Compact and matrix operators on the space $\left|A_f^\theta\right|_k$

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Abstract

In this study, we introduce a new space $\left|A_f^\theta\right|_k$ by using factorable matrix and investigate its certain topological and algebraic structures where θ is a positive sequence. Also, we characterize some matrix operators on this space and determine their norms and the Hausdorff measure of noncompactness. In the particular case, we get some well known results.

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1 Introduction

Let ω be the set of all complex sequences. Any vector subspace of ω is called a sequence space. We write c, l_∞, c_s, l_k $(k \geq 1)$ for the sequence space of all convergent and bounded sequences, for the space of all convergent and k-absolutely convergent series, respectively. A sequence space X with complete norm is called a BK-space provided that linear functional $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \geq 0$, where $\mathbb{C}$ denotes the complex field. A sequence (b_n) is a Schauder base of the normed space X, if, for every $x \in X$, there is a unique sequence (x_n) of scalars such that

$$\left\| x - \sum_{n=0}^{m} x_n b_n \right\| \to 0 \text{ as } m \to \infty.$$

Let X and Y be two subspaces of ω and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series converges for $n \geq 0$. Then, we say that A defines a matrix transformation from X into Y, it is denoted by $A \in (X, Y)$, or $A : X \to Y$, if $A(x) = (A_n(x))$ exists and in Y for all $x \in X$. The domain of the matrix A in X and β- duals of X are defined by

$$X_A = \{x = (x_n) \in \omega : A(x) \in X\}, \tag{1.1}$$

and

$$X^\beta = \{\varepsilon \in w : (\varepsilon_n x_n) \in c_s \text{ for all } x \in X\},$$

respectively.

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Let Σx_v be an infinite series with nth partial sum s_n, (θ_n) be a sequence of positive terms and A be a triangular matrix. The series Σx_v is summable $|A,\theta_n|_k$, $1 \leq k < \infty$, if (see [11])

$$\sum_{n=0}^{\infty} \theta_n^{k-1} \left| A_n(s) - A_{n-1}(s) \right|^k < \infty. \tag{1.2}$$

If we take the weighted mean matrix (with $\theta = P_n/p_n$ and $\theta = n$) instead of A in (1.2), the summability $|A,\theta_n|_k$ is reduced to the summability $\left|\overline{N},p_n,\theta_n\right|_k$ [15] ($\left|\overline{N},p_n\right|_k$ [1] and $|R,p_n|_k$ [13], respectively) and we get the set $\left|\overline{N}_p^\theta\right|_k$ of all series summable by this method [10] i.e.,

$$\left|\overline{N}_p^\theta\right|_k = \left\{ x = (x_v) : \sum_{n=1}^{\infty} \theta_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v \right|^k < \infty \right\}.$$

Here, the weighted mean matrix is defined by

$$a_{nv} = \begin{cases} p_v/P_n, & 0 \leq v \leq n \\ 0, & v > n, \end{cases}$$

where (p_n) is a positive sequence with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, $(P_{-1} = p_{-1} = 0)$. Also, by a factorable matrix, we mean the following matrix

$$a_{nv} = \begin{cases} \hat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v \geq n, \end{cases} \tag{1.3}$$

where $(\hat{a}_n)$ and (a_v) are any sequence of real numbers.

In this study, we define a new space $\left|A_f^\theta\right|_k$ that extends the space $\left|\overline{N}_p^\theta\right|_k$ using by factorable matrix in following way

$$\left|A_f^\theta\right|_k = \left\{ x \in w : \sum_{n=0}^{\infty} \theta_n^{k-1} \left| \hat{a}_n \sum_{v=0}^{n} a_v x_v \right|^k < \infty \right\},$$

and characterize some matrix operators on that space, and also determine their norms and the Hausdorff measure of noncompactness. Thus we extend some well known results.

Also, according to (1.1), we note that it can be stated by $\left|A_f^\theta\right|_k = (l)_{T_A^{(k)}}$, where the matrix $T_A^{(k)}$ is defined by

$$t_{nv}^{(k)} = \begin{cases} \theta_n^{1/k^*} \hat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v \geq n, \end{cases} \tag{1.4}$$

and $S_A^{(k)}$, the inverse of $T_A^{(k)}$, is given by

$$s_{nv}^{(k)} = \begin{cases} \dfrac{1}{\theta_n^{1/k^*} a_n \hat{a}_n}, & v = n \\ \dfrac{1}{\theta_{n-1}^{1/k^*} a_n \hat{a}_{n-1}}, & v = n-1 \\ 0, & v \neq n-1, n \end{cases} \tag{1.5}$$

where k^* is conjugate of k, i.e., $1/k + 1/k^* = 1$ for $k > 1$, and $1/k^* = 0$ for $k = 1$.

Now, we point out some well known lemmas which play important roles the proof of theorems.

Lemma 1.1 [14] Let $1 < k < \infty$. Then, $A \in (l_k, l)$ if and only if

$$\|A\|'_{(l_k,l)} = \sup_F \left\{ \sum_{v=0}^{\infty} \left| \sum_{n\in F} a_{nv} \right|^{k^*} \right\}^{1/k^*}$$

where F denotes the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.1 exposes a rather difficult condition to apply in applications. So, the following lemma which gives equivalent norm is more pratical in many cases.

