A note on the new set operator ψ_r

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Abstract

Recently many published works made on local function used in ideal topological spaces can be found in related literature. "Semi Local Functions in Ideal Topological Spaces", "Closure Local Functions", and "$()^p$ and ψ_p-Operator" can be mentioned among such works those aim to define such functions. In general, the researchers prefer using the generalized open sets instead of topology in ideal topological spaces. Obtaining a Kuratowski closure operator with the help of local functions is an important detail in ideal topological space. However, it is not possible to obtain a Kuratowski closure operator from many of these local functions proposed by the above mentioned works. In order to address the lack of such an operator, the goal of this paper is to introduce another local function to give possibility of obtaining a Kuratowski closure operator. On the other hand, regular local functions defined for ideal topological spaces have not been found in the current literature. Regular local functions for the ideal topological spaces has been described within this work. Moreover, with the help of regular local functions Kuratowski closure operators cl_I^{*r} and τ^{*r} topology are obtained. Many theorems in the literature have been revised according to the definition of regular local functions.

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1 Introduction

The studies about ideal topological space has been enriched by so many mathematicians. Hamlett and Jankovic [13], Modak and Bandyopadhyay [12] were able to define a closure operator with the help of local function, and hence defined a new topology.
Lately, local functions on a spaces in which topology is replaced by its generalized open sets worked by many mathematicians [11], [7], [3].

This paper deals with a space in which topology is replaced by the family of regular open sets.

2 Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $cl(A)$ and $Int(A)$ denote the closure and interior of A in (X, τ), respectively. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties [6]:

1. $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity),

2. $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity).

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An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I).
For a subset $A \subseteq X$, $A^*(I, \tau) = \{x \in X : A \cap U \notin I$ for every $U \in \tau(X, x)\}$ is called the local function of A with respect to I and τ, where $\tau(X, x) = \{U \in \tau : x \in U\}$ [4]. We simply write A^* instead of $A^*(I, \tau)$ in case there is no chance for confusion.
For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by the base $\beta(I, \tau) = \{U - J : U \in \tau$ and $J \in I\}$. It is known in [4] that $\beta(I, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(I)$ is denoted by τ^*. For a subset $A \subseteq X$, $cl^*(A)$ and $int^*(A)$ will, respectively, denote the closure and interior of A in (X, τ^*).

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be regular open [2] if $A = int(cl(A))$. The complement of a regular open set is said to be regular closed. The collection of all regular open (resp. regular closed) sets in X is denoted by RO(X) (resp. RC(X)). The regular closure of A in (X, τ) is denoted by the intersection of all regular closed sets containing A and is denoted by $rcl(A)$.

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be semi open [9] if there exist an open set U in X such that $U \subseteq A \subseteq cl(U)$. The other definition of semi open set is that: A subset A of X is said to be semi open [8] if $A \subseteq cl(int(A))$. The complement of a semi open set is said to be semi closed. The collection of all semi open (resp. semi closed) sets in X is denoted by SO(X) (resp. SC(X)).

Definition 2.3. Let (X, τ, I) be a ideal space and A a subset of X. Then $A_*(I, \tau) = \{x \in X : A \cap U \notin I$ for every $U \in$ SO $(X, x)\}$ is called the semi local function of A with respect to I and τ where SO$(X, x) = \{U \in$ SO$(X) : x \in U\}$ [7].

When there is no ambiguity, we will write simply A_* for $A_*(I, \tau)$.

3 Regular local functions

In this section we shall introduce regular ideal space and $()^{*r}$ operator and discuss various properties of this operator.
Let (X, τ) be a topological space and I be an ideal on X, then $(X$,RO $(X, \tau), I)$ is called regular ideal space.
Now we shall define the operator $()^{*r}$.

Definition 3.1. Let $(X$,RO $(X, \tau), I)$ be a regular ideal space and A a subset of X. Then $A^{*r}(I,$ RO $(X, \tau)) = \{x \in X : A \cap U \notin I$ for every $U \in$ RO $(X, x)\}$ is called the regular local function of A with respect to I and τ where RO$(X, x) = \{U \in$ RO$(X) : x \in U\}$.

