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Generic distance-squared mappings on plane curves

Shunsuke Ichiki

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Abstract. A distance-squared function is one of the most significant functions in the application of singularity theory to differential geometry. Moreover, distance-squared mappings are naturally extended mappings of distance-squared functions, wherein each component is a distance-squared function. In this paper, compositions of a given plane curve and generic distance-squared mappings on the plane into the plane are investigated from the viewpoint of stability.

1. Introduction

Throughout this paper, let ℓ and n stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary. Let $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ be a given point. The mapping $d_q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_q(x) = \sum_{i=1}^{n} (x_i - q_i)^2$$

is called a distance-squared function, where $x = (x_1, \dots, x_n)$. In [5], the following notion is investigated.

Definition 1. Let $p_1, \dots, p_\ell$ be ℓ given points in $\mathbb{R}^n$. Set $p = (p_1, \dots, p_\ell) \in (\mathbb{R}^n)^\ell$. The mapping $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ defined by

$$D_p = (d_{p_1}, \dots, d_{p_\ell})$$

is called a *distance-squared mapping*.

We have the following motivation for investigating distance-squared mappings. Height functions and distance-squared functions have been investigated in detail so far, and they are useful tools in the applications of singularity theory to differential geometry (see [1]). A mapping in which each

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component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far. For example, in [6] (resp., [2]), compositions of generic projections and embeddings (resp., stable mappings) are investigated from the viewpoint of stability (for the definition of stability, refer to [3]). On the other hand, a mapping in which each component is a distance-squared function is a distance-squared mapping. Therefore, it is natural to investigate distance-squared mappings as well as projections.

In this paper, compositions of a given plane curve and generic distance-squared mappings on the plane into the plane are investigated from the viewpoint of stability.

A mapping $f : \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *$\mathcal{A}$-equivalent* to a mapping $g : \mathbb{R}^n \to \mathbb{R}^\ell$ if there exist diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\psi \circ f \circ \varphi^{-1} = g$. For given points $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, set

$$\overrightarrow{xy} = (y_1 - x_1, \dots, y_n - x_n).$$

Given ℓ points $p_1, \dots, p_\ell \in \mathbb{R}^n$ $(1 \leq \ell \leq n+1)$ are said to be *in general position* if $\ell = 1$ or $\overrightarrow{p_1 p_2}, \dots, \overrightarrow{p_1 p_\ell}$ $(2 \leq \ell \leq n+1)$ are linearly independent.

In [5], a characterization of distance-squared mappings is given as follows:

Proposition 1 ([5]). (1) *Let ℓ, n be integers such that $2 \leq \ell \leq n$, and let $p_1, \dots, p_\ell \in \mathbb{R}^n$ be in general position. Then, $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the mapping defined by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{\ell-1}, x_\ell^2 + \cdots + x_n^2)$.*

(2) *Let ℓ, n be integers such that $1 \leq n < \ell$, and let $p_1, \dots, p_\ell \in \mathbb{R}^n$ be ℓ points such that $p_1, \dots, p_{n+1}$ are in general position. Then, $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the inclusion $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$.*

In the following, by N, we denote a manifold of dimension 1. A mapping $f : N \to \mathbb{R}^2$ is called a *mapping with normal crossings* if the mapping f satisfies the following conditions.

(1) For any $y \in \mathbb{R}^2$, $|f^{-1}(y)| \leq 2$, where $|A|$ is the number of elements of the set A.

(2) For any two distinct points $q_1, q_2 \in N$ satisfying $f(q_1) = f(q_2)$, we have $\dim(df_{q_1}(T_{q_1}N) + df_{q_2}(T_{q_2}N)) = 2$.

From Corollary 8 in [4], we have the following.

Proposition 2 ([4]). *Let $\gamma : N \to \mathbb{R}^2$ be an injective immersion, where N is a manifold of dimension 1. Then, the set*

$$\{p \in \mathbb{R}^2 \times \mathbb{R}^2 \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is an immersion with normal crossings}\}$$

is dense in $\mathbb{R}^2 \times \mathbb{R}^2$.

