Stability and non-stability of generalized radical cubic functional equation in quasi-β-Banach spaces

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Abstract

The object of this paper is to solve the generalized radical cubic functional equation, and discuss the stability problem in quasi-β-Banach spaces and then the stability by using subadditive and subquadratic functions in (β, p)-Banach spaces for the generalized radical cubic functional equation. Also certain non-stability results are investigated via specific counterexamples. Our results are generalization of the main results which are established by Z. Alizadeh and A. G. Ghazanfari in 2016.

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1 Introduction and preliminaries

In this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers and complex numbers, respectively; we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. Moreover, E and F always stand for normed spaces.

The study of stability problems for functional equations is related to the famous question of S.M. Ulam (see [36]) in 1940, concerning the stability of group homomorphisms.

Ulam's question. *Let $(G_1, *)$, $(G_2, \star)$ be two groups and $d : G_2 \times G_2 \to [0, \infty)$ be a metric. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality*

$$d(f(x * y), f(x) \star f(y)) \leq \delta$$

for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), h(x)) \leq \varepsilon \ \text{ for all } \ x \in G_1\,?$$

In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H. Hyers [21] gave the first affirmative answer to the question of Ulam for Banach spaces.

The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \qquad x, y \in E. \tag{1.1}$$

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Theorem 1.1. Let $f : E \to F$ satisfy the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in E\backslash\{0\}$, where θ and p are real constants with $\theta > 0$ and $p \neq 1$. Then the following two statements are valid.

(a) If $p \geq 0$ and F is complete, then there exists a unique solution $T : E \to F$ of (1.1) such that

$$\|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p, \qquad x \in E\backslash\{0\}. \tag{1.3}$$

(b) If $p < 0$, then f is additive, i.e., (1.1) holds.

Note that Theorem 1.1 reduces to the first result of stability due to D.H. Hyers [21] if $p = 0$, T. Aoki [2] for $0 < p < 1$ (see also [31]). Afterwards, Gajda [18] obtained this result for $p > 1$ and gave an example to show that Theorem 1.1 fails whenever $p = 1$. Also, Rassias [32] proved Theorem 1.1 for $p < 0$ (see [33, page 326] and [5]). In particular, D. G. Bourgin [4] had commended the stability bounded by function on C^*-algebra. Now, it is well-known that the statement (b) is valid, i.e., f must be additive in that case, which has been proved for the first time in [28] and next in [6] on the restricted domain. In 1978, Gruber [20] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. Gajda and Ger [19] showed that one can get analogous stability results for subadditive multifunctions. In 1994, Găvruta [17] obtained a generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. During the past few years several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see, for instance, [7–12, 15, 16, 26, 27, 29, 30, 34, 37]). Jun and Kim [24] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.4}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.4). The $f(x) = cx^3$ satisfies the functional equation (1.4), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. The stability problem of the radical functional equations in various spaces was proved in [13, 14, 22, 23].

Recently, interesting results concerning the radical cubic functional equation

$$f\left(\sqrt[3]{x^3 + y^3}\right) = f(x) + f(y), \tag{1.5}$$

have been obtained in [1].

We consider some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K\big(\|x\| + \|y\|\big)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a ***quasi-β-normed space*** if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A ***quasi-β-Banach space*** is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a *(β, p)-norm* $(0 < p \leq 1)$ if

$$\|x+y\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a *(β, p)-Banach space*.

By the Aoki-Rolewicz theorem [35] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, we restrict our attention mainly to p-norms.

Example 1.2. For $x = (x_1, x_2) \in \mathbb{R}^2$, we define

$$\|x\|_{p,\,\beta} = \begin{cases} \big(|x_1|^{\beta p} + |x_2|^{\beta p}\big)^{\frac{1}{p}}, & \text{if } x_2 \neq 0 \\ 2|x_1|^\beta, & \text{if } x_2 = 0 \end{cases}$$

where $0 < p,\ \beta \leq 1$. Then $(\mathbb{R}^2, \|\cdot\|_{p,\,\beta})$ is a $(\beta,\ p)$-norm space.

Example 1.3. If X is a quasi-β-norm space with the quasi-β-norm $\|x\|_\beta$, then it is a quasi-norm space with the quasi-norm $\|x\| = \|x\|_\beta^{\frac{1}{\beta}}$.

In this paper, we introduce the general solutions of the following radical cubic functional equation:

$$f\left(\sqrt[3]{ax^3 + by^3}\right) = af(x) + bf(y), \tag{1.6}$$

where a, b are nonzero fixed reals with $a + b \neq 0$. We use a direct method to prove the generalized Ulam stability, in the spirit of Găvruta (see [17]), of the functional equation (1.6) in quasi-β-normed spaces. Moreover, we generalize Ulam stability results controlled by more general mappings, by considering approximately mappings satisfying conditions much weaker than Hyers and Rassias conditions on approximately mappings. In fact, we investigate new theorems about the generalized Ulam stability by using subadditive and subquadratic functions in (β, p)-Banach spaces for the functional equation (1.6). Our results are the generalization of the main results which are in [1]. Note that J.M. Rassias [29] introduced the pioneering fuctional equation

$$f(x+2y) - 3f(x+y) - 3f(x) - f(x-y) = 6f(y), \tag{1.7}$$

satisfied by $f(x) = cx^3$.

