Chebyshev wavelet method for solving radiative transfer equation in a slab medium

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Abstract

In this paper, a numerical method for solving the radiative transfer equation in a slab medium with isotropic scattering is presented. By employing the properties of Chebyshev wavelets together with the collocation method, the problem is reduced into a system of algebraic equations and the approximate solutions are computed. Moreover, numerical examples are included to demonstrate the validity and applicability of this method and a comparison is made with the existing results.

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1 Introduction

Mathematical modelling of systems usually consists of solving differential and integro-differential equations (see, e.g., [10, 17]). Many numerical methods have been applied for solving these equations, which include finite difference methods (see, e.g., [36, 44]), finite element methods (see, e.g., [11, 37]), Laplace transform methods (see, e.g., [42]), meshless methods (see, e.g., [26, 27, 28, 7, 24]), spectral methods (see, e.g., [19, 23, 18, 21, 20]) and other numerical techniques. Finite difference methods are conditionally stable, i.e, we have restriction on step sizes. In finite element methods, large amounts of CPU time for building a body fitted mesh in two and three-dimensional problems will be needed. Moreover, the boundary element method requires a domain node distribution and spectral methods are not flexible with the domain of problem.

Recently, the issue of wavelets has influenced major areas of pure and applied mathematics, especially in the numerical analysis of integro-differential equations (see, e.g., [30, 2, 4, 47]). Also, Wavelets are considered as a strong mathematical tool with a wide range of applications (see, e.g., [6]).

In order to represent the efficiency of wavelets, the Chebyshev wavelet method is applied for solving the radiative transfer equation (RTE).

The RTE in a slab medium is defined as (see, e.g.,[41, 43])

$$\frac{x}{t_0}\frac{\partial I(y,x)}{\partial y} + I(y,x) = S(y) + \frac{\omega}{2}\int_{-1}^{1} P(x,\widehat{x})I(y,\widehat{x})d\widehat{x}, \tag{1}$$

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which is irradiated by an isotropic radiation field $I_0(y,x)$, with azimuthal symmetry. In this problem, $I(y,x)$ is the angle distribution of intensity normalized to $I_0(y,x)$, y is the distance normalized by the optical depth t_0 of slab $(0 \le y \le 1)$, ω is the albedo of a single scattering, x is the direction cosine of angle made by the specific intensity at any depth y with the direction of increasing y and $S(y)$ is the dimensionless emission source. The boundary conditions of problem are assumed to be

$$I(0,x) = f^0(x), \qquad 0 < x \le 1 \tag{2}$$

and

$$I(1,x) = f^1(x), \qquad -1 \le x < 0, \tag{3}$$

where $f^0(x)$ and $f^1(x)$ are known (see, e.g., [34]). The scattering phase function $P(x,\widehat{x})$ is represented in terms of the Legendre polynomials of first kind $P_n(x)$ by the expansion (see, e.g., [40, 41]),

$$P(x,\widehat{x}) = 1 + \sum_{n=1}^{\infty} c_n P_n(x) P_n(\widehat{x}) = \sum_{n=0}^{\infty} c_n P_n(x) P_n(\widehat{x}),$$

where c_n's are the expansion coefficients with $c_0 = 1$.
The RTE has many practical applications in coal-fired combustion and conversion systems, lightweight fibrous insulation, study of atmospheres and remote sensing (see, e.g., [1, 41]). In the applications of remote sensing at optical wavelengths to different surfaces from satellite borne and high-resolution instruments, an understanding of the various physical mechanisms that contribute to the measured data is important. Accordingly, the solution of radiative transfer equation (RTE) has been utilized in several applications such as pattern recognition, target information retrieval techniques, and the bidirectional reflectance model in the remote sensing (see, e.g., [33]).

Many numerical methods for solving this problem have been developed (see, e.g., [5, 14, 35, 29, 22, 38, 39, 9, 41, 43, 25]). Examples of such methods are: iterative, two-flux, spherical harmonic, series expansion, variational, eigenfunction expansion, Padé approximation, Generalized Eddington approximation and linear spline approximation (see, e.g., [33]).

In this paper, the Chebyshev wavelet method (CWM) is presented to the solution of equations (1)-(3). In this procedure, the Chebyshev wavelets are used as the basis functions to approximate the solution $I(y,x)$ and then the RTE is converted into a system of algebraic equations. The properties of Chebyshev wavelets, such as orthogonality, compact support and the ability to represent functions at different levels of resolution, together with the collocation method are employed to solve the obtained system. The root points of Chebyshev wavelets are used as the collocation points to evaluate the unknown coefficients and then the approximate solutions of equations (1)-(3) are identified.

This paper is organized as follows. The properties of Chebyshev wavelets and the operational matrices required for our problem are described in Section 2. In Section 3, the application of current method to the solution of the RTE is discussed. The proposed method is applied to numerical examples in Section 4. Moreover, a comparison is made between the exact and approximate solutions obtained by other methods and the ability of proposed method is also discussed.