On the other hand, the purpose of this paper is to investigate whether the set

$$\{p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is an immersion with normal crossings}\}$$

is dense in $\gamma(N) \times \gamma(N)$ or not. Here, note that O is an open set of $\gamma(N) \times \gamma(N)$ if there exists an open set O' of $\mathbb{R}^2 \times \mathbb{R}^2$ satisfying $O = O' \cap (\gamma(N) \times \gamma(N))$.

Let $\gamma : N \to \mathbb{R}^2$ be an immersion. We say that $\kappa : U \to \mathbb{R}$ is called the *curvature* of γ on a coordinate neighborhood (U, t) of N if

$$\kappa(t) = \frac{\det\begin{pmatrix} \dfrac{d\gamma_1}{dt}(t) & \dfrac{d^2\gamma_1}{dt^2}(t) \\ \dfrac{d\gamma_2}{dt}(t) & \dfrac{d^2\gamma_2}{dt^2}(t) \end{pmatrix}}{\left(\left(\dfrac{d\gamma_1}{dt}(t)\right)^2 + \left(\dfrac{d\gamma_2}{dt}(t)\right)^2\right)^{3/2}},$$

where $\gamma = (\gamma_1, \gamma_2)$. Note that for a given point $q \in N$, whether $\kappa(q) = 0$ or not does not depend on the choice of a coordinate neighborhood.

Definition 2. Let N be a manifold of dimension 1. We say that an immersion $\gamma : N \to \mathbb{R}^2$ *satisfies* $(*)$ if for any non-empty open set U of N, there exists a point $q \in U$ satisfying $\kappa(q) \neq 0$, where κ is the curvature of γ on a coordinate neighborhood around q.

The main result in this paper is the following.

Theorem 1. *Let $\gamma : N \to \mathbb{R}^2$ be an injective immersion satisfying $(*)$, where N is a manifold of dimension* 1. *Then, the set*

$$\{p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \textit{ is an immersion with normal crossings}\}$$

is dense in $\gamma(N) \times \gamma(N)$.

If we drop the hypothesis $(*)$ in Theorem 1, then the conclusion of Theorem 1 does not necessarily hold (see Examples 1 and 2 in Section 2).

In Theorem 1, if the mapping $D_p \circ \gamma : N \to \mathbb{R}^2$ is proper, then the immersion with normal crossings $D_p \circ \gamma : N \to \mathbb{R}^2$ is necessarily stable (see [3], p. 86). Thus, from Theorem 1, we get the following.

Corollary 1. *Let N be a compact manifold of dimension* 1. *Let $\gamma : N \to \mathbb{R}^2$ be an embedding satisfying $(*)$. Then, the set*

$$\{p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \textit{ is stable}\}$$

is dense in $\gamma(N) \times \gamma(N)$.

In Section 2, Examples 1 and 2 are given. In Section 3, preliminaries for the proof of Theorem 1 are given. Section 4 is devoted to the proof of Theorem 1.

2. Dropping the hypothesis $(*)$ in Theorem 1

In this section, we will give two examples such that Theorem 1 without the hypothesis $(*)$ does not hold (see Examples 1 and 2).

Firstly, we prepare the following proposition, which is used in Example 1.

Proposition 3. *Let $\gamma : N \to \mathbb{R}^2$ be a mapping, where N is a manifold of dimension 1. Let p_1, p_2 be two points of $\mathbb{R}^2$. Then, a point $q \in N$ is a singular point of the mapping $D_p \circ \gamma : N \to \mathbb{R}^2$ $(p = (p_1, p_2))$ if and only if*

$$\overrightarrow{p_1\gamma(q)} \cdot \frac{d\gamma}{dt}(q) = 0 \qquad \textit{and} \qquad \overrightarrow{p_2\gamma(q)} \cdot \frac{d\gamma}{dt}(q) = 0,$$

where t is a local coordinate around the point q and "$\cdot$" stands for the inner product in $\mathbb{R}^2$, that is, p_1 and p_2 are on the line normal to the curve $\gamma(N)$ at $\gamma(q)$.