2 Solution of generalized radical cubic functional equation (1.6)

In this section, we give the general solution of functional equation (1.6).

Theorem 2.1 ([1]). Let A be a linear space. If a function $f:\mathbb{R}\to A$ satisfies the functional equation (1.5), then f is a cubic function.

Theorem 2.2. Let A be a linear space. If a function $f:\mathbb{R}\to A$ satisfies the functional equation (1.6) with $a,b\in\mathbb{R}\backslash\{0\}$ and $a+b\neq 1$, then f is a cubic function.

Proof. We first assume that f satisfies (1.6). Substituting $x=y=0$ in (1.6) to obtain $f(0)=0$ since $a+b\neq 1$. Letting $y=0$ in (1.6), we get

$$f\big(\sqrt[3]{a}x\big)=af(x)$$

for all $x\in\mathbb{R}$. Putting $x=0$ in (1.6), we obtain

$$f\big(\sqrt[3]{b}y\big)=bf(y),$$

for all $y\in\mathbb{R}$. So, we have

$$f\big(\sqrt[3]{ab}x\big)=abf(x) \tag{2.1}$$

for all $x\in\mathbb{R}$. Replacing (x,y) by $\big(\sqrt[3]{b}x,\sqrt[3]{a}y\big)$ in (1.6), we get

$$f\left(\sqrt[3]{abx^3+aby^3}\right)=af(\sqrt[3]{b}x)+bf(\sqrt[3]{a}y), \tag{2.2}$$

for all $x,y\in\mathbb{R}$. It follows from (2.1) and (2.2) that f satisfies (1.5). Hence, by Theorem 2.1, f is cubic. This completes the proof. Q.E.D.

3 Approximation of generalized radical cubic functional equation (1.6)

In this section, we investigate the Hyers-Ulam stability of the generalized radical cubic functional equation (1.6) in quasi-β-normed spaces and (β,p)-Banach spaces, respectively.

Let X be a quasi-β-normed space and $\varphi:\mathbb{R}^2\to[0,\infty)$ be a function. A function $f:\mathbb{R}\to X$ is called a φ-approximatively radical cubic function if

$$\left\|f\left(\sqrt[3]{ax^3+by^3}\right)-af(x)-bf(y)\right\|\leq\varphi(x,y) \tag{3.1}$$

for all $x,y\in\mathbb{R}$, where $a,b\in\mathbb{R}\backslash\{0\}$ are such that $a+b\neq 1$.

Theorem 3.1. Let X be a quasi-β-Banach space and $f:\mathbb{R}\to X$ be a φ-approximatively radical cubic function with $f(0)=0$. If a function $\Phi:\mathbb{R}^2\to[0,\infty)$ satisfies the following

$$\Phi(x,y):= \tag{3.2}$$
$$\sum_{j=0}^{\infty}\left(\frac{K}{2^\beta}\right)^j\left\{\varphi\left(2^{\frac{j}{3}}x,\sqrt[3]{\frac{a}{b}}2^{\frac{j}{3}}y\right)+\varphi\left(2^{\frac{j+1}{3}}x,0\right)+\varphi\left(2^{\frac{j}{3}}x,0\right)+\varphi\left(0,\sqrt[3]{\frac{a}{b}}2^{\frac{j}{3}}y\right)\right\}<\infty$$

where

$$\lim_{n\to\infty}2^{-n\beta}\varphi\left(2^{\frac{n}{3}}x,2^{\frac{n}{3}}y\right)=0 \tag{3.3}$$

for all $x,y\in\mathbb{R}$, then there exists a unique cubic mapping $C:\mathbb{R}\to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x)-C(x)\|\leq\frac{K^3}{2^\beta|a|^\beta}\Phi(x,x) \tag{3.4}$$

for all $x\in\mathbb{R}$.

Proof. Replacing (x, y) by $\left(\frac{x}{\sqrt[3]{a}}, \frac{y}{\sqrt[3]{b}}\right)$ in (3.1), we get

$$\left\| f\left(\sqrt[3]{x^3+y^3}\right) - af\left(\frac{x}{\sqrt[3]{a}}\right) - bf\left(\frac{y}{\sqrt[3]{b}}\right) \right\| \le \varphi\left(\frac{x}{\sqrt[3]{a}}, \frac{y}{\sqrt[3]{b}}\right) \tag{3.5}$$

for all $x, y \in \mathbb{R}$. Setting $x = y = \sqrt[3]{a}x$ in (3.5), we obtain

$$\left\| f\left(\sqrt[3]{2a}x\right) - af(x) - bf\left(\sqrt[3]{\frac{a}{b}}x\right) \right\| \le \varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) \tag{3.6}$$

for all $x \in \mathbb{R}$. Replacing (x, y) by $(\sqrt[3]{2a}x, 0)$ in (3.5), we get

$$\left\| f\left(\sqrt[3]{2a}x\right) - af(\sqrt[3]{2}x) \right\| \le \varphi\left(\sqrt[3]{2}x, 0\right) \tag{3.7}$$

