On projective QTAG-modules

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Abstract

In this paper we introduce the new class of QTAG-modules namely ω_1-$(\omega+n)$-projective modules, which is an amalgamation of three important classes of modules: n-bounded modules, the direct sum of uniserial modules and the countably generated modules. This class is given many equivalent characterizations including being the smallest class containing the $(\omega+n)$-projective modules that is closed with respect to ω_1-bijective homomorphisms.

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1 Introduction

Many concepts for groups like purity, projectivity, injectivity, height etc. have been generalized for modules. To obtain results of groups which are not true for modules either conditions have been applied on modules or upon the underlying rings. We imposed the condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the rings are associative with unity. After these conditions many elegant results of groups can be proved for QTAG-modules which are not true in general. Many results of this paper are the generalization of the paper [2].

The study of QTAG-modules was initiated by Singh [10]. Khan, Mehdi, Abbasi etc. worked a lot on these modules [3, 4] etc. They studied different notions and structures of QTAG-modules, developed the theory of these modules by introducing several notions and investigated some interesting properties and characterized them. Yet there is much to explore.

A module M over an associative ring R with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules [11]. All the rings R considered here are associative with unity and modules M are unital QTAG-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ [3] and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A $QTAG$-module M is said to be separable, if $M^1 = 0$. Let M be a module, then the sum of all simple

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submodules of M is called the socle of M and is denoted by $\mathrm{Soc}(M)$. If M, M' are QTAG-modules then a homomorphism $f : M \to M'$ is an isometry if it is 1-1, onto and $H_{M'}(f(x)) = H_M(x)$, for all $x \in M$. A submodule N of a QTAG-module M is a nice submodule if every nonzero coset $a+N$ is proper with respect to N *i.e.* for every nonzero $a+N$ there is an element $b \in N$ such that $H_M(a+b) = H_{M/N}(a+N)$. A family $\mathcal{N}$ of submodules of M is called a nice system in M if

(i) $0 \in \mathcal{N}$;

(ii) If $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$;

(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [4].

Every submoddule in a nice system is nice submodule. A h-reduced $QTAG$-module M is called totally projective if it has a nice system and direct sums and direct summands of totally projective modules are also totally projective. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A QTAG- module M is $(\omega+n)$- projective, if there exists a submodule $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules or equivalently, if and only if there is a direct sum of uniserial module K with a submodule $L \subseteq H^n(K)$ such that $M \cong K/L$. M is ω-projective if and only if it is a direct sum of uniserial modules. Also two $(\omega+n)$-projective QTAG-modules M_1 , M_2 are isometric if and only if there is a height preserving isomorphism between $H^n(M_1)$ and $H^n(M_2)$ [4]. For any QTAG- module M, $g(M)$ denotes the smallest cardinal number λ such that M admits a generating set X of uniform elements of cardinality λ *i.e* :, $|X| = \lambda$. A homomorphism $f : M \to N$ is said to be ω_1-bijective if $g(ker\ f),\ g(N/f(M)) < \omega_1$.

2 Some characterizations of the class $\mathcal{W}_n$

The three important classes of modules that are closed under arbitrary submodules are

(i) The class of n-bounded modules, which we denote by $\mathcal{B}_n$.

(ii) The class of countably generated modules, which we denote by $\mathcal{C}$.

(iii) The class of direct sum of uniserial modules, which we denote by $\mathcal{D}$.

Here we study a class of modules which combines all three of these classes and denoted by $\mathcal{W}_n$ and we call these modules as ω_1-$(\omega+n)$-projective modules.

In this section, we will establish some interesting equivalent characterizations of this class and we also show that the class $\mathcal{W}_n$ is the smallest class of modules containing the $(\omega+n)$-projective modules that is closed under ω_1-bijective homomorphisms.

If $\mathcal{M}$ and $\mathcal{N}$ are classes of modules, a module M is said to be an extension of a module in $\mathcal{M}$ by a quotient in $\mathcal{N}$, denoted by $\mathcal{E}(\mathcal{M},\mathcal{N})$, if and only if there is a short exact sequence $0 \to L \to M \to N \to 0$ where $L \in \mathcal{M}$ and $N \in \mathcal{N}$. Such module is said to be an elongation of a module in $\mathcal{N}$ modulo a submodule in $\mathcal{M}$. Dually, M is said to be a quotient of a module in $\mathcal{M}$ by a submodule in $\mathcal{N}$, denoted by $M \in Q(\mathcal{M},\mathcal{N})$, if and only if there is a short exact sequence $0 \to L \to N \to M \to 0$ where $L \in \mathcal{M}$ and $N \in \mathcal{N}$.

We start with the following lemma:

Lemma 2.1. (cf. [2], Lemma 1.1) Suppose M is a QTAG-module.

(a) $M \in \mathcal{E}(\mathcal{B}_n, \mathcal{D})$ if and only if $M \in Q(\mathcal{B}_n, \mathcal{D})$ if and only if M is $(\omega + n)$-projective.

(b) $M \in \mathcal{E}(\mathcal{B}_n, \mathcal{C})$ if and only if $M \in \mathcal{E}(\mathcal{C}, \mathcal{B}_n)$ if and only if $H_n(M) \in \mathcal{C}$ if and only if M is isomorphic to the direct sum of an H^n-bounded and a countably generated submodule.

(c) $M \in \mathcal{E}(\mathcal{C}, \mathcal{D})$ if and only if $M \in \mathcal{E}(\mathcal{D}, \mathcal{C})$ if and only if $M \in Q(\mathcal{C}, \mathcal{D})$ if and only if M is isomorphic to the direct sum of a countably generated module and a module which is direct sum of uniserial modules.

Proof. (a) follows trivially by characterization of $(\omega + n)$-projective modules.

