensor is simplified as

$$C_{ij} = \frac{1}{2}C_{nmi}\varepsilon^{nmj}, \qquad i,j = 1,2,3,$$

where $C_{ijk} = C(e_i,e_j)e_k$. It is clear that $C_{ijk} = -C_{jik}$ and $C_{iik} = 0$ for all $i$, $j$, $k$. Thus

$$\begin{aligned} C_{11} &= \frac{1}{2}C_{nm1}\varepsilon^{nm1} = \frac{1}{2}C_{1m1}\varepsilon^{1m1} + \frac{1}{2}C_{2m1}\varepsilon^{2m1} + \frac{1}{2}C_{3m1}\varepsilon^{3m1} \\ &= \frac{1}{2}C_{231}\varepsilon^{231} + \frac{1}{2}C_{321}\varepsilon^{321} = C_{231}. \end{aligned}$$

Analogously, we have

$$C_{12} = C_{311}, \qquad C_{13} = C_{121}, \qquad C_{22} = C_{312}, \qquad C_{23} = C_{122}, \qquad C_{33} = C_{123}.$$

Since $\xi$ is an eigenvector of $Q$, by the third term of (7) we have $A = B = 0$. Next, making use of (17) and Lemma 1, we directly compute the components of $C$ as follows:

$$
\begin{aligned}
C_{11} &= (\nabla_{e_2} S)(e_3, e_1) - (\nabla_{e_3} S)(e_2, e_1) = (\nabla_{\phi e}\ Ric)(\xi, e) - (\nabla_\xi\ Ric)(\phi e, e) \\
&= -Ric(\nabla_{\phi e}\xi, e) - Ric(\xi, \nabla_{\phi e} e) - \xi(Z) + Ric(\nabla_\xi \phi e, e) + Ric(\phi e, \nabla_\xi e) \\
&= -(1-\lambda)\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) - 2(\lambda - 1)(1 - \lambda^2) \\
&\quad - \xi(Z) - a\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) + a\left(\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda\right) \\
&= -(1-\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 - 2a\lambda\right) - \xi(Z) + 4a^2\lambda,
\end{aligned}
$$

$$
\begin{aligned}
C_{12} &= (\nabla_{e_3} S)(e_1, e_1) - (\nabla_{e_1} S)(e_3, e_1) = (\nabla_\xi\ Ric)(e, e) - (\nabla_e\ Ric)(\xi, e) - \frac{1}{4}\xi(r) \\
&= \xi\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) - 2\ Ric(\nabla_\xi e, e) + Ric(\nabla_e \xi, e) + Ric(\xi, \nabla_e e) - \frac{1}{4}\xi(r) \\
&= \xi\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) - 2aZ - (1+\lambda)Z - \frac{1}{4}\xi(r) \\
&= -2\lambda\xi(a) - 4aZ - (1-\lambda)Z + \frac{1}{4}\xi(r),
\end{aligned}
$$

$$
\begin{aligned}
C_{13} &= (\nabla_{e_1} S)(e_2, e_1) - (\nabla_{e_2} S)(e_1, e_1) = (\nabla_e\ Ric)(\phi e, e) - (\nabla_{\phi e}\ Ric)(e, e) + \frac{1}{4}\phi e(r) \\
&= e(Z) - Ric(\nabla_e \phi e, e) - Ric(\phi e, \nabla_e e) - \phi e\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) \\
&\quad + 2\ Ric(\nabla_{\phi e} e, e) + \frac{1}{4}\phi e(r) \\
&= e(Z) - 4ab\lambda - \phi e\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) - 2cZ + \frac{1}{4}\phi e(r) \\
&= e(Z) - 4ab\lambda - \phi e(\lambda^2 - 2a\lambda) - 2cZ - \frac{1}{4}\phi e(r),
\end{aligned}
$$

$$
\begin{aligned}
C_{22} &= (\nabla_{e_3} S)(e_1, e_2) - (\nabla_{e_1} S)(e_3, e_2) = (\nabla_\xi\ Ric)(e, \phi e) - (\nabla_e\ Ric)(\xi, \phi e) \\
&= \xi(Z) - Ric(\nabla_\xi e, \phi e) - Ric(e, \nabla_\xi \phi e) + Ric(\nabla_e \xi, \phi e) + Ric(\xi, \nabla_e \phi e) \\
&= \xi(Z) - 4a^2\lambda - (1+\lambda)\left(\frac{1}{2}r + 2a\lambda\right) + 3(1+\lambda)(1-\lambda^2),
\end{aligned}
$$

$$
\begin{aligned}
C_{23} &= (\nabla_{e_1} S)(e_2, e_2) - (\nabla_{e_2} S)(e_1, e_2) \\
&= (\nabla_e\ Ric)(\phi e, \phi e) - (\nabla_{\phi e}\ Ric)(e, \phi e) - \frac{1}{4}e(r)
\end{aligned}
$$

$$= e\left(\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda\right) - 2\, Ric(\nabla_e \phi e, \phi e) - \phi e(Z) + Ric(\nabla_{\phi e} e, \phi e)$$

$$+ Ric(e, \nabla_{\phi e} \phi e) - \frac{1}{4} e(r)$$

$$= e(\lambda^2 + 2a\lambda) + 2bZ - \phi e(Z) - 4ac\lambda + \frac{1}{4} e(r),$$

$$C_{33} = (\nabla_{e_1} S)(e_2, e_3) - (\nabla_{e_2} S)(e_1, e_3) = (\nabla_e\, Ric)(\phi e, \xi) - (\nabla_{\phi e}\, Ric)(e, \xi)$$

$$= -Ric(\nabla_e \phi e, \xi) - Ric(\phi e, \nabla_e \xi) + Ric(\nabla_{\phi e} e, \xi) + Ric(e, \nabla_{\phi e} \xi)$$

$$= \left(\frac{1}{2}r + 2a\lambda\right)(1 + \lambda) - 6(1 - \lambda^2) + (1 - \lambda)\left(\frac{1}{2}r - 2a\lambda\right)$$

$$= r + 4a\lambda^2 - 6(1 - \lambda^2).$$

This completes the proof. □

## 3. Contact metric 3-manifolds with $Q\xi = \rho\xi$

First we assume that the function $\rho$ is constant along Reeb vector field $\xi$ and prove the following conclusion.

THEOREM 1. *Let $(M^3, \phi, \xi, \eta, g)$ be a contact metric manifold such that $Q\xi = \rho\xi$, where $\rho$ is a smooth function on $M^3$ constant along Reeb vector field $\xi$. If $M$ admits a Cotton soliton with potential vector field being collinear with Reeb vector field $\xi$, then $M$ either is Sasakian, or has constant sectional curvature* 0 *or* 1.

