Properties and applications of the Gould-Hopper-Frobenius-Euler polynomials

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Abstract

This article deals with the introduction of Gould-Hopper based Frobenius-Euler polynomials and derivation of their properties. The summation formulae and operational rule for these polynomials are derived. Also, certain identities for these polynomials are established by using operational formalism.

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1 Introduction and preliminaries

Frobenius [4, 11] studied in great detail the Frobenius-Euler polynomials $F_n(x|u)$ satisfying the following exponential generating function:

$$\frac{1-u}{e^t-u}\ e^{xt} = \sum_{n=0}^{\infty} F_n(x|u)\frac{t^n}{n!},\ \forall\ u \in \mathbb{C};\ u \neq 1. \tag{1.1}$$

In particular, $F_n(u) = F_n(0|u)$ are called the Frobenius-Euler numbers defined by the following generating relation:

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} F_n(0|u)\frac{t^n}{n!},\ \forall\ u \in \mathbb{C};\ u \neq 1. \tag{1.2}$$

In fact, the Frobenius-Euler polynomials can also be defined recursively by the Frobenius-Euler numbers, as follows:

$$F_n(x|u) = \sum_{k=0}^{n} \binom{n}{k} F_k(u) x^{n-k}, \qquad n \geq 0, \tag{1.3}$$

where, Frobenius-Euler numbers satisfy the recurrence relation

$$F_0(u) = 1,$$

$$(F(u)+1)^n - F_n(u) = \begin{cases} 1-u, & n=0, \\ 0, & n \geq 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing $F^n(u)$ by $F_n(u)$.

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Some analogues of the Frobenius-Euler polynomials are the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$. They are given by the following generating relations:

$$\frac{t}{e^t-1}\ e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{1.5}$$

and

$$\frac{2}{e^t+1}\ e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{1.6}$$

respectively.

Especially, the rational numbers $B_n=B_n(0)$ and integers $E_n=2^nE_n(\frac{1}{2})$ are called the classical Bernoulli numbers and Euler numbers, respectively. These numbers and polynomials play important roles in many different areas of mathematics including number theory, combinatorics, special functions and analysis. Obviously, the Frobenius-Euler polynomials give the classical Euler polynomials when $u=-1$ in equation (1.1).

The special polynomials of two variables are important from the point of view of applications. These polynomials allow the derivation of a number of useful identities in a fairly straight forward way and help in introducing new families of special polynomials. For example, Bretti *et al.* [3] introduced general classes of the Appell polynomials of two variables by using properties of an iterated isomorphism, related to the Laguerre-type exponentials. The two variable forms of the Hermite, Laguerre and truncated exponential polynomials as well as their generalizations are considered by several authors, see for example [2, 6–8].

To solve the problems arising in many branches of mathematics, going from the theory of partial differential equations to abstract group theory, requirement of multi-index and multi-variable special functions are realized. The theory of multi-index and multi-variable Hermite polynomials was initially developed by Hermite himself [13]. The Hermite polynomials turn up in combinatorics, as an example of an Appell sequence, obeying the umbral calculus; in numerical analysis as Gaussian quadrature; in physics, where they give rise to the eigen states of the quantum harmonic oscillator and also turn up in the solution of the Schrödinger equation for the harmonic oscillator.

The Gould-Hopper polynomials $g_n^{(m)}(x,y)$ [12] are defined by the following series expansion:

$$g_n^{(m)}(x,y)=n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x^{n-mr}y^r}{(n-mr)!r!}, \tag{1.7}$$

where m is a positive integer. These polynomials possess the following generating function:

$$e^{xt+yt^m}=\sum_{n=0}^{\infty}g_n^{(m)}(x,y)\frac{t^n}{n!} \tag{1.8}$$

and are the solutions of the generalized heat equation [9]

$$\frac{\partial}{\partial y}\{f(x,y)\}=\frac{\partial^m}{\partial x^m}\{f(x,y)\},\qquad f(x,0)=x^n. \tag{1.9}$$