Proof. Let q be a point of N. The composition of $\gamma : N \to \mathbb{R}^2$ and $D_p : \mathbb{R}^2 \to \mathbb{R}^2$ is given as follows:

$$D_p \circ \gamma(q) = ((\gamma_1(q) - p_{11})^2 + (\gamma_2(q) - p_{12})^2, (\gamma_1(q) - p_{21})^2 + (\gamma_2(q) - p_{22})^2),$$

where $p_1 = (p_{11}, p_{12})$, $p_2 = (p_{21}, p_{22})$ and $\gamma = (\gamma_1, \gamma_2)$.

Then, we have

$$\begin{aligned}\frac{dD_p \circ \gamma}{dt}(q) &= 2\biggl((\gamma_1(q) - p_{11})\frac{d\gamma_1}{dt}(q) + (\gamma_2(q) - p_{12})\frac{d\gamma_2}{dt}(q), \\ &\qquad (\gamma_1(q) - p_{21})\frac{d\gamma_1}{dt}(q) + (\gamma_2(q) - p_{22})\frac{d\gamma_2}{dt}(q)\biggr) \\ &= 2\left(\overrightarrow{p_1\gamma(q)} \cdot \frac{d\gamma}{dt}(q), \overrightarrow{p_2\gamma(q)} \cdot \frac{d\gamma}{dt}(q)\right),\end{aligned}$$

where t is a local coordinate around the point q. Hence, a point q is a singular point of the mapping $D_p \circ \gamma$ if and only if

$$\left(\overrightarrow{p_1\gamma(q)} \cdot \frac{d\gamma}{dt}(q), \overrightarrow{p_2\gamma(q)} \cdot \frac{d\gamma}{dt}(q)\right) = (0, 0).$$

Example 1. In this example, we use Proposition 3. Let $\gamma : S^1 \to \mathbb{R}^2$ be an embedding such that $\gamma(S^1)$ is given by Figure 1. Here, note that there

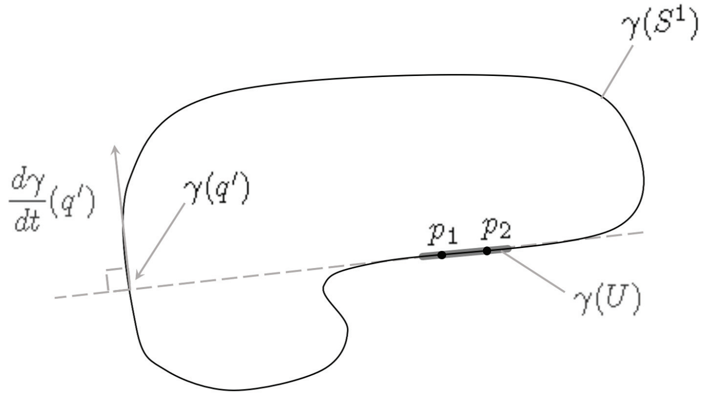

Fig. 1. Curve γ of Example 1

exists an open set U of N such that for any $q \in U$, $\kappa(q) = 0$ (see $\gamma(U)$ in Figure 1). Namely, γ does not satisfy $(*)$.

Let $p = (p_1, p_2) \in \gamma(U) \times \gamma(U)$ be any point. Then, we will show that the mapping $D_p \circ \gamma$ is not an immersion. From Figure 1, it is clearly seen that

$$\overrightarrow{p_1\gamma(q')} \cdot \frac{d\gamma}{dt}(q') = 0 \qquad \text{and} \qquad \overrightarrow{p_2\gamma(q')} \cdot \frac{d\gamma}{dt}(q') = 0,$$

where $\gamma(q')$ is the point in Figure 1 and t is a local coordinate around the point q'. By Proposition 3, the point q' is a singular point of $D_p \circ \gamma$. Namely, for any $p = (p_1, p_2) \in \gamma(U) \times \gamma(U)$, the mapping $D_p \circ \gamma$ is not an immersion. Since $\gamma(U) \times \gamma(U)$ is a non-empty open set of $\gamma(S^1) \times \gamma(S^1)$, the conclusion of Theorem 1 does not hold.