2 Chebyshev wavelets and their properties

In this section, the Chebyshev polynomials are presented to derive the Chebyshev wavelets. Also, the main properties of these wavelets are also discussed.

2.1 Chebyshev polynomials

The Chebyshev polynomial of first kind of order m is defined as follows:

$$T_m(t) = cos[mcos^{-1}(t)], \qquad t \in [-1,1], \quad m = 0,1,2,....$$

Hence

$$T_m(cos\theta) = cos(m\theta) \qquad \theta \in [0,\pi], \quad m = 0,1,2,....$$

2.2 Properties of the Chebyshev polynomials

The Chebyshev polynomials $T_m(t)$, $m \geq 1$, satisfy the following properties:

$$\begin{aligned} T_0(t) &= 1, \\ T_1(t) &= t, \\ T_{m+1}(t) &= 2tT_m(t) - T_{m-1}(t), \qquad m = 1,2,..., \end{aligned} \tag{4}$$

which are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1,1]$.

2.3 Fundamentals of wavelet theory

Consider a complex-valued function ψ which satisfies the following properties

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty, \tag{5}$$

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty, \tag{6}$$

where Ψ is the Fourier transform of ψ. The first condition implies finite energy of the function ψ, and the second condition, the admissibility condition, implies that if $\Psi(\omega)$ is smooth then $\Psi(0) = 0$. The function ψ, is called the mother wavelet.

2.4 One-dimensional Chebyshev wavelets

In recent years, wavelets have attracted much interest in the field of science and technology. Wavelets are a family of functions derived from the dilation and translation of a single function $\psi(t)$.

The continuous wavelets can be presented as:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi(\frac{t-b}{a}), \tag{7}$$

where $\psi(t)$ is a single wavelet function and $a \neq 0$ and b are the dilation and translation parameters which are real numbers. For discrete values of a and b,

$$\begin{aligned} a &= a_0^{-k}, \qquad\qquad a_0 > 1, \quad b_0 > 0, \\ b &= nb_0 a_0^{-k}, \end{aligned} \tag{8}$$

the family of discrete wavelets are then shown as follows:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0), \tag{9}$$

which is a wavelet basis for $L^2(R)$ with integers k and n. Also, an orthonormal basis is constructed for $a_0 = 2$ and $b_0 = 1$.

Consequently, Chebyshev Wavelets are in the following form:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \overline{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $m = 0, 1, 2, ..., M-1$ is the order of Chebyshev polynomials of first kind, $n = 1, 2, ..., 2^{k-1}$ and k is any positive integer. Moreover,

$$\overline{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0. \end{cases} \tag{11}$$

Then, by (9)-(10), the wavelets $\{\psi_{n,m}\}$ form an orthonormal basis for $L^2([0,1])$ (see, e.g., [3]). Because of the orthogonality, in this form of Chebyshev wavelets, the weight function $\overline{w}(t) = w(2t-1)$ should be dilated and transformed to $w_n(t) = w(2^k t - 2n + 1)$.

In Chebyshev wavelet method, a given function $u(t)$ on the domain $[0,1]$ is approximated as:

$$u(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) \tag{12}$$

where $c_{n,m} = (u(t), \psi_{n,m}(t))$ and $(\ , \ )$ is the inner product in $L^2([0,1])$.
The infinite series in equation (12) can be truncated as follows:

$$u(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t) \tag{13}$$

where C and $\Psi(t)$ are matrices of size $(2^{k-1}M \times 1)$ as follows:

$$C = [c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, ..., c_{2^{k-1},0}, c_{2^{k-1},1}, ..., c_{2^{k-1},M-1}]^T,$$

$$\Psi = [\psi_{1,0}, \psi_{1,1}, ..., \psi_{1,M-1}, \psi_{2,0}, ..., \psi_{2,M-1}, ..., \psi_{2^{k-1},0}, \psi_{2^{k-1},1}, ..., \psi_{2^{k-1},M-1}]^T.$$

Theorem 2.1. Let $\Psi(t)$ be the one-dimensional Chebyshev wavelets vector defined in (13), we have

$$\frac{d\Psi(t)}{dt} = D\Psi(t)$$

where D is $2^{k-1}M \times 2^{k-1}M$, and

$$D = \begin{bmatrix} F & O & \cdots & O \\ O & F & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F \end{bmatrix} \tag{14}$$

in which F and O are $M \times M$, and the elements are

$$F_{i,j} = \begin{cases} 2^{k+1}(i-1)\sqrt{\frac{\sigma_{i-1}}{\sigma_{j-1}}}, , & i = 2, 3, ..., M, j = 1, 2, ..., i-1, i+j \text{ odd}, \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

and

$$\sigma_j = \begin{cases} 2, & j = 0, \\ 1 & j \geq 1. \end{cases} \tag{16}$$

Proof. A proof can be found in [45]. Q.E.D.

2.5 Error analysis

The convergence of Chebyshev wavelet approximation in (12) has been proved in [45].