(b) Suppose M is isomorphic to the direct sum of a n-bounded and a countably generated module *i.e.* $M \cong L \oplus K$, where $L \in \mathcal{B}_n,\ K \in \mathcal{C}$; then $H_n(M) \cong H_n(K)$ is countably generated and the first two statements hold as well. Conversely, if $H_n(M)$ is countably generated, let K be a maximal n-bounded submodule of M. It follows that $M \cong K \oplus L$ and as $H_n(M)$ is countably generated it gives L is countably generated.

Let $M \in \mathcal{E}(\mathcal{B}_n, \mathcal{C})$ and let N be an n-bounded submodule of M with $M/N = K$ countably generated. Let H be a countably generated submodule of M such that $M = N + H$. Therefore $H_n(M) = H_n(N) + H_n(H) \subseteq H$ is countably generated as desired. Further, suppose that $M \in \mathcal{E}(\mathcal{C}, \mathcal{B}_n)$. Since $H_n(M) \subseteq N$ is countably generated submodule of M for which $M/N = K$ is n-bounded, which completes the proof of part (b).

(c) Suppose M is isomorphic to the direct sum of countably generated module L and a module N, which is direct sum of uniserial modules *i.e.* $M \cong L \oplus N$, then let D be the direct sum of countably generated modules with a submodule F such that $D/F \cong L$. If $E = D \oplus N$, then $E \in \mathcal{D}$ with exact sequence $0 \to F \to E \to M \to 0$ implying $M \in Q(\mathcal{C}, \mathcal{D})$. On the other hand, if $M \in Q(\mathcal{C}, \mathcal{D})$, then let L be a countably generated module contained in $E \in \mathcal{D}$ such that $E/L \cong M$ implying $E = T \oplus P$ with T countably generated and $L \subseteq T$. If $C = T/L$, then $P \in \mathcal{D}, C \in \mathcal{C}$ and $M \cong C \oplus P$ as desired.

If $M \in \mathcal{E}(\mathcal{C}, \mathcal{D})$, then it has a countably generated submodule N such that $P = M/N \in \mathcal{D}$. Take the maximal submodule N' of M such that $N' \cap N = \{0\}$. As $M \to P$ is injective on N' implies $N' \in \mathcal{D}$. Hence $M \in \mathcal{E}(\mathcal{D}, \mathcal{C})$ as N maps to an essential submodule of $P' = M/N'$ implying $N' \in \mathcal{C}$.

Lastly, if $M \in \mathcal{E}(\mathcal{D}, \mathcal{C})$ then there is a submodule $N \in \mathcal{D}$ such that $P = M/N \in \mathcal{C}$. If we choose a countably generated module T in such a way that $M = T + N$, then $N = T' \oplus E$ where T' is countably generated module contains $T \cap N$ and $E \in \mathcal{D}$. If $A = T + T'$, then $A \in \mathcal{C}$ and $M = A \oplus E$, which completes the proof.

Q.E.D.

Let us recall a module M is said to be a Σ-module if every high submodule of M is a direct sum of uniserial modules. In general, submodules of Σ-module need not be Σ-module. We will say M is a totally Σ-module if every submodule of M is also a Σ-module. M is said to be ω-totally ω-projective if every separable submodule of M is a direct sum of uniserial modules [9].

We will use the following notations in the subsequent results:

$\mathcal{F}_n$: The class of $(\omega+n)$-projective modules.
$\mathcal{C}_n$: The class of n^{th}-level countably generated modules.
$\mathcal{W}$: The class of ω-totally ω-projective modules.

We will say that a module $M \in \mathcal{F}_n$ if it satisfies Lemma 2.1 (a) and $M \in \mathcal{C}_n$ if it satisfies Lemma 2.1 (b). Now we have the following interesting result:

Theorem 2.2. (cf. [2], Theorem 1.2) For the module M, the following are equivalent:

(i) M is an extension of a n-bounded module by an ω-totally ω-projective quotient *i.e.* $M \in \mathcal{E}(\mathcal{B}_n, \mathcal{W})$.

(ii) M is a quotient of an ω-totally ω-projective module modulo an n-bounded submodule *i.e.* $M \in Q(\mathcal{B}_n, \mathcal{W})$.

(iii) M is an extension of a n^{th}-level countably generated module by a quotient which is a direct sum of uniserial modules *i.e.* $M \in \mathcal{E}(\mathcal{C}_n, \mathcal{D})$.

(iv) M is a quotient of a direct sum of uniserial modules modulo an n^{th}-level countably generated submodule *i.e.* $M \in Q(\mathcal{C}_n, \mathcal{D})$.

(v) M is an extension of a countably generated module by a $(\omega+n)$-projective quotient *i.e.* $M \in \mathcal{E}(\mathcal{F}_n, \mathcal{C})$.

(vi) M is a quotient of $(\omega+n)$-projective module modulo a countably generated submodule *i.e.* $M \in Q(\mathcal{C}, \mathcal{F}_n)$.

(vii) M is an extension of a $(\omega+n)$-projective module by a countably generated quotient.

Proof. Let N be a submodule of M with $K = M/N$ and P be a submodule of Q with $Q/P = M$.

$(i) \Rightarrow (iii)$: If N be an n-bounded submodule and K be ω-totally ω-projective module, then there is a countably generated submodule S of K such that $K/S = K' \in \mathcal{D}$. If we let N' be the submodule of M containing N such that $N'/N = S$ and $M/N' \cong K' \in \mathcal{D}$, as desired.