Example 2. Let I_1, I_2 and I_3 be open intervals $(0, 1)$, $(1, 2)$ and $(2, 3)$ of $\mathbb{R}$, respectively. Let $\gamma : I_1 \cup I_2 \cup I_3 \to \mathbb{R}^2$ be the mapping given by

$$\gamma(t) = \begin{cases} (t, -1), & t \in I_1, \\ (t - 1, 0), & t \in I_2, \\ (t - 2, 1), & t \in I_3. \end{cases}$$

For the image of γ, see Figure 2. Here, note that γ does not satisfy $(*)$. Let $p = (p_1, p_2) \in \gamma(I_2) \times \gamma(I_2)$ be any point. Then, we will show that $D_p \circ \gamma$ is not a mapping with normal crossings. Since $p_1 = (p_{11}, p_{12}), p_2 = (p_{21}, p_{22}) \in \gamma(I_2)$, we have $p_{12} = p_{22} = 0$. Thus, we obtain

$$D_p(x_1, x_2) = ((x_1 - p_{11})^2 + x_2^2, (x_1 - p_{21})^2 + x_2^2).$$

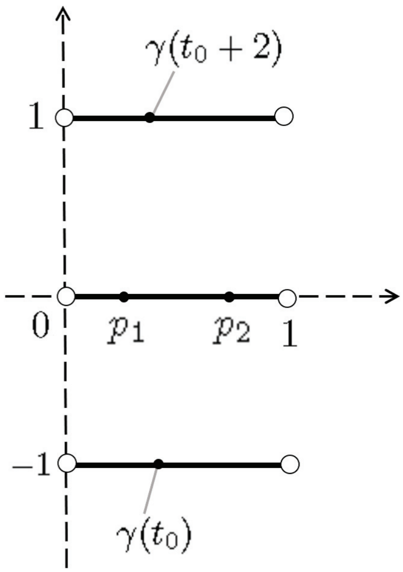

Fig. 2. Image of the mapping γ of Example 2

Let $t_0 \in I_1$ be any element. Then, it follows that $t_0 + 2 \in I_3$ and

$$(D_p \circ \gamma)(t_0) = (D_p \circ \gamma)(t_0 + 2).$$

Since

$$(D_p \circ \gamma)|_{I_1}(t) = ((t - p_{11})^2 + 1, (t - p_{21})^2 + 1),$$
$$(D_p \circ \gamma)|_{I_3}(t) = ((t - 2 - p_{11})^2 + 1, (t - 2 - p_{21})^2 + 1),$$

we get

$$d(D_p \circ \gamma)_{t_0} = 2\begin{pmatrix} t - p_{11} \\ t - p_{21} \end{pmatrix}_{t=t_0},$$
$$d(D_p \circ \gamma)_{t_0+2} = 2\begin{pmatrix} t - 2 - p_{11} \\ t - 2 - p_{21} \end{pmatrix}_{t=t_0+2}.$$

Since the rank of the 2×2 matrix $(d(D_p \circ \gamma)_{t_0}, d(D_p \circ \gamma)_{t_0+2})$ is less than two, $D_p \circ \gamma$ is not a mapping with normal crossings. Hence, for any $p = (p_1, p_2) \in \gamma(I_2) \times \gamma(I_2)$, $D_p \circ \gamma$ is not a mapping with normal crossings.

Remark 1. There is an example such that Theorem 1 without the hypothesis $(*)$ holds. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $\gamma(t) = (t, 0)$. Set

$$A = \{p \in \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \mid D_p \circ \gamma : \mathbb{R} \to \mathbb{R}^2 \text{ is an immersion with normal crossings}\}.$$

We will show that A is dense in $\gamma(\mathbb{R}) \times \gamma(\mathbb{R})$. Let $p_1 = (p_{11}, p_{12}), p_2 = (p_{21}, p_{22}) \in \gamma(\mathbb{R})$ $(= \mathbb{R} \times \{0\})$ be arbitrary points. Then, we have

$$D_p \circ \gamma(t) = ((t - p_{11})^2, (t - p_{21})^2),$$

where $p = (p_1, p_2)$. It is not hard to see that if $p_{11} \neq p_{21}$, then there exists a diffeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H \circ D_p \circ \gamma(t) = (t, 0)$. Namely, if $p_{11} \neq p_{21}$, then $D_p \circ \gamma$ is an immersion with normal crossings. On the other hand, if $p_{11} = p_{21}$, then $D_p \circ \gamma$ is not an immersion with normal crossings. Hence,

$$A = \{p \in \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \mid p_{11} \neq p_{21}\}.$$

Thus, A is dense in $\gamma(\mathbb{R}) \times \gamma(\mathbb{R})$.