Theorem 2.2. A function $u(t) \in L^2_\omega([0,1])$, with bounded second derivative, say $|u''(t)| \leq N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to u(t), that is,

$$u(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t).$$

Since the truncated Chebyshev wavelets series is an approximate solution, so one has an error function E(t) for u(t) as follows:

$$E(t) = |u(t) - C^T\Psi(t)|.$$

The error bound of approximate solution by using Chebyshev wavelets series is given by the following theorem.

Theorem 2.3. Suppose that $u(t) \in C^m[0,1]$ and $C^T\Psi(t)$ is the approximate solution using the CWM. Then the error bound would be obtained as follows:

$$E(t) \leq \| \frac{2}{m!4^m 2^{m(k-1)}} max_{t\in[0,1]}|u^m(t)|\|^2. \tag{17}$$

Proof. Applying the definition of norm in the inner product space, we have

$$\|E(t)\|^2 = \int_0^1 [u(t) - C^T\Psi(t)]^2 dt.$$

Because the interval $[0,1]$ is divided into 2^{k-1} subintervals $I_n = [\frac{(n-1)}{2^{k-1}}, \frac{n}{2^{k-1}}]$, $n = 1, 2, ..., 2^{k-1}$, then we can obtain

$$\begin{aligned} \|E(t)\|^2 &= \int_0^1 [u(t) - C^T\Psi(t)]^2 dt, \\ &= \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [u(t) - C^T\Psi(t)]^2 dt, \\ &\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [u(t) - P_m(t)]^2 dt, \end{aligned} \tag{18}$$

where $P_m(t)$ is the interpolating polynomial of degree m which agrees with $u(t)$ at the Chebyshev nodes on I_n with the following error bound for interpolating:

$$|u(t) - P_m(t)| \leq \frac{2}{m!4^m 2^{m(k-1)}} max_{t\in I_n}|u^m(t)|.$$

Therefore, using the above equation, we would get

$$
\begin{aligned}
\|E(t)\|^2 &\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [u(t) - P_m(t)]^2 dt \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [\tfrac{2}{m!4^m 2^{m(k-1)}} max_{t\in I_n} |u^m(t)|]^2 dt \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [\tfrac{2}{m!4^m 2^{m(k-1)}} max_{t\in[0,1]} |u^m(t)|]^2 dt, \\
&= \int_0^1 [\tfrac{2}{m!4^m 2^{m(k-1)}} max_{t\in[0,1]} |u^m(t)|]^2 dt, \\
&= \| \tfrac{2}{m!4^m 2^{m(k-1)}} max_{t\in[0,1]} |u^m(t)| \|^2.
\end{aligned}
\tag{19}
$$

Q.E.D.

2.6 Two-dimensional Chebyshev wavelets

The generalization of CWM for two dimensional problems on a closed domain $\Omega = \Omega_y \times \Omega_t$ is considered where $\Omega_y = [y_1, y_2]$ and $\Omega_t = [t_1, t_2]$. The subscript y is used to denote that the wavelet basis and all the parameters associated with it $(a_{y_0}, a_y, n_y, m_1, b_{y_0}, b_y)$ are defined for the domain Ω_y. Thus, relation (8) can be written as:

$$
\begin{aligned}
a_y &= a_{y_0}^{-k_y}, \quad a_{y_0} > 1, \quad b_{y_0} > 1, \\
b_y &= n_y b_{y_0} a_{y_0}^{-k_y}.
\end{aligned}
\tag{20}
$$

Similarly the subscript t and parameters $(a_{t_0}, a_t, n_t, m_2, b_{t_0}, b_t)$ is used for the domain $\Omega_t = [0,1]$. Thus, the two dimensional basis $\{\psi_{n,m_1,m_2}(y,t)\}$ can be constructed as a combination of two one-dimensional translation and a dilation of a truly two-dimensional wavelet $\psi(y,t)$.

Consider $k = max\{k_y, k_t\}$, consequently, a two-dimensional wavelet basis on $[0,1] \times [0,1]$ from the relation (10) can be written as (see, e.g., [12]):

$$
\psi_{n,m_1,m_2}(y,t) = \begin{cases} 2^k \overline{T}_{m_1}(2^k y - 2n + 1)\overline{T}_{m_2}(2^k t - 2n + 1), & \frac{n_y - 1}{2^{k_y - 1}} \leq y < \frac{n_y}{2^{k_y - 1}}, \\ & \frac{n_t - 1}{2^{k_t - 1}} \leq t < \frac{n_t}{2^{k_t - 1}}, \\ 0 & \text{otherwise}, \end{cases}
\tag{21}
$$

where $m_1 = 0, 1, ..., M_1 - 1$, $m_2 = 0, 1, ..., M_2 - 1$, k_y and k_t are positive integers and n_y and n_t are defined similarly to n. Then the following relation is also constituted:

$$
\{(b_{y_0}, b_{t_0})\} = \{b_{y_0}\} \times \{b_{t_0}\},
$$

and the wavelets $\psi_{n,m_1,m_2}(y,t)$ form a basis for $L^2([0,1]^2)$.
A function $I(y,t)$ defined on a closed domain $\Omega = [0,1] \times [0,1]$ can be expanded as (see. e.g., [15, 46]):

$$
I(y,t) \cong \sum_{n=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c_{n,m_1,m_2} \psi_{n,m_1,m_2}(y,t).
\tag{22}
$$