These polynomials are also defined by the following operational rule:

$$g_n^{(m)}(x,y) = \exp\left(y\frac{\partial^m}{\partial x^m}\right)\{x^n\}, \tag{1.10}$$

and satisfy the following multiplicative and derivative operators:

$$\hat{M}_g := x + my\frac{\partial^{m-1}}{\partial x^{m-1}} \tag{1.11}$$

and

$$\hat{P}_g := \frac{\partial}{\partial x}. \tag{1.12}$$

The idea of monomiality traces back in 1941, when J. F. Steffensen [16] suggested the concept of poweroid. The monomiality principle is reformulated and developed by Dattoli [6], according to the monomiality principle, there exist two operators $\hat{M}$ and $\hat{P}$ playing, respectively, the role of multiplicative and derivative operators for a polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$, that is, $\hat{M}$ and $\hat{P}$ satisfy the following identities, for all $n \in \mathbb{N}$:

$$\hat{M}\{p_n(x)\} = p_{n+1}(x) \tag{1.13}$$

and

$$\hat{P}\{p_n(x)\} = n\ p_{n-1}(x). \tag{1.14}$$

The polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is then called a quasi-monomial. These multiplicative and derivative operators must satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1} \tag{1.15}$$

and therefore exhibits a Weyl group structure.
If the expressed polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can be established from those of the $\hat{M}$ and $\hat{P}$ operators. In fact the following holds:

(i) If $\hat{M}$ and $\hat{P}$ have differential realizations, then the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{p_n(x)\} = n\ p_n(x). \tag{1.16}$$

(ii) Assuming that $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \hat{M}^n\ \{1\}. \tag{1.17}$$

(iii) In view of identity (1.17), the exponential generating function of $p_n(x)$ can be cast in the form

$$e^{t\hat{M}}\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!},\ \ |t| < \infty\ . \tag{1.18}$$

Most properties of polynomial families, identified as quasi-monomials, can be deduced by making use of operational rules associated with the relevant multiplicative and derivative operators. The concept of quasi-monomiality has been exploited within different contexts to deal with isospectral problems [15] and to study the properties of new families of special functions, see for example [6]. Thus, the families of isospectral problems can be defined by using the following correspondence:

$$\begin{array}{lcl} \hat{M} & \Leftrightarrow & x, \\ \hat{P} & \Leftrightarrow & \dfrac{\partial}{\partial x}, \\ p_n(x) & \Leftrightarrow & x^n. \end{array} \tag{1.19}$$

There is continuous use of operational methods in research fields like quantum and classical optics and in these respective areas, the operational methods provide powerful and efficient means of investigation.

The article is organized as follows: In Section 2, Gould-Hopper based Frobenius-Euler polynomials are introduced and certain properties of this family of polynomials are derived. In Section 3, Summation formulae for the Gould-Hopper based Frobenius-Euler polynomials are derived . In the last Section, certain identities for these polynomials are established by using the operational formalism.

2 Gould-Hopper based Frobenius-Euler polynomials

In this section Gould-Hopper based Frobenius-Euler polynomials are introduced and their quasi-monomial properties are derived.

To generate the Gould-Hopper based Frobenius-Euler polynomials denoted by ${}_{g^{(m)}}F_n(x,y|u)$, we prove the following result:

Theorem 2.1. For the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$, the following generating function holds true:

$$\frac{1-u}{(e^t-u)}\exp(xt+yt^m)=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x,y|u)\frac{t^n}{n!}. \tag{2.1}$$

Proof. Replacing x in equation (1.1) by the multiplicative operator $\hat{M}_g$ of the Gould-Hopper polynomials $g_n^{(m)}(x,y)$, we have

$$\frac{1-u}{(e^t-u)}\exp(\hat{M}_g t)\{1\}=\sum_{n=0}^{\infty}F_n(\hat{M}_g|u)\frac{t^n}{n!}. \tag{2.2}$$

Using the expression of $\hat{M}_g$ given in equation (1.11) and then decoupling the exponential operator in the l.h.s of the resultant equation by using the Crofton-type identity

$$f\left(y+m\lambda\frac{d^{m-1}}{dy^{m-1}}\right)\{1\}=\exp\left(\lambda\frac{d^m}{dy^m}\right)\Big\{f(y)\Big\}, \tag{2.3}$$

we find

$$\frac{1-u}{(e^t-u)}\ \exp\left(y\frac{\partial^m}{\partial x^m}\right)\exp(xt)=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}|u)\frac{t^n}{n!}. \tag{2.4}$$

Which on using equations (1.10) and (1.8) in the l.h.s. of above equation and denoting the Gould-Hopper based Frobenius-Euler polynomials in the r.h.s. of equation (2.4) by ${}_{g^{(m)}}F_n(x,y|u)$, that is

$${}_{g^{(m)}}F_n(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}|u)={}_{g^{(m)}}F_n(x,y|u) \tag{2.5}$$

we get assertion (2.1).