3. Preliminaries for the proof of Theorem 1

For the proof of Theorem 1, we prepare Proposition 4 and Lemma 1.

Proposition 4. *Let L be a straight line of $\mathbb{R}^2$. For any $p_1, p_2 \in L$ $(p_1 \neq p_2)$ and for any $\tilde{p}_1, \tilde{p}_2 \in L$ $(\tilde{p}_1 \neq \tilde{p}_2)$, there exists an affine transformation $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$H \circ D_p = D_{\tilde{p}},$$

where $p = (p_1, p_2)$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$.

Proof. Set $p_1 = (p_{11}, p_{12})$, $p_2 = (p_{21}, p_{22})$, $\tilde{p}_1 = (\tilde{p}_{11}, \tilde{p}_{12})$ and $\tilde{p}_2 = (\tilde{p}_{21}, \tilde{p}_{22})$.

Let $H_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$H_1(X_1, X_2) = (X_1, X_1 - X_2).$$

Then, we have

$$H_1 \circ D_p(x_1, x_2) = ((x_1 - p_{11})^2 + (x_2 - p_{12})^2, \\ 2((p_{21} - p_{11})x_1 + (p_{22} - p_{12})x_2) + c_1),$$

where c_1 is a constant term.

Let $H_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation defined by

$$H_2(X_1, X_2) = (X_1, X_2 - c_1).$$

Then, we get

$$H_2 \circ H_1 \circ D_p(x_1, x_2) = ((x_1 - p_{11})^2 + (x_2 - p_{12})^2, \\ 2((p_{21} - p_{11})x_1 + (p_{22} - p_{12})x_2)).$$

Since $p_1, p_2, \tilde{p}_1, \tilde{p}_2 \in L$ and $p_1 \neq p_2$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying

$$\tilde{p}_1 = p_1 + \lambda_1 \overrightarrow{p_1 p_2}, \tag{1}$$

$$\tilde{p}_2 = p_1 + \lambda_2 \overrightarrow{p_1 p_2}. \tag{2}$$

Since $\tilde{p}_1 \neq \tilde{p}_2$, we get $\lambda_1 \neq \lambda_2$.

Let $H_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$H_3(X_1, X_2) = (X_1 - \lambda_1 X_2, X_1 - \lambda_2 X_2).$$

Then, we get

$$\begin{aligned} &H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\ &\quad = (x_1^2 - 2(p_{11} + \lambda_1(p_{21} - p_{11}))x_1 + x_2^2 - 2(p_{12} + \lambda_1(p_{22} - p_{12}))x_2 + d_1, \\ &\qquad x_1^2 - 2(p_{11} + \lambda_2(p_{21} - p_{11}))x_1 + x_2^2 - 2(p_{12} + \lambda_2(p_{22} - p_{12}))x_2 + d_2), \end{aligned}$$

where d_1, d_2 are constant terms. By (1) and (2), we also get

$$\begin{aligned} &H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\ &\quad = (x_1^2 - 2\tilde{p}_{11}x_1 + x_2^2 - 2\tilde{p}_{12}x_2 + d_1, x_1^2 - 2\tilde{p}_{21}x_1 + x_2^2 - 2\tilde{p}_{22}x_2 + d_2) \\ &\quad = ((x_1 - \tilde{p}_{11})^2 + (x_2 - \tilde{p}_{12})^2 + d_1', (x_1 - \tilde{p}_{21})^2 + (x_2 - \tilde{p}_{22})^2 + d_2'), \end{aligned}$$

where d_1', d_2' are constant terms.

Let $H_4 : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation defined by

$$H_4(X_1, X_2) = (X_1 - d_1', X_2 - d_2').$$

Then, we have

$$\begin{aligned} &H_4 \circ H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\ &\quad = ((x_1 - \tilde{p}_{11})^2 + (x_2 - \tilde{p}_{12})^2, (x_1 - \tilde{p}_{21})^2 + (x_2 - \tilde{p}_{22})^2) \\ &\quad = D_{\tilde{p}}(x_1, x_2). \end{aligned}$$

This completes the proof of Proposition 4.