$(iii) \Rightarrow (v)$: If N be an n^{th}-level countably generated module and K be a submodule which is the direct sum of uniserial modules and there is a countably generated submodule N' of N such that $N/N' = S \in \mathcal{B}_n$, which is a submodule of $K' = M/N'$ such that $K'/S \cong K \in \mathcal{D}$ implies $K' \in \mathcal{F}_n$, which is (v).

$(v) \Rightarrow (vii)$: Let N be a countably generated submodule and K be a $(\omega+n)$-projective module. Choose a submodule N' of M which is maximal with the property that $N' \cap N = \{0\}$. N' is

$(\omega+n)$-projective as it maps injectively into K. Since N is isomorphic to an essential submodule of $K' = M/N'$ implying that K' is countably generated, which is (vii).

$(vii) \Rightarrow (vi)$: Let N be an $(\omega+n)$-projective submodule and K be a countably generated submodule of M. Let S be a countably generated module which is a direct sum of uniserial module and $f : S \to M$ be a homomorphism such that $M = N + f(S)$. If $Q = N \oplus S$, then Q is $(\omega+n)$-projective module. Let $N \to M$ be the identity map, then $Q \to M$ be a surjective homomorphism. If P is its kernel, then $P \cap N = \{0\}$ so that P is isomorphic to a submodule of S and hence P will be countably generated, which is (vi).

$(vi) \Rightarrow (iv)$: Suppose Q be an $(\omega+n)$-projective and P be a countably generated module. There is a module Q', which is the direct sum of uniserial modules, having an n-bounded submodule S such that $Q = Q'/S$. If P' be the submodule of Q' such that $P'/S = P$, then P' is n^{th}-level countably generated and $Q'/P' \cong M$, which is (iv).

$(iv) \Rightarrow (ii)$: Suppose Q be the direct sum of uniserial modules and P be n^{th}-level countably generated module, then there is a submodule S of Q such that S is countably generated with $P/S = P'$ is n-bounded module. As $Q/S = Q'$ be ω-totally ω-projective module and $Q'/P' \cong M$, which is (ii).

$(ii) \Rightarrow (i)$: Let Q be ω-totally ω-projective and P be n-bounded module and $N = H^n(Q)/P \subseteq M$ with $K = M/N$. Since N is n-bounded and

$$K = \frac{(Q/P)}{(H^n(Q)/P)} \cong \frac{Q}{H^n(Q)} \cong H_n(Q) \in \mathcal{W}$$

which completes the proof of the theorem. Q.E.D.

Now we have the following:

Definition 2.3. A module M is said to be ω_1-$(\omega+n)$-projective if it satisfies any of the properties of Theorem 2.2.

Proposition 2.4. (cf. [2], Proposition 1.4) A module M is ω_1-$(\omega+n)$-projective if and only if it has a countably generated submodule $N \subseteq H_\omega(M)$ such that M/N is $(\omega+n)$-projective.

Proof. Suppose N is a countably generated submodule of $H_\omega(M)$ such that M/N is $(\omega+n)$-projective. Then the module M satisfies Theorem 2.2(v) and hence M is ω_1-$(\omega+n)$-projective.
For the reverse implication, suppose M is ω_1-$(\omega+n)$-projective, then by Theorem 2.2 (ii), there is an ω-totally ω-projective module P with an n-bounded submodule Q such that $M = P/Q$. Let $N = [H_\omega(P) + Q]/Q$, which is a countably generated submodule of $H_\omega(M)$. Now

$$\begin{aligned} M/N = \frac{(P/Q)}{([H_\omega(P) + Q]/Q)} &\cong \frac{P}{(H_\omega(P) + Q)} \\ &\cong \frac{(P/H_\omega(P))}{([(H_\omega(P) + Q)]/H_\omega(P))}. \end{aligned}$$

As $P/H_\omega(P)$ is a direct sum of uniserial modules and $[H_\omega(P) + Q]/H_\omega(P)$ is n-bounded, it imply that M/N is $(\omega+n)$-projective and we are done. Q.E.D.

We will discuss an interesting characterization of ω_1-$(\omega+n)$-projective modules as follows:

Theorem 2.5. (cf. [2], Theorem 1.5) For a QTAG-module M, the following ar equivalent:

(a) M is ω_1-$(\omega+n)$-projective.

(b) M is a submodule of the direct sum P of a countably generated module Q and a $(\omega+n)$-projective module S.

(c) M is h-pure submodule of a module P (as in part (b)), where P/M is countably generated.

Proof. $(b) \Rightarrow (a)$: Suppose $P = Q \oplus S$ is a QTAG-module where Q is the direct sum of countably generated module and S an $(\omega+n)$-projective module such that $M \subseteq P$. If $\varphi : P \to S$ is the projection, then we can find a short exact sequence $0 \to M \cap Q \to M \to \varphi(M) \to 0$. As $\varphi(M) \subseteq S$, so it is $(\omega+n)$-projective and $M \cap Q (\subseteq Q)$ is countably generated. Therefore M is an extension of a countably generated module by a $(\omega+n)$-projective quotient. Now by Theorem 2.2 (v), M is ω_1-$(\omega+n)$-projective.

$(a) \Rightarrow (c)$: Suppose M is ω_1-$(\omega+n)$-projective. By Proposition 2.4, we may find countably generated submodule $N \subseteq H_\omega(M)$ such that M/N is $(\omega+n)$-projective. If D is the h-divisible hull of N, then we may find a commutative diagram with short exact sequences and columns

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & D/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & D/N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since D is h-divisible, we have $P \cong D \oplus M/N$. This implies that $P/M \cong D/N$, which is countably generated and M is mapped as an h-pure submodule of P and we are done.

$(c) \Rightarrow (a)$ is trivial.

Q.E.D.

As an immediate consequence, we have the following:

Corollary 2.6. (cf. [2], Corollary 1.6) If P is ω_1-$(\omega+n)$-projective module and M is an arbitrary submodule of P, then M is also ω_1-$(\omega+n)$-projective module.