Lemma 1.2 [8] Let $1 < k < \infty$. Then, $A \in (l_k, l)$ if and only if

$$\|A\|_{(l_k,l)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty.$$

Moreover since $\|A\|'_{(l_k,l)} \leq \|A\|_{(l_k,l)} \leq 4\|A\|'_{(l_k,l)}$, there exists $1 \leq \xi \leq 4$ such that $\|A\|_{(l_k,l)} = \xi \|A\|'_{(l_k,l)}$.

Lemma 1.3 [2] Let $1 \leq k < \infty$. Then, $A \in (l, l_k)$ if and only if

$$\|A\|_{(l,l_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{1/k}.$$

Lemma 1.4 [14] Let $1 < k < \infty$. Then,

$$A \in (l,c) \Leftrightarrow (i) \lim_n a_{nv} \text{ exists for } v \geq 0, (ii) \sup_{n,v} |a_{nv}| < \infty,$$

$$A \in (l_k,c) \Leftrightarrow (i) \text{ holds, } (iii) \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty.$$

2 The Hausdorff measure of noncompactness

Let $\varepsilon > 0$, S and H are subsets of a metric space (X, d). Then S is called an ε-net of H, if, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-net of H.

If Q is a bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } X\},$$

and χ is called the Hausdorff measure of noncompactness.

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact if and only if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. By $C(X, Y)$, we denote the class of these operators.

The following lemma is an important tool for determining the Hausdorff measure of noncompactness of a bounded subset of l_k which is a BK-space for $k \geq 1$.

Lemma 2.1 [6] Let Q be a bounded subset of the normed space X where $X = c_0, l_k$ for $1 \leq k < \infty$. If $P_n : X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, ...x_n, 0, 0, ...)$ for all $x \in X$, then

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I - P_r(x))\|\right).$$

Let X and Y be Banach spaces and χ_1 and χ_2 be the Hausdorff measures on X and Y, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Y and there exists a positive constant M such that $\chi_2(L(Q)) \leq M\chi_1(Q)$ for every bounded subset Q of X. If L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded set } Q \subset X\}$$

is called the (χ_1, χ_2)-measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$ then $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 2.2 [4] Let X and Y be Banach spaces, $L \in B(X, Y)$ and $S_x = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere in X. Then,

$$\|L\|_\chi = \chi(L(S_x))$$

and

$$L \in C(X, Y) \Leftrightarrow \|L\|_\chi = 0.$$

Lemma 2.3 [3] Let X be a normed sequence space, $T = (t_{nv})$ be an infinite triangle matrix, χ_T and χ denote the Hausdorff meausures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.

3 Main results

In this section, by giving some properties of $\left|A_f^\theta\right|_k$ we characterize some matrix operators on that space and also determine their norms and the Hausdorff measure of noncompactness.

Theorem 3.1. Let $k \geqslant 1$. Then, the set $\left|A_f^\theta\right|_k$ becomes a linear space with the coordinatewise addition and scalar multiplication, and also it is a BK-space under the following norm

$$\|x\|_{\left|A_f^\theta\right|_k} = \left\|T_A^{(k)}(x)\right\|_{l_k}. \tag{3.1}$$

Moreover, $\left|A_f^\theta\right|_k$ has a Schauder base.

Proof. The first part is a routine verification, so it is omitted. Let us consider the transformation $T_A^{(k)}$ defined by (1.4). Then, $T_A^{(k)}$ defines a matrix map from ω into ω since it is triangle matrix. Further, since l_k is BK space and $\left|A_f^\theta\right|_k = (l_k)_{T_A^{(k)}}$, then $\left|A_f^\theta\right|_k$ is a BK-space from Wilansky's Theorem 4.3.2 [16].

To prove that $\left|A_f^\theta\right|_k$ has Schauder base, let us consider the base $\left(e^{(n)}\right)$ of l_k, where $e^{(n)}$ is a sequence whose only non-zero term is one in nth place for $n \geq 0$. Let $x \in \left|A_f^\theta\right|_k$, then $y = T_A^{(k)}(x) \in l_k$. Also, since $y \in l_k$ and l_k has Schauder base, we get

$$\left\| y-\sum_{v=0}^{n} y_v e^{(v)} \right\|_{l_k} = \left\| x-\sum_{v=0}^{n} x_v S_A^{(k)}\left(e^{(v)}\right) \right\|_{\left|A_f^\theta\right|_k} \to 0 \text{ as } (n\to\infty)$$

which means that $\left|A_f^\theta\right|_k$ has Schauder base. This completes the proof.

Theorem 3.2. Let $1 \le k < \infty$. Then the space $\left|A_f^\theta\right|_k$ is linearly isomorphic to the space l_k.