Lemma 1. *Let $\gamma : N \to \mathbb{R}^2$ be an immersion satisfying $(*)$, where N is a manifold of dimension* 1. *Then, for any non-empty open set $U_1 \times U_2$ of $N \times N$,* *there exists an element* $(q_1, q_2) \in U_1 \times U_2$ *such that*

$$\det \begin{pmatrix} \frac{d\gamma_1}{dt_1}(q_1) & \gamma_1(q_2) - \gamma_1(q_1) \\ \frac{d\gamma_2}{dt_1}(q_1) & \gamma_2(q_2) - \gamma_2(q_1) \end{pmatrix} \neq 0,$$

where $\gamma = (\gamma_1, \gamma_2)$ *and* t_1 *is a local coordinate around* q_1.

Proof. Let $U_1 \times U_2$ be any non-empty open set of $N \times N$. Then, there exists a coordinate neighborhood $(U_1' \times U_2', (t_1, t_2))$ satisfying $U_1' \times U_2' \subset U_1 \times U_2$. Fix $q_1' \in U_1'$.

Now, suppose that for any point $t_2 \in U_2'$,

$$\det \begin{pmatrix} \frac{d\gamma_1}{dt_1}(q_1') & \gamma_1(t_2) - \gamma_1(q_1') \\ \frac{d\gamma_2}{dt_1}(q_1') & \gamma_2(t_2) - \gamma_2(q_1') \end{pmatrix} = 0, \tag{3}$$

where $\gamma = (\gamma_1, \gamma_2)$. By (3), we have

$$\frac{d\gamma_1}{dt_1}(q_1')(\gamma_2(t_2) - \gamma_2(q_1')) - \frac{d\gamma_2}{dt_1}(q_1')(\gamma_1(t_2) - \gamma_1(q_1')) = 0,$$

for any point $t_2 \in U_2'$. Hence, we get

$$\frac{d\gamma_1}{dt_1}(q_1')\frac{d\gamma_2}{dt_2}(t_2) - \frac{d\gamma_2}{dt_1}(q_1')\frac{d\gamma_1}{dt_2}(t_2) = 0, \tag{4}$$

$$\frac{d\gamma_1}{dt_1}(q_1')\frac{d^2\gamma_2}{dt_2^2}(t_2) - \frac{d\gamma_2}{dt_1}(q_1')\frac{d^2\gamma_1}{dt_2^2}(t_2) = 0, \tag{5}$$

for any point $t_2 \in U_2'$. By (4) and (5), we have

$$\begin{pmatrix} \frac{d\gamma_2}{dt_2}(t_2) & -\frac{d\gamma_1}{dt_2}(t_2) \\ \frac{d^2\gamma_2}{dt_2^2}(t_2) & -\frac{d^2\gamma_1}{dt_2^2}(t_2) \end{pmatrix} \begin{pmatrix} \frac{d\gamma_1}{dt_1}(q_1') \\ \frac{d\gamma_2}{dt_1}(q_1') \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6}$$

for any point $t_2 \in U_2'$. Since γ is an immersion, it follows that

$$\begin{pmatrix} \frac{d\gamma_1}{dt_1}(q_1') \\ \frac{d\gamma_2}{dt_1}(q_1') \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7}$$

By (6) and (7), we have

$$\det\begin{pmatrix} \dfrac{d\gamma_2}{dt_2}(t_2) & -\dfrac{d\gamma_1}{dt_2}(t_2) \\ \dfrac{d^2\gamma_2}{dt_2^2}(t_2) & -\dfrac{d^2\gamma_1}{dt_2^2}(t_2) \end{pmatrix} = 0$$

for any point $t_2 \in U_2'$. This contradicts the hypothesis that γ satisfies $(*)$.

Remark 2. It is clearly seen that Lemma 1 does not depend on the choice of a coordinate neighborhood containing a point q_1 of N.

4. Proof of Theorem 1

Let O be any non-empty open set of $\gamma(N) \times \gamma(N)$. Then, there exist non-empty open sets O_1 and O_2 of $\gamma(N)$ satisfying $O_1 \times O_2 \subset O$. For the proof, it is sufficient to show that there exist points $p_1 \in O_1$ and $p_2 \in O_2$ such that $D_p \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings, where $p = (p_1, p_2)$. Since γ is continuous, there exist coordinate neighborhoods (U_1, t_1) and (U_2, t_2) of N such that $\gamma(U_1) \subset O_1$ and $\gamma(U_2) \subset O_2$.