for all $x \in \mathbb{R}$. It follows from (3.6) and (3.7) that

$$\left\| af\left(\sqrt[3]{2}x\right) - af(x) - bf\left(\sqrt[3]{\frac{a}{b}}x\right) \right\| \le K\varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + K\varphi(\sqrt[3]{2}x, 0) \tag{3.8}$$

for all $x \in \mathbb{R}$. Substituting $x = \sqrt[3]{a}x$ and $y = 0$ in (3.5), we get

$$\left\| f\left(\sqrt[3]{a}x\right) - af(x) \right\| \le \varphi\left(x, 0\right) \tag{3.9}$$

for all $x \in \mathbb{R}$. Also, substituting $y = \sqrt[3]{a}x$ and $x = 0$ in (3.5), we obtain

$$\left\| f\left(\sqrt[3]{a}x\right) - bf\left(\sqrt[3]{\frac{a}{b}}x\right) \right\| \le \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right) \tag{3.10}$$

for all $x \in \mathbb{R}$. It follows from (3.9) and (3.10) that

$$\left\| bf\left(\sqrt[3]{\frac{a}{b}}x\right) - af(x) \right\| \le K\varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right) + K\varphi(x, 0) \tag{3.11}$$

for all $x \in \mathbb{R}$. It follows from (3.8) and (3.11) that

$$\left\| af(\sqrt[3]{2}x) - 2af(x) \right\| \le K^2\Big\{\varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + \varphi(\sqrt[3]{2}x, 0) + \varphi(x, 0) + \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right)\Big\} \tag{3.12}$$

for all $x \in \mathbb{R}$. Then we have

$$\begin{aligned}\left\| \frac{f(\sqrt[3]{2}x)}{2} - f(x) \right\| &\le \frac{K^2}{|2a|^\beta}\Big\{\varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + \varphi(\sqrt[3]{2}x, 0) + \varphi(x, 0) + \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right)\Big\} \\ &:= \psi(x, x)\end{aligned} \tag{3.13}$$

for all $x \in \mathbb{R}$. Then, by the iterative method, we get

$$\left\| \frac{f(2^{\frac{n}{3}}x)}{2^n} - f(x) \right\| \le K\sum_{j=0}^{n-1}\left(\frac{K}{2^\beta}\right)^j \psi\left(2^{\frac{j}{3}}x, 2^{\frac{j}{3}}x\right) \tag{3.14}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. For all $m, k \in \mathbb{N}_0$, with $m > k$, we have

$$\left\| \frac{f\left(2^{\frac{k}{3}}x\right)}{2^k} - \frac{f\left(2^{\frac{m}{3}}x\right)}{2^m} \right\| \leq K \sum_{j=k}^{m-1} \left(\frac{K}{2^\beta}\right)^j \psi\left(2^{\frac{j}{3}}x, 2^{\frac{j}{3}}x\right) \tag{3.15}$$

for all $x \in \mathbb{R}$. By (3.2) and (3.15), the sequence $\left\{2^{-n} f\left(2^{\frac{n}{3}}x\right)\right\}$ is a Cauchy sequence for all $x \in \mathbb{R}$. Since X is quasi-β-Banach space, it converges for all $x \in \mathbb{R}$. We can define a mapping $C : \mathbb{R} \to X$ by

$$C(x) := \lim_{n\to\infty} \frac{f\left(2^{\frac{n}{3}}x\right)}{2^n}$$

for all $x \in \mathbb{R}$. Also, we have

$$\begin{aligned} &\left\| C\left(\sqrt[3]{ax^3+by^3}\right) - aC(x) - bC(y) \right\| \\ &\qquad = \lim_{n\to\infty} 2^{-\beta n} \left\| f\left(2^{\frac{n}{3}} \sqrt[3]{ax^3+by^3}\right) - af\left(2^{\frac{n}{3}}x\right) - bf\left(2^{\frac{n}{3}}y\right) \right\| \\ &\qquad \leq \lim_{n\to\infty} 2^{-\beta n} \varphi\left(2^{\frac{n}{3}}x, 2^{\frac{n}{3}}y\right) = 0 \end{aligned}$$

for all $x, y \in \mathbb{R}$. Hence

$$C\left(\sqrt[3]{ax^3+by^3}\right) = aC(x) + bC(y)$$

for all $x, y \in \mathbb{R}$, and by Theorem 2.2, the mapping $C : \mathbb{R} \to X$ is cubic. Taking $m \to \infty$ in (3.15) with $k = 0$, we find that the mapping C satisfies (3.4) near $f : \mathbb{R} \to X$ of the functional equation (1.6).

Next, we assume that there is another cubic mapping $C' : \mathbb{R} \to X$ which satisfies (1.6) and (3.4) . Since C' satisfies (1.6), we have $C'\left(2^{\frac{n}{3}}x\right) = 2^n C'(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Thus, we get

$$\begin{aligned} \left\| 2^{-n} f\left(2^{\frac{n}{3}}x\right) - C'(x) \right\| &= 2^{-n\beta} \left\| f\left(2^{\frac{n}{3}}x\right) - C'\left(2^{\frac{n}{3}}x\right) \right\| \\ &\leq \frac{K^3}{2^{(n+1)\beta}|a|^\beta} \Phi\left(2^{\frac{n}{3}}x, 2^{\frac{n}{3}}x\right) \end{aligned}$$

for all $x \in \mathbb{R}$. Letting $n \to \infty$, we establish $C(x) = C'(x)$ for all $x \in \mathbb{R}$. This completes the proof. Q.E.D.