Proof. To prove the theorem, we should show that there exists a linear bijection between the spaces $\left|A_f^\theta\right|_k$ and l_k for $1 \le k < \infty$. Let us consider the map $T_A^{(k)} : \left|A_f^\theta\right|_k \to l_k$, $y = T_A^{(k)}(x)$, defined by

$$y_n = \theta_n^{1/k^*} \hat{a}_n \sum_{v=0}^{n} a_v x_v, \text{ for } n \ge 0. \tag{3.2}$$

It is a bijective isomorphism preserving the norm, linear and injective transformation. In fact, it is clear that it is linear and injective. Now, for surjective, take $y \in l_k$. Then, by the definition, there is a sequence $x = (x_n) \in \left|A_f^\theta\right|_k$ such that

$$x_n = \frac{1}{a_n}\left(\frac{y_n}{\theta_n^{1/k^*}\hat{a}_n} - \frac{y_{n-1}}{\theta_{n-1}^{1/k^*}\hat{a}_{n-1}} \right). \tag{3.3}$$

Also,

$$\|x\|_{\left|A_f^\theta\right|_k} = \left\| T_A^{(k)}(x) \right\|_{l_k}.$$

So the space $\left|A_f^\theta\right|_k$ is linearly isomorphic to the space l_k.

Now we define the following notations.

$$D_1 = \left\{ \varepsilon = (\varepsilon_v) : \sup_m \left| \frac{\varepsilon_m}{a_m \hat{a}_m} \right| < \infty \right\},$$

$$D_2 = \left\{ \varepsilon = (\varepsilon_v) : \sup_r \left| \frac{1}{\hat{a}_r} \Delta\left(\frac{\varepsilon_r}{a_r}\right) \right| < \infty \right\},$$

$$D_3 = \left\{ \varepsilon = (\varepsilon_v) : \sup_m \left| \frac{\theta_m^{-1/k^*}\varepsilon_m}{a_m \hat{a}_m} \right| < \infty \right\},$$

$$D_4 = \left\{ \varepsilon = (\varepsilon_v) : \sup_m \sum_{r=0}^{m-1} \left| \frac{\theta_r^{-1/k^*}}{\hat{a}_r} \Delta\left(\frac{\varepsilon_r}{a_r}\right) \right|^{k^*} < \infty \right\}.$$

Theorem 3.3. Let $1 < k < \infty$, A be a factorable matrix defined by (1.3) and (θ_n) be a positive sequence. Then

$$\left\{ \left|A_f^\theta\right|_k \right\}^\beta = D_3 \cap D_4 \text{ and } \{|A_f|\}^\beta = D_1 \cap D_2.$$

Proof. Now, $\varepsilon \in \left\{\left|A_f^\theta\right|_k\right\}^\beta$ if and only if $\left(\sum_{\upsilon=0}^{n} \varepsilon_\upsilon x_\upsilon\right)$ is convergent for every $x \in \left|A_f^\theta\right|_k$. Note that, if $x \in \left|A_f^\theta\right|_k$, then $y \in l_k$, where

$$y_n = \theta_n^{1/k^*} \hat{a}_n \sum_{\upsilon=0}^{n} a_\upsilon x_\upsilon \text{ for } n \geq 0.$$

So we can write from (3.3) that

$$\begin{aligned} \sum_{r=0}^{m} \varepsilon_r x_r &= \sum_{r=0}^{m} \frac{\varepsilon_r}{a_r}\left(\frac{y_r}{\theta_r^{1/k^*}\hat{a}_r} - \frac{y_{r-1}}{\theta_{r-1}^{1/k^*}\hat{a}_{r-1}}\right) \\ &= \frac{\varepsilon_m y_m}{a_m \theta_m^{1/k^*}\hat{a}_m} + \sum_{r=0}^{m-1} \frac{1}{\theta_r^{1/k^*}\hat{a}_r}\left(\frac{\varepsilon_r}{a_r} - \frac{\varepsilon_{r+1}}{a_{r+1}}\right) y_r \\ &= \sum_{r=0}^{m} h_{mr} y_r \end{aligned}$$

where

$$h_{mr} = \begin{cases} \dfrac{1}{\theta_r^{1/k^*}\hat{a}_r}\left(\dfrac{\varepsilon_r}{a_r} - \dfrac{\varepsilon_{r+1}}{a_{r+1}}\right), & 0 \leq r \leq m-1 \\ \dfrac{\varepsilon_m}{a_m \theta_m^{1/k^*}\hat{a}_m}, & r = m \\ 0, & r > m, \end{cases} \tag{3.4}$$

which implies that $\varepsilon \in \left\{\left|A_f^\theta\right|_k\right\}^\beta \Leftrightarrow H \in (l_k, c)$. Therefore it follows from Lemma 1.4 that $\varepsilon \in \left\{\left|A_f^\theta\right|_k\right\}^\beta$ iff $\varepsilon \in D_3 \cap D_4$. This completes the proof. The proof of other part can similarly be proved, so we leave the reader.