If M is a module, then by a countable system on M we will mean a collection $\mathcal{M}$ of countably generated submodules of M such that (i) if $\{N_i\}_{i<\omega} \subseteq \mathcal{M}$ is an increasing sequence of submodules in $\mathcal{M}$, then $N = \bigcup\limits_{i<\omega} N_i \in \mathcal{M}$ and (ii) if $K \subseteq M$ is any countably generated subset then there is a submodule $N \in \mathcal{M}$ such that $K \subseteq N$. Clearly the intersection of two countable system is again countable system.

Lemma 2.7. (cf. [2], Lemma 1.7) Let Q be $(\omega+n)$-projective module, then there is a countable system $\mathcal{M}$ on P consisting of submodules $P \subseteq Q$ such that Q/P is $(\omega+n)$-projective.

Proof. Let S be an n-bounded submodule of Q such that $T = Q/S$ is direct sum of uniserial modules. Fix a decomposition $T = \bigoplus_{k\in K} T_k$, where each T_k is a uniserial module and $\varphi : Q \to T$ be surjection. Let $\mathcal{M}$ be the collection of all countably generated submodules $P \subseteq Q$ such that $\varphi(P) = \bigoplus_{k\in I} T_k$, where $I \subseteq K$. Clearly $\mathcal{M}$ is a countably generated system. If $P \in \mathcal{M}$, then there is a short exact sequence $0 \to (P+S)/P \to Q/P \to Q/(P+S) \to 0$. As $(P+S)/P$ is n-bounded and
$$Q/(P+S) \cong \frac{(Q/S)}{(P+S)/S} \cong (\bigoplus_{k\in K} T_k)/(\bigoplus_{k\in I} T_k) \cong \bigoplus_{k\in K-I} T_k$$
is a direct sum of uniserial modules and each Q/P is $(\omega+n)$-projective as required. Q.E.D.

If $\mathcal{J}$ and $\mathcal{K}$ are classes of modules, let $P\mathcal{E}(\mathcal{J},\mathcal{K})$ be the class of all modules M such that there is a pure submodule J of M with $K = M/J \in \mathcal{Y}$. Define $P\mathcal{Q}(\mathcal{J},\mathcal{K})$ in the similar way. Theorem 2.5 suggests the question of whether in the various parts of Theorem 2.2, we can replace $\mathcal{E}$ or $\mathcal{Q}$ by $P\mathcal{E}$ or $P\mathcal{Q}$ respectively. As a bounded pure submodule is a summand, we have $P\mathcal{E}(\mathcal{B}_n,\mathcal{W}) = P\mathcal{Q}(\mathcal{B}_n,\mathcal{W})$, so Theorem 2.2 (i) and (ii) gives nothing new and since a module which is direct sum of uniserial module is a summand, we have $P\mathcal{E}(\mathcal{C}_n,\mathcal{D}) = \mathcal{W}$, so also Theorem 2.2 (iii) gives nothing new. As to Theorem 2.2 (iv), if $M \in P\mathcal{Q}(\mathcal{C}_n,\mathcal{D})$ then there is a module $N \in \mathcal{D}$ with a pure submodule $H \in \mathcal{C}_n$ such that $N/H = M$. Note that $H = A \oplus B$, where A is countably generated and B is n-bounded. It follows that B is also pure in N and hence a summand. Also, $H' = H/B$ is a countably generated submodule $N' = N/B$ such that $N'/H' \cong N/H = M$. Since N' is isomorphic to a summandof N, it is in $\mathcal{P}$, so that $M \in \mathcal{W}$ or we can say that $P\mathcal{Q}(\mathcal{C}_n,\mathcal{D})$ also agree with $\mathcal{W}$.

Following is an easy consequence of the above discussion:

Theorem 2.8. (cf. [2], Theorem 1.8) We have $\mathcal{W}_n = P\mathcal{E}(\mathcal{C},\mathcal{F}_n) = P\mathcal{Q}(\mathcal{C},\mathcal{F}_n) = P\mathcal{E}(\mathcal{F}_n,\mathcal{C})$.

Let us recall that a homomorphism $f : M \to N$ is said to ω_1-bijective if and only if its kernel and cokernel are countably generated and a class of modules $\mathcal{M}$ is closed under ω_1-bijective homomorphism if whenever $M \to N$ is an ω_1-bijective homomorphism, then $N \in \mathcal{M}$ if and only if $M \in \mathcal{M}$.

Lemma 2.9. (cf. [2], Lemma 1.9) If $\mathcal{M}$ is a class of modules, then the following are equivalent:

(i) $\mathcal{M}$ is closed under ω_1-bijective homomorphisms.

(ii) Whenever M is a submodule of N with N/M countably generated, then $M \in \mathcal{M}$ if and only if $N \in \mathcal{M}$.

(iii) Whenever L is a countably generated submodule of M, then $M \in \mathcal{M}$ if and only if $M/L \in \mathcal{M}$.

Proof. If $\mathcal{M}$ is closed with respect to ω_1-bijections and N is a submodule of M with M/N is countably generated, then the inclusion map is ω_1-bijective, implying $M \in \mathcal{M}$ if and only if $N \in \mathcal{M}$.