PROOF. We can denote $U'$ and $U$ as follows:

$$U' = \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\},$$

$$U = \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\}.$$

If $M = U'$, then $M$ is Sasakian. In the following we assume that $U$ is not empty, and let $\{e, \phi e, \xi\}$ be a $\phi$-basis in $U$.

The assumption that $Q\xi = \rho\xi$ and (7) imply $A = B = 0$ and $\rho = 2(1 - \lambda^2)$, where $\xi(\rho) = 0$. From this we know $Z = \xi(\lambda) = 0$.

If $V = 0$ then Cotton equation (1) becomes $C = \sigma g$. Since the $(0, 2)$-tensor $C$ is trace-free, we see that $\sigma$ must vanish, thus $M$ is locally conformally flat. By Theorem 4.1 of [8], $M$ has constant sectional curvature 0 or 1.

Next we suppose that $V = f\xi$ for some non-zero smooth function $f$. Then in view of (4), for any $X, Y \in \mathfrak{X}(M)$, Cotton soliton equation (1) may be expressed as

$$-2fg(\phi hX, Y) + X(f)\eta(Y) + Y(f)\eta(X) + C(X, Y) = \sigma g(X, Y). \tag{18}$$

Letting $X = Y = e$ in (18) and recalling (11) imply

$$-(1-\lambda)\left(\frac{1}{2}r - 2a\lambda\right) + 4a^2\lambda + 3(1-\lambda)(1-\lambda^2) = \sigma. \tag{19}$$

Similarly, letting $X = Y = \phi e$ in (18) and recalling (14) give

$$-4a^2\lambda - (1+\lambda)\left(\frac{1}{2}r + 2a\lambda\right) + 3(1+\lambda)(1-\lambda^2) = \sigma \tag{20}$$

and putting $X = e$ and $Y = \phi e$ in (18) and using (12) give

$$-2\lambda\xi(a) + \frac{1}{4}\xi(r) = 2\lambda f. \tag{21}$$

Now using (19) to plus (20) implies

$$2\sigma = -r - 4a\lambda^2 + 6(1-\lambda^2). \tag{22}$$

Comparing (22) with (19), we conclude

$$2a(2a + 1 - \lambda^2) = \sigma.$$

Moreover, differentiating this along $\xi$ implies

$$(4a + 1 - \lambda^2)\xi(a) = 0 \tag{23}$$

since $\sigma$ is constant and $\xi(\lambda) = 0$.

If $\xi(a) = 0$ then differentiating (22) along $\xi$ yields $\xi(r) = 0$. By (21), we have $f = 0$ since $\lambda > 0$. This shows that Cotton soliton is trivial.

If $\xi(a) \neq 0$ on some open subset $\mathcal{O} \subset U$, then $\lambda^2 = 1 + 4a$ by (23). Therefore, by differentiating this along $\xi$, we see $\xi(a) = 0$. This is a contradiction.

We complete the proof theorem. □

For a gradient Cotton soliton on $M^3$, we prove the following conclusion.

Theorem 2. *Let $(M^3, \phi, \xi, \eta, g)$ be a contact metric manifold such that $Q\xi = \rho\xi$, where $\rho$ is a smooth function on $M$ constant along Reeb vector field $\xi$. If $M$ admits a gradient Cotton soliton, then $M$ either is Sasakian, or has constant sectional curvature* 0 *or* 1.

Proof. As before if $M = U'$ then $M$ is Sasakian. Let $\{e, \phi e, \xi\}$ be a $\phi$-basis in non-empty set $U$. First we write the potential vector field

$$V = \nabla f = f_1 e + f_2\phi e + f_3\xi,$$

where $f_1$, $f_2$, $f_3$ are three smooth functions on $M$. Since $C$ is divergence-free, we have $Q\nabla f = 0$ (see [18, Remark 3]). Hence we derive from (7) that

$$f_1\left(\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda\right) = 0, \qquad f_2\left(\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda\right) = 0,$$
$$f_3(1 - \lambda^2) = 0. \tag{24}$$

If $V = 0$ then Cotton soliton is trivial as in the proof of Theorem 1. Now we assume that at least one of $f_1$, $f_2$, $f_3$ is nonzero. Next we will divide into two cases to discuss.

**Case I:** If $\lambda = 1$ then $b = c = 0$ by (5) and (6). Moreover, Equation (8) implies $r = 4a$, thus it follows from the second term of Equation (24) that $af_2 = 0$.

For every Riemannian manifold we recall the following well-known formula:

$$\frac{1}{2}\nabla r = \operatorname{div} Q.$$

Making use of (7) and the above formula, a direct computation deduces that $\nabla a = 0$, i.e. $a$ is constant. If $a = 0$ all components of $C$ are zero, that means that $M$ is locally conformally flat. If $a \neq 0$, then $f_2 = 0$. By Proposition 1, the components of $C$ become

$$C_{11} = 4a^2, \qquad C_{12} = 0, \qquad C_{13} = 0,$$
$$C_{22} = -4a^2 - 8a, \qquad C_{23} = 0, \qquad C_{33} = 8a. \tag{25}$$

For any $X, Y \in \mathfrak{X}(M)$, the gradient Cotton soliton equation (3) is expressed as

$$2g(\nabla_X \nabla f, Y) + C(X, Y) = \sigma g(X, Y). \tag{26}$$

By taking $X = Y = e$ in (26) and using (25), we get

$$2e(f_1) + 4a^2 = \sigma$$

and taking $X = \xi$ and $Y = e$ in (26) gives $\xi(f_1) = 0$. Finally, putting $X = \phi e$ and $Y = e$ in (26) implies $\phi e(f_1) = 0$ since $\lambda = 1$. By the third term of (9) acting on $f_1$, we find $e(f_1) = 0$, which shows $\sigma = 4a^2$. Moreover, putting $X = Y = \phi e$ in (26) gives

$$-4a^2 - 8a = \sigma = 4a^2,$$

which shows $a = -1$.

Similarly, we can obtain from (26) that $\phi e(f_3) = 0$, $e(f_3) = 0$ and $\xi(f_3) + 8a = \sigma = 4a^2$, i.e. $\xi(f_3) = 12$ as $a = -1$. However, the first term of (9) acting on $f_3$ implies $\xi(f_3) = 0$ because $b = c = 0$. This leads to a contradiction.