Theorem 3.4. Let A and B be factorable matrices as in (1.3), (θ_n) be a positive sequence and $1 < k < \infty$. Then, $C \in \left(\left|A_f^\theta\right|_k, |B_f|\right)$ if and only if, for all n,

$$\sup_m \left|\frac{c_{nm}}{a_m \theta_m^{1/k^*}\hat{a}_m}\right| \text{ exists,} \tag{3.5}$$

$$\sup_m \sum_{r=0}^{m-1} \left|\frac{1}{\theta_r^{1/k^*}\hat{a}_r}\left(\frac{c_{nr}}{a_r} - \frac{c_{n,r+1}}{a_{r+1}}\right)\right|^{k^*} < \infty, \tag{3.6}$$

$$\sum_{\upsilon=0}^{\infty}\left(\sum_{n=0}^{\infty} |d_{n\upsilon}|\right)^{k^*} < \infty \tag{3.7}$$

where $D = (d_{n\upsilon})$ is defined by

$$d_{n\upsilon}=\theta_{\upsilon}^{-1/k^{*}}\frac{\hat{b}_{n}}{\hat{a}_{\upsilon}}\sum_{r=0}^{n}b_{r}\left(\frac{c_{r\upsilon}}{a_{\upsilon}}-\frac{c_{r,\upsilon+1}}{a_{\upsilon+1}}\right);\upsilon,n\geq 0. \tag{3.8}$$

Moreover, if $C\in\left(\left|A_{f}^{\theta}\right|_{k},|B_{f}|\right)$, then it is a bounded linear operator and there exists $1\leq\xi\leq 4$ such that

$$\|C\|_{\left(\left|A_{f}^{\theta}\right|_{k},|B_{f}|\right)}=\frac{1}{\xi}\|D\|_{(l_{k},l)}$$

and also

$$\|C\|_{\chi}=\frac{1}{\xi}\lim_{r\to\infty}\sum_{\upsilon=0}^{\infty}\left\{\sum_{n=r+1}^{\infty}|d_{n\upsilon}|\right\}^{k^{*}}.$$

Proof. Let $1<k<\infty$. $C\in\left(\left|A_{f}^{\theta}\right|_{k},|B_{f}|\right)$ iff $(c_{n\upsilon})_{\upsilon=0}^{\infty}\in\left\{\left|A_{f}^{\theta}\right|_{k}\right\}^{\beta}$ and $C(x)\in|B_{f}|$ for every $x\in\left|A_{f}^{\theta}\right|_{k}$. Also, by Theorem 3.3, $(c_{n\upsilon})_{\upsilon=0}^{\infty}\in\left\{\left|A_{f}^{\theta}\right|_{k}\right\}^{\beta}$ iff the conditions (3.5) and (3.6) hold. On the other hand, as in Theorem 3.3 we get that

$$\sum_{r=0}^{m}c_{nr}x_{r}=\sum_{r=0}^{m}h_{mr}^{(n)}y_{r}$$

where $H^{(n)}=\left(h_{mr}^{(n)}\right)$ is defined by (3.4). Additionally, if any matrix $R=(r_{nv})\in(l_{k},c)$, then, the series $R_{n}(x)=\Sigma_{\upsilon}r_{nv}x_{v}$ converges uniformly in n, since, by Lemma 1.4, the remaining term tends to zero uniformly in n. In fact, using Hölder's inequality, we get

$$\left|\sum_{\upsilon=m}^{\infty}r_{n\upsilon}x_{\upsilon}\right|\leq\sup_{n}\left(\sum_{\upsilon=0}^{\infty}|r_{n\upsilon}|^{k^{*}}\right)^{1/k^{*}}\left(\sum_{\upsilon=m}^{\infty}|x_{\upsilon}|^{k}\right)^{1/k}$$

and also right side of this inequality tends to zero as $m\to\infty$, since $x\in l_{k}$, which implies

$$\lim_{n}R_{n}(x)=\sum_{\upsilon=0}^{\infty}\lim_{n}r_{n\upsilon}x_{\upsilon}. \tag{3.9}$$

Hence, it follows from (3.5) and (3.6) that $H^{(n)}=\left(h_{mr}^{(n)}\right)\in(l_{k},c)$, so by using (3.9), we get

$$C_{n}(x)=\sum_{r=0}^{\infty}\lim_{m}h_{mr}^{(n)}y_{r}=\sum_{r=0}^{\infty}\tilde{h}_{nr}y_{r}=\tilde{H}_{n}(y),n\geq 0.$$

This means that $C(x)\in|B_{f}|$ for every $x\in\left|A_{f}^{\theta}\right|_{k}$ iff $\tilde{H}(y)\in|B_{f}|$ for every $y\in l_{k}$. Further, if we define matrix $D=(d_{n\upsilon})$ by $D=T_{B}^{(1)}\tilde{H}$, we get that $\tilde{H}(y)\in|B_{f}|$ for every $y\in l_{k}$ iff $D\in(l_{k},l)$ since $|B_{f}|=(l)_{T_{B}^{(1)}}$, where

$$d_{n\upsilon}=\sum_{r=0}^{n}\frac{\hat{b}_{n}b_{r}}{\theta_{\upsilon}^{1/k^{*}}\hat{a}_{\upsilon}}\left(\frac{c_{r\upsilon}}{a_{\upsilon}}-\frac{c_{r,\upsilon+1}}{a_{\upsilon+1}}\right)$$

and so it follows from applying Lemma 1.2 to the matrix D that $D \in (l_k, l)$ iff (3.7) is satisfied, and this proves the first part.

Since $\left|A_f^\theta\right|_k$ and $|B_f|$ are BK-spaces by Theorem 3.1, the second part of Theorem is result of Theorem 4.2.8 of Wilansky [16].