When there is no ambiguity, we will write simply A^{*r} for $A^{*r}(I, \tau)$.

Theorem 3.2. Let $(X$,RO $(X, \tau), I)$ be a regular ideal space and A a subset of X.

(i) $A_* \subseteq A^* \subseteq A^{*r}$ for every $A \subseteq X$.

(ii) $A_* = A^{*r}$ if SO(X, τ) =RO(X, τ).

(iii) If $A \in I$, then $A^{*r} = \varnothing$.

(iv) $(\varnothing)^{*r} = \varnothing$.

Proof. (i) Let $x \in A^*(I,\tau)$. Then, $A \cap U \notin I$ for every $U \in \tau$. Since every regular open set is open, therefore $x \in A^{*r}(I,\tau)$. Converse is not true in general, it is shown in Example 3.3.

(ii) It is obvious from definition of regular local and semi local functions.

(iii) Let $A \in I$ and $x \in A^{*r}$. Then for every regular open set U containing x, $U \cap A \notin I$. On the other hand X is also regular open set. So $X \cap A = A \notin I$. It is contradiction.

(iv) Because of (iii) it is obvious.

Q.E.D.

Example 3.3. Let $X = \{a,b,c,d\}$ and $\tau = \{\varnothing, X, \{c,d\}\}$ with $I = \{\varnothing, \{c\}\}$. Let $A = \{a,b\}$ then $A^* = \{a,b\} = cl(A^*)$ and $A^{*r} = X = rcl(A^{*r})$. So $A^{*r} \nsubseteq A^*$.

Remark 3.4. Let $(X,\text{RO }(X,\tau), I)$ be a regular ideal space and A a subset of X. Neither $A \subseteq A^{*r}$ nor $A^{*r} \subseteq A$ in general.

The following is an example that supports this remark.

Example 3.5. Let $X = \{a,b,c,d\}$ and $\tau = \{\varnothing, X, \{b\}, \{c,d\}, \{b,c,d\}\}$ with $I = \{\varnothing, \{a\}\}$. For $A = \{a\}$, $A^{*r} = \varnothing$ and so $A^{*r} \subset A$. For $A = \{a,c\}$, $A^{*r} = \{a,c,d\}$ and so $A \subset A^{*r}$. For $A = \{a,b\}$, $A^{*r} = \{a,b\}$ and so $A^{*r} = A$.

Theorem 3.6. Let (X,τ,I) be an ideal topological space and A, B subsets of X. Then, for regular local functions, the following properties hold:

(i) If $A \subseteq B$, then $A^{*r} \subseteq B^{*r}$,

(ii) If I, J ideal on X and $I \subseteq J$, then $A^{*r}(J) \subseteq A^{*r}(I)$.

Proof. (i) Let $x \in A^{*r}$. Then for every regular open set U_x containing x, $U_x \cap A \notin I$. Since $U_x \cap A \subseteq U_x \cap B$, then $U_x \cap B \notin I$.

(ii) Let $x \in A^{*r}(J)$. Then $A \cap U \notin J$, for every $U \in \text{RO}(X,x)$. Since $J \supseteq I$, $A \cap U \notin I$ and hence $x \in A^{*r}(I)$.

Q.E.D.

Theorem 3.7. Let $(X,\text{RO}(X,\tau), I)$ be an regular ideal space and A, B subsets of X. Then, for regular local functions, the following properties hold:

(i) $A^{*r} = \text{cl}(A^{*r}) \subseteq \text{rcl}(A)$ and A^{*r} is closed in (X,τ),

(ii) $(A^{*r})^{*r} \subseteq A^{*r}$,

(iii) $A^{*r} \cup B^{*r} = (A \cup B)^{*r}$,

(iv) $(A \cap B)^{*r} \subseteq A^{*r} \cap B^{*r}$,

(v) $A^{*r} \setminus B^{*r} = (A \setminus B)^{*r} \setminus B^{*r} \subseteq (A \setminus B)^{*r}$.