**Case II:** If $\lambda \neq 1$ in some open set $\mathcal{O} \subset U$ then $f_3 = 0$ by the third term of (24). Putting $X = Y = \xi$ in (26) and using (16) we have

$$r + 4a\lambda^2 - 6(1 - \lambda^2) = \sigma. \tag{27}$$

Letting $X = e$ and $Y = \xi$ in (26), we conclude from (13) and (5) that

$$2f_2(1 + \lambda) - (4b\lambda^2 - 2\lambda\phi e(a)) - \frac{1}{4}\phi e(r) = 0. \tag{28}$$

Similarly, letting $X = \phi e$ and $Y = \xi$ in (26), we conclude from (6) and (15) that

$$2f_1(\lambda - 1) + 4c\lambda^2 + 2\lambda e(a) + \frac{1}{4}e(r) = 0. \tag{29}$$

Next we consider the following open sets:

$$\mathcal{O}_1 = \left\{ p \in \mathcal{O} : \frac{1}{2}r - 1 + \lambda^2 - 2a\lambda \neq 0 \text{ in a neighborhood of } p \right\},$$

$$\mathcal{O}_2 = \left\{ p \in \mathcal{O} : \frac{1}{2}r - 1 + \lambda^2 - 2a\lambda = 0 \text{ in a neighborhood of } p \right\},$$

where the set $\mathcal{O}_1 \cup \mathcal{O}_2$ is open and dense in the closure of $\mathcal{O}$. In the set $\mathcal{O}_1$, it implies $f_1 = 0$ from the first term of (24). Since $f_3 = 0$, we must have that $f_2 \neq 0$ in $\mathcal{O}_1$. Hence the second term of (24) yields

$$\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda = 0.$$

By comparing it with (27), we get

$$-4(1 - \lambda^2) - 4a(\lambda - \lambda^2) = \sigma. \tag{30}$$

Since Poincare Lemma $d^2 f = 0$, i.e. the relation

$$g(\nabla_X \nabla f, Y) = g(\nabla_Y \nabla f, X) \tag{31}$$

holds for any $X, Y \in \mathfrak{X}(M)$, letting $X = \xi$ and $Y = e$ in (31) and using Lemma 1, we obtain

$$a = -1 - \lambda.$$

Substituting this into (30) implies that $\lambda$ and $a$ are constants. Thus $b = c = 0$ by (5) and (6). Furthermore, it follows from (27) that $r$ is also constant. Recalling (28), we find $f_2(1+\lambda) = 0$. This shows $f_2 = 0$ since $\lambda > 0$ in $\mathcal{O}$. The contradiction means that $\mathcal{O}_1$ is empty.

In $\mathcal{O}_2$, the following relation holds:

$$\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda = 0. \tag{32}$$

Then $af_2 = 0$ by the second term of (24). Write

$$\mathcal{V}_1 = \{p \in \mathcal{O}_2 : a \neq 0\} \qquad \text{and} \qquad \mathcal{V}_2 = \{p \in \mathcal{O}_2 : a = 0\}.$$

Here $\mathcal{V}_1 \cup \mathcal{V}_2$ is the open and dense in the closure of $\mathcal{O}_2$. Then $f_2 = 0$ in $\mathcal{V}_1$. Letting $X = \xi$ and $Y = \phi e$ in (31) and using Lemma 1, we obtain

$$a = -1 + \lambda$$

since $f_1 \neq 0$ in $\mathcal{V}_1$. Adopting analogous method as before, we can prove that $b = c = 0$ and $a$, $r$ are constants. Thus (29) implies $f_1 = 0$. The contradiction shows that $\mathcal{V}_1$ is empty. Thus $a = 0$ in $\mathcal{O}_2$ and it implies from (32) that

$$r = 2(1 - \lambda^2).$$

Inserting this into (27) implies $\sigma = -4(1-\lambda^2)$. This shows that $r$ is constant and $b = c = 0$. However, Equations (28) and (29) yield $f_1 = f_2 = 0$ since $\lambda \neq 1$. It is impossible.

We complete the proof of theorem. □

Furthermore, for the potential vector field $V$ being orthogonal to $\xi$, we need more strong hypothesis that $\rho$ is constant.

Theorem 3. *Let $(M^3, \phi, \xi, \eta, g)$ be a contact metric manifold such that $Q\xi = \rho\xi$, where $\rho$ is constant. If $M$ admits a Cotton soliton with potential vector field being orthogonal to Reeb vector field $\xi$, then $M$ is either*

(a) *Sasakian,*
(b) *flat,*
(c) *locally isometric to one of the following Lie groups equipped with a left invariant metric: $SU(2)$ or $SO(3)$.*

Proof. Under the assumption, by the main theorem of [12], the Ricci operator is expressed as

$$Q = \alpha I + \beta \eta \otimes \xi + \gamma h,$$

where $\alpha = \frac{1}{2}(r - 2k)$, $\beta = \frac{1}{2}(6k - r)$, $\gamma = -\alpha$ and $k = \frac{1}{2}\operatorname{trace}(\ell)$. Moreover, $r$ and $\lambda = \sqrt{1-k}$ are constants. Thus we have $b = c = A = B = Z = 0$ and $a = \frac{1}{2}\alpha$ is also constant from (7).

When $\lambda = 0$, $M$ is Sasakian. In the following we assume $\lambda > 0$. By Proposition 1, the components of $C$ become

$$\begin{cases} C_{11} = (1-\lambda)(\beta + \alpha\lambda) + \alpha^2\lambda, \\ C_{12} = 0, \quad C_{13} = 0, \\ C_{22} = (1+\lambda)(\beta - \alpha\lambda) - \alpha^2\lambda, \\ C_{23} = 0, \quad C_{33} = 2\alpha + 2\alpha\lambda^2 - 4k. \end{cases} \tag{33}$$

Set $V = f_1 e + f_2 \phi e$, where $f_1$, $f_2$ are smooth functions on $M$. For any $X, Y \in \mathfrak{X}(M)$, Cotton soliton equation (1) is rewritten as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + C(X, Y) = \sigma g(X, Y). \tag{34}$$

Putting $X = Y = e$ in (34), it follows from Lemma 1 and (33) that

$$2e(f_1) + (1-\lambda)(\beta + \alpha\lambda) + \alpha^2\lambda = \sigma. \tag{35}$$

Putting $X = Y = \phi e$ in (34), it follows from Lemma 1 that

$$2\phi e(f_2) + (1+\lambda)(\beta - \alpha\lambda) - \alpha^2\lambda = \sigma. \tag{36}$$

Similarly, putting $X = Y = \xi$ in (34) and using (33) we have

$$\sigma = 2\alpha + 2\alpha\lambda^2 - 4k. \tag{37}$$

Letting $X = e$ and $Y = \xi$ in (34), it implies from Lemma 1 that

$$f_2(1 + \lambda - a) + \xi(f_1) = 0. \tag{38}$$

Letting $X = e$ and $Y = \phi e$ in (34) implies

$$e(f_2) + \phi e(f_1) = 0 \tag{39}$$

and letting $X = \phi e$ and $Y = \xi$ in (34) implies

$$f_1(\lambda - 1 + a) + \xi(f_2) = 0. \tag{40}$$

Now differentiating (38) along $e$ and using (39), we have

$$-\phi e(f_1)(1 + \lambda - a) + e(\xi(f_1)) = 0.$$

Since $e(f_1)$ is constant by (35), applying the second term of (9) in $f_1$ provides

$$e(\xi(f_1)) = \xi(e(f_1)) - (a + \lambda + 1)\phi e(f_1) = -(a + \lambda + 1)\phi e(f_1).$$

Substituting this into previous formula gives $\phi e(f_1) = 0$, which implies $e(f_2) = 0$ from (39).