Also, considering that $T_A^{(k)} : \left|A_f^\theta\right|_k \to l_k$ and $T_B^{(1)} : |B_f| \to l$ are norm isomorphism, as in (3.2), then we get that $C = S_B^{(1)} \circ D \circ T_A^{(k)}$. So, by (3.1), we obtain that

$$\begin{aligned} \|C\|_{\left(\left|A_f^\theta\right|_k, |B_f|\right)} &= \sup_{x\neq\theta} \frac{\left\|S_B^{(1)}(D(T_A^{(k)}(x)))\right\|_{|B_f|}}{\|x\|_{\left|A_f^\theta\right|_k}} = \sup_{x\neq\theta} \frac{\left\|\left(D\left(T_A^{(k)}(x)\right)\right)\right\|_l}{\left\|T_A^{(k)}(x)\right\|_{l_k}} \\ &= \|D\|'_{(l_k,l)} = \frac{1}{\xi}\|D\|_{(l_k,l)}. \end{aligned}$$

Finally, let $S = \left\{x \in \left|A_f^\theta\right|_k : \|x\| \leq 1\right\}$ and define the matrix $D^{(r)} = \left(\bar{d}_{nv}^{(r)}\right)$ by

$$\bar{d}_{nv}^{(r)} = \begin{cases} 0, & 1 \leq n \leq r \\ d_{nv}, & n > r. \end{cases}$$

Then, by Lemma 2.1-Lemma 2.3 and the definition of $D^{(r)}$ we get that

$$\begin{aligned} \|C\|_\chi &= \chi(CS) = \chi\left(T_B^{(1)}CS\right) = \chi\left(DT_A^{(k)}S\right) \\ &= \lim_{r\to\infty} \sup_{y\in T_A^{(k)}S} \|(I-P_r)D(y)\|_l = \lim_{r\to\infty}\left\|D^{(r)}\right\|_{(l_k,l)} \\ &= \lim_{r\to\infty}\sum_{v=0}^{\infty}\left\{\sum_{n=r+1}^{\infty}|d_{nv}|\right\}^{k^*} \end{aligned}$$

where $P_r : l_k \to l_k$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$. So, the proof is completed together with Lemma 1.2.

Theorem 3.5. Let A, B be factorable matrices defined by (1.3), (θ_n) be a positive sequence and $1 \leq k < \infty$. Then, $C \in \left(|A_f|, \left|B_f^\theta\right|_k\right)$ if and only if, for all n,

$$\sup_m \left|\frac{c_{nm}}{a_m\hat{a}_m}\right| < \infty, \tag{3.10}$$

$$\sup_v \left|\frac{1}{\hat{a}_v}\left(\frac{c_{nv}}{a_v} - \frac{c_{n,v+1}}{a_{v+1}}\right)\right| < \infty, \tag{3.11}$$

$$\sup_v \sum_{n=0}^{\infty}|f_{nv}|^k < \infty, \tag{3.12}$$

where the matrix $F=(f_{n\upsilon})$ is defined by

$$f_{n\upsilon}=\theta_n^{1/k^*}\hat{b}_n\sum_{r=0}^{n}\frac{b_r}{\hat{a}_\upsilon}\left(\frac{c_{r\upsilon}}{a_\upsilon}-\frac{c_{r,\upsilon+1}}{a_{\upsilon+1}}\right),\upsilon,n\geq 0. \tag{3.13}$$

Moreover, if $C\in\left(|A_f|,\left|B_f^\theta\right|_k\right)$, then C is a bounded linear operator,

$$\|C\|_{\left(|A_f|,\left|B_f^\theta\right|_k\right)}=\|F\|_{(l,l_k)}$$

and

$$\|C\|_\chi=\lim_{r\to\infty}\sup_j\left\{\sum_{n=r+1}^{\infty}|f_{nj}|^k\right\}^{1/k}.$$

Proof. Let $1<k<\infty$ and consider the operators $T_A^{(1)}:|A_f|\to l$ and $T_B^{(k)}:\left|B_f^\theta\right|_k\to l_k$ defined by (1.4), also the inverse $S_B^{(k)}$ of $T_B^{(k)}$ is given by (1.5). Then $C\in\left(|A_f|,\left|B_f^\theta\right|_k\right)$ if and only if $(c_{n\upsilon})_{\upsilon=0}^\infty\in\{|A_f|\}^\beta$ and $C(x)\in\left|B_f^\theta\right|_k$ for every $x\in|A_f|$. Also, by Theorem 3.3, $(c_{n\upsilon})_{\upsilon=0}^\infty\in\{|A_f|\}^\beta$ iff the conditions (3.10) and (3.11) hold. On the other hand, if any matrix $R=(r_{n\upsilon})\in(l,c)$, then a series

$$\sum_{\upsilon=0}^{\infty}r_{n\upsilon}x_\upsilon$$

converges uniformly in n. In fact, the remaining term of the series tends to zero uniformly in n, because, considering $x\in l$, by Lemma 1.4 we get

$$\left|\sum_{\upsilon=m}^{\infty}r_{n\upsilon}x_\upsilon\right|\leq\sup_{n,\upsilon}|r_{n\upsilon}|\sum_{\upsilon=m}^{\infty}|x_\upsilon|\to 0\ \ as\ \ (m\to\infty)$$

which leads us

$$\lim_n R_n(x)=\sum_{\upsilon=0}^{\infty}\lim_n r_{n\upsilon}x_\upsilon. \tag{3.14}$$