On the other hand, if $\mathcal{M}$ satisfies (ii), suppose $\alpha : M \to N$ is an ω_1-bijective homomorphism. Let H be the kernel of α and J be the image of α. Let P be a submodule of M which is maximal with respect to the property $P \cap H = \{0\}$. Clearly H is isomorphic to an essential submodule of M/P, and since H is countably generated so is M/P implying $M \in \mathcal{M}$ if and only if $P \in \mathcal{M}$. As $P \to M/H \cong J$ is also injective with a countably generated cokernel, gives that $P \in \mathcal{M}$ if and only if $J \in \mathcal{M}$. Lastly, as the inclusion $J \subseteq N$ has a countably generated cokernel, $J \in \mathcal{M}$ if and only if $N \in \mathcal{M}$. Which shows that the first two conditions are equivalent. In the same fashion we can establish the equivalence of third condition, completing the proof. Q.E.D.

If $\mathcal{H}$ is a class of modules, let $\overline{\mathcal{H}}$ be the smallest class of modules containing $\mathcal{H}$ which is closed under ω_1-bijective homomorphisms. Now we have the following:

Proposition 2.10. (cf. [2], Proposition 1.10) If $\mathcal{H}$ is a class of modules, then the following are equivalent:

(*i*) $M \in \overline{\mathcal{H}}$.

(*ii*) There is a module $M' \in \mathcal{H}$ and a module K containing both M and M' for which K/M and K/M' are countably generated.

(*iii*) There is a module $M'' \in \mathcal{H}$ and countably generated submodules $N \subseteq M$, $N'' \subseteq M''$ such that $M/N \cong M''/N''$.

Proof. Let $\mathcal{K}$ be the collection of all M for which there is module K containing both M, $M' \in \mathcal{H}$ such that K/M and K/M' are countably generated. If $M \in \mathcal{K}$, so that M' and K exist, then since $M' \subseteq K$ is ω_1-bijective implies that $K \in \overline{\mathcal{H}}$ and as $M \subseteq K$ is also ω_1-bijective conclude that $M \in \overline{\mathcal{H}}$. Hence $K \subseteq \overline{\mathcal{H}}$.

On the other hand, if we are able to show that $\mathcal{K}$ is closed under ω_1-bijective homomorphisms, we are done as $\mathcal{H} \subseteq \mathcal{K}$. Let M be a submodule of P such that P/M is countably generated. If $P \in \mathcal{K}$, then we take K containing P and $P' \in \mathcal{H}$ such that K/P and K/P' are countably generated. Then M is also a submodule of K with K/M is countably generated as K/P and P/M are countably generated implying $M \in \mathcal{K}$. For completing the proof of this part by Lemma 2.9 (*ii*), it remains to show that if $M \in \mathcal{K}$, then $P \in \mathcal{K}$. For showing this suppose that $M \in \mathcal{K}$; and let $M' \in \mathcal{H}$ such that M and M' are contained in a module K with K/M and K/M' countably generated. Let $K' = (P \oplus K)/\{(x, -x) : \ x \in M\}$ be the sum of P and K along M. If P identifies with the image of $P/ \oplus \{0\}$ in K' and K with the image of $\{0\} \oplus K$ in K', then $P + K = K'$ and $P \cap K = M$. It is easy to see that K/M and P/M are countably generated as $K'/P \cong K/(P \cap K) = K/M$ and $K'/K \cong P/(P \cap K) = P/M$ are countably generated and as K/M' is countably generated implies that K'/M' is countably generated. Therefore $P \in \mathcal{K}$, which ensures by Lemma 2.9 (*ii*), now that $\mathcal{K}$ is closed under ω_1-bijective homomorphisms, which establishes the equivalence of the first two conditions; their equivalence to the third condition can be establish in the same fashion, completing the proof of the proposition. Q.E.D.

Now we have interesting characterization of $\mathcal{W}_n$:

Theorem 2.11. (cf. [2], Theorem 1.11) We have $\mathcal{W}_n = \overline{\mathcal{F}}_n$.

Proof. Clearly $\mathcal{W}_n$ is closed under ω_1-bijective homomorphisms. So by Lemma 2.9, suppose P is a module and M is a submodule of P such that P/M is countably generated. If $P \in \mathcal{W}_n$, then $M \in \mathcal{W}_n$ by Corollary 2.6. On the other hand, if $M \in \mathcal{W}_n$, then by Theorem 2.2 (*vii*), there is an

$(\omega+n)$-projective submodule D of M such that M/D is countably generated implying that P/D is also countably generated, so that $\overline{\mathcal{F}}_n \subseteq \mathcal{W}_n$. Reverse implication follows immediately by Theorem 2.2 (v), (vi) or (vii) *i.e.* any class containing $\mathcal{F}_n$ which is closed under ω_1-bijective homomorphism necessarily contains $\mathcal{W}_n$, which completes the proof. Q.E.D.

Now we investigate that under what conditions the factor module $M/H_{\omega+n}(M)$ is ω_1-$(\omega+n)$-projective.

Proposition 2.12. (cf. [2], Corollary 2.1) If M is a module, then M is ω_1-$(\omega+n)$-projective module if and only if the following holds:

(i) $H_{\omega+n}(M)$ is countably generated, and

(ii) $M/H_{\omega+n}(M)$ is ω_1-$(\omega+n)$-projective module.

Proof. Suppose M is ω_1-$(\omega+n)$-projective module, then by Theorem 2.2 (v), we have a countably generated submodule N of M such that $M/N = P$ is $(\omega+n)$-projective implying $H_{\omega+n}(P) = 0$, so $H_{\omega+n}(M) \subseteq N$, which ensures that $H_{\omega+n}(M)$ is countably generated, which is (i). Therefore $M \to M/H_{\omega+n}(M)$ will be ω_1-bijective, so by Theorem 2.11, $M/H_{\omega+n}(M)$ is ω_1-$(\omega+n)$-projective, which is (ii).

On the other hand if M satisfies (i) and (ii), then $M \to M/H_{\omega+n}(M)$ will be ω_1-bijective. Therefore by Theorem 2.11 and (ii) imply that $M \in \mathcal{W}_n$, which completes the proof. Q.E.D.