Further, applying the first term of (9) in $f_1$ and $f_2$ respectively provides $\xi(f_1) = \xi(f_2) = 0$. Therefore (38) and (40) become

$$f_2(1 + \lambda - a) = 0 \qquad \text{and} \qquad f_1(\lambda - 1 + a) = 0. \tag{41}$$

If $1 + \lambda - a = 0$ then $f_1 = 0$. Applying the second term of (9) in $f_2$ provides

$$e(\xi(f_2)) - \xi(e(f_2)) = -(a + \lambda + 1)\phi e(f_2) = -2(\lambda + 1)\phi e(f_2).$$

This shows that $\phi e(f_2) = 0$, i.e. $f_2$ is constant. Moreover, (35) and (36) become

$$(1 + \lambda)(\beta - \alpha\lambda) - \alpha^2\lambda = \sigma,$$

$$(1 - \lambda)(\beta + \alpha\lambda) + \alpha^2\lambda = \sigma.$$

The above equations, combining the relation $\beta = 2k - \alpha$ and (37), imply

$$2\alpha - 2k + \alpha^2 = 0 \qquad \text{and} \qquad 2\alpha - 2k - \alpha k = 0.$$

That is, $\alpha = -k$. Because $1 + \lambda = a = \frac{1}{2}\alpha$ and $\lambda^2 = 1 - k$, we get $\alpha = 8$ and $\lambda = 3$. Equation (9) becomes

$$[e, \phi e] = 2\xi, \qquad [\xi, e] = 8\phi e, \qquad [\phi e, \xi] = 2e.$$

Thus $M$ is locally isometric to $SU(2)$ or $SO(3)$ according to [12, Theorem 3].

If $\lambda - 1 + a = 0$ then $f_2 = 0$ by (41). Applying the third term of (9) in $f_1$ provides

$$0 = \phi e(\xi(f_1)) - \xi(\phi e(f_1)) = (a - \lambda + 1)e(f_1) = -2(\lambda - 1)e(f_1). \tag{42}$$

For $\lambda = 1$, then $k = 0$ and $a = 0$. Therefore, by (35) and (37), we have $e(f_1) = 4\alpha - \alpha^2 = 0$. In this case $M$ is flat and $f_1$ is constant. When $\lambda \neq 1$, Equation (42) shows that $e(f_1) = 0$, i.e. $f_1$ is constant. As before, from (35), (36) and (37) we can obtain $\alpha = 8$ and $\lambda = -3$. It is impossible.

Summing up the above discussion, we thus complete the proof of theorem. □

## 4. $(\kappa, \mu, \nu)$-contact metric 3-manifolds

Definition 1 ([13]). A contact metric manifold $(M^3, \phi, \xi, \eta, g)$ is called a $(\kappa, \mu, \nu)$-*contact metric manifold* if the curvature tensor satisfies the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY)$$

for any vector fields $X$, $Y$, where $\kappa$, $\mu$ and $\nu$ are smooth functions on $M$.

In particular, if $\nu = 0$ and $\kappa$, $\mu$ are constants, $M$ is said to be a $(\kappa,\mu)$-*contact metric space* (cf. [4]).

LEMMA 2 ([15, Lemma 4.3]). *For every $p \in U$, there exists an open neighborhood $W$ of $p$ and orthonormal local vector fields $e$, $\phi e$, $\xi$, defined on $W$, such that*

$$he = \lambda e, \qquad h\phi e = -\lambda\phi e, \qquad h\xi = 0,$$

*where $\lambda = \sqrt{1-\kappa}$.*

LEMMA 3. *Let $(M^3, \phi, \xi, \eta, g)$ be a $(\kappa,\mu,\nu)$-contact metric manifold. Then*

$$\xi(r) = 2\xi(\kappa) = -4(1-\kappa)\nu.$$

PROOF. For a $(\kappa,\mu,\nu)$-contact metric manifold the Ricci operator may be expressed as (see [1, Eq. (3.3)]):

$$Q = \left(\frac{1}{2}r - \kappa\right)I + \left(-\frac{1}{2}r + 3\kappa\right)\eta\otimes\xi + \mu h + \nu\phi h. \tag{43}$$

Taking the basis $\{e, \phi e, \xi\}$, by Lemma 2, we thus have

$$Q\xi = 2\kappa\xi,$$

$$Qe = \left(\frac{1}{2}r - \kappa + \lambda\mu\right)e + \lambda\nu\phi e,$$

$$Q\phi e = \left(\frac{1}{2}r - \kappa - \lambda\mu\right)\phi e + \lambda\nu e.$$

It implies from (7) that $Z = \lambda\nu$. Now using Lemma 1, we obtain

$$(\nabla_\xi Q)\xi = 2\xi(\kappa)\xi,$$

$$(\nabla_e Q)e = \nabla_e(Qe) - Q\nabla_e e = e\left(\frac{1}{2}r - \kappa + \lambda\mu\right)e + b\left(\frac{1}{2}r - \kappa + \lambda\mu\right)\phi e$$

$$+ e(\lambda\nu)\phi e + \lambda\nu(-be + (1+\lambda)\xi) - bQ\phi e$$

$$= e\left(\frac{1}{2}r - \kappa + \lambda\mu\right)e + b\left(\frac{1}{2}r - \kappa + \lambda\mu\right)\phi e$$

$$+ e(\lambda v)\phi e + \lambda v(-be + (1+\lambda)\xi) - b\left(\left(\frac{1}{2}r - \kappa - \lambda\mu\right)\phi e + \lambda v e\right)$$

$$= \left\{e\left(\frac{1}{2}r - \kappa + \lambda\mu\right) - 2b\lambda v\right\}e + \{2b\lambda\mu + e(\lambda v)\}\phi e + \lambda v(1+\lambda)\xi,$$

$$(\nabla_{\phi e}Q)\phi e = \nabla_{\phi e}(Q\phi e) - Q\nabla_{\phi e}\phi e = \phi e\left(\frac{1}{2}r - \kappa - \lambda\mu\right)\phi e + c\left(\frac{1}{2}r - \kappa - \lambda\mu\right)e$$

$$+ \phi e(\lambda v)e + \lambda v(-c\phi e + (\lambda - 1)\xi) - cQe$$

$$= \phi e\left(\frac{1}{2}r - \kappa - \lambda\mu\right)\phi e + c\left(\frac{1}{2}r - \kappa - \lambda\mu\right)e$$

$$+ \phi e(\lambda v)e + \lambda v(-c\phi e + (\lambda - 1)\xi) - c\left(\left(\frac{1}{2}r - \kappa + \lambda\mu\right)e + \lambda v\phi e\right)$$

$$= \{\phi e(\lambda v) - 2c\lambda\mu\}e + \left\{\phi e\left(\frac{1}{2}r - \kappa - \lambda\mu\right) - 2c\lambda v\right\}\phi e + \lambda v(\lambda - 1)\xi.$$