Q.E.D.

In order to frame the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ within the context of monomiality principle formalism, we prove the following result:

Theorem 2.2. For the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$, the following multiplicative and derivative operators hold true:

$$\hat{M}_{g^{(m)}F}=x+myD_x^{m-1}-\frac{e^{D_x}}{e^{D_x}-u} \tag{2.6}$$

and

$$\hat{P}_{g^{(m)}F}=D_x, \tag{2.7}$$

respectively.

Proof. Differentiating (2.1) partially with respect to t, we have

$$\left(x+myt^{m-1}-\frac{e^t}{e^t-u}\right)\frac{1-u}{e^t-u}e^{xt+yt^m}=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_{n+1}(x,y|u)\frac{t^n}{n!}. \tag{2.8}$$

Using the identity

$$D_x\Big\{\frac{1-u}{e^t-u}e^{xt+yt^m}\Big\}=t\Big\{\frac{1-u}{e^t-u}e^{xt+yt^m}\Big\} \tag{2.9}$$

and generating relation (2.1) in the equation (2.8), we find

$$\left(x+myD_x^{m-1}-\frac{e^{D_x}}{e^{D_x}-u}\right)\left(\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x,y|u)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_{n+1}(x,y|u)\frac{t^n}{n!}. \tag{2.10}$$

Equating the coefficients of same powers of t on both sides of above equation, we are lead to assertion (2.6).

Again making use of generating function (2.1) on both sides of identity equation (2.9), we have

$$D_x\Big\{\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x,y|u)\frac{t^n}{n!}\Big\}=\Big\{\sum_{n=0}^{\infty}{}_{g^{(m)}}F_{n-1}(x,y|u)\frac{t^n}{(n-1)!}\Big\}. \tag{2.11}$$

Equating the coefficients of same powers of t on both sides of the above equation, we are lead to assertion (2.7).

Q.E.D.

Remark 2.1. Using expressions (2.6) and (2.7) in equation (1.16), we find the following differential equation for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$:

$$\left(xD_x + myD_x^m - \frac{e^{D_x}}{e^{D_x}-u}D_x - n\right){}_{g^{(m)}}F_n(x,y|u) = 0. \tag{2.12}$$

Remark 2.2. On substituting $y = 0$, the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ reduces to the Frobenius-Euler polynomials $F_n(x|u)$, thus taking $y = 0$ in equations (2.1), (2.6), (2.7) and (2.12), the following generating function, multiplicative and derivative operators and differential equation for the Frobenius-Euler polynomials $F_n(x|u)$ given by

$$\frac{1-u}{(e^t-u)}\exp(xt) = \sum_{n=0}^{\infty} F_n(x|u)\frac{t^n}{n!}, \tag{2.13}$$

$$\hat{M}_F = x - \frac{e^{D_x}}{e^{D_x}-u}, \tag{2.14}$$

$$\hat{P}_F = D_x \tag{2.15}$$

and

$$\left(xD_x - \frac{e^{D_x}}{e^{D_x}-u}D_x - n\right)F_n(x|u) = 0, \tag{2.16}$$

are obtained respectively.

Next, we find the series definitions for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ by proving the following result:

Theorem 2.3. The Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ are defined by the following series expansions:

$${}_{g^{(m)}}F_n(x,y|u) = \sum_{k=0}^{n}\binom{n}{k}F_k(u)g_{n-k}^{(m)}(x,y) \tag{2.17}$$

and

$${}_{g^{(m)}}F_n(x,y|u) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{F_{n-k}(x|u)\ y^k}{k!\ (n-mk)!}, \tag{2.18}$$

respectively.

Proof. Using equations (1.2) and (1.8) in the l.h.s. of the equation (2.1), we have

$$\sum_{k=0}^{\infty}F_k(u)\frac{t^k}{k!}\sum_{n=0}^{\infty}g_n^{(m)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x,y|u), \tag{2.19}$$

which on using the cauchy product rule in the l.h.s. of the above equation and equating the coefficients of like powers of t on both sides, assertion (2.17) follows.