Theorem 3.2. Let X and f be same as in Theorem 3.1. If a function $\Psi : \mathbb{R}^2 \to [0, \infty)$ satisfies the following

$$\begin{aligned} &\Psi(x,y) := \\ &\sum_{j=1}^{\infty} \left(2^\beta K\right)^j \left\{ \varphi\left(2^{-\frac{j}{3}}x, \sqrt[3]{\frac{a}{b}} 2^{-\frac{j}{3}}y\right) + \varphi\left(2^{\frac{1-j}{3}}x, 0\right) + \varphi\left(2^{-\frac{j}{3}}x, 0\right) + \varphi\left(0, \sqrt[3]{\frac{a}{b}} 2^{-\frac{j}{3}}y\right) \right\} < \infty \end{aligned} \tag{3.16}$$

where

$$\lim_{n\to\infty} 2^{n\beta} \varphi\left(2^{-\frac{n}{3}}x, 2^{-\frac{n}{3}}y\right) = 0 \tag{3.17}$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x) - C(x)\| \le \frac{K^2}{2^\beta |a|^\beta} \Psi(x, x) \tag{3.18}$$

for all $x \in \mathbb{R}$.

Proof. If x is replaced by $\frac{x}{\sqrt[3]{2}}$ in (3.13), then the proof of Theorem 3.2 follows from the proof of Theorem 3.1. Q.E.D.

From Theorems 3.1 and 3.2, we obtain the following corollaries concerning the stability for approximate mappings controlled by a sum of powers of norms and a product of powers of norms and a mixed product-sum of powers of norms.

Corollary 3.1. Let X be a quasi-β-Banach space, $r, s, \varepsilon \in \mathbb{R}_+$ such that $r + s < 3(\beta - \log_2 K)$ and let $f : \mathbb{R} \to X$ be a function satisfying the following inequality

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \le \varepsilon |x|^r |y|^s$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and

$$\|f(x) - C(x)\| \le \frac{\varepsilon |\frac{a}{b}|^{\frac{s}{3}} K^3}{|a|^\beta \left(2^\beta - K 2^{\frac{r+s}{3}}\right)} |x|^{r+s}$$

for all $x \in \mathbb{R}$.

Corollary 3.2. Let X be a quasi-β-Banach space, $r, s, \varepsilon \in \mathbb{R}_+$ such that $r, s < 3(\beta - \log_2 K)$ and let $f : \mathbb{R} \to X$ be a function satisfying the following inequality

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \le \varepsilon (|x|^r + |y|^s) \tag{3.19}$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and

$$\|f(x) - C(x)\| \le \frac{\varepsilon K^3}{|a|^\beta} \left\{ \frac{\left(2 + 2^{\frac{r}{3}}\right) |x|^r}{2^\beta - K 2^{\frac{r}{3}}} + \frac{2 |\frac{a}{b}|^{\frac{s}{3}} |x|^s}{2^\beta - K 2^{\frac{s}{3}}} \right\} \tag{3.20}$$

for all $x \in \mathbb{R}$.

Corollary 3.3. Let X be a quasi-β-Banach space, $r, s, \varepsilon \in \mathbb{R}_+$ such that $r + s < 3(\beta - \log_2 K)$ and let $f : \mathbb{R} \to X$ be a function satisfying the following inequality

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \le \varepsilon \left(|x|^r |y|^s + |x|^{r+s} + |y|^{r+s} \right)$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and

$$\|f(x) - C(x)\| \leq \frac{\varepsilon K^3 |x|^{r+s}}{|a|^\beta \left(2^\beta - K 2^{\frac{r+s}{3}}\right)} \left\{ \left|\frac{a}{b}\right|^{\frac{s}{3}} + 2\left|\frac{a}{b}\right|^{\frac{r+s}{3}} + 2^{\frac{r+s}{3}} + 2 \right\}$$

for all $x \in \mathbb{R}$.

Corollary 3.4. Let X be a quasi-β-Banach space let $r, s \in \mathbb{R}_+ \cup \{0\}$, $\varepsilon \geq 0$ and $f : \mathbb{R} \to X$ be a function satisfying the following inequality