If the infinite series in (22) is truncated, then we have

$$I(y,t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m_1=0}^{M_1-1} \sum_{m_2=0}^{M_2-1} c_{n,m_1,m_2} \psi_{n,m_1,m_2}(y,t), \tag{23}$$

where

$$c_{n,m_1,m_2} = \langle I(y,t), \psi_{n,m_1,m_2}(y,t)\rangle = \int_0^1 \int_0^1 I(y,t)\psi_{n,m_1,m_2}(y,t)dydt.$$

The equation (23) can be expressed as

$$I(y,t) = C^T \Psi(y,t), \tag{24}$$

where C and $\Psi(y,t)$ are as follows

$$\begin{aligned} C = \; & [c_{1,0,0}, ..., c_{1,0,M_2-1}, c_{1,1,0}, ..., c_{1,1,M_2-1}, ..., c_{1,M_1-1,0}, ..., c_{1,M_1-1,M_2-1}, ..., \\ & c_{2,0,0}, ..., c_{2,0,M_2-1}, c_{2,1,0}, ..., c_{2,1,M_2-1}, ..., c_{2,M_1-1,0}, ..., c_{2,M_1-1,M_2-1}, ..., \\ & c_{2^{k-1},0,0}, ..., c_{2^{k-1},0,M_2-1}, ..., c_{2^{k-1},M_1-1,0}, ..., c_{2^{k-1},M_1-1,M_2-1}]^T, \end{aligned} \tag{25}$$

$$\begin{aligned} \Psi = \; & [\psi_{1,0,0}, ..., \psi_{1,0,M_2-1}, \psi_{1,1,0}, ..., \psi_{1,1,M_2-1}, ..., \psi_{1,M_1-1,0}, ..., \psi_{1,M_1-1,M_2-1}, ..., \\ & \psi_{2,0,0}, ..., \psi_{2,0,M_2-1}, \psi_{2,1,0}, ..., \psi_{2,1,M_2-1}, ..., \psi_{2,M_1-1,0}, ..., \psi_{2,M_1-1,M_2-1}, ..., \\ & \psi_{2^{k-1},0,0}, ..., \psi_{2^{k-1},0,M_2-1}, ..., \psi_{2^{k-1},M_1-1,0}, ..., \psi_{2^{k-1},M_1-1,M_2-1}]^T. \end{aligned} \tag{26}$$

Theorem 2.4. Let $\Psi(y,t)$ be the two-dimensional Chebyshev wavelets vector defined in (24), we have

$$\frac{\partial \Psi(y,t)}{\partial y} = D_y \Psi(y,t)$$

where D_y is $2^{k-1}M_1M_2 \times 2^{k-1}M_1M_2$, and we have

$$D_y = \begin{bmatrix} D & O' & \cdots & O' \\ O' & D & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & D \end{bmatrix} \tag{27}$$

in which D and O' are $M_1M_2 \times M_1M_2$. The elements are

$$D_{i,j} = \begin{cases} 2^{k+1}(i-1)\sqrt{\frac{\sigma_{i-1}}{\sigma_{j-1}}}I, , & i = 2,3,...,M_1, j = 1,2,...,i-1, i+j \text{ odd}, \\ \text{O}, & \text{otherwise}, \end{cases} \tag{28}$$

where I and O are $M_2 \times M_2$ matrix, and

$$\sigma_j = \begin{cases} 2, & j = 0, \\ 1 & j \geq 1. \end{cases} \tag{29}$$

Proof. A proof can be found in [45, 46]. Q.E.D.

Theorem 2.5. Let $\Psi(y,t)$ be the two-dimensional Chebyshev wavelets vector defined in (24). In general, the $r-$times derivative of $\Psi(y,t)$ for $r \in N$ can be expressed as:

$$\frac{\partial^r \Psi(y,t)}{\partial y^r} = D_y^r \Psi(y,t)$$

where D_y^r is the $r-$th power of matrix D_y.

Similarly, by using the operational matrix D_t, we can obtain the derivatives with respect to t. In this section, the operational matrix of integration P_t will be derived.