Similarly, using equation (1.1) and expanding the term e^{yt^m} in the l.h.s. of the equation (2.1), we have

$$\sum_{n=0}^{\infty} F_n(x|u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{y^k\ t^{mk}}{k!} = \sum_{n=0}^{\infty} {}_{g^{(m)}}F_n(x,y|u)\frac{t^n}{n!}, \tag{2.20}$$

which on using the cauchy product rule in the l.h.s. of the above equation, assertion (2.18) follows.

Q.E.D.

Finally, we find the operation connection between the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ and Frobenius-Euler polynomials $F_n(x|u)$ by proving the following result:

Theorem 2.4. The Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ are defined by the following operational rule:

$$\exp\left(y\frac{\partial^m}{\partial x^m}\right)F_n(x|u) = {}_{g^{(m)}}F_n(x,y|u). \tag{2.21}$$

Proof. Differentiating generating function (2.1) partially m times with respect to x and y, we have

$$\frac{\partial^m}{\partial x^m}\left({}_{g^{(m)}}F_n(x,y|u)\right) = n(n-1)(n-2)\cdots(n-m)\left({}_{g^{(m)}}F_{n-m}(x,y|u)\right) \tag{2.22}$$

and

$$\frac{\partial}{\partial y}\left({}_{g^{(m)}}F_n(x,y|u)\right) = n(n-1)(n-2)\cdots(n-m)\left({}_{g^{(m)}}F_{n-m}(x,y|u)\right). \tag{2.23}$$

Thus from equations (2.22) and (2.23), it follows that

$$\frac{\partial^m}{\partial x^m}\left({}_{g^{(m)}}F_n(x,y|u)\right) = \frac{\partial}{\partial y}\left({}_{g^{(m)}}F_n(x,y|u)\right) \tag{2.24}$$

and in view of the initial condition:

$${}_{g^{(m)}}F_n(x,0|u) = F_n(x|u), \tag{2.25}$$

assertion (2.21) follows.

Q.E.D.

In the next section, certain summation formulae for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ are derived.

3 Summation formulae

In order to obtain the implicit summation formula for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$, the following result is proved:

Theorem 3.1. The Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ satisfies the following implicit summation formula:

$${}_{g^{(m)}}F_n(x+\nu,y|u) = \sum_{k=0}^{n}\binom{n}{k}\nu^k\,{}_{g^{(m)}}F_{n-k}(x,y|u). \tag{3.1}$$

Proof. Replacing $x \to x + \nu$ in generating function (2.1), we have

$$\frac{1-u}{(e^t-u)}\exp\big((x+\nu)t+yt^m\big)=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x+\nu,y|u)\frac{t^n}{n!}. \tag{3.2}$$

Expanding the exponential in the l.h.s. of the above equation and then using the Cauchy product rule in the resultant equation, it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\nu^k{}_{g^{(m)}}F_{n-k}(x,y|u)\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x+\nu,y|u)\frac{t^n}{n!}. \tag{3.3}$$

Exchanging the sides and equating the coefficients of same powers of t in the resultant equation, assertion (3.1) follows.

Q.E.D.

Corollary 3.1. For, $\nu = 1$, the following implicit summation formula for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ holds true:

$${}_{g^{(m)}}F_n(x+1,y|u)=\sum_{k=0}^{n}\binom{n}{k}{}_{g^{(m)}}F_{n-k}(x,y|u). \tag{3.4}$$

Theorem 3.2. For the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$, the following implicit summation formula holds true:

$${}_{g^{(m)}}F_{n+k}(\eta,y|u)=\sum_{l,s=0}^{n,k}\binom{n}{l}\binom{k}{s}(\eta-x)^{l+s}{}_{g^{(m)}}F_{n+k-l-s}(x,y|u). \tag{3.5}$$

Proof. Replacing $t \to t + w$ in generating function (2.1), we have

$$\frac{1-u}{(e^{t+w}-u)}\exp\big(x(t+w)+y(t+w)^m\big)=\sum_{n=0}^{\infty}{}_{g^{(m)}}F_n(x,y|u)\frac{(t+w)^n}{n!}. \tag{3.6}$$

Using

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{l,m=0}^{\infty}f(l+m)\frac{x^l\ y^m}{l!\ m!} \tag{3.7}$$

in equation (3.6) and shifting the exponential term to the r.h.s in the resultant equation, we have

$$\frac{1-u}{(e^{t+w}-u)}\exp\big(y(t+w)^m\big)=\exp\big(-x(t+w)\big)\sum_{n,k=0}^{\infty}{}_{g^{(m)}}F_{n+k}(x,y|u)\frac{t^n\ w^k}{n!\ k!}. \tag{3.8}$$