Since $\frac{1}{2}\nabla r = \operatorname{div} Q$, which, in the basis $\{e, \phi e, \xi\}$, is written as

$$\frac{1}{2}\{e(r)e + \phi e(r)\phi e + \xi(r)\xi\} = (\nabla_e Q)e + (\nabla_{\phi e}Q)\phi e + (\nabla_\xi Q)\xi,$$

we conclude

$$\frac{1}{2}\xi(r) = 2\lambda^2 v + 2\xi(\kappa).$$

Since $\xi(\lambda) = Z = \lambda v$ and $\lambda = \sqrt{1-\kappa}$, we get the desired conclusion. □

In the following we use the above two lemmas to prove our conclusions.

THEOREM 4. *Let $(M^3, \phi, \xi, \eta, g)$ be a $(\kappa, \mu, v)$-contact metric manifold. If $M$ admits a Cotton soliton such that the potential vector field $V$ is the Reeb vector field $\xi$, then $M$ is Sasakian.*

PROOF. As before if $M = U'$ then $M$ is Sasakian. Cotton soliton equation (1), for any $X, Y \in \mathfrak{X}(M)$, is expressed as

$$-2g(\phi hX, Y) + C(X, Y) = \sigma g(X, Y). \tag{44}$$

The relation $Q\xi = 2\kappa\xi$ shows $A = B = 0$ from the third term of (7). Furthermore, $Z = \lambda v$ and $\mu = -2a$ by (7) and (43).

Letting $X = Y = e$ in (44) and using (11) imply

$$-(1-\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 - 2a\lambda\right) - \xi(Z) + 4a^2\lambda = \sigma$$

and letting $X = Y = \phi e$ in (44) and using (14) give

$$\xi(Z) - 4a^2\lambda - (1+\lambda)\left(\frac{1}{2}r + 2a\lambda\right) + 3(1+\lambda)(1-\lambda^2) = \sigma.$$

The previous two formulas yield

$$2\sigma = -r - 4a\lambda^2 + 6(1-\lambda^2). \tag{45}$$

Putting $X = Y = \xi$ in (44) and using (16), we have

$$r - 2\mu(1-\kappa) - 6\kappa = \sigma. \tag{46}$$

This yields $\sigma = 0$ by comparing (45) with (46). That is,

$$r = 2\mu\lambda^2 + 6\kappa. \tag{47}$$

Putting $X = e$ and $Y = \phi e$ in (44) gives

$$\lambda\xi(\mu) + 2\mu\lambda v - (1-\lambda)\lambda v + \frac{1}{4}\xi(r) = 2\lambda. \tag{48}$$

Similarly, using (13) and (15) respectively, we deduce

$$\lambda e(v) - \lambda(4b\lambda + \phi e(\mu)) - \frac{1}{4}\phi e(r) = 0, \tag{49}$$

$$\lambda(4c\lambda - e(\mu)) - \lambda\phi e(v) + \frac{1}{4}e(r) = 0. \tag{50}$$

Here we have used $Z = \lambda v$, $a = -\frac{1}{2}\mu$ and Equations (5) and (6).

Because $\xi(\lambda) = Z = \lambda v$, differentiating (47) with respect to $\xi$ gives

$$\xi(r) = 2\xi(\mu\lambda^2) + 6\xi(\kappa) = 2\lambda^2\xi(\mu) + 4\mu v\lambda^2 + 6\xi(\kappa).$$

By Lemma 3, we see

$$\xi(\mu) = 4v - 2\mu v. \tag{51}$$

Substituting (51) into (48), we obtain

$$v = \frac{2}{3}. \tag{52}$$

For a $(\kappa, \mu, v)$-contact metric manifold, we recall the following equations (see [13, Eq. (4-18)]):

$$e(\kappa) - \lambda e(\mu) - \lambda\phi e(v) = 0, \tag{53}$$

$$-\phi e(\kappa) - \lambda\phi e(\mu) + \lambda e(v) = 0. \tag{54}$$

Making use of (49) and (50), we obtain from (47) that

$$\phi e(\mu) = -4b - 4b\mu,$$

$$e(\mu) = -4c - 4c\mu.$$

Hence, by (52), inserting this into (53) and (54) respectively gives

$$0 = \phi e(v) = -4c\lambda + 4c + 4c\mu, \tag{55}$$

$$0 = e(v) = -4b\lambda - 4b - 4b\mu. \tag{56}$$

Next we decompose three cases to discuss.

**Case I:** If $b = c = 0$ then $e(\mu) = \phi e(\mu) = 0$, and further $e(\kappa) = \phi e(\kappa) = 0$ from (53) and (54). However, the first term of (9) acting on $\kappa$ implies $\xi(\kappa) = 0$. It is a contradiction since $\xi(\kappa) = -2\lambda^2 v \neq 0$ by (52).

**Case II:** If $b \neq 0$ in some open set $\mathcal{O} \subset U$ then $\lambda + 1 = -\mu$ by (56). Inserting this into (55) gives $c(\mu + 1) = 0$. For $\mu = -1$, it follows from (53) and (54) that $e(\kappa) = \phi e(\kappa) = 0$. It is impossible as before. Thus $c = 0$, i.e. $e(\mu) = 0$ in $\mathcal{O}$. Using the second term of (9) and (51), we have

$$0 = e(\xi(\mu)) - \xi(e(\mu)) = [e, \xi]\mu = -(a + \lambda + 1)\phi e(\mu),$$

which yields $a + \lambda + 1 = 0$, i.e. $\lambda + 1 = \frac{1}{2}\mu$ since if $\phi e(\mu) = 0$ it will lead to a contradiction as Case I. Recalling the previous relation $\lambda + 1 = -\mu$, we derive that $\mu = 0$. That means that $\lambda = -1$. It is impossible.

**Case III:** If $c \neq 0$ in some open subset of $U$ then $\lambda - 1 = \mu$ by (55). Inserting this into (56) gives $b(\mu + 1) = 0$. In the same way as Case II, we can prove that it is impossible.