Theorem 2.6. The operational matrix of integration P_t of the chebyshev wavelet $\Psi(y,t)$ is in the following form

$$\int_0^t \Psi(y,s)ds \simeq P_t \Psi(y,t)$$

where $\Psi(y,t)$ is given before and P_t is a $2^{k-1}M_1M_2 \times 2^{k-1}M_1M_2$ matrix, given by:

$$P_t = \begin{bmatrix} C & S & S & \cdots & S \\ O' & C & S & \cdots & S \\ O' & O' & C & \cdots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ O' & O' & O' & \cdots & C \end{bmatrix} \tag{30}$$

and S and C are $M_1M_2 \times M_1M_2$

$$S = \frac{\sqrt{2}}{2^{k-1}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{31}$$

and

$$C = \frac{1}{2^{k-1}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{4(M-2)} & 0 \end{bmatrix} \tag{32}$$

Proof. A proof can be found in [13]. Q.E.D.

3 Application of the CWM for solving radiative transfer equation

In this section, the CWM is employed for solving the RTE.
With the transformation $x = 2t-1$, the shifted Legendre polynomials $P_n(t,\widehat{t})$ are obtained on $[0,1]$ by shifting the defining domain of $P_n(x,\widehat{x})$ which is $[-1,1]$. We have

$$\frac{2t-1}{t_0}\frac{\partial I(y,t)}{\partial y} + I(y,t) = S(y) + \frac{\omega}{2}\int_0^1 P(t,\widehat{t})I(y,\widehat{t})d\widehat{t}, \tag{33}$$

with the following initial conditions;

$$I(0,t) = f^0(t), \qquad \frac{1}{2} < t \leq 1,$$

$$I(1,t) = f^1(t), \qquad 0 \leq t < \frac{1}{2}.$$

Then with the collocation method, for solving (33), we define;

$$F(I(y,t)) = (F_1(I(y,t)), F_2(I(y,t)), F_3(I(y,t))), \tag{34}$$

and we have to solve a system of differential equations in the following form

$$F(I(y,t)) = 0, \tag{35}$$

where;

$$\begin{array}{lll} F_1(I(y,t)) & = \frac{2t-1}{t_0}\frac{\partial I(y,t)}{\partial y} + I(y,t) - S(y) - \frac{\omega}{2}\int_0^1 P(t,\widehat{t})I(y,\widehat{t})d\widehat{t}, & \\ F_2(I(0,t)) & = I(0,t) - f^0(t), & \frac{1}{2} < t \leq 1, \\ F_3(I(1,t)) & = I(1,t) - f^1(t), & 0 \leq t < \frac{1}{2}, \end{array} \tag{36}$$

and $\frac{\partial I(y,t)}{\partial y}$ and $\int_0^1 P(t,\widehat{t})I(y,\widehat{t})d\widehat{t}$ are obtained by using the Theorems 4 and 6.

By collocating the function F at points $\{(y_i,t_j)|i = 0,1,2,...,2^{k-1}M_1, j = 0,1,2,...,M_2\}$, the following system of equations is obtained:

$$\begin{array}{lll} F_1(I(y_i,t_j)) & = 0, & i = 1,2,...,2^{k-1}M_1-1, j = 1,2,...,M_2, \\ F_2(I(0,t_j)) & = 0, & j = \frac{M_2}{2}+1,...,M_2, \\ F_3(I(1,t_j)) & = 0, & j = 0,...,\frac{M_2}{2}-1, \end{array} \tag{37}$$

where $(y_i,t_j)_{i,j}$ are the roots of Chebyshev wavelets on Ω. These equations generate an $2^{k-1}M_1M_2$ set of algebraic equations. Thus, by evaluating the coefficients $\{c_{n,m_1,m_2}\}$, approximate solution $I(y,t)$ can be computed at every chosen point and the solutions of RTE are obtained.

4 Numerical examples and results

In this section, the proposed method is used for solving two examples with $k = 3$, $M_1 = M_2 = 3$ and the numerical results are also presented. Moreover, the approximate solutions are compared with the exact and approximated solutions obtained by other methods and the validity of proposed method is also demonstrated.

Example 4.1. The RTE can be given as (see, e. g., [31])

$$\frac{x}{t_0}\frac{\partial I(y,x)}{\partial y}+I(y,x)=\frac{1}{2}\int_{-1}^{1}(1+c_1P_1(x)P_1(\widehat{x}))I(y,\widehat{x})d\widehat{x}, \tag{38}$$

with the following conditions,

$$I(0,x)=1 \qquad\qquad 0<x\leq 1, \tag{39}$$

$$I(1,x)=0 \qquad\qquad -1\leq x<1. \tag{40}$$

By using the CWM for equations (38)–(40), the radiative fluxes are evaluated as

$$F^{+}(y)=2\int_0^1 I(y,x)xdx,$$

which have been presented in Tables 1-3 for different values of c_1 and t_0. Moreover by using the generalized Eddington approximation (GEA) (see, e.g., [43]) and the linear spline approximation (LSA) (see, e.g., [41]), pseudospectral Legendre method (PLM) (see, e.g., [31]) and multiquadric radial basis functions (MQ-RBF) (see, e.g., [25]) together with the exact values (see, e.g., [8]), a comparison is made for these results.

