Special functions, integral transforms with applications

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Abstract

In this study, the author used integral transforms and a method of the exponential nature to deal with the families of fractional differential equations, Stieltjes type singular integral equations and boundary value problems. Certain integrals involving special functions are evaluated. Constructive examples are also provided throughout the paper. The main purpose of this article is to present mathematical results that are useful to researchers in a variety of fields.

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1 Introduction

In this work, the authors implement transform method for solving singular integral equation and partial fractional differential equation which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1] , [2] , [3], [6] , the Fourier transform method [14], the iteration method [13] and operational method [7]. However most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. More detailed information about some of these results can be found in a survey paper by Kilbas and Trujillo [8]. Atanackovic and Stankovic [4],[5]and Stankovic [17] used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for a viscoelastic rod. Oldham and Spanier I, [11] and [12] , studied respectively, by reducing a boundary value problem involving Ficks second law in electroanalytic chemistry to a formulation based on the partial Riemann Liouville fractional with half derivative. Wyss [18] and Schneider[16] considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions.
In this study, we develop and present a general method for evaluating certain integrals involving special functions such as Bessel's functions, modified Bessels function, parabolic cylinder functions, Airy function, non- linear fractional differential equations, solution to a Stieltjes type singular integral equations and boundary value problems. The obtained results reveal that the combined use of integral transforms and exponential operators provides a powerful tool to deal with fractional differential equations and boundary value problems. Until now, two methods, have been more extensively used for solving PDEs, the Laplace and Fourier transforms on the one hand and separation of variables on the other hand. Let us mention also solution in the form of a series of functions. Laplace transform is the best known and most widely used of the integral transforms, particularly in boundary value problems. We consider some methods consisting Laplace, Stieljes transforms to evaluate certain integrals and also to find the solution of singular integral equations. The author has already studied several methods to evaluate integrals and solve fractional differential equations, specially the popular Laplace transform method.
The structure of the present paper is as follows. In section one, we will give certain useful definitions and lemmas for solving space fractional PDEs in three dimensions and the Lamb–Bateman singular integral equation, we evaluate certain integrals involving the Bessel's functions via the Laplace Transforms. In section 2, we evaluate certain integrals involving the modified Bessel's functions . In section 3, solution to a Stieltjes type singular integral equations is given . In section 4, we used the exponential operator method to solve a partial differential equation with non - constant coefficients, a non - linear fractional differential equation and a variety of the Lamb - Bateman singular integral equation. Finally, the conclusion is drawn

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in section 5.
Definition 1.1. The Laplace transform of the function $f(t), t>0$ is defined as [21], [23]

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt := F(s). \tag{1}$$

If $\mathcal{L}\{f(t)\}=F(s)$, then $\mathcal{L}^{-1}\{F(s).\}$, is given by

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{2}$$

where $F(s)$ is analytic in the region $\mathrm{R}e(s)>c$.

Definition 1.2. Let us define the left Riemann-Liouville fractional derivative of order $0<\alpha<1$ as follows[24], [21],[14]

$$D_a^{RL,\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\Phi(\xi)}{(x-\xi)^\alpha}d\xi. \tag{3}$$

Many problems of applied mathematics and physical interest lead to the Laplace transform whose inverses are not readily expressed in terms of elementary functions. Therefore, it is highly desirable to have methods for inversion. In the following example an algorithm to invert the Laplace transform is presented.

Example 1.1. By using an appropriate integral representation for the modified Bessel's functions of the second kind of order ν, $K_\nu(s)$, show that

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})K_0(b\sqrt{s})\}=\int_0^t \frac{e^{-\frac{a^2}{4(t-\eta)}-\frac{b^2}{4\eta}}}{4ab\eta(t-\eta)}d\eta.$$

Proof. It is well known that$K_\nu(a\sqrt{s})$ has the following integral representation [24]

$$K_\nu(a\sqrt{s})=\frac{(a\sqrt{s})^\nu}{2^{\nu+1}}\int_0^\infty e^{-\xi-\frac{a^2s}{4\xi}}\frac{d\xi}{\xi^{\nu+1}}.$$

By using inversion formula for the Laplace transform and the above integral representation we get

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})\}=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}(\frac{e^{ts}}{2}(\int_0^\infty e^{-\xi-\frac{a^2s}{4\xi}}\frac{d\xi}{\xi^{\eta+1}}))ds.$$