So, using (3.14), we obtain

$$C_n(x)=\lim_{m\to\infty}\sum_{r=0}^{m}c_{nr}x_r=\lim_{m\to\infty}\sum_{r=0}^{m}\hat{f}_{mr}^{(n)}y_r=\sum_{r=0}^{\infty}\widetilde{f}_{nr}y_r=\widetilde{F}_n(y)$$

where the matrices $\hat{F}^{(n)}=\left(\hat{f}_{mr}^{(n)}\right)$ and $\widetilde{F}=\left(\widetilde{f}_{nr}\right)$ are given as

$$\hat{f}_{mr}^{(n)}=\begin{cases}\dfrac{1}{\hat{a}_r}\left(\dfrac{c_{nr}}{a_r}-\dfrac{c_{n,r+1}}{a_{r+1}}\right), & 0\leq r\leq m-1\\ \dfrac{c_{nm}}{a_m\hat{a}_m}, & r=m\\ 0, & r>m\end{cases}$$

and $\widetilde{f}_{nr} = \lim_{m\to\infty} \hat{f}^{(n)}_{mr}$. Hence, we get that $C(x) \in \left|B^{\theta}_f\right|_k$ for every $x \in |A_f|$ iff $\widetilde{F}(y) \in \left|B^{\theta}_f\right|_k$ for every $y \in l$, that is, $F = T^{(k)}_B \widetilde{F} \in (l, l_k)$, and so it follows from applying Lemma 1.3 to the matrix F that (3.12) holds. This proves the first part.

Since $|A_f|$ and $\left|B^{\theta}_f\right|_k$ are BK-spaces by Theorem 3.1, C is a bounded operator by Theorem 4.2.8 of Wilansky [16].

Now, to determine operator norm of C, consider the isomorphism $T^{(1)}_A : |A_f| \to l$ and $T^{(k)}_B : \left|B^{\theta}_f\right|_k \to l_k$. Then, it is easy to see that $C = S^{(k)}_B \circ F \circ T^{(1)}_A$, and also, by (3.1),

$$\begin{aligned}
\|C\|_{\left(|A_f|,\left|B^{\theta}_f\right|_k\right)} &= \sup_{x\neq\theta} \frac{\left\|S^{(k)}_B \left(F\left(T^{(1)}_A(x)\right)\right)\right\|_{\left|B^{\theta}_f\right|_k}}{\|x\|_{|A_f|}} = \sup_{x\neq\theta} \frac{\left\|F\left(T^{(1)}_A(x)\right)\right\|_{l_k}}{\left\|T^{(1)}_A(x)\right\|_l} \\
&= \|F\|_{(l,l_k)} .
\end{aligned}$$

Finally, let $S = \{x \in |A_f| : \|x\| \le 1\}$. Then, by Lemma 2.1-Lemma 2.3, we get that

$$\begin{aligned}
\|C\|_\chi &= \chi(CS) = \chi\left(T^{(k)}_B CS\right) = \chi\left(FT^{(1)}_A S\right) \\
&= \lim_{r\to\infty} \sup_{y\in T^{(1)}_A S} \|(I - P_r) F(y)\|_{l_k} \\
&= \lim_{r\to\infty} \left\|F^{(r)}\right\|_{(l,l_k)} \\
&= \lim_{r\to\infty} \sup_{\upsilon} \sum_{n=r+1}^{\infty} |f_{n\upsilon}|^k < \infty,
\end{aligned}$$

where $P_r : l_k \to l_k$ is given by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$ and the matrix $F^{(r)} = \left(\overline{f}^{(r)}_{n\upsilon}\right)$ is given by

$$\overline{f}^{(r)}_{n\upsilon} = \begin{cases} 0, & 1 \le n \le r \\ f_{n\upsilon}, & n > r. \end{cases}$$

Thus the proof is completed by Lemma 1.3.

4 Applications

Making use of Theorem 3.4 and Theorem 3.5, we can characterize the compact operators in the classes $\left(\left|A^{\theta}_f\right|_k, |B_f|\right)$ and $\left(|A_f|, \left|B^{\theta}_f\right|_k\right)$.

Corollary 4.1. Under conditions of Theorem 3.4, $C \in \left(\left|A^{\theta}_f\right|_k, |B_f|\right)$ is compact if and only if

$$\lim_{r\to\infty} \sum_{\upsilon=0}^{\infty} \left\{ \sum_{n=r+1}^{\infty} \left| \frac{\theta_\upsilon^{-1/k^*} \hat{b}_n}{\hat{a}_\upsilon} \sum_{r=0}^{n} b_r \left(\frac{c_{r\upsilon}}{a_\upsilon} - \frac{c_{r,\upsilon+1}}{a_{\upsilon+1}} \right) \right| \right\}^{k^*} = 0.$$

Corollary 4.2. Under conditions of Theorem 3.5, $C \in \left(|A_f|, \left|B_f^\theta\right|_k\right)$ is compact if and only if

$$\lim_{r\to\infty} \sup_{\upsilon} \sum_{n=r+1}^{\infty} \left| \theta_n^{1/k^*} \hat{b}_n \sum_{r=0}^{n} \frac{b_r}{\hat{a}_\upsilon} \left(\frac{c_{r\upsilon}}{a_\upsilon} - \frac{c_{r,\upsilon+1}}{a_{\upsilon+1}} \right) \right|^k = 0.$$

Also, if one takes $\hat{a}_n = \frac{p_n}{P_n P_{n-1}}$, $a_n = P_{n-1}$, then the space $\left|A_f^\theta\right|_k$ reduces to the space $\left|\bar{N}_p^\theta\right|_k$. Thus we get the following results of Sarıgöl [10].