Replacing $x \to \eta$ in above equation and then equating the resultant equation to the above equation, we find

$$\sum_{n,k=0}^{\infty}{}_{g^{(m)}}F_{n+k}(\eta,y|u)\frac{t^n\ w^k}{n!\ k!}=\exp\big((\eta-x)(t+w)\big)\sum_{n,k=0}^{\infty}{}_{g^{(m)}}F_{n+k}(x,y|u)\frac{t^n\ w^k}{n!\ k!}. \tag{3.9}$$

In view of equation (3.7), the above equation gives

$$\sum_{n,k=0}^{\infty} {}_{g^{(m)}}F_{n+k}(\eta,y|u)\frac{t^n\ w^k}{n!\ k!} = \sum_{l,s=0}^{\infty}(\eta-x)^{l+s}\frac{t^l\ w^s}{l!\ s!}\ \sum_{n,k=0}^{\infty} {}_{g^{(m)}}F_{n+k}(x,y|u)\frac{t^n\ w^k}{n!\ k!}. \tag{3.10}$$

Applying the Cauchy product rule in the r.h.s. of the above equation and equating the coefficients of same powers of t in the resultant equation, assertion (3.5) follows.

Q.E.D.

Corollary 3.2. For, $n = 0$ in Theorem 3.2, Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ satisfies the following implicit summation formula:

$${}_{g^{(m)}}F_k(\eta,y|u) = \sum_{s=0}^{k}\binom{k}{s}(\eta-x)^s\,{}_{g^{(m)}}F_{k-s}(x,y|u). \tag{3.11}$$

Corollary 3.3. Replace $\eta \to \eta + x$ and $y = 0$ in Theorem 3.2, the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ satisfies the following implicit summation formula:

$${}_{g^{(m)}}F_{n+k}(\eta+x,0|u) = \sum_{l,s=0}^{n,k}\binom{n}{l}\binom{k}{s}(\eta)^{l+s}\,{}_{g^{(m)}}F_{n+k-l-s}(x,0|u). \tag{3.12}$$

Corollary 3.4. For, $\eta = 0$ in Theorem 3.2, the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ satisfies the following implicit summation formula:

$${}_{g^{(m)}}F_{n+k}(0,y|u) = \sum_{l,s=0}^{n,k}\binom{n}{l}\binom{k}{s}(-x)^{l+s}\,{}_{g^{(m)}}F_{n+k-l-s}(x,y|u). \tag{3.13}$$

Theorem 3.3. The following implicit summation formula for the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ holds true:

$${}_{g^{(m)}}F_n(x+v,y+w|u) = \sum_{k=0}^{n}\binom{n}{k}{}_{g^{(m)}}F_{n-k}(x,y|u)\ g_k^{(m)}(v,w). \tag{3.14}$$

Proof. Replacing $x \to x + v$ and $y \to y + w$ in generating function (2.1) and using equation (1.8), we find

$$\sum_{n=0}^{\infty} {}_{g^{(m)}}F_n(x+v,y+w|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{g^{(m)}}F_n(x,y|u)\ g_k^{(m)}(v,w)\frac{t^{n+k}}{n!\ k!}. \tag{3.15}$$

Applying the Cauchy product rule in the r.h.s. of the above equation and then comparing the coefficients of the like powers of t in the resultant equation yields assertion (3.14).

Q.E.D.

In the next section, some examples are considered to give applications of the operational formalism developed here.

4 Applications

Several identities involving Frobenius-Euler polynomials are known. The operational formalism developed in the previous section can be used to obtain the corresponding identities involving the Gould-Hopper based Frobenius-Euler polynomials. To achieve this, we perform the following operation:

($\mathcal{O}$) operating $\exp\left(y\frac{\partial^m}{\partial x^m}\right)$ on both sides of a given relation.

We recall the following functional equations involving Frobenius-Euler polynomials $F_n(x|u)$ [5]:

$$F_k(x|u)\ F_n(x|v) = F_{kn}(x|uv)\frac{(1-u)(1-v)}{1-uv} + \frac{u(1-v)}{1-uv}\sum_{r=0}^{k}\binom{k}{r}F_k(u)F_{k+n-r}(x|uv)$$
$$+\ \frac{v(1-v)}{1-uv}\sum_{s=0}^{n}\binom{n}{s}F_s(v)F_{k+n-s}(x|uv), \tag{4.1}$$

where $u, v \in \mathbb{C}$, $u \neq 1$, $v \neq 1$ and $uv \neq 1$.