Hence we complete the proof. □

Theorem 5. *Let $(M^3, \phi, \xi, \eta, g)$ be a $(\kappa, \mu, v)$-contact metric manifold. If $M$ admits a nontrivial gradient Cotton soliton, then one of the following statements holds:*

(a) *for $\kappa = 1$, $M$ is Sasakian,*

(b) *for $\kappa = 0$, $M$ is either flat or $(0, -4)$-contact metric space. In the second case $M$ is locally isometric to one of the following Lie groups: $SU(2)$ or $SO(3)$,*

(c) *for $\kappa < 1$ and $\kappa \neq 0$, $M$ is a contact metric $(\kappa, 0)$-space. In this case, $M$ is locally isometric to one of the following Lie groups equipped with a left invariant metric: $SU(2)$ if $0 < \kappa < 1$, $SL(2, \mathbb{R})$ if $\kappa < 0$.*

Proof. If $M = U'$ then a $(\kappa, \mu, v)$-contact metric manifold is Sasakian with $\kappa = 1$, $\mu \in \mathbb{R}$ and $h = 0$. Next we assume that $U$ is not empty and $\{e, \phi e, \xi\}$ is a $\phi$-basis as before.

Write the potential vector field

$$V = \nabla f = f_1 e + f_2 \phi e + f_3 \xi,$$

where $f_1$, $f_2$, $f_3$ are three smooth functions on $M$. For any $X, Y \in \mathfrak{X}(M)$, the gradient Cotton soliton equation (3) is written as Equation (26). Since $Q\nabla f = 0$, we have

$$f_1\left(\frac{1}{2}r - \kappa + \lambda\mu\right) + f_2\lambda\nu = 0, \qquad f_2\left(\frac{1}{2}r - \kappa - \lambda\mu\right) + f_1\lambda\nu = 0,$$
$$f_3\kappa = 0. \tag{57}$$

If $\kappa \equiv 0$ in $U$ then $\lambda = \sqrt{1-\kappa} = 1$. We get $Z = \xi(\lambda) = \lambda\nu = 0$, equivalently, $\nu = 0$. Further it is easy to see that $r = 4a$ and $\mu = -2a$ are constants. From (57), $af_2 = 0$. If $a = 0$, i.e. $\mu = 0$ and in this case $M$ is flat. If $a \neq 0$ then $f_2 = 0$. Putting $X = Y = \xi$ in (26) we have

$$2\xi(f_3) + 8a = \sigma. \tag{58}$$

Letting $X = e$ and $Y = \xi$ in (26) implies $e(f_3) = 0$. Moreover, letting $X = \phi e$ and $Y = \xi$ in (26) implies $\phi e(f_3) = 0$. Because $b = c = 0$, applying the first term of (9) on $f_3$ gives $\xi(f_3) = 0$. Thus (58) implies $\sigma = 8a$.

On the other hand, since $g(\nabla_\xi \nabla f, \phi e) = g(\nabla_{\phi e}\nabla f, \xi)$, we obtain $af_1 = \phi e(f_3) = 0$, i.e. $f_1 = 0$. Letting $X = Y = e$ in (26) implies $2e(f_1) + 4a^2 = \sigma$, i.e. $\sigma = 4a^2$. Therefore we find $a = 2$, i.e. $\mu = -4$. According to [4, Theorem 3], $M$ is locally isometric to one of the following Lie groups: $SU(2)$ or $SO(3)$.

In the following we consider the case where $\kappa < 1$ and $\kappa \neq 0$. Denote by

$$U_1 = \{p \in U : \kappa(p) \neq 0 \text{ and } \kappa(p) < 1\}.$$

Then $f_3 = 0$ in $U_1$. Putting $X = Y = \xi$ in (26) we have

$$r - 2\mu(1-\kappa) - 6\kappa = \sigma. \tag{59}$$

Since $\sigma$ is constant, differentiating (59) along $\xi$ and using Lemma 2, we also obtain Equation (51).

Because at least one of $f_1$ and $f_2$ is nonzero, the first and second terms of (57) imply

$$(1-\kappa)(\mu^2 + \nu^2) = \left(\frac{1}{2}r - \kappa\right)^2. \tag{60}$$

Next we prove $v \equiv 0$ in $U_1$. Since $Z = \lambda v$ and $a = -\frac{1}{2}\mu$, letting $X = e$ and $Y = \phi e$ in (26) gives

$$2bf_1 + 2e(f_2) + \lambda\xi(\mu) + 2\mu\lambda v - (1-\lambda)\lambda v + \frac{1}{4}\xi(r) = 0.$$

In terms of (51) and Lemma 3, the above formula becomes

$$2bf_1 + 2e(f_2) + 3\lambda v = 0. \tag{61}$$

Letting $X = e$ and $Y = \xi$ in (26) implies

$$2f_2(1+\lambda) - 8\lambda^2 b = \frac{1}{4}\phi e(r). \tag{62}$$

Moreover, letting $X = \phi e$ and $Y = \xi$ in (26) implies

$$2f_1(\lambda - 1) + 8c\lambda^2 = -\frac{1}{4}e(r). \tag{63}$$

Here we have used Equations (53) and (54).

Using (62) and (63), we conclude from the second term of (9) that

$$\begin{aligned} -be(r) + c\phi e(r) + 2\xi(r) = [e, \phi e]r &= e(\phi e(r)) - \phi e(e(r)) \\ &= 8e(f_2)(1+\lambda) + 16f_2c\lambda - 32\lambda^2 e(b) \\ &\quad + 8\phi e(f_1)(\lambda - 1) + 16f_1 b\lambda + 32\lambda^2\phi e(c). \end{aligned}$$

It follows from Lemma 3 that

$$-\lambda^2 v = [e(f_2) + bf_1](1+\lambda) + [\phi e(f_1) + cf_2](\lambda - 1) - 4\lambda^2 e(b) + 4\lambda^2\phi e(c). \tag{64}$$

Since $g(\nabla_{\phi e}\nabla f, e) = g(\nabla_e \nabla f, \phi e)$, using Lemma 1 we see that

$$\phi e(f_1) + cf_2 = e(f_2) + bf_1, \tag{65}$$

thus recalling (61) we obtain from (64) that

$$v = -2e(b) + 2\phi e(c). \tag{66}$$

Since $A = B = 0$, it follows from (5) and (6) that

$$e(b) = e\left(\frac{\phi e(\lambda)}{2\lambda}\right) = \frac{e(\phi e(\lambda))\lambda - \phi e(\lambda)e(\lambda)}{2\lambda^2},$$

$$\phi e(c) = \phi e\left(\frac{e(\lambda)}{2\lambda}\right) = \frac{\phi e(e(\lambda))\lambda - e(\lambda)\phi e(\lambda)}{2\lambda^2}.$$

Hence using the first term of (9) we have

$$\phi e(c) - e(b) = \frac{[\phi e, e](\lambda)}{2\lambda} = \frac{be(\lambda) - c\phi e(\lambda) - 2\xi(\lambda)}{2\lambda} = -\nu.$$

Substituting this into (66), we find that $\nu = 0$ on $U_1$. This shows $\xi(\kappa) = \xi(\mu) = 0$ from (51). Moreover, by (60) we know that either $\frac{1}{2}r - \kappa = \lambda\mu$ or $\frac{1}{2}r - \kappa = -\lambda\mu$.