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \leq \begin{cases} \varepsilon |x|^r |y|^s, & r + s > 3(\beta + \log_2 K) \\ \varepsilon(|x|^r + |y|^s), & r, s > 3(\beta + \log_2 K) \\ \varepsilon(|x|^r |y|^s + |x|^r + |y|^s), & r + s > 3(\beta + \log_2 K) \end{cases}$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and

$$\|f(x) - C(x)\| \leq \begin{cases} \frac{\varepsilon \left|\frac{a}{b}\right|^{\frac{s}{3}} K^3}{|a|^\beta \left(2^{\frac{r+s}{3}} - K 2^\beta\right)} |x|^{r+s}, & r + s > 3(\beta + \log_2 K) \\ \frac{\varepsilon K^3}{|a|^\beta} \left\{ \frac{\left(2 + 2^{\frac{r}{3}}\right)|x|^r}{2^{\frac{r}{3}} - K 2^\beta} + \frac{2\left|\frac{a}{b}\right|^{\frac{s}{3}} |x|^s}{2^{\frac{s}{3}} - K 2^\beta} \right\}, & r, s > 3(\beta + \log_2 K) \\ \frac{\varepsilon K^3 |x|^{r+s}}{|a|^\beta \left(2^{\frac{r+s}{3}} - 2^\beta K\right)} \left\{ \left|\frac{a}{b}\right|^{\frac{s}{3}} + 2\left|\frac{a}{b}\right|^{\frac{r+s}{3}} + 2^{\frac{r+s}{3}} + 2 \right\}, & r + s > 3(\beta + \log_2 K) \end{cases}$$

for all $x \in \mathbb{R}$.

Now, we give an example to illustrate that the functional equation (1.6) is not stable for $r = s = 3$ with $\beta = 1$ and $K = 1$ in Corollary 3.2. This example is a modification of the example of Gajda [18] for the additive functional inequality (see also [25]).

Example 3.3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \gamma, & x \geq 1; \\ \gamma x^3, & |x| < 1; \\ -\gamma, & x \leq -1 \end{cases}$$

where $\gamma > 0$ is a constant. and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{n=0}^{\infty} \frac{\varphi(2^n x)}{8^n}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$|f\left(\sqrt[3]{3x^3 + y^3}\right) - 3f(x) - f(y)| \leq \frac{5\gamma \times 8^3}{7}(|x|^3 + |y|^3) \tag{3.21}$$

for all $x, y \in \mathbb{R}$, but there do not exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that $|f(x) - C(x)| \leq \lambda |x|^3$ for all $x \in \mathbb{R}$.

Proof. It is clear that f is bounded by $\frac{8}{7}\gamma$ on $\mathbb{R}$. We are going to prove that f satisfies (3.21).

If $x = y = 0$ then (3.21) is trivial. If $|x|^3 + |y|^3 \geq \frac{1}{8}$, then the left hand side of (3.21) is less than $\frac{5\gamma\times 8}{7}$. Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{8}$. Then there exists a nonnegative integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^3 + |y|^3 < \frac{1}{8^{k+1}}. \tag{3.22}$$

Hence $8^k|x|^3 < 1/8$, $8^k|y|^3 < 1/8$, and $2^{k-1}x, 2^{k-1}y, 2^{k-1}\sqrt[3]{3x^3+y^3} \in (-1,1)$. Hence, for $n = 0, 1, \cdots, k-1$,

$$\varphi(2^n\sqrt[3]{3x^3+y^3}) - 3\varphi(2^n x) - \varphi(2^n y) = 0.$$

Using (3.22) and the definition of f, we obtain that

$$\begin{aligned}
|f(\sqrt[3]{3x^3+y^3}) - 3f(x) - f(y)| &= \left|\sum_{n=0}^{\infty}\frac{\varphi(2^n\sqrt[3]{3x^3+y^3})}{8^n} - 3\sum_{n=0}^{\infty}\frac{\varphi(2^n x)}{8^n} - \sum_{n=0}^{\infty}\frac{\varphi(2^n y)}{8^n}\right| \\
&\leq \sum_{n=0}^{\infty} 8^{-n}\left|\varphi(2^n\sqrt[3]{3x^3+y^3}) - 3\varphi(2^n x) - \varphi(2^n y)\right| \\
&\leq\ 5\gamma\sum_{n=k}^{\infty} 8^{-n} = \frac{3\gamma\times 8^3}{8^{k+2}\times 7} \leq \frac{5\gamma\times 8^3}{7}(|x|^3+|y|^3)
\end{aligned}$$

for all $x, y \in \mathbb{R}$ with $0 < |x|^3 + |y|^3 < \frac{1}{8}$. Therefore, f satisfies (3.21) for all $x, y \in \mathbb{R}$.

Next, we claim that a radical quintic functional equation (1.6) is not stable for $r = s = 3$, $\beta = 1$ and $K = 1$ in Corollary 3.2. Suppose on the contrary that there exists a quintic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that $|f(x) - C(x)| \leq \lambda|x|^3$ for all $x \in \mathbb{R}$. Then there exists a constant $c \in \mathbb{R}$ such that $C(x) = dx^3$ for all rational numbers x (see [25]). So, we have

$$|f(x)| \leq (\lambda + |d|)|x|^3 \quad \text{for all} \ \ x \in \mathbb{Q}. \tag{3.23}$$

But we can choose a positive integer m with $m\gamma > \lambda + |d|$. If x is a rational number in $\in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for $n = 0, \ldots, m-1$ and for this x we get

$$f(x) = \sum_{n=0}^{\infty}\frac{\varphi(2^n x)}{8^n} \geq \sum_{n=0}^{m-1}\frac{\gamma(2^n x)^3}{8^n} = \gamma m x^3 > (\lambda + |d|)x^3,$$

which contradicts (3.23).