Corollary 4.3. Let $C = (c_{n\upsilon})$ be a triangular matrix and (θ_n) be a positive sequence. Then $C \in \left(\left|\bar{N}_p^\theta\right|_k, \left|\bar{N}_q\right|\right)$ if and only if

$$\sum_{\upsilon=1}^{\infty} \frac{1}{\theta_\upsilon p_\upsilon^{k^*}} \left(\sum_{n=\upsilon}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left| \sum_{m=\upsilon}^{n} Q_{m-1} \left(P_\upsilon c_{m\upsilon} - P_{\upsilon-1} c_{m,\upsilon+1} \right) \right| \right)^{k^*} < \infty. \tag{4.1}$$

Proof. If we take $\hat{a}_n = \frac{p_n}{P_n P_{n-1}}$, $a_n = P_{n-1}$, $\hat{b}_n = \frac{q_n}{Q_n Q_{n-1}}$, $b_n = Q_{n-1}$ in Theorem 3.4, then (3.5) and (3.6) are satisfied since $C = (c_{n\upsilon})$ is triangular matrix, and also (3.7) reduces to (4.1), which completes the proof.

Corollary 4.4. Let $C = (c_{n\upsilon})$ be a triangular matrix and (θ_n) be a sequence of positive terms. Then $C \in \left(\left|\bar{N}_p\right|, \left|\bar{N}_q^\theta\right|_k\right)$ if and only if

$$\frac{P_\upsilon q_\upsilon}{p_\upsilon Q_\upsilon} c_{\upsilon\upsilon} = O\left(\theta_\upsilon^{-1/k^*}\right), \tag{4.2}$$

$$\sum_{n=\upsilon+1}^{\infty} \left| \frac{\theta_n^{1/k^*} q_n}{Q_n Q_{n-1}} \sum_{m=\upsilon+1}^{n} Q_{m-1} c_{m,\upsilon+1} \right|^k = O(1), \tag{4.3}$$

$$\sum_{n=\upsilon+1}^{\infty} \left| \frac{\theta_n^{1/k^*} q_n}{Q_n Q_{n-1}} \sum_{m=\upsilon}^{n} Q_{m-1} \left(c_{m\upsilon} - c_{m,\upsilon+1} \right) \right|^k = O\left\{ \left(\frac{p_\upsilon}{P_\upsilon} \right)^k \right\} \text{ as } \upsilon \to \infty. \tag{4.4}$$

Proof. Take $\hat{a}_n = \frac{p_n}{P_n P_{n-1}}$, $a_n = P_{n-1}$, $\hat{b}_n = \frac{q_n}{Q_n Q_{n-1}}$, $b_n = Q_{n-1}$ in the Theorem 3.5. With a few calculations, (3.10) and (3.11) are satisfied since $C = (c_{n\upsilon})$ is triangular matrix, and (3.12) reduces to $(4.2), (4.3)$ and (4.4).

Corollary 4.5 [5]. $I \in \left(|R_p|, |R_q|_k\right), k \geq 1$, if and only if

$$\frac{P_\upsilon q_\upsilon}{p_\upsilon Q_\upsilon} = O\left(\upsilon^{-1/k^*}\right), \qquad W_\upsilon = O\left(\frac{p_\upsilon}{P_\upsilon q_\upsilon}\right), \qquad Q_\upsilon W_\upsilon = O(1),$$

where

$$W_\upsilon = \left\{ \sum_{n=\upsilon+1}^{\infty} \left(\frac{n^{1/k^*} q_n}{Q_n Q_{n-1}} \right)^k \right\}^{1/k}.$$

Note that there is a close relation between the problems of absolute summability factors and comparison of these methods and special matrix transformations such as an identity matrix I and a matrix $W=(w_{nv})$ defined by $w_{nv}=\varepsilon_v$ for $v=n$ otherwise $w_{nv}=0$. So if we take the matrix I instead of matrix C in the Theorem 3.4 and Theorem 3.5, we obtain the followings.

Corollary 4.6. Suppose that (θ_n) is a positive sequence and $k>1$. Then $I\in\left(\left|A_f^\theta\right|_k,|B_f|\right)$ if and only if

$$\sum_{v=0}^{\infty}\frac{1}{\theta_v}\left(\sum_{n=v+1}^{\infty}\left|\frac{\hat{b}_n}{\hat{a}_v}\Delta\left(\frac{b_v}{a_v}\right)\right|+\left|\frac{\hat{b}_v}{\hat{a}_v}\frac{b_v}{a_v}\right|\right)^{k^*}<\infty \tag{4.5}$$

where $\Delta\left(\frac{b_v}{a_v}\right)=\frac{b_v}{a_v}-\frac{b_{v+1}}{a_{v+1}}$, for all $v\geq 0$.