Proof. (i) We have $A^{*r} \subseteq cl(A^{*r})$ in general. Let $x \in cl(A^{*r})$. Also given the set $U \in$RO(X,x). Then $A^{*r} \cap T \neq \varnothing, T \in \tau(x)$. Since U is open, $A^{*r} \cap U \neq \varnothing$. Therefore, there exists some $y \in (A^{*r} \cap U)$ and $U \in$RO(X,y). Since $y \in A^{*r}, A \cap U \notin I$ and hence $x \in A^{*r}$. Hence we have $cl(A^{*r}) \subseteq A^{*r}$ and hence $A^{*r} = cl(A^{*r})$. Again, let $x \in A^{*r} = cl(A^{*r})$, then $A \cap U \notin I$, for every $U \in$RO(X,x). This implies $A \cap U \neq \varnothing$ for every $U \in$RO(X,x). Therefore, $x \in$rcl(A). This shows that $A^{*r} = cl(A^{*r}) \subseteq rcl(A)$. Since $A^{*r} = cl(A^{*r})$, A^{*r} is closed.

(ii) Let $x \in (A^{*r})^{*r}$. Then for every $U \in$RO$(X,x), U \cap A^{*r} \notin I$ and hence $U \cap A^{*r} \neq \varnothing$. Let $y \in U \cap A^{*r}$. Then $U \in$RO(X,y) and $y \in A^{*r}$. Hence we have $U \cap A \notin I$ and $x \in A^{*r}$. This shows that $(A^{*r})^{*r} \subseteq A^{*r}$.

(iii) By theorem 3.6 (i), we have $A^{*r} \cup B^{*r} \subseteq (A \cup B)^{*r}$. To prove the reverse inclusion, let $x \notin A^{*r} \cup B^{*r}$. Then x belongs neither to A^{*r} nor to B^{*r}. Therefore there exist $U_x, V_x \in$RO(X,x) such that $A \cap U_x \in I$ and $B \cap V_x \in I$. Since I is additive, $(A \cap U_x) \cup (B \cap V_x) \in I$.

$$\begin{aligned}(A \cap U_x) \bigcup (B \cap V_x) &= [(A \cap U_x) \bigcup V_x] \bigcap [(A \cap U_x) \bigcup B] \\ &= (U_x \cup V_x) \bigcap (A \cup V_x) \bigcap (U_x \cup B) \bigcap (A \cup B)\end{aligned}$$

On the other hand since $U_x \cap V_x \subseteq U_x \cup V_x, V_x \subseteq A \cup V_x$ and $U_x \subseteq B \cup U_x$, we have

$$(U_x \cup V_x) \bigcap (A \cup V_x) \bigcap (U_x \cup B) \bigcap (A \cup B) \supseteq (U_x \cap V_x) \bigcap (A \cup B)$$

Since I is heredity, $(U_x \bigcap V_x) \bigcap (A \cup B) \in I$. Since regular open sets closed under the finite intersections, $U_x \cap V_x \in$RO(X,x) and so $x \notin (A \cup B)^{*r}$. Hence $(X \setminus A^{*r}) \bigcap (X \setminus B^{*r}) \subseteq X \setminus (A \cup B)^{*r}$ or $(A \cup B)^{*r} \subseteq A^{*r} \cup B^{*r}$.

(iv) By theorem 3.6 (i), $(A \cap B)^{*r} \subseteq A^{*r}$ and $(A \cap B)^{*r} \subseteq B^{*r}$ so $(A \cap B)^{*r} \subseteq A^{*r} \cap B^{*r}$.

(v) We have By theorem 3.7 (iii), $A^{*r} = [(A \setminus B) \bigcup (A \cap B)]^{*r} = (A \setminus B)^{*r} \bigcup (A \cap B)^{*r} \subseteq (A \setminus B)^{*r} \bigcup B^{*r}$. Thus $A^{*r} \setminus B^{*r} \subseteq (A \setminus B)^{*r} \setminus B^{*r}$.
On the other hand, by theorem 3.6 (i), $(A \setminus B)^{*r} \subseteq A^{*r}$ and hence $(A \setminus B)^{*r} \setminus B^{*r} \subseteq A^{*r} \setminus B^{*r}$.
Hence $A^{*r} \setminus B^{*r} = (A \setminus B)^{*r} \setminus B^{*r} \