Q.E.D.

The following corollaries are particular cases of Theorems 3.1 and 3.2 in the case $a = b = 1$.

Corollary 3.5 ([1]). Let X be a quasi-β-Banach space and $f : \mathbb{R} \to X$ be a φ-approximatively radical cubic function with $f(0) = 0$ and $a = b = 1$. If a function $\Phi_1 : \mathbb{R}^2 \to [0,\infty)$ satisfies the following

$$\Phi_1(x,y) := \sum_{j=0}^{\infty}\left(\frac{K}{2^\beta}\right)^j \varphi\left(2^{\frac{j}{3}}x, 2^{\frac{j}{3}}y\right)$$

and

$$\lim_{n\to\infty} 2^{-n\beta}\varphi\left(2^{\frac{n}{3}}x, 2^{\frac{n}{3}}y\right) = 0$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the following inequality

$$\|f(x) - C(x)\| \leq \frac{K}{2^\beta}\Phi_1(x, x)$$

Corollary 3.6 ([1]). Let X and f be the same as Corollary 3.5. If a function $\Psi_1 : \mathbb{R}^2 \to [0, \infty)$ satisfies the following

$$\Psi_1(x, y) := \sum_{j=1}^{\infty} \left(2^\beta K\right)^j \varphi\left(2^{-\frac{j}{3}}x, 2^{-\frac{j}{3}}y\right)$$

and

$$\lim_{n\to\infty} 2^{n\beta}\varphi\left(2^{-\frac{n}{3}}x, 2^{-\frac{n}{3}}y\right) = 0$$

for all $x, y \in \mathbb{R}$, then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the following inequality

$$\|f(x) - C(x)\| \leq \frac{\Psi_1(x, x)}{2^\beta}$$

for all $x \in \mathbb{R}$.

We recall that a subadditive (superadditive) function is a function $\varphi : A \to B$ having a domain A and a codomain $(B, \leq)$ that are both closed under addition with the following property:

$$\varphi(x + y) \leq (\geq)\ \varphi(x) + \varphi(y)$$

for all $x, y \in A$. Also, a subquadratic (superquadratic) function is a function $\varphi : A \to B$ with $\varphi(0) = 0$ and the following property:

$$\varphi(x + y) + \varphi(x - y) \leq (\geq)\ 2\varphi(x) + 2\varphi(y)$$

for all $x, y \in A$.

Let $\ell \in \{-1, 1\}$ be fixed. Assume that there exists a constant L with $0 < L < 1$ such that a function $\varphi : A \to B$ satisfies

$$\ell\varphi(x + y) \leq \ell L^\ell\big(\varphi(x) + \varphi(y)\big)$$

for all $x, y \in A$. Then we say that φ is contractively subadditive if $\ell = 1$ and φ is expansively superadditive if $\ell = -1$. It follows by the last inequality that φ satisfies the following properties:

$$\varphi\big(2^\ell x\big) \leq 2^\ell L\varphi(x)$$

$$\varphi\big(2^{\ell k} x\big) \leq \big(2^\ell L\big)^k\varphi(x)$$

for all $x \in A$ and $k \geq 1$.

Similarly, if there exists a constant L with $0 < L < 1$ such that a function $\varphi : A \to B$, with $\varphi(0) = 0$ satisfies

$$\ell\varphi(x+y) + \ell\varphi(x-y) \leq 2\ell L^{\ell}\big(\varphi(x) + \varphi(y)\big)$$

for all $x, y \in A$. Then we say that φ is contractively subquadratic if $\ell = 1$ and φ is expansively superquadratic if $\ell = -1$. It follows by the last inequality that φ satisfies the following properties:

$$\varphi\big(2^{\ell}x\big) \leq 4^{\ell}L\varphi(x)$$

$$\varphi\big(2^{\ell k}x\big) \leq \big(4^{\ell}L\big)^{k}\varphi(x)$$

for all $x \in A$ and $k \geq 1$. From now on, we investigate the generalized Hyers-Ulam stability of generalized radical cubic functional equations (1.6) in (β, p)-Banach spaces.

Theorem 3.4. Let X be a (β, p)-Banach space and $f : \mathbb{R} \to X$ be a φ-approximatively radical cubic function with $f(0) = 0$. Assume that the function φ is contractively subadditive with a constant L satisfying $2^{1-3\beta}L < 1$. Then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x) - C(x)\| \leq \frac{\widehat{\widehat{\Phi}}(x)}{|a|^{\beta}\sqrt[p]{8^{\beta p} - (2L)^{p}}}, \tag{3.24}$$

for all $x \in \mathbb{R}$, where

$$\widehat{\Phi}(x) := \varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + \varphi\big(\sqrt[3]{2}x, 0\big) + \varphi(x, 0) + \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right)$$

and

$$\widehat{\widehat{\Phi}}(x) := K^{3}\big(4^{\beta}K\widehat{\Phi}(x) + 2^{\beta}K\widehat{\Phi}(\sqrt[3]{2}x) + \widehat{\Phi}(\sqrt[3]{4}x)\big).$$