At this point, changing the order of integration and simplifying to obtain

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})\}=\int_0^\infty \frac{e^{-\xi}}{\xi}(\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{e^{ts-\frac{a^2s}{4\xi}}}{2}ds)d\xi.$$

The value of inner integral is $\delta(t-\frac{a^2}{4\xi})$, we arrive at

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})\}=\int_0^\infty \frac{e^{-\xi}}{\xi}\delta(t-\frac{a^2}{4\xi})d\xi,$$

making a change of variable$(t-\frac{a^2}{4\xi})=u$, and using elementary properties of Dirac - delta function, we get the following result

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})\}=\frac{e^{-\frac{a^2}{4t}}}{(2at)}.$$

Finally, we obtain

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})K_0(b\sqrt{s})\} = \frac{e^{-\frac{a^2}{4t}}}{2at} * \frac{e^{-\frac{b^2}{4t}}}{2bt} = \int_0^t \frac{e^{-\frac{a^2}{4(t-\eta)}}}{2a(t-\eta)} \frac{e^{-\frac{b^2}{4\eta}}}{2b\eta} d\eta,$$

after simplifying, we arrive at

$$\mathcal{L}^{-1}\{K_0(a\sqrt{s})K_0(b\sqrt{s})\} = \int_0^t \frac{e^{-\frac{a^2}{4(t-\eta)}-\frac{b^2}{4\eta}}}{4ab\eta(t-\eta)} d\eta.$$

Let us consider the special case $a = b = \beta$, we get the following relation

$$\mathcal{L}^{-1}\{K_0^2(\beta\sqrt{s})\} = \int_0^t \frac{e^{-\frac{\beta^2 t}{4\eta(t-\eta)}}}{4\beta^2\eta(t-\eta)} d\eta.$$

The Laplace transform is used in a variety of applications. An interesting application of the Laplace transform involves the evaluation of certain integrals, particularly those containing the special functions. Let us first recall the following useful Lemma.

Lemma 1.1. Let $\mathcal{L}\{f(t)\} = F(s)$, then we have the following identities
1. $\mathcal{L}\{f(t^2)\} = \int_0^\infty \sqrt{\frac{1}{4\pi\xi}} e^{-\frac{s^2}{4\xi}} F(\xi) d\xi$,
2. $\mathcal{L}\{f(\frac{1}{t})\} = \int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi$,
3. $\mathcal{L}\{f(t^3)\} = \int_0^\infty \sqrt{\frac{s}{\xi}} K_{\frac{1}{3}}((\frac{s}{3\sqrt[3]{\xi}})^{\frac{2}{3}}) F(\xi) d\xi$.

Proof 2. We may show that

$$f(\frac{1}{t}) = \mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi).$$

the right hand side of the above relation can be re - written as follows

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} e^{st} (\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi)) ds,$$

at this point, using the fact that$\mathcal{L}\{f(t)\} = F(s)$, we get the following relation

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} e^{st} (\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) (\int_0^\infty e^{-\xi u} f(u) du) d\xi)) ds$$

changing the order of integraion, yields

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} e^{st} (\int_0^\infty f(u) (\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) e^{-u\xi} d\xi) du) ds$$

after evaluation of inner integral, we arrive at

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} e^{st} (\int_0^\infty f(u) \frac{e^{-\frac{s}{u}}}{u^2} du) ds$$

changing the order of integration, leads to

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) F(\xi) d\xi) = \int_0^\infty \frac{f(u)}{u^2} (\frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} e^{s(t-\frac{1}{u})} ds) du.$$

The value of the inner integral is $\delta(t-\frac{1}{u})$, we get

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}}J_1(2\sqrt{s\xi})F(\xi)d\xi) = \int_0^\infty \frac{f(u)}{u^2}\delta(t-\frac{1}{u})du,$$

by making a change of variable $\eta = \frac{1}{u}$ and using elementary properties of Dirac - delta fuction, we obtain

$$\mathcal{L}^{-1}(\int_0^\infty \sqrt{\frac{\xi}{s}}J_1(2\sqrt{s\xi})F(\xi)d\xi) = \int_0^\infty f(\frac{1}{\eta})\delta(t-\eta)d\eta = f(\frac{1}{t}).$$