If $\frac{1}{2}r - \kappa = \lambda\mu$ then Equation (57) implies $f_1\mu = 0$. Consider

$$\mathscr{V}_1 = \{p \in U_1 : f_1(p) = 0\} \qquad \text{and} \qquad \mathscr{V}_2 = \{p \in U_1 : f_1(p) \neq 0\}.$$

Thus $\mathscr{V}_1 \cup \mathscr{V}_2$ is dense in the closure of $U_1$. In $\mathscr{V}_1$, we have $f_2 \neq 0$. Then (61) yields $e(f_2) = 0$, which further implies $c = 0$ from (65). Recalling (6) we get $e(\lambda) = 0$.

Now by using the second term of (9) on $\lambda$ we obtain $(a + \lambda + 1)\phi e(\lambda) = 0$. If $\phi e(\lambda) \neq 0$ in some open set $\mathscr{V}_1' \subset \mathscr{V}_1$ then $a = -\lambda - 1$, i.e. $\frac{1}{2}\mu = \lambda + 1$. Recalling $\kappa = 1 - \lambda^2$, we derive from (59) that

$$4\lambda(\lambda + 1) - 4\lambda^3 - 4 = \sigma.$$

This shows that $\lambda$ is constant since $\sigma$ is constant. Consequently, $\phi e(\lambda) = 0$ in $\mathscr{V}_1'$. The contradiction gives $\phi e(\lambda) = 0$ in $\mathscr{V}_1$. Namely, $\lambda$ is constant, hence it is easy to see that $r$ is constant and $b = 0$. However, Equation (62) yields $\lambda = -1$, which is impossible since $f_2 \neq 0$ and $\lambda$ is positive. This shows that $\mathscr{V}_1$ is empty and $\mu = 0$ in $U_1$. We conclude from (53) and (54) that $\kappa$ is constant.

For $\frac{1}{2}r - \kappa = -\lambda\mu$, we have $\mu f_2 = 0$ from (57). In the same way, we can prove that $\mu = 0$ and $\kappa$ is constant.

Summing up the above discussion, we complete the proof. □

Since the condition that $\nu$ is constant does not imply that the other functions $\kappa$ and $\nu$ are constants (see [13, Remark 5.3]), we consider the case where $\nu$ is constant.

Theorem 6. *Let $(M^3, \phi, \xi, \eta, g)$ be a contact metric $(\kappa, \mu, \nu)$-manifold such that $\nu$ is constant. If $M$ admits a Cotton soliton with potential vector field $V$ being orthogonal to Reeb vector field $\xi$, then $M$ is either*

(a) *Sasakian,*

(b) *a contact metric $(\kappa, \mu)$-space. Moreover, by Theorem 3, in this case $M$ is either flat, or locally isometric to one of the following Lie groups equipped with a left invariant metric: $SU(2)$ or $SO(3)$.*

Proof. We know that $Q\xi = 2\kappa\xi$ implies $A = B = 0$, and $Z = \lambda\nu$, $\mu = -2a$. Then $\xi(Z) = \lambda^2\nu$, $e(Z) = 2\lambda c\nu$ and $\phi e(Z) = 2b\lambda\nu$.

As before, we may set $V = f_1 e + f_2 \phi e$. Using Cotton soliton equation (1) we derive from Lemma 1 and Proposition 1 the following equations:

$$\begin{cases} bf_1 + e(f_2) + \phi e(f_1) + f_2 c + 3\lambda v = 0, \\ f_2(1+\lambda-a) + \xi(f_1) - 8b\lambda^2 = \frac{1}{4}\phi e(r), \\ f_1(\lambda-1+a) + \xi(f_2) + 8c\lambda^2 = -\frac{1}{4}e(r), \\ 2e(f_1) - 2bf_2 - (1-\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 + \mu\lambda\right) + 4a^2\lambda = \sigma, \\ 2\phi e(f_2) - 2cf_1 - 4a^2\lambda - (1+\lambda)\left(\frac{1}{2}r - 3 + 3\lambda^2 - \mu\lambda\right) = \sigma, \\ r - 2\mu(1-\kappa) - 6\kappa = \sigma. \end{cases} \tag{67}$$

Here the first equation has used (51) and Lemma 3 and the second and third equations have used Equations (53) and (54).

Moreover, differentiating the last equation of (67) along $\xi$, we can also obtain (51). Since $v$ is constant, by (53) and (54), we have

$$e(\mu) = \frac{e(\kappa)}{\lambda} = \frac{e(1-\lambda^2)}{\lambda} = -4c\lambda \tag{68}$$

and

$$\phi e(\mu) = -\frac{\phi e(\kappa)}{\lambda} = -\frac{\phi e(1-\lambda^2)}{\lambda} = 4b\lambda. \tag{69}$$

Here we have used (5) and (6). Using (51) and the second term of (9), we get

$$\begin{aligned} -(a+\lambda+1)\phi e(\mu) = [e,\xi]\mu &= e(\xi(\mu)) - \xi(e(\mu)) \\ &= -2e(\mu)v + 4\xi(c)\lambda + 4c\lambda v. \end{aligned}$$

Since $\mu = -2a$ and (69) imply $\phi e(\lambda) = -\phi e(a)$, we derive from the first term of (10) that $\xi(c) = b(a+\lambda+1)$. Hence inserting (68) and (69) into the previous relation gives

$$3cv = -2b(a+\lambda+1). \tag{70}$$

Using the similar method with above, we can obtain

$$3bv = 2c(a-\lambda+1). \tag{71}$$

Next we consider four open subsets

$$U_1 = \{p \in U : b(p) \neq 0,\ c(p) \neq 0\},$$

$$U_2 = \{p \in U : b(p) = 0,\ c(p) \neq 0\},$$

$$U_3 = \{p \in U : b(p) \neq 0,\ c(p) = 0\},$$
$$U_4 = \{p \in U : b(p) = 0,\ c(p) = 0\}$$

of $U$. Clearly, $U_1 \cup U_2 \cup U_3 \cup U_4$ is dense in the closure of $U$.

**Case I.** For $p \in U$, if $p \in U_1$ then the previous two formulas (70) and (71) yield

$$9v^2 = -4(a + \lambda + 1)(a - \lambda + 1). \tag{72}$$

Differentiating this along $e$ gives $a - \lambda + 1 = 0$ since $e(a) = e(\lambda) \neq 0$ obtained from (6) and (68). On the other hand, differentiating (72) along $\phi e$ gives $a + \lambda + 1 = 0$ since $\phi e(a) = -\phi e(\lambda) \neq 0$. Thus we obtain $\lambda = 0$, which is impossible.