Proof. It follows from (3.13) in the proof of Theorem 3.1 that

$$\left\|f\big(\sqrt[3]{2}x\big) - 2f(x)\right\| \leq \frac{K^{2}}{|a|^{\beta}}\left\{\varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + \varphi\big(\sqrt[3]{2}x, 0\big) + \varphi(x, 0) + \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right)\right\}. \tag{3.25}$$

Let $\widehat{\Phi}(x) := \varphi\left(x, \sqrt[3]{\frac{a}{b}}x\right) + \varphi\big(\sqrt[3]{2}x, 0\big) + \varphi(x, 0) + \varphi\left(0, \sqrt[3]{\frac{a}{b}}x\right)$. Then, we obtain

$$\left\|f(x) - \frac{f(2x)}{8}\right\| \leq \frac{\widehat{\widehat{\Phi}}(x)}{|8a|^{\beta}} \tag{3.26}$$

for all $x \in \mathbb{R}$, where $\widehat{\widehat{\Phi}}(x) := K^{3}\big(4^{\beta}K\widehat{\Phi}(x) + 2^{\beta}K\widehat{\Phi}(\sqrt[3]{2}x) + \widehat{\Phi}(\sqrt[3]{4}x)\big)$. It follows from (3.26) with

$2^j x$ in the place of x and the iterative method that

$$\begin{aligned}\left\|\frac{f(2^k x)}{8^k}-\frac{f(2^m x)}{8^m}\right\|^p &= \left\|\sum_{j=k}^{m-1}\left(\frac{f(2^j x)}{8^j}-\frac{f(2^{j+1}x)}{8^{j+1}}\right)\right\|^p \\ &\leq \sum_{j=k}^{m-1}\frac{1}{8^{\beta jp}}\left\|\left(f(2^j x)-\frac{f(2^{j+1}x)}{8}\right)\right\|^p \\ &\leq \frac{1}{|8a|^{\beta p}}\sum_{j=k}^{m-1}\frac{1}{8^{\beta jp}}\widehat{\widehat{\Phi}}(2^j x)^p \\ &\leq \left(\frac{\widehat{\widehat{\Phi}}(x)}{|8a|^{\beta}}\right)^p\sum_{j=k}^{m-1}\left(2^{1-3\beta}L\right)^{jp}, \end{aligned} \tag{3.27}$$

for all $x \in \mathbb{R}$ and $m, k \in \mathbb{N}_0$ with $m > k$. Then the sequence $\{8^{-n} f(2^n x)\}$ is a Cauchy sequence in a (β, p)-Banach space X and so we can define a mapping $C : \mathbb{R} \to X$ by

$$C(x) := \lim_{n\to\infty}\frac{f(2^n x)}{8^n}$$

for all $x \in \mathbb{R}$. Then, we get

$$\left\|C\left(\sqrt[3]{ax^3+by^3}\right)-aC(x)-bC(y)\right\|^p \leq \varphi(x,y)^p \lim_{n\to\infty}\left(2^{1-3\beta}L\right)^{np}=0,$$

for all $x, y \in \mathbb{R}$. Hence $C\left(\sqrt[3]{ax^3+by^3}\right) = aC(x) + bC(y)$, that is, $C : \mathbb{R} \to X$ is cubic. Taking $m \to \infty$ in (3.27) with $k = 0$, we can show that the mapping C satisfies (3.24) near the approximate $f : \mathbb{R} \to X$ of the functional equation (1.6).

Next, we assume that there is another cubic mapping $C' : \mathbb{R} \to X$ which satisfies (1.6) and (3.24). Thus, we have

$$\left\|8^{-n} f(2^n x)-C'(x)\right\|^p \leq \frac{\widehat{\widehat{\Phi}}(x)^p}{|8a|^{p\beta}-\left(2|a|^{\beta}L\right)^p}\left(2^{1-3\beta}L\right)^{np},$$

for all $x \in \mathbb{R}$. Letting $n \to \infty$, the uniqueness of C follows. This completes the proof. Q.E.D.

Theorem 3.5. Let X, f, $\widehat{\Phi}(x)$ be same as in Theorem 3.4. Assume that the function φ is expansively superadditive with a constant L satisfying $2^{3\beta-1}L < 1$. Then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x)-C(x)\| \leq \frac{\widehat{\widehat{\Phi}}(x)}{|a|^{\beta}\sqrt[p]{(2L^{-1})^p-8^{\beta p}}}, \tag{3.28}$$

for all $x \in \mathbb{R}$, where

$$\widehat{\widehat{\Phi}}(x) := K^3\left(4^{\beta}K\widehat{\Phi}(x)+2^{\beta}K\widehat{\Phi}(\sqrt[3]{2}x)+\widehat{\Phi}(\sqrt[3]{4}x)\right).$$

Proof. It follows from (3.26) in the proof of Theorem 3.4 that

$$\left\| f(x) - 8f\left(\frac{x}{2}\right) \right\| \leq \frac{\widehat{\widehat{\Phi}}\left(\frac{x}{2}\right)}{|a|^{\beta}}, \tag{3.29}$$

for all $x \in \mathbb{R}$. Then, in (3.29), replacing x by $2^{-j}x$ and using the iterative method, we have

$$\begin{aligned} \left\|8^k f\left(2^{-k}x\right) - 8^m f\left(2^{-m}x\right)\right\|^p &\leq \sum_{j=k}^{m-1} \left\|8^j f\left(2^{-j}x\right) - 8^{j+1} f\left(2^{-(j+1)}x\right)\right\|^p \\ &\leq \left(\frac{\widehat{\widehat{\Phi}}(x)}{|8a|^{\beta}}\right)^p \sum_{j=k+1}^{m} \left(2^{3\beta-1}L\right)^{jp}, \end{aligned} \tag{3.30}$$

for all $x \in \mathbb{R}$ and $m, k \in \mathbb{N}_0$ with $m > k$. The remains follow the proof of Theorem 3.4. This completes the proof.