Proof. If we take the identity matrix I instead of C in Theorem 3.4, (3.5) and (3.6) hold directly, and also the last condition gives (4.5).

Corollary 4.7. Assume that (θ_n) is a positive sequence and $1\leq k<\infty$. Then $I\in\left(|A_f|,\left|B_f^\theta\right|_k\right)$ if and only if

$$\sup_v\left\{\sum_{n=v+1}^{\infty}\left|\theta_n^{1/k^*}\frac{\hat{b}_n}{\hat{a}_v}\Delta\left(\frac{b_v}{a_v}\right)\right|^k+\left|\theta_v^{1/k^*}\frac{\hat{b}_v}{\hat{a}_v}\frac{b_v}{a_v}\right|^k\right\}<\infty.$$

Corollary 4.8. Let (a_n), (b_n), $(\hat{a}_n)$ and $\left(\hat{b}_n\right)$ be sequences of positive numbers connected by

$$Y_n^*=\hat{a}_n\sum_{v=1}^{n}a_{v-1}x_v\ ,X_n^*=\hat{b}_n\sum_{v=1}^{n}b_{v-1}\varepsilon_v x_v$$

where (ε_v) is a sequence of complex numbers and $k\geq 1$. Then, $\sum_{n=1}^{\infty}|Y_n^*|<\infty\Longrightarrow\sum_{n=1}^{\infty}|X_n^*|^k<\infty$ if and only if

$$\left|\frac{\hat{b}_v b_v\varepsilon_v}{a_v\hat{a}_v}\right|=O(1) \tag{4.6}$$

and

$$\left|\frac{1}{\hat{a}_v}\Delta\left(\frac{b_v\varepsilon_v}{a_v}\right)\right|\left(\sum_{n=v+1}^{\infty}\hat{b}_n^k\right)^{1/k}=O(1)\ \text{ as } v\to\infty. \tag{4.7}$$

Proof. Take $\theta_n=1$ and C as diagonal matrix with $c_{vv}=\varepsilon_v$ in Theorem 3.5. Note that $x\in|A_f|$ iff $\sum_{n=1}^{\infty}|Y_n^*|<\infty$ and $\varepsilon x\in\left|B_f^\theta\right|_k$ iff $\sum_{n=1}^{\infty}|X_n^*|^k<\infty$. Further, (3.11) and (3.12) are automatically satisfied and (3.13) reduces to

$$\sup_v\sum_{n=v}^{\infty}\left|\frac{\hat{b}_n}{\hat{a}_v}\sum_{r=v}^{n}b_r\left(\frac{c_{rv}}{a_v}-\frac{c_{r,v+1}}{a_{v+1}}\right)\right|^k<\infty,$$

which is equivalent to (4.6) and (4.7) .This completes the proof.

This is the main result of [9].

References

[1] H. Bor, *On two summability methods,* Math. Proc. Cambridge Philos. Soc., **97(1)** (1985), 147-149.

[2] I. J. Maddox, *Elements of functinal analysis*, Cambridge University Press, London, New York, 1970.

[3] E. Malkowsky and V. Rakocevic, *On matrix domains of triangles*, Appl. Math. Comp. **189(2)**(2007), 1146-1163.

[4] E. Malkowsky and V. Rakocevic, *An introduction into the theory of sequence space and measures of noncompactness*, Zb. Rad.(Beogr), **17** (2000), 143-234.

[5] C. Orhan and M. A. Sarıgöl, *On absolute weighted mean summability*, Rocky Moun. J. Math., **23(3)** (1993), 1091-1097.

[6] V. Rakocevic, *Measures of noncompactness and some applications*, Filomat, **12(2)** (1998), 87-120.

[7] M. A. Sarigöl, *Norms and compactness of operators on absolute weighted mean summable series*, Kuwait J. Sci., **43(4)** (2016), 68-74.

[8] M. A. Sarigöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci., **42(3)** (2015), 28-35.

[9] M. A. Sarigöl, *Characterization of general summability factors and applications*, Comp. Math. Appl., **62(6)** (2011), 2665-2670.

[10] M. A. Sarigöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar., **48(3)** (2011), 331-341.

[11] M. A. Sarigöl, *On local properties of factored Fourier series*, Appl. Math. Comp., **216(11)** (2010), 3386-3390.

[12] M. A. Sarigöl and H. Bor, *Characterization of absolute summability factors*, J. Math. Anal. Appl., **195(2)** (1995), 537-545.

[13] M. A. Sarigöl, *On the absolute Riesz summability factors of infinite series,* Indian Jour. Pure Appl. Math., **23(12)** (1992), 881-881.

[14] M. Stieglitz and H. Tietz, *Matrix transformationen von Folgenraumen Eine Ergebnisübersicht*, Math. Z., **154(1)** (1977), 1-16.

[15] W. T. Sulaiman, *On some summability factors of infinite series*, Proc. Amer. Math. Soc., (1992), 313-317.

[16] A. Wilansky, *Summability Through Functional Analysis*, Mathematics Studies. 85. North Holland , Amsterdam, 1984.