**Case II.** If $p \in U_2$, we have $v = 0$ and $a - \lambda + 1 = 0$ from (70) and (71). Moreover, it is easy to prove that $e(\mu) \neq 0$ and $\phi e(\mu) = \phi e(\lambda) = \phi e(\kappa) = \phi e(r) = 0$. By (51) and $v = 0$, we know $\xi(\mu) = 0$. Moreover, it is easy to see that $\xi(\lambda) = \xi(\kappa) = \xi(c) = 0$. Recalling $\kappa = 1 - \lambda^2$, Equation (67) becomes

$$\begin{cases} e(f_2) + \phi e(f_1) + f_2 c = 0, \\ f_2(1 + \lambda - a) + \xi(f_1) = 0, \\ f_1(\lambda - 1 + a) + \xi(f_2) + 8c\lambda^2 = -\frac{1}{4}e(r), \\ 2e(f_1) - \kappa\lambda\mu + \mu^2\lambda = \sigma + \frac{1}{2}(1 - \lambda)\sigma, \\ 2\phi e(f_2) - 2cf_1 - \mu^2\lambda + \kappa\lambda\mu = \sigma + \frac{1}{2}(1 + \lambda)\sigma. \end{cases} \tag{73}$$

Differentiating the third term of (73) with respect to $\phi e$ implies $\phi e(\xi(f_2)) = -\phi e(f_1)(\lambda - 1 + a)$ and differentiating the last term of (73) with respect to $\xi$ gives $\xi(\phi e(f_2)) = c\xi(f_1)$. Hence applying the third term of (9) in $f_2$ implies

$$0 = [\phi e, \xi](f_2) = \phi e(\xi(f_2)) - \xi(\phi e(f_2)) = -\phi e(f_1)(\lambda - 1 + a) - c\xi(f_1).$$

Recalling the first and second terms of (73) we obtain

$$e(f_2)(\lambda - 1) + cf_2\lambda = 0. \tag{74}$$

On the other hand, differentiating the second term of (73) along $e$ gives $e(\xi(f_1)) = -(\lambda + 1 - a)e(f_2) - (2c\lambda - e(a))f_2$ and differentiating the fourth term of (73) along $\xi$ gives $\xi(e(f_1)) = 0$. Hence applying the second term of (9) in $f_1$ implies

$$\begin{aligned} -(a + \lambda + 1)\phi e(f_1) &= [e, \xi](f_1) = e(\xi(f_1)) - \xi(e(f_1)) \\ &= -(\lambda + 1 - a)e(f_2) - (2c\lambda - e(a))f_2. \end{aligned}$$

Recalling the first term of (73) we get

$$(\lambda + 1)e(f_2) = -\lambda c f_2. \tag{75}$$

By comparing (74) with (75), we find $2c\lambda f_2 = 0$, which shows $f_2 = 0$ since $\lambda > 0$. Thus Equation (73) is simplified as

$$\begin{cases} 2f_1 a = 2c\lambda^2(a-\mu) = 6ac\lambda^2, \\ 2e(f_1) - \kappa\lambda\mu + \mu^2\lambda = \frac{1}{2}(3-\lambda)\sigma, \\ -2cf_1 - \mu^2\lambda + \kappa\lambda\mu = \frac{1}{2}(3+\lambda)\sigma, \\ r - 2\mu(1-\kappa) - 6\kappa = \sigma. \end{cases} \tag{76}$$

Here we have used

$$e(r) = e(2\mu\lambda^2 + 6\kappa) = -8c\lambda^2(\lambda - \mu + 3).$$

We know $a \neq 0$ in $U_2$, otherwise, if $a = 0$ then $\lambda = 1$ which implies $c = 0$ from (6). By the first term of (76), we obtain $f_1 = 3c\lambda^2$. Inserting this into the third term of (76) gives

$$-6c^2\lambda^2 - \mu^2\lambda + \kappa\lambda\mu = \frac{1}{2}(3+\lambda)\sigma. \tag{77}$$

Differentiating $f_1 = 3c\lambda^2$, we have

$$e(f_1) = 3\lambda^2 e(c) + 12c^2\lambda^2.$$

Substituting this into the second term of (76), we conclude

$$6\lambda^2 e(c) + 24c^2\lambda^2 - \kappa\lambda\mu + \mu^2\lambda = \frac{1}{2}(3-\lambda)\sigma. \tag{78}$$

Furthermore, since $r = 2\mu\lambda^2 + 6\kappa + \sigma$, it follows from (8) that

$$e(c) = (1+\lambda^2)\mu + 2\kappa + c^2 + \frac{\sigma}{2}. \tag{79}$$

From (77), (78) and (79), we can eliminate the function $c$. We remark that $\kappa = 1 - \lambda^2$ and $\mu = -2a = -2(\lambda - 1)$. Therefore we see that $\lambda$ must be constant since $\sigma$ is constant. It shows that $c = 0$ from (6), which is contradictory with $p \in U_2$.

**Case III.** If $p \in U_3$ then we have $v = 0$ and $a + \lambda + 1 = 0$. Moreover, $\phi e(\mu) \neq 0$ and $e(\mu) = e(\lambda) = e(\kappa) = e(r) = 0$. Also, we have $\xi(\lambda) = \xi(\kappa) = \xi(c) = 0$. In the same way as Case II, we can obtain from the above formulas that $f_1 = 0$. Thus Equation (67) is simplified as

$$\begin{cases} f_2(1+\lambda-a) - 8\lambda^2 b = \frac{1}{4}\phi e(r), \\ -2bf_2 - \kappa\lambda\mu + \mu^2\lambda = \frac{1}{2}(3-\lambda)\sigma, \\ 2\phi e(f_2) - \mu^2\lambda + \kappa\lambda\mu = \frac{1}{2}(3+\lambda)\sigma, \\ r - 2\mu(1-\kappa) - 6\kappa = \sigma. \end{cases}$$

As Case II, making use of (5), (8) and the above formulas, we can also prove that $\lambda$ is constant, which is contradictory with $p \in U_3$.

**Case IV.** If $p \in U_4$ then $e(\mu) = \phi e(\mu) = 0$. Applying the first term of (9) on $\mu$, we get $\xi(\mu) = 0$, which shows that $\mu$ and $a$ are constants. Moreover, it is easy to prove that $\lambda, \kappa$ are constants and $\nu = 0$. That shows that $M$ is a contact metric $(\kappa, \mu)$-space, equivalently, $M$ satisfies $Q\xi = \rho\xi$ with $\rho = 2\kappa$ is constant.

By Theorem 3, we complete the proof. □

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