Q.E.D.

Theorem 3.6. Let X, f, $\widehat{\Phi}(x)$ be same as in Theorem 3.4. Assume that the function φ is contractively subquadratic with a constant L satisfying $2^{2-3\beta}L < 1$. Then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x) - C(x)\| \leq \frac{\widehat{\widehat{\Phi}}(x)}{|a|^{\beta} \sqrt[p]{8^{\beta p} - (4L)^p}} \tag{3.31}$$

for all $x \in \mathbb{R}$, where

$$\widehat{\widehat{\Phi}}(x) := K^3\left(4^{\beta} K \widehat{\Phi}(x) + 2^{\beta} K \widehat{\Phi}(\sqrt[3]{2}x) + \widehat{\Phi}(\sqrt[3]{4}x)\right).$$

Proof. Similar to Theorem 3.4, we get

$$\begin{aligned} \left\| \frac{f\left(2^k x\right)}{8^k} - \frac{f\left(2^m x\right)}{8^m} \right\|^p &\leq \frac{1}{|8a|^{\beta p}} \sum_{j=k}^{m-1} \frac{1}{8^{\beta j p}} \widehat{\widehat{\Phi}}\left(2^j x\right)^p \\ &\leq \left(\frac{\widehat{\widehat{\Phi}}(x)}{|8a|^{\beta}}\right)^p \sum_{j=k}^{m-1} \left(2^{2-3\beta}L\right)^{jp}, \end{aligned} \tag{3.32}$$

for all $x \in \mathbb{R}$ and $m, k \in \mathbb{N}_0$ with $m > k$. Then the sequence $\left\{8^{-n} f\left(2^n x\right)\right\}$ is a Cauchy sequence in a (β, p)-Banach space X and so we can define a mapping $C : \mathbb{R} \to X$ by

$$C(x) := \lim_{n \to \infty} \frac{f\left(2^n x\right)}{8^n},$$

for all $x \in \mathbb{R}$. Then, we get

$$\left\| C\left(\sqrt[3]{ax^3 + by^3}\right) - aC(x) - bC(y) \right\|^p \leq \varphi(x, y)^p \lim_{n \to \infty} \left(2^{2-3\beta}L\right)^{np} = 0,$$

for all $x, y \in \mathbb{R}$. Hence $C\left(\sqrt[3]{ax^3+by^3}\right) = aC(x) + bC(y)$; that is, $C : \mathbb{R} \to X$ is a cubic mapping. Taking $m \to \infty$ in (3.32) with $k = 0$, we have

$$\|f(x) - C(x)\| \leq \frac{\widehat{\widehat{\widehat{\Phi}}}(x)}{|a|^\beta \sqrt[p]{8^{\beta p} - (4L)^p}},$$

for all $x \in \mathbb{R}$. Next, we assume that there is another cubic mapping $C' : \mathbb{R} \to X$ which satisfies (1.6) and (3.31). Then, we have

$$\left\|8^{-n} f\left(2^n x\right) - C'(x)\right\|^p \leq \frac{\widehat{\widehat{\widehat{\Phi}}}(x)^p}{|8a|^{p\beta} - \left(4|a|^\beta L\right)^p}\left(2^{2-3\beta}L\right)^{np},$$

for all $x \in \mathbb{R}$. Letting $n \to \infty$, the uniqueness of C follows. This completes the proof. Q.E.D.

Analogously to the proof of Theorem 3.5, we prove the following Theorem.

Theorem 3.7. Let X, f, $\widehat{\Phi}(x)$ be same as in Theorem 3.4. Assume that the function φ is expansively superquadratic with a constant L satisfying $2^{3\beta-2}L < 1$. Then there exists a unique cubic mapping $C : \mathbb{R} \to X$ satisfying the functional equation (1.6) and the following inequality

$$\|f(x) - C(x)\| \leq \frac{\widehat{\widehat{\widehat{\Phi}}}(x)}{|a|^\beta \sqrt[p]{\left(4L^{-1}\right)^p - 8^{\beta p}}}, \tag{3.33}$$

for all $x \in \mathbb{R}$, where

$$\widehat{\widehat{\widehat{\Phi}}}(x) := K^3\left(4^\beta K\widehat{\Phi}(x) + 2^\beta K\widehat{\Phi}(\sqrt[3]{2}x) + \widehat{\Phi}(\sqrt[3]{4}x)\right).$$

Remark 3.8. In the case $a = b = 1$ in the above theorems, we will get the main results which are in [1].

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