Proof 3. [22] The above Lemma has some interesting applications as below

Lemma 1.2. The following integral relations hold true.
1.$\int_0^\infty K_{\frac{1}{3}}((\frac{1}{3\sqrt[3]{\xi}})^{\frac{2}{3}})(\frac{\gamma+ln\xi}{\xi\sqrt{\xi}})d\xi = -3\gamma$,
2.$\int_0^\infty J_1(2\sqrt{\xi})(\frac{\gamma+ln\xi}{\sqrt{\xi}})d\xi = -\gamma$,
3.$\int_0^\infty \sqrt{\frac{1}{4\pi}}e^{-\frac{1}{4\xi}}\frac{d\xi}{\xi^{\nu+1+0.5}} = \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}$.

Proof 1. Let us take$f(t) = \ln t$, then we get $F(s) = -\frac{\gamma+\ln s}{s}$, on the other hand we have

$$\mathcal{L}\{f(t^3)\} = \mathcal{L}\{3\ln t\} = -\frac{3(\gamma+\ln s)}{s}.$$

By setting all of the information in part 3 of the Lemma 1.1., we get the following

$$\mathcal{L}\{f(t^3)\} = \frac{1}{3\pi}\int_0^\infty \sqrt{\frac{s}{\xi}}K_{\frac{1}{3}}((\frac{s}{3\sqrt[3]{\xi}})^{\frac{2}{3}})(\frac{\gamma+\ln\xi}{\xi})d\xi = -\frac{3(\gamma+\ln s)}{s}.$$

Now, by choosing $s = 1$ and after some simple manipulations we arrive at

$$\int_0^\infty K_{\frac{1}{3}}((\frac{1}{3\sqrt[3]{\xi}})^{\frac{2}{3}})(\frac{\gamma+\ln\xi}{\xi\sqrt{\xi}})d\xi = -3\gamma.$$

Proof 2. Let us take $f(t) = \ln t$, then we have$F(s) = -\frac{\gamma+\ln s}{s}$. On the other hand $\mathcal{L}f(\frac{1}{t}) = \mathcal{L}(-\ln t) = \frac{\gamma+\ln s}{s}$, using second part of the Lemma1.1. we obtain

$$\mathcal{L}\{f(\frac{1}{t})\} = \int_0^\infty \sqrt{\frac{\xi}{s}}J_1(2\sqrt{s\xi})(-\frac{\gamma+\ln\xi}{\xi})d\xi = \frac{\gamma+\ln s}{s}.$$

Now, by choosing $s = 1$ and after simplifying, we arrive at

$$\int_0^\infty J_1(2\sqrt{\xi})(\frac{\gamma+\ln\xi}{\sqrt{\xi}})d\xi = -\gamma.$$

In the above relation $J_1(.)$ stands for the Bessel's function of the first kind and of order one. **Proof.3.** Let us take $f(t) = t^\nu$, then we have$F(s) = \frac{\Gamma(\nu+1)}{s^{\nu+1},}$ using second part of the Lemma1.1. we obtain

$$\mathcal{L}\{f(t^2)\} = \int_0^\infty \sqrt{\frac{1}{4\pi\xi}}e^{-\frac{s^2}{4\xi}}\frac{\Gamma(\nu+1)}{\xi^{\nu+1}}d\xi = \frac{\Gamma(2\nu+1)}{s^{2\nu+1}}.$$

Now, by choosing$s = 1$ and after simplifying, we arrive at

$$\int_0^\infty \sqrt{\frac{1}{4\pi}}e^{-\frac{1}{4\xi}}\frac{d\xi}{\xi^{\nu+1+0.5}} = \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}.$$

Definition 1.3. A solution of the second order differential equation $y'' + (\nu+\frac{1}{2}-\frac{x^2}{4})y = 0$ is denoted by $D_\nu(x)$ and is called parabolic cylinder function, with the following integral representation

$$D_\nu(x) = \frac{e^{\frac{-x^2}{4}}}{\Gamma(-\nu)}\int_0^\infty \xi^{-(\nu+1)}e^{-\frac{\xi^2}{2}-x\xi}d\xi, \qquad \nu \neq 0,1,2,3,.....$$