Proposition 2.13. (cf. [2], Corollary 2.2) If M is ω_1-$(\omega+n)$-projective module, then its *Ulm* factor $M/H_\omega(M)$ is $(\omega+n)$-projective.

Proof. Let N be a countably generated submodule of $H_\omega(M)$ such that M/N is $(\omega+n)$-projective implying that $M/H_\omega(M) \cong \dfrac{(M/N)}{H_\omega(M/N)}$ is $(\omega+n)$-projective. Q.E.D.

The following example ensures the existence of a module M such that $H_\omega(M)$ is a uniserial module of length n, M is not $(\omega+n)$-projective but $M/H_\omega(M)$ is $(\omega+1)$-projective.

Example 1 Let $M = \overline{B}$ be an unbounded closed QTAG-module [7], such that M is the complete direct sum of B_i's where $B = \bigoplus_{i=1}^{\infty} B_i$ is a basic submodule of M. Here each B_i is the direct sum of the uniserial modules of length i. Consider a h-pure and h-dense submodule N of M such that $g(M/N) = 1$. Let us put $K = M/H^n(N)$, $T = H^{n+1}(M)/H^n(N)$ and $X = \{x \in T \mid H_T(x) = H_M(x) \geq \omega\}$. This is possible because the elements of T have finite exponents but their heights are not bounded and we have $X = H^n(M)/H^n(N) \cong Y$, where Y is a uniserial module of length n. Now T/X is isometric to $H^{n+1}(M)/H^n(M) \cong Soc(H_n(M))$, which is closed in the h-topology. Let P be a module such that $P \supseteq T$ and the heights of elements of T in T are same as their heights in P and P/T is a direct sum of uniserial modules. Now $H_\omega(P) = X \cong Y$. Moreover, $P/H_\omega(P) \cong P/X$ has a 1-bounded submodule $Q = T/X$ for which $\dfrac{(P/H_\omega(P))}{Q} \cong P/T$ is a direct sum of uniserial modules. Therefore $P/H_\omega(P)$ is $(\omega+1)$-projective.

Let A be a direct sum of uniserial modules and $f : P \to A$ be a homomorphism. Now, $f(X) = 0$ so that f determines a height preserving isomorphism $\overline{f} : T/X \to A$ and T/X is complete. Therefore there exists an integer k such that $f(T \cap H_k(P)) = 0$. Since $H_k(T)$ is not n-bounded, there is no n-bounded submodule D of P such that P/D is a direct sum of uniserial modules. Therefore T is not $(\omega+n)$-projective.

Proposition 2.14. (cf. [2], Proposition 2.4) Suppose M_k be a module for each $k \in K$. Then $M = \bigoplus_{k\in K} M_k \in \mathcal{W}_n$ if and only if for each $k \in K$, $M_k \in \mathcal{W}_n$ and there is a countable subset $J \subseteq K$ such that $M_k \in \mathcal{F}_n$ for all $k \in K - J$.

Proof. Suppose $M = \bigoplus_{k\in K} M_k \in \mathcal{W}_n$, then by Corollary 2.6, each $M_k \in \mathcal{W}_n$. If N is a countably generated submodule of M such that $M/N \in \mathcal{F}_n$, then let J be a countable subset of K such that $N \subseteq \bigoplus_{k\in J} M_k$. Since for $k \in K - J$, M_k embeds in M/N implies that all these M_k are $(\omega+n)$-projective.

For the reverse implication, suppose the given conditions hold. For each $k \in J$, we can find a countably generated submodule N_k such that M_k/N_k is $(\omega+n)$-projective. If $N = \bigoplus_{k\in J} N_k \subseteq M$, then N is countably generated and $M/N \cong (\bigoplus_{k\in J}(M_k/N_k)) \oplus (\bigoplus_{k\in K-J} M_k)$ is $(\omega+n)$-projective, which completes the proof. Q.E.D.

One of the interesting property of $(\omega+n)$-projective modules is that two modules of this class are isomorphic if and only if they have isometric $H^n(M)$-socles. But the obvious generalization of this to the class of ω_1-$(\omega+n)$-projective modules does not hold.

Proposition 2.15. (cf. [2], Example 2.6) If $n < \omega$, then there are non-isomorphic $\omega+n$-bounded ω_1-$(\omega+n)$-projective modules M_1 and M_2, whose $H^n(M)$-socles $H^n(M_1)$ and $H^n(M_2)$ are isometric.

Proof. Let P be a module for which $H_\omega(P)$ is isomorphic to a uniserial module of length n. If N is a ω-high submodule of P, then there is an isomorphism $H^n(P) \cong H^n(N) \oplus H_\omega(P)$. This isomorphism is an isometry as the nonzero elements of $H_\omega(P)$ have infinite height. It is easy to see that P is not $(\omega+n)$-projective but since N embeds in $P/H_{\omega+n}(P)$, which is $(\omega+1)$-projective, we infer that N is $(\omega+1)$-projective.
Further, let S be a countably generated module such that $H_\omega(S)$ is isomorphic to the uniserial module of length n and Q be a $H_\omega(M)$-high submodule of S. From the above arguments there is an isometry $H^n(S) \cong H^n(Q) \oplus H_\omega(S)$. Since Q embeds in $S/H_\omega(S)$ which is direct sum of uniserial modules, we conclude that Q is also direct sum of uniserial modules. Set $M_1 = P \oplus Q$ and $M_2 = N \oplus S$. Since P, N, S and Q are ω_1-$(\omega+n)$-projective so M_1 and M_2 also. Now

$$\begin{aligned} H^n(M_1) &\cong H^n(P) \oplus H^n(Q) \cong H^n(N) \oplus H_\omega(P) \oplus H^n(Q) \\ &\cong H^n(N) \oplus H_\omega(S) \oplus H^n(Q) \cong H^n(N) \oplus H^n(S) \\ &\cong H^n(M_2) \end{aligned}$$

Since P is not $(\omega+n)$-projective neither is M_1; but since both N and S are $(\omega+n)$-projective, M_2 is as well. In particular, M_1 and M_2 are not isomorphic. Q.E.D.