Now, let I_1 (resp., I_2) be an open interval containing 0 (resp., 1) of $\mathbb{R}$, and let $\Phi : U_1 \times U_2 \times I_1 \times I_2 \to \mathbb{R}^4$ be the mapping defined by

$$\begin{aligned}\Phi(t_1, t_2, s_1, s_2) &= (\gamma(t_1) + s_1\overrightarrow{\gamma(t_1)\gamma(t_2)}, \gamma(t_1) + s_2\overrightarrow{\gamma(t_1)\gamma(t_2)}) \\ &= ((1 - s_1)\gamma_1(t_1) + s_1\gamma_1(t_2), (1 - s_1)\gamma_2(t_1) + s_1\gamma_2(t_2), \\ &\qquad (1 - s_2)\gamma_1(t_1) + s_2\gamma_1(t_2), (1 - s_2)\gamma_2(t_1) + s_2\gamma_2(t_2)),\end{aligned}$$

where $\gamma = (\gamma_1, \gamma_2)$. Then, we get

$$J\Phi_{(t_1,t_2,s_1,s_2)} = \begin{pmatrix} (1-s_1)\dfrac{d\gamma_1}{dt_1}(t_1) & s_1\dfrac{d\gamma_1}{dt_2}(t_2) & \gamma_1(t_2) - \gamma_1(t_1) & 0 \\ (1-s_1)\dfrac{d\gamma_2}{dt_1}(t_1) & s_1\dfrac{d\gamma_2}{dt_2}(t_2) & \gamma_2(t_2) - \gamma_2(t_1) & 0 \\ (1-s_2)\dfrac{d\gamma_1}{dt_1}(t_1) & s_2\dfrac{d\gamma_1}{dt_2}(t_2) & 0 & \gamma_1(t_2) - \gamma_1(t_1) \\ (1-s_2)\dfrac{d\gamma_2}{dt_1}(t_1) & s_2\dfrac{d\gamma_2}{dt_2}(t_2) & 0 & \gamma_2(t_2) - \gamma_2(t_1) \end{pmatrix}.$$

Set $s_1 = 0$ and $s_2 = 1$. Then, we have

$$J\Phi_{(t_1,t_2,0,1)} = \begin{pmatrix} \dfrac{d\gamma_1}{dt_1}(t_1) & 0 & \gamma_1(t_2)-\gamma_1(t_1) & 0 \\ \dfrac{d\gamma_2}{dt_1}(t_1) & 0 & \gamma_2(t_2)-\gamma_2(t_1) & 0 \\ 0 & \dfrac{d\gamma_1}{dt_2}(t_2) & 0 & \gamma_1(t_2)-\gamma_1(t_1) \\ 0 & \dfrac{d\gamma_2}{dt_2}(t_2) & 0 & \gamma_2(t_2)-\gamma_2(t_1) \end{pmatrix}.$$

Let us first show that there exists an element $(\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2$ such that $\det d\Phi_{(\tilde{t}_1,\tilde{t}_2,0,1)} \neq 0$. Let $\varphi_1 : U_1 \times U_2 \to \mathbb{R}$ and $\varphi_2 : U_1 \times U_2 \to \mathbb{R}$ be the functions defined by

$$\varphi_1(t_1,t_2) = \det \begin{pmatrix} \dfrac{d\gamma_1}{dt_1}(t_1) & \gamma_1(t_2)-\gamma_1(t_1) \\ \dfrac{d\gamma_2}{dt_1}(t_1) & \gamma_2(t_2)-\gamma_2(t_1) \end{pmatrix},$$

$$\varphi_2(t_1,t_2) = \det \begin{pmatrix} \dfrac{d\gamma_1}{dt_2}(t_2) & \gamma_1(t_2)-\gamma_1(t_1) \\ \dfrac{d\gamma_2}{dt_2}(t_2) & \gamma_2(t_2)-\gamma_2(t_1) \end{pmatrix},$$