Lemma 1.3. Assume that $\mathcal{L}\{f(t)\} = F(s)$, then we have the following identity

$$\mathcal{L}\{t^{\nu} f(t^2)\} = \frac{1}{4\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{s^2\xi^2}{2}} D_{\nu}(s\xi) F(\frac{1}{2\xi^2}) d\xi.$$

Proof. (See [22], [25]) The above Lemma has some interesting applications as below

Theorem 1.1. The following integral identity holds true

$$\frac{1}{2\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{\xi^2}{2}-\frac{1}{2\xi^2}} D_{\nu}(\xi) d\xi = e^{-1}.$$

Proof. Let us take $f(t) = \delta(t - \lambda)$, then we have

$$\mathcal{L}(f(t)) = F(s) = \mathcal{L}(\delta(t - \lambda)) = e^{-\lambda s}.$$

At this point, we can evaluate $\mathcal{L}(t^{\nu}\delta(\frac{1}{t^2} - \lambda))$, in two differnt ways as follows first, by the definition of the Laplace transforms we have

$$\mathcal{L}(t^{\nu}\delta(\frac{1}{t^2} - \lambda)) = \int_0^{+\infty} e^{-st} t^{\nu} \delta(\frac{1}{t^2} - \lambda) dt,$$

in order to evaluate the above integral, we introduce a change of variable $w = \frac{1}{t^2}$, we get

$$\mathcal{L}(t^{\nu}\delta(\frac{1}{t^2} - \lambda)) = \int_0^{+\infty} e^{-\frac{s}{\sqrt{w}}} w^{-\frac{\nu}{2}} \delta(w - \lambda) \frac{1}{2} w^{-\frac{3}{2}} dw = e^{-\frac{s}{\sqrt{\lambda}}} \lambda^{-\frac{\nu+3}{2}},$$

second, by using the Lemma 1.3., we get

$$\mathcal{L}(t^{\nu}\delta(\frac{1}{t^2} - \lambda)) = \frac{1}{4\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{s^2\xi^2}{2}} D_{\nu}(s\xi) e^{-\frac{\lambda}{2\xi^2}} d\xi,$$

consequently, we obtain the following result

$$\mathcal{L}(t^{\nu}\delta(\frac{1}{t^2} - \lambda)) = \frac{1}{4\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{s^2\xi^2}{2}} D_{\nu}(s\xi) e^{-\frac{\lambda}{2\xi^2}} d\xi = \frac{1}{2} \lambda^{-\frac{\nu+3}{2}} e^{-\frac{s}{\sqrt{\lambda}}},$$

simplifying leads to the following relation

$$\frac{1}{4\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{s^2\xi^2}{2}} D_{\nu}(s\xi) e^{-\frac{\lambda}{2\xi^2}} d\xi = \frac{1}{2} \lambda^{-\frac{\nu+3}{2}} e^{-\frac{s}{\sqrt{\lambda}}},$$

in the above integral relation, let us choose $s = \lambda = 1$, after simplifying ,
we obtain

$$\frac{1}{2\sqrt{\pi}} \int_0^{\infty} \xi^{\nu-2} e^{-\frac{\xi^2}{2}-\frac{1}{2\xi^2}} D_{\nu}(\xi) d\xi = e^{-1}.$$

In the above relation $D_{\nu}(.)$ stands for the Parabolic Cylinder function of order ν [4] .

Lemma 1.4. We have the following relations .

1. $\exp(\pm\lambda \frac{d}{dt})\Phi(t) = \Phi(t\pm\lambda)$,
2. $\exp(\pm\lambda t \frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda})$,
3. $\exp(\lambda q(t) \frac{d}{dt})\Phi(t) = \Phi(Q(F(t) + \lambda))$.

Where $F(t)$ is a primitive function of $(q(t))^{-1}$, $Q(t)$ is the inverse function of $F(t)$.

Proof. See [23], [24].

Example 1.2. The following relation holds true.

1. $\exp(\pm\frac{t^2}{\lambda}\frac{d}{dt})\Phi(t) = \Phi(\frac{\lambda t}{t\mp\lambda})$.