3 Other classes of modules

Let us recall a module M is $(\omega+n)$-totally $(\omega+n)$-projective if every $H^{\omega+n}(M)$-bounded submodule of M is $(\omega+n)$-projective. For $n > 0$, $(\omega+n)$-projective module is proper if it fails to be $(\omega+n-1)$-projective. If $n < \omega$, then a QTAG-module M has the core class property if either it is $(\omega+n)$-projective or it contains a proper $(\omega+n-1)$-projective submodule *i.e.* the submodule is $(\omega+n-1)$-projective but not $(\omega+n)$-projective. If this is true $\forall\ n < \omega$, M has the generalized core class property.

Proposition 3.1. (cf. [2], Proposition 3.1) If M is $(\omega+n)$-totally $(\omega+n)$-projective, then it is ω_1-$(\omega+n)$-projective.

Proof. We know that if $n < \omega$ and M is $(\omega+n)$-totally $(\omega+n)$-projective, then $H_{\omega+n}(M)$ is countably generated. If K is $H_{\omega+n}(M)$-high in M, then M/K will be countably generated as $H_{\omega+n}(M)$ embed in it as an essential submodule. As $H_{\omega+n}(K) = \{0\}$, K is $(\omega+n)$-projective and the result follows from Theorem 2.11. Q.E.D.

Proposition 3.2. (cf. [2], Proposition 3.2) A module M fails to be ω-totally ω-projective if and only if it has a separable submodule which is a proper $(\omega+1)$-projective module.

Proof. Suppose M is not ω-totally ω-projective module. Then it has a submodule P which is not a direct sum of uniserial modules. By extended core class theorem for QTAG-modules P has a separable submodule Q which is a proper $(\omega+1)$-projective module. For the converse part, if M has a separable submodule which is a proper $(\omega+1)$-projective, then it is not an ω-totally ω-projective module, which completes the proof. Q.E.D.

This result leads us to the investigation of the relationship between ω_1-$(\omega+n)$-projective modules and ω-totally $(\omega+n)$-projective modules. For exploring this, we start with the following:

Theorem 3.3. (cf. [2], Theorem 3.3) The class of ω-totally $(\omega+n)$-projective modules is closed under ω_1-bijections.

Proof. For proving this result we will use Lemma 2.9. Suppose P is a module and M is a submodule of P such that P/M is countably generated. If P is ω-totally $(\omega+n)$-projective, then all its separable submodules are $(\omega+n)$-projective implying that M has the same property, so it is ω-totally $(\omega+n)$-projective. For the reverse implication, suppose M is ω-totally $(\omega+n)$-projective. If Q is a separable submodule of P, then $S = Q \cap M$ is a separable submodule of M. By hypothesis, S is a separable $(\omega+n)$-projective. Clearly, Q/S embeds in P/M, which is countably generated, therefore Q/S is also countably generated. By [4], Q is $(\omega+n)$-projective so that P is ω-totally $(\omega+n)$-projective. Q.E.D.

Corollary 3.4. (cf. [2], Corollary 3.4) A module M is ω-totally $(\omega+n)$-projective if it is ω_1-$(\omega+n)$-projective.

Proof. Since any $(\omega+n)$-projective module is ω-totally $(\omega+n)$-projective. Also $\mathcal{W}_n$ is the smallest class of modules containing $\mathcal{F}_n$, which by Theorem 2.11, is closed under ω_1-bijective homomorphisms and hence the result follows from Theorem 3.3. Q.E.D.

Corollary 3.5. (cf. [2], Proposition 3.5) A module M is ω-totally $(\omega+n)$-projective if and only if

(a) $H_{\omega+n}(M)$ is countably generated; and

(b) $M/H_{\omega+n}(M)$ is ω-totally $(\omega+n)$-projective.

Proof. If M is ω-totally $(\omega+n)$-projective, then by [9], $H_{\omega+n}(M)$ is countably generated, so this is true on both sides of the "if and only if " statement. Hence $M \to M/H_{\omega+n}(M)$ is an ω_1-bijective homomorphism and hence the result follows from Theorem 3.3. Q.E.D.

Now we will show that the class of ω_1-$(\omega+n)$-projective modules and the class of ω-totally $(\omega+n)$-projective modules coincide in a simple case.

Theorem 3.6. (cf. [2], Theorem 3.6) Suppose M is a module and $H_\omega(M)$ is countably generated. Then the following are equivalent:

(a) M is ω_1-$(\omega+n)$-projective.

(b) M is ω-totally $(\omega+n)$-projective.

(c) $M/H_\omega(M)$ is $(\omega+n)$-projective.

(d) Some ω-high submodule of M is $(\omega+n)$-projective.

Proof. (a) implies (b) by Corollary 3.4. Suppose (b) holds. As $M \to M/H_\omega(M)$ is ω_1-bijective, by Theorem 3.3, $M/H_\omega(M)$ is ω-totally $(\omega+n)$-projective, which implies (c) immediately. Now, suppose (c) holds and let N be ω-high submodule in M, then it embeds in $M/H_\omega(M)$ so that it is $(\omega+n)$-projective, which is (d). Lastly, suppose (d) holds, and let N be an ω-high submodule of M that is $(\omega+n)$-projective and hence ω_1-$(\omega+n)$-projective. As $H_\omega(M)$ is countably generated and it embeds as an essential submodule of M/N implying that M/N is countably generated. By Theorem 2.11, (a) follows immediately. Q.E.D.