TABLE 1. Approximated and exact values of $F^{+}(1)$ with $c_1=0.7$

method	$t_0=0.1$	$t_0=0.5$	$t_0=1$	$t_0=3$
GEA [43]	Not reported	0.753	0.615	0.369
LSA[41]	Not reported	0.7498	0.6112	0.3547
Collocation-type method (PLM) [31]	0.93187	0.75035	0.61123	0.35806
MQ-RBF [25]	0.93071	0.75049	0.61211	0.35834
Galerkin-type method [32]	0.93187	0.75035	0.61123	0.35806
Hybrid functions method [33]	0.93187	0.75035	0.61123	0.35806
Tau method [34]	0.9317	0.7503	0.6112	0.3580
Chebyshev Wavelet (present method)	0.9314	0.7504	0.6114	0.3580
Exact	0.931	0.750	0.611	0.358

TABLE 2. Approximated and exact values of $F^{+}(1)$ with $c_1=0$

method	$t_0=0.1$	$t_0=0.5$	$t_0=1$	$t_0=3$
GEA [43]	Not reported	0.707	0.555	0.315
LSA [41]	Not reported	0.7036	0.5520	0.2989
Collocation-type method (PLM) [31]	0.91710	0.70434	0.55340	0.30131
MQ-RBF [25]	0.91581	0.70427	0.55351	0.30132
Galerkin-type method [32]	0.91630	0.70423	0.55330	0.30121
Hybrid functions method [33]	0.91710	0.70434	0.55340	0.30131
Tau method [34]	0.9170	0.7043	0.5533	0.3013
Chebyshev Wavelet (present method)	0.91618	0.70420	0.55333	0.30131
Exact	0.916	0.704	0.553	0.301

Table 3. Approximated and exact values of $F^{+}(1)$ with $c_1 = -0.7$

method	$t_0 = 0.1$	$t_0 = 0.5$	$t_0 = 1$	$t_0 = 3$
GEA [43]	Not reported	0.666	0.507	0.274
LSA [41]	Not reported	0.6628	0.5033	0.2583
Collocation-type method (PLM) [31]	0.90242	0.66327	0.50483	0.26007
MQ-RBF [25]	0.901372	0.663414	0.504659	0.260349
Galerkin-type method [32]	0.90142	0.66325	0.50522	0.26007
Hybrid functions method [33]	0.90242	0.66327	0.50483	0.26007
Tau method [34]	0.9024	0.6633	0.5048	0.2601
Chebyshev Wavelet (present method)	0.90154	0.66328	0.50510	0.26007
Exact	0.901	0.663	0.505	0.260

Clearly, the approximations in Tables 1-3, show the efficiency of CWM for solving this kind of problem. Moreover, numerical results indicate the efficiency and accuracy of the proposed method in comparison with the other methods.

Example 4.2. Let RTE be given as (see, e.g., [31])

$$x\frac{\partial I(y,x)}{\partial y} + I(y,x) = \frac{0.8}{2}\int_{-1}^{1}[1+\sum_{n=1}^{4} c_n p_n(x) p_n(\widehat{x})] I(y,\widehat{x}) d\widehat{x}, \tag{41}$$

which it was considered by Mengüç and Viskanta in [16]. In this example, the

Table 4. Approximated and exact values of $F^{+}(1)$

method	$t_0 = 0.1$
Generalized Eddington algorithm [43]	0458
Modified two-flux method	0.471
Spherical harmonic method (P_1)	0.456
Spherical harmonic method (P_3)	0.456
F_N method (F_1)	0.455
F_N method (F_3)	0.456
F_N method (F_9)	0.456
Collocation-type method (PLM) ($M_1 = M_2 = 7$) [31]	0.4569
Galerkin-type method ($M_1 = M_2 = 7$) [32]	0.4569
Hybrid functions method ($M_1 = M_2 = 7$) [33]	0.4564
Tau method ($M_1 = M_2 = 7$) [34]	0.4569
MQ-RBF ($M_1 = M_2 = 10$) [25]	0.4577
Chebyshev Wavelet (present method)	0.4561

values $c_1 = 0.6438$, $c_2 = 0.5542$, $c_3 = 0.1036$, and $c_4 = 0.0105$ are given and boundary conditions are the same as those in equations (39)–(40). In Table 4, the values of $F^{+}(y)$ have been approximated by the CWM. The obtained results show the efficiency of the presented method in compare with the other techniques.

5 Conclusions

In this paper, the CWM was applied for the RTE in a slab medium. By using this method, the differential and integral expressions which arise in the radiative transfer equation are converted into a system of linear algebraic equations and the solutions are found by determining the corresponding coefficients that satisfy in the algebraic equations. By comparing the numerical results obtained in Tables 1- 4, the high ability of proposed method was proved in comparison with the other techniques. The simplicity of Chebyshev wavelets and their efficiency together with the convergence of this method make this procedure very attractive for solving this kind of problems.

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References

[1] S. P. Ahmad and D. W. Deering, A simple analytical function for bidirectional reectance, Journal of Geophysical Research: Atmospheres, vol. 97, pp. 18867-18886 (1992).