Proof. Let us take $t^{-1} = \xi$, then we have the following sequence of relations

$$\exp(\pm\frac{t^2}{\lambda}\frac{d}{dt})\Phi(t) = \exp(\mp\frac{1}{\lambda}\frac{d}{d\xi})\Phi(\frac{1}{\xi}) = \varphi(\frac{1}{\xi\mp\frac{1}{\lambda}}) = \varphi(\frac{1}{\frac{1}{t}\mp\frac{1}{\lambda}}) = \varphi(\frac{\lambda t}{\lambda\mp t}).$$

Example 1.3. Let us show the following identity.

1. $(t\frac{d}{dt})^{-\frac{k}{n}}\varphi(t) = \frac{1}{\Gamma(\frac{k}{n})}\int_0^{+\infty}\xi^{\frac{k}{n}-1}\varphi(te^{-\xi})d\xi, k = 1, 2, ..., n-1.$

Proof. In order to show the above identity, let us recall the following elementary integral relation

$$s^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\xi^{\alpha-1}e^{-s\xi}d\xi,$$

in the above integral, let us put $s = t\frac{d}{dt}, \alpha = \frac{k}{n}$, we get

$$(t\frac{d}{dt})^{-\frac{k}{n}}\varphi(t) = \frac{1}{\Gamma(\frac{k}{n})}\int_0^{+\infty}\xi^{\frac{k}{n}-1}(e^{-\xi(t\frac{d}{dt})}\varphi(t))d\xi,$$

in view of second part of the Lemma 1.4., we get

$$(t\frac{d}{dt})^{-\frac{k}{n}}\varphi(t) = \frac{1}{\Gamma(\frac{k}{n})}\int_0^{+\infty}\xi^{\frac{k}{n}-1}\varphi(te^{-\xi})d\xi.$$

2 Airy functions

George Biddell Airy (1801-1892), he introduced the function defined by the following integral representations

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{it(x+\frac{t^2}{3})}dt = \frac{1}{\pi}\int_0^{+\infty}\cos\left(\frac{t^3}{3}+xt\right)dt,$$

which is the solution of the differential equation called Airy differential equation[19],[27].

$$y'' - xy = 0.$$

The Airy functions are related to the modified Bessel's fuctions of the first and second kind of order $\pm\frac{1}{3}$ as below[27]

$$Ai(x) = \frac{\sqrt{x}}{\sqrt{3}\pi}K_{\pm\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) = \frac{\sqrt{x}}{\sqrt{3}\pi}(I_{-\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) - I_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3})). \quad (4)$$

Lemma 2.1. The following identities hold true

$$1-\int_0^{+\infty}z^{\mu}K_{\nu}^{*}(z)dz = 2^{\mu-2}(\Gamma'(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}) - \Gamma(\frac{\mu+\nu+1}{2})\Gamma'(\frac{\mu-\nu+1}{2})).$$

$$2-\int_0^{+\infty}\sqrt{z}\,K_{\frac{1}{2}}^{*}(z)dz = -\frac{1}{2\sqrt{2}}(\ln 2 + \sqrt{\pi}\gamma + \Gamma'(\frac{1}{2})).$$

Where $\gamma = \int_1^{\infty}(\frac{1}{[x]} - \frac{1}{x})dx = .577....$ is known as Euler- Maschroni constant.

1 - Let us use an integral representation for the modified Bessel function of order ν, to obtain [19]

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = \int_0^{+\infty} z^\mu \left\{\frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^{+\infty} \exp(-t-\frac{z^2}{4t})\frac{dt}{t^{\nu+1}}\right\} dz.$$

Changing the order of integration to get

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = \left(\frac{1}{2}\right)^{\nu+1} \int_0^{+\infty} e^{-t}\left(\int_0^{+\infty} z^{\nu+\mu}e^{-\frac{z^2}{4t}}dz\right)\frac{dt}{t^{\nu+1}}.$$

Let us introduce a change of variable in the inner integral as $w = \frac{z^2}{4\,t}$

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = 2^{\mu-1}\{\int_0^{+\infty} w^{\frac{\mu+\nu-1}{2}}e^{-w}dw\}\{\int_0^{+\infty} t^{\frac{\mu-\nu-1}{2}}e^{-t}dt\}.$$

By definition of the Laplace transform, we can evaluate the integrals, hence

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = 2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}).$$

Remark. For the special case $\nu = \pm\frac{1}{3}$ and in view of relation (4), we get certain intergrals involving the Airy function [27].
At this point, taking derivative with respect to ν, leads to the following result

$$\int_0^{+\infty} z^\mu K^*_\nu(z)dz = 2^{\mu-2}(\Gamma'(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}) - \Gamma(\frac{\mu+\nu+1}{2})\Gamma'(\frac{\mu-\nu+1}{2}).)$$

By choosing $\mu = \nu = 0.5$ we get the following

$$\int_0^{+\infty} \sqrt{z}\, K^*_{\frac{1}{2}}(z)dz = -\frac{1}{2\sqrt{2}}(\sqrt{\pi}\gamma + \Gamma'(\frac{1}{2}))$$

In the above relation, $K^*_\nu(.)$ is the derivative of $K_\nu(.)$ with respect to order ν.