Now, we define

Definition 3.7. A module satisfying the conditions in Theorem 3.6 will be called an ω_1-separable $(\omega+n)$-projective module.

In fact $(\omega+1)$-projective modules necessarily decompose into a direct sum $P \oplus Q$, where P is a separable $(\omega+1)$-projective module and Q is an $(\omega+1)$-bounded direct sum of countably generated modules. We now show that this extends to the current scenario.

Theorem 3.8. (cf. [2], Theorem 3.7) A module M is ω_1-$(\omega+1)$-projective if and only if it is isomorphic to the direct sum of an ω_1-separable $(\omega+1)$-projective module and $(\omega+1)$-bounded direct sum of countably generated modules.

Proof. Suppose M has a pure and $(\omega+1)$-projective submodule Q such that M/Q is countably generated. Let P be a countably generated submodule of M such that $M = P+Q$. Let $Q = B \oplus C$, where B is separable and $(\omega+1)$-projective and C is an $(\omega+1)$-bounded direct sum of countably generated modules. Let $C = \bigoplus\limits_{k\in K} C_k$ be the decomposition such that each C_k is countably generated $(\omega+1)$-bounded module. If $J \subseteq K$, let $C_J = \bigoplus\limits_{k\in J} C_k \subseteq C$. Let J be a countable subset of K such that $P \cap Q \subseteq B \oplus C_J$. If $N = P + (B \oplus C_J)$ and $T = C_{I-J}$, then obviously $M = N \oplus T$ and T is

$H^{\omega+1}(M)$-bounded direct sum of countably generated modules. Therefore, to complete the proof it remains to show that $H_\omega(N)$ is countably generated. If $S = B \oplus C_J$, then S is pure in Q, which is pure in M, so that S is also pure in N. Also, if $D = N/S$ then $D = Soc(P + S)/S \cong P/(P \cap S)$ is countably generated.

Clearly the pure exact sequence $0 \to S \to N \to D \to 0$ induces a left exact sequence $0 \to H_\omega(S) \to H_\omega(N) \to H_\omega(D)$. As $H_\omega(D) \subseteq D$ is countably generated and $H_\omega(D) = H_\omega(C_J)$ is also countably generated implying $H_\omega(N)$ is countably generated, which completes the proof. Q.E.D.

Recall that for any QTAG-module M, $g(M)$ denotes the smallest cardinal number λ such that M admits a generating set X of uniform elements of cardinality λ and the final $g(M)$ or $fin\ g(M)$ of a QTAG-module M is defined as the infimum of $g(H_k(M))$ for $k = 0, 1, 2, \ldots, \infty$ *i.e.* $fin\ g(M) = \inf g(H_k(M))$. A QTAG-module M is U-decomposable if $M \cong N \oplus U$, where U is the direct sum of uniserial modules and $fing(U) = fing(M)$ [6]. It is useful to recall that $(\omega + 1)$-projective modules are always U-decomposable. Now we are able to prove:

Theorem 3.9. (cf. [2], Theorem 3.8) A reduced ω_1-$(\omega{+}1)$-projective module M is U-decomposable.

Proof. Let M be an h-reduced module. It is easy to verify that M is U-decomposable for countable $fin\ g(M)$. We are done if we can show that M is U-decomposable with M has uncountable $fin\ g(M) = \ell$ (say). Without loss of generality, we may assume that M has $g(M) = \ell$ as well. Let N be a pure and $(\omega + 1)$-projective submodule of M such that M/N is countably generated. It is easy to see that $fin\ g(N)$ and $g(N)$ are ℓ as well. Let $N = P \oplus Q$, where P is direct sum of uniserial modules summand with $fin\ g(P) = \ell$. There is a countably generated submodule S of M such that $M = N + S$, and a decomposition $P = P_0 \oplus P_1$, where P_1 is countably generated and $S \cap N \subseteq P_1 \oplus Q$. So we have $M = P_0 \oplus (S + (P_1 \oplus Q))$, and P_0 will also have $fin\ g(P_0) = \ell$, which completes the proof. Q.E.D.

We end this article with the discussion of the interesting question: describe when an ω_1-$(\omega{+}n)$-projective module is an $(\omega + n)$-module. Let us recall some definitions: Let α denote the class of all $QTAG$- modules M such that $M/H_\beta(M)$ is totally projective for all ordinals $\beta < \alpha$. These module are called α- modules [5]. In addition we say that a $QTAG$-module M is n-summable, if $\mathrm{Soc}(H_n(M))$ is isometric to a valuated direct sum of countably generated modules.

Following results will also be useful:

Theorem 4.2.3 [8] A module M is n-summable if and only $H_\omega(M)$ is n-summable and it is $(\omega{+}n)$-module.

Remark 4.3.1 [8] Let M be a h-reduced $QTAG$-module, which is a nice-$\aleph_0$-elongation of K by N. Then K is totally projective if and only if M is totally projective.

Remark 4.3.2 [8] For $\alpha \leq \omega_1$, if M is a nice-$\aleph_0$- elongation of K, then M is an α-module if and only if K is an α-module.

Now we are able to prove the following:

Theorem 3.10. (cf. [2], Theorem 3.9) If M is ω_1-$(\omega+n)$-projective module with $n > 0$, then following are equivalent:

(i) M is direct sum of countably generated modules.

(ii) M is an $(\omega+n)$-module.

(iii) M is n-summable module.

Proof. Trivially (i) and (ii) are equivalent and their equivalence to (iii) follows directly from Theorem 4.2.3 [8].

Q.E.D.

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