Performing the operation ($\mathcal{O}$) on above equation and using operational rule (2.21) on the resultant equation, the following identity involving the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ is obtained:

$${}_{g^{(m)}}F_k(x,y|u)\ F_n(x|v) = {}_{g^{(m)}}F_{kn}(x,y|uv)\frac{(1-u)(1-v)}{1-uv} + \frac{u(1-v)}{1-uv}\sum_{r=0}^{k}\binom{k}{r}F_k(u)$$
$$\times\ {}_{g^{(m)}}F_{k+n-r}(x,y|uv) + \ \frac{v(1-v)}{1-uv}\sum_{s=0}^{n}\binom{n}{s}F_s(v)\ {}_{g^{(m)}}F_{k+n-s}(x,y|uv), \tag{4.2}$$

where $u, v \in \mathbb{C}$, $u \neq 1$, $v \neq 1$ and $uv \neq 1$.

Also, consider the following identities for the Frobenius-Euler polynomials $F_n^{(\alpha)}(x;u)$ from [14]:

$$u\ F_n(x|u^{-1}) + F_n(x|u) = (1+u)\sum_{k=0}^{n}\binom{n}{k}\ F_{n-k}(u^{-1})\ F_k(x|u), \tag{4.3}$$

$$\frac{1}{n+1}F_k(x|u) + F_{n-k}(x|u) = \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\Big((-u)F_{l-k}(u)\ F_{n-l}(u)$$
$$+\ 2u\ F_{n-k}(u)\Big)\ F_k(x|u)\ F_n(x|u), \tag{4.4}$$

$$F_n^{(\alpha)}(x|u) = \sum_{k=0}^{n}\binom{n}{k}F_{n-k}^{(\alpha-1)}(u)\ F_k(x|u)\ (n \in \mathbb{Z}_+), \tag{4.5}$$

$$F_n(x|u) = \frac{1}{(1-u)^\alpha}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-u)^{\alpha-j}F_{n-k}(j|u)\right)F_k^{(\alpha)}(x|u)\ (n\in\mathbb{Z}_+). \tag{4.6}$$

Again performing the same operation ($\mathcal{O}$) on equation (4.3) - (4.6) and using operational rule (2.21) on the resultant equation, the following identity involving the Gould-Hopper based Frobenius-Euler polynomials ${}_{g^{(m)}}F_n(x,y|u)$ are obtained:

$$u\ {}_{g^{(m)}}F_n(x,y|u^{-1}) + {}_{g^{(m)}}F_n(x,y|u) = (1+u)\sum_{k=0}^{n}\binom{n}{k}\ F_{n-k}(u^{-1})\ {}_{g^{(m)}}F_k(x,y|u), \tag{4.7}$$

$$\begin{aligned}\frac{1}{n+1}{}_{g^{(m)}}F_k(x,y|u) + {}_{g^{(m)}}F_{n-k}(x,y|u) = \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\Big((-u)F_{l-k}(u)\ F_{n-l}(u)\\ + 2u\ F_{n-k}(u)\Big)\ F_k(x|u)\ {}_{g^{(m)}}F_n(x,y|u),\end{aligned} \tag{4.8}$$

$${}_{g^{(m)}}F_n^{(\alpha)}(x,y|u) = \sum_{k=0}^{n}\binom{n}{k}F_{n-k}^{(\alpha-1)}(u)\ {}_{g^{(m)}}F_k(x,y|u)\ (n\in\mathbb{Z}_+), \tag{4.9}$$

$${}_{g^{(m)}}F_n(x,y|u) = \frac{1}{(1-u)^\alpha}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-u)^{\alpha-j}F_{n-k}(j|u)\right){}_{g^{(m)}}F_k^{(\alpha)}(x,y|u)\ (n\in\mathbb{Z}_+). \tag{4.10}$$

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of functions. The importance of the use of operational techniques in the study of special functions are aimed at providing explicit solutions for families of partial differential equations including those of Heat and D′Alembert type and their applications has been recognized by Dattoli and his co-workers, see for example [6, 9, 10]. In the case of multi-variable generalized special functions, The method proposed in this article can be used in combination with the monomiality principle as a useful tool in new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems.

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