3 Stieltjes transform

It is well - known that the second iterate of Laplace transform is the Stieltjes transform, therefore Stieltjes transform of a given function is defined as follows[22],[26]

$$\mathcal{S}\{f(t); t \to s\} = \mathcal{L}(\mathcal{L}f(t) : t \to s) = \int_0^\infty \frac{f(t)}{t+s}dt.$$

Lemma 3.1. Assume that $\mathcal{S}\{f(x); x \to s\} = F(s)$ then we have the following relation

$$f(x) = \frac{1}{2\pi i}\{F(xe^{-i\pi}) - F(xe^{i\pi})\}.$$

Proof. See [22], [26].

To illustrate the method mentioned above, let us consider the following examples

Example 3.1. Let us evaluate the inverse Stieltjes transform of the following function

$$\mathcal{S}^{-1}(\frac{1}{\sqrt[k]{s}(\sqrt[m]{s}+a)}),$$

for $a \in R$.

Proof. By using the inverse Stietjes transform of the convolution of functions, we arrive at[26]

$$\mathcal{S}^{-1}\left\{\frac{1}{\sqrt[k]{s}(\sqrt[m]{s}+a)}; s \to t\right\} = \mathcal{S}^{-1}\left\{\frac{1}{\sqrt[k]{s}}\right\} \otimes \mathcal{S}^{-1}\left\{\frac{1}{\sqrt[m]{s}+a}\right\},$$

it is well - known that, the second iteration of the Laplace transform is the Stieltjes transform, hence

$$\mathcal{S}^{-1}\left\{\frac{1}{\sqrt[k]{s}}\right\}=\frac{1}{\pi\sqrt[k]{t}},\ \mathcal{S}^{-1}\left\{\frac{1}{\sqrt[m]{s}+a}\right\}=\frac{\sqrt[m]{t}\sin(\frac{\pi}{m})}{(\sqrt[m]{t}e^{\frac{-\pi}{m}}+a)(\sqrt[m]{t}e^{\frac{\pi}{m}}+a)},$$

therefore the final result will be

$$\mathcal{S}^{-1}\left\{\frac{1}{\sqrt[k]{s}(\sqrt[m]{s}+a)};s\to t\right\}=...$$

$$\frac{\sqrt[m]{t}\sin(\frac{\pi}{m})}{(\sqrt[m]{t}e^{\frac{-\pi}{m}}+a)(\sqrt[m]{t}e^{\frac{\pi}{m}}+a)}\int_0^\infty\frac{1}{\pi\sqrt{u}(u-t)}du+..$$

$$..\frac{1}{\pi\sqrt{t}}\int_0^\infty\frac{\frac{\sqrt[m]{u}\sin(\frac{\pi}{m})}{(\sqrt[m]{u}e^{\frac{-\pi}{m}}+a)(\sqrt[m]{u}e^{\frac{\pi}{m}}+a)}}{(u-t)}du,$$

by a change of variable $u^2=xw$

$$\mathcal{S}^{-1}\left\{\frac{1}{\sqrt[k]{s}(\sqrt[m]{s}+a)};s\to t\right\}=-\frac{\frac{\sqrt[m]{t}\sin(\frac{\pi}{m})}{(\sqrt[m]{t}e^{\frac{-\pi}{m}}+a)(\sqrt[m]{t}e^{\frac{\pi}{m}}+a)}}{\sqrt{t}}+..$$

$$+\frac{1}{\pi\sqrt{t}}\int_0^\infty\frac{\frac{\sqrt[m]{u}\sin(\frac{\pi}{m})}{(\sqrt[m]{u}e^{\frac{-\pi}{m}}+a)(\sqrt[m]{u}e^{\frac{\pi}{m}}+a)}}{(u-t)}du.$$