[2] J. M. Alam, N. K. R. Kevlahan, and O. V. Vasilyev, Simultaneous space-time adaptive wavelet solution of nonlinear parabolic differential equations, Journal of Computational Physics, vol. 214, no. 2, pp. 829-857 (2006).

[3] E. Babolian and F. Fattahzadeh, Numerical computation method in solving integral equations by using chebyshev wavelet operational matrix of integration, Applied Mathematics and Computation, vol. 188, no. 1, pp. 1016-1022 (2007).

[4] G. Beylkin and J. M. Keiser, On the adaptive numerical solution of nonlinear partial differential equations in wavelet bases, Journal of Computational Physics, vol. 132, no. 2, pp. 233-259 (1997).

[5] J. Canosa and H. R. Penafiel, A direct solution of the radiative transfer equation: Application to rayleigh and mie atmospheres, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 13, no. 1, pp. 21-39 (1973).

[6] C. K. Chui, Wavelets: A mathematical tool for signal analysis, SIAM e-books, Society for Industrial and Applied Mathematics(SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), (1997).

[7] B. Dai, B. Zheng, Q. Liang, and L. Wang, Numerical solution of transient heat conduction problems using improved meshless local petrov-galerkin method, Applied Mathematics and Computation, vol. 219, no. 19, pp. 10044-10052 (2013).

[8] A. Dayan and C. L. Tien, Heat transfer in a gray planar medium with linear anisotropic scattering, ASME Transactions Journal of Heat Transfer, vol. 97, pp. 391-396 (1975).

[9] S. A. El Wakil, M. H. Haggag, H. M. Machali and E. A. Saad, Padé approximant in radiative transfer, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 32, no. 2, pp. 173-177 (1984).

[10] P. D. I. Fletcher, S. J. Haswell, and V. N. Paunov, Theoretical considerations of chemical reactions in micro-reactors operating under electroosmotic and electrophoretic control, Analyst, vol. 124, pp. 1273-1282 (1999).

[11] A. Gachpazan, M. Kerayechian and h. zeidabadi, Finite element method for solving linear volterra integro-differential equations of the second kind, Journal of Information and Computing Science, vol. 9, no. 4, pp. 289-297 (2014).

[12] E. Hesameddini, S. Shekarpaz, Wavelet Solutions of the Klein-Gordon Equation, Journal of Mahani Mathematical Research Center, vol. 1, no. 1, pp. 29-45 (2012).

[13] M. Tavassoli Kajania, A. Hadi Vencheha and M. Ghasemib, The Chebyshev wavelets operational matrix of integration and product operation matrix, International Journal of Computer Mathematics, vol. 86, no. 7, pp. 11181125 (2009).

[14] S. Kumar, A. Majumdar and C. L. Tien, The differential-discrete-ordinate method for solutions of the equation of radiative transfer, Journal of Heat Transfer, vol. 112, pp. 424-429 (1990). vol. 15, no. 9, pp. 2284-2292 (2010).

[15] E-B. Lin and Y. Al-Jarrah ,Wavelet Based Methods for Numerical Solutions of Two Dimensional Integral Equations, Mathematica Aeterna, vol. 4, no. 8, pp. 839-853 (2014).

[16] M. P. Mengüç and R. Viskanta, Comparison of radiative transfer approximations for a highly forward scattering planar medium, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 29, no. 5, pp. 381-394 (1983).

[17] S. Müller, Adaptive multiscale schemes for conservation laws, Lecture Notes in Computational Science and Engineering, vol. 27, Springer, (2003).

[18] K. Parand, Sayyed A. Hossayni, and J. A. Rad, An operation matrix method based on bernstein polynomials for riccati differential equation and volterra population model, Applied Mathematical Modelling, vol. 40, no. 2, pp. 9931011 (2016).

[19] K. Parand, M. Dehghan, and A. Pirkhedri, The sinc-collocation method for solving the thomas-fermi equation, Journal of Computational and Applied Mathematics, vol. 237, no. 1, pp. 244-252 (2013).

[20] K. Parand, M. Dehghan, and A. Taghavi, Modified generalized laguerre function tau method for solving laminar viscous flow: The blasius equation, International Journal of Numerical Methods for Heat and Fluid Flow, vol. 20, no. 7, pp. 728-743 (2010).

[21] K. Parand, M. Shahini and M. Dehghan, Solution of a laminar boundary layer flow via a numerical method, Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 2, pp. 360-367 (2010).

[22] G. C. Pomraning, The milne problem in a statistical medium, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 41, no. 2, pp. 103-115 (1989).

[23] J. A. Rad, S. Kazem, M. Shaban, K. Parand and A. Yildirim, Numerical solution of fractional differential equations with a tau method based on legendre and bernstein polynomials, Mathematical Methods in the Applied Sciences, vol. 37, no. 3, pp. 329-342 (2014).