respectively. Note that the function φ_1 (resp., φ_2) is defined by the entries of the 1st column vector and the 3rd column vector of $J\Phi_{(t_1,t_2,0,1)}$ (resp., the 2nd column vector and the 4th column vector of $J\Phi_{(t_1,t_2,0,1)}$). In order to show that there exists an element $(\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2$ such that $\det d\Phi_{(\tilde{t}_1,\tilde{t}_2,0,1)} \neq 0$, it is sufficient to show that there exists an element $(\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2$ satisfying $\varphi_1(\tilde{t}_1, \tilde{t}_2) \neq 0$ and $\varphi_2(\tilde{t}_1, \tilde{t}_2) \neq 0$. By Lemma 1, there exists $(t_1', t_2') \in U_1 \times U_2$ such that $\varphi_1(t_1', t_2') \neq 0$. Since the function φ_1 is continuous, there exists an open neighborhood $U_1' \times U_2'$ $(\subset U_1 \times U_2)$ of (t_1', t_2') satisfying $\varphi_1(t_1, t_2) \neq 0$ for any $(t_1, t_2) \in U_1' \times U_2'$. Moreover, by Lemma 1, there exists $(\tilde{t}_1, \tilde{t}_2) \in U_1' \times U_2'$ such that $\varphi_2(\tilde{t}_1, \tilde{t}_2) \neq 0$. Namely, there exists an element $(\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2$ such that $\det d\Phi_{(\tilde{t}_1,\tilde{t}_2,0,1)} \neq 0$.

Now, by the inverse function theorem, there exists an open neighborhood V of $(\tilde{t}_1, \tilde{t}_2, 0, 1) \in U_1 \times U_2 \times I_1 \times I_2$ such that $\Phi : V \to \Phi(V)$ is a diffeomorphism. Let $\Sigma \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the set consisting of points $p = (p_1, p_2) \in \mathbb{R}^4$ such that $D_p \circ \gamma : N \to \mathbb{R}^2$ is not an immersion with normal crossings. Note that by Proposition 2, the set $\mathbb{R}^4 - \Sigma$ is dense in $\mathbb{R}^4$. Set

$$\Delta = \{(y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid y_1 = y_2\}.$$

Since $\Phi(V)$ is an open set of $\mathbb{R}^4$ and the set Δ is a proper algebraic set of $\mathbb{R}^4$, there exists an element $p' = (p_1', p_2') \in \Phi(V) - \Sigma \cup \Delta$. As $p' \notin \Sigma$, the composition $D_{p'} \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings. Set $(t_1', t_2', s_1', s_2') = (\Phi|_V)^{-1}(p_1', p_2')$. Then, we have

$$p_1' = \gamma(t_1') + s_1'\overrightarrow{\gamma(t_1')\gamma(t_2')},$$

$$p_2' = \gamma(t_1') + s_2'\overrightarrow{\gamma(t_1')\gamma(t_2')}.$$

Since $p_1' \neq p_2'$, we get $\gamma(t_1') \neq \gamma(t_2')$. Let L be the straight line defined by

$$L = \{\gamma(t_1') + s\overrightarrow{\gamma(t_1')\gamma(t_2')} \mid s \in \mathbb{R}\}.$$

Set $\tilde{p}_1 = \gamma(t_1')$ and $\tilde{p}_2 = \gamma(t_2')$. Then, it is clearly seen that $\tilde{p}_1 \in O_1$ and $\tilde{p}_2 \in O_2$. Since $p_1', p_2' \in L$ $(p_1' \neq p_2')$ and $\tilde{p}_1, \tilde{p}_2 \in L$ $(\tilde{p}_1 \neq \tilde{p}_2)$, by Proposition 4, there exists an affine transformation $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$H \circ D_{p'} = D_{\tilde{p}},$$

where $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$. Since $D_{p'} \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings, $D_{\tilde{p}} \circ \gamma : N \to \mathbb{R}^2$ is also an immersion with normal crossings. $\square$

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Shunsuke Ichiki
Department of Mathematical and Computing Science
School of Computing, Tokyo Institute of Technology
Tokyo 152-8552, *Japan*
E-mail: ichiki@c.titech.ac.jp