Example 3.2. Let us solve the following singular integral equation of Stieltjes type (this type of singular integral equation is not considered in the literature).

$$\int_0^\infty\frac{\varphi(t)}{t+s}dt=H(\eta(s)).$$

Solution. By using the inverse of Stieltjes transform we obtain the formal solution as below

$$\varphi(t)=\frac{1}{2\pi i}\{H(\eta(te^{-i\pi}))-H(\eta(te^{i\pi}))\}.$$

Special case: Let us solve the following Stieltjes type singular integral equation

$$\int_0^\infty\frac{\varphi(t)}{t+s}dt=\frac{1}{s\sqrt[n]{s}},$$

therefore, we have

$$\varphi(t)=\frac{1}{2\pi i}\left\{\frac{1}{(te^{-i\pi})^{\frac{n+1}{n}}}-\frac{1}{(te^{i\pi})^{\frac{n+1}{n}}}\right\}=\frac{\sin(\frac{\pi(n+1)}{n})}{\pi t\sqrt[n]{t}}.$$

4 Main results

In this section, the author used the exponential operator method to solve a partial differential equation with non - constant coefficients, a non - linear fractional differential equation and a variety of the Lamb - Bateman singular integral equation. The procedure as described in this paper should be generally applicable to the most fractional partial differential equations [20], [25].

Proposition 4.1. Let us show that the following first order partial differential equation with variable coefficients with the given initial condition has a formal solution

$$\frac{\partial u}{\partial t}+\lambda t^{\lambda-1}x^2\frac{\partial u}{\partial x}=-\alpha t^{\alpha-1}u,\qquad 0<\alpha,\lambda<1. \tag{5}$$

$$u(x,0)=\varphi(x). \tag{6}$$

Proof. The above partial differential equation can be written as follows

$$\frac{\partial u}{\partial t}=-(\lambda t^{\lambda-1}(x^2\frac{\partial}{\partial x})-\alpha t^{\alpha-1})u, \qquad 0<\alpha,\lambda<1. \tag{7}$$

At this point, we solve the above first order differential equation with respect to variable t and using initial condition to obtain

$$u(x,t)=e^{t^{-\lambda}}e^{-(\frac{x^2}{t^{-\alpha}}\frac{\partial}{\partial x})}\varphi(x),$$

by using Lemma 1.5., we may find the result of the action of exponential operator over the function, thereforewe get the solution to PDE.

$$u(x,t)=e^{t^{-\lambda}}\varphi(\frac{xt^{-\alpha}}{x+t^{-\alpha}})=e^{t^{-\lambda}}\varphi(\frac{x}{xt^{-\alpha}+1}),$$

it is easy to check that $u(x,0)=\varphi(x)$.

Proposition 4.2. Let us consider the following non - linear fractional differential equation

$$\sqrt[n]{\lambda-\frac{t^2}{\beta}\frac{d}{dt}}y(t)=\varphi(t). \tag{8}$$

The above fractional differential equation has the following formal solution

$$y(t)=\frac{1}{\Gamma(\frac{1}{n})}\int_0^{+\infty}\xi^{-\frac{1}{n}}e^{-\lambda\xi}\varphi(\frac{\beta t}{\xi t-\beta})d\xi.$$

Proof. We can rewrite the above equation as follows

$$y(t)=(\frac{1}{\sqrt[n]{\lambda-\frac{t^2}{\beta}\frac{d}{dt}}})\varphi(t)=(\lambda-\frac{t^2}{\beta}\frac{d}{dt})^{-\frac{1}{n}}\varphi(t),$$

at this point, let us recall the following well - known identity

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^{+\infty}e^{-a\xi}\xi^{\nu-1}d\xi,\nu>0,$$

now, in the above integral we set $a=\lambda-\frac{t^2}{\beta}\frac{d}{dt},\nu=\frac{1}{n}$, we get

$$(\lambda-\frac{t^2}{\beta}\frac{d}{dt})^{-\frac{1}{n}}\varphi(t)=\frac{1}{\Gamma(\frac{1}{n})}\int_0^{+\infty}\xi^{-\frac{1}{n}}(e^{-(\lambda-\frac{t^2}{\beta}\frac{d}{dt})\xi}\varphi(t))d\xi,$$

after using the Lemma 1.5. and simplifying, we obtain the formal solution

$$y(t)=\frac{1}{\Gamma(\frac{1}{n})}\int_0^{+\infty}\xi^{-\frac{1}{n}}e^{-\lambda\xi}\varphi(\frac{\beta t}{\xi t-\beta})d\xi.$$

Proposition 4.3. Let us show that the following generalized Lamb - Bateman integral equation has a formal solution. This singular integral equation was introduced to study the solitary waves diffraction.

$$\int_{-\infty}^{+\infty}e^{-\eta y^2}\varphi(\xi-ay^2+by-c)dy=g(\xi). \tag{9}$$

Proof. The above singular integral equation can be rewritten in the following form

$$\int_{-\infty}^{+\infty}e^{-(y^2-\frac{by}{a}+\frac{c}{a})(\eta+a\partial_\xi)}\varphi(\xi)dy=g(\xi),$$

equivalently

$$(\int_{-\infty}^{+\infty} e^{-(\eta+a\partial_\xi)(y-\frac{b}{2a})^2}dy)(e^{-(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}\varphi(\xi)) = g(\xi),$$

evaluation of the integral leads to the following

$$\sqrt{\frac{\pi}{\eta+a\partial_\xi}}(e^{-(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}\varphi(\xi)) = g(\xi),$$

in view of the Lemma 1.4., we arrive at the following relation

$$\varphi(\xi) = e^{(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}\sqrt{\frac{\eta+a\partial_\xi}{\pi}}g(\xi).$$

At this point, in order to find the result of the action of fractional operator, we may re-write the above relation as follows

$$\varphi(\xi) = \frac{1}{\pi}e^{(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}(\eta+a\partial_\xi)(\eta+a\partial_\xi)^{-\frac{1}{2}}g(\xi),$$

finally we get

$$\varphi(\xi) = \frac{1}{\pi}e^{(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}(\eta+a\partial_\xi)\int_0^{+\infty}\frac{1}{\sqrt{u}}e^{-u(\eta+a\partial_\xi)}g(\xi)du,$$

equivalently

$$\varphi(\xi) = \frac{1}{\pi}e^{(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}(\eta+a\partial_\xi)\int_0^{+\infty}\frac{e^{-\eta u}}{\sqrt{\pi u}}g(\xi-au)du,$$

or,

$$\varphi(\xi) = \frac{1}{\pi}e^{(\frac{c}{a}-\frac{b^2}{4a^2})(\eta+a\partial_\xi)}\int_0^{+\infty}\frac{e^{-\eta u}}{\sqrt{\pi u}}(\eta g(\xi-au)+ag^{'}(\xi-au))du.$$

Let us introduce a change of parameter $\frac{c}{a}-\frac{b^2}{4a^2}=w$, we get the following

$$\varphi(\xi) = \frac{e^{\eta w}}{\pi}\int_0^{+\infty}\frac{e^{-\eta u}}{\sqrt{\pi u}}(\eta g(\xi-au+aw)+ag^{'}(\xi-au+aw))du.$$

Let us consider the special case $a=1, b=c=\eta=0$, we obtain the following singular integral equation with the solution as below[23]

$$\int_{-\infty}^{+\infty}\varphi(\xi-y^2)dy = g(\xi). \tag{10}$$

$$\varphi(\xi) = \frac{1}{\pi}\int_0^{+\infty}\frac{1}{\sqrt{\pi u}}g^{'}(\xi-u)du.$$

Let us make a change of variable $\xi-u=\tau$, after simplifying we get the solution as follows

$$\varphi(\xi) = \frac{1}{\pi}\int_{-\infty}^{\xi}\frac{1}{\sqrt{\pi(\xi-\tau)}}g^{'}(\tau)d\tau.$$

5 Conclusions

We implement the Laplace and Stieltjes transforms for solving differential and integral equations and evaluation of certain integrals. Lastly, we develop a method for finding analytic solutions to partial differential equation with non - constant coefficients, non - linear fractional differential equation and generalized Lamb - Bateman singular integral equation. The article is intended for scientists and researchers of different disciplines of engineering and science dealing with the solutions of fractional integro - differential and fractional PDEs. The results reveal that the integral transforms and exponential operators method is very convenient and effective.

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