[24] J. A. Rad, K. Parand and S. Abbasbandy, Local weak form meshless techniques based on the radial point interpolation (RPI) method and local boundary integral equation (LBIE) method to evaluate european and american options, Communications in Nonlinear Science and Numerical Simulation, vol. 22, no. 1, pp. 1178-1200 (2015).

[25] J. A. Rad, S. Kazem and K. Parand, The meshless method for solving radiative transfer problems in a slab medium based on radial basis functions, arXiv preprint arXiv:1408.2209 (2014).

[26] J. A. Rad, K. Parand and S. Abbasbandy, Pricing european and american options using a very fast and accurate scheme: The meshless local petrov-galerkin method, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, vol. 85, no. 3, pp. 337-351 (2015).

[27] J. A. Rad, K. Parand and L. V. Ballestra, Pricing european and american options by radial basis point interpolation, Applied Mathematics and Computation, vol. 251, pp. 363-377 (2015).

[28] K. Rashedi, H. Adibi, J. A. Rad and K. Parand, Application of meshfree methods for solving the inverse one-dimensional stefan problem, Engineering Analysis with Boundary Elements, vol. 40, pp. 1-21 (2014).

[29] K. Razi Naqvi, Milne's problem for a non-capturing medium: Accurate analytic approximations for particle density and emergent angular distribution, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 50, no. 1, 59-64 (1993).

[30] M. Razzaghi and S. Yousefi, Legendre wavelets method for the solution of nonlinear problems in the calculus of variations, Mathematical and Computer Modelling, vol. 34, no. 1, pp. 45-54 (2001).

[31] M. Razzaghi, S. Oppenheimer and F. Ahmad, A collocation-type method for radiative transfer problems in a slab medium, Microwave and Optical Technology Letters, vol. 28, no. 5, pp. 307-311 (2001).

[32] M. Razzaghi, S. Oppenheimer and F. Ahmad, Numerical solution of radiative transfer problems in a slab medium by galerkin-type approximation techniques, Physica Scripta, vol. 64, no. 2, 97 (2001).

[33] M. Razzaghi, On the applications of orthogonal functions in pattern recognition, Smart Structures and Materials, International Society for Optics and Photonics, pp. 543-552 (2005).

[34] M. Razzaghi, S. Oppenheimer and F. Ahmad, Tau method approximation for radiative transfer problems in a slab medium, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 72, no. 4, pp. 439-447 (2002).

[35] C. E. Siewert, J. R. Maiorino and M. N. Özişik, The use of the the F_N method for radiative transfer problems with reactive boundary conditions, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 23, no. 6, pp. 565-573 (1980).

[36] K. Styś and T. Styś, A higher-order finite difference method for solving a system of integro-differential equations, Journal of Computational and Applied Mathematics, vol. 126, no. 12, pp. 33-46 (2000).

[37] V. Thomée and N. Y. Zhang, Error estimates for semidiscrete finite element methods for parabolic integro-differential equations, Mathematics of Computation, vol. 53, no. 187, pp. 121-139 (1989).

[38] S. T. Thynell and M. N. Özişik, A new efficient method of solution to radiation transfer in absorbing, emitting, isotropically scattering, homogeneous, finite or semi-infinite, plane-parallel media, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 36, no. 1, pp. 39-50 (1986).

[39] T. W. Tong and C. L. Tien, Resistance-network representation of radiative heat transfer with particulate scattering, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 24, no. 6, pp. 491-503 (1980).

[40] J. R. Tsai, M. N. Özişik and F. Santarelli, Radiation in spherical symmetry with anisotropic scattering and variable properties, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 42, no. 3, pp. 187-199 (1989).

[41] Y. Wang, Y. Mu, and P. Ding, A linear spline approximation for radiative transfer problems in slab medium, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 55, no. 1, pp. 1-5 (1996).

[42] A. M. Wazwaz, The combined laplace transform adomian decomposition method for handling nonlinear volterra integro-differential equations, Applied Mathematics and Computation, vol. 216, no. 4, pp. 1304-1309 (2010).

[43] S. J. Wilson and K. K. Sen, Generalized eddington approximation method for radiative transfer problems in slab medium, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 35, no. 6, pp. 467-472 (1986).

[44] X. T. Xiong, C. L. Fu and Z. Qian, Two numerical methods for solving a backward heat conduction problem, Applied Mathematics and Computation, vol. 179, no. 1, pp. 370-377 (2006).

[45] C. Yang and J. Hou, Chebyshev wavelets method for solving Bratu's problem, Boundary value problems, no. 1, pp. 1-9 (2013). Science (2012).

[46] F. Yin, T. Tian, J. Song and M. Zhu, Spectral methods using Legendre wavelets for nonlinear Klein Sine-Gordon equations, Journal of Computational and Applied Mathematics, vol. 275, pp. 321334 (2015).

[47] S. Yousefi and M. Razzaghi, Legendre wavelets method for the nonlinear volterra-fredholm integral equations, Mathematics and Computers in Simulation, vol. 70, no. 1, pp. 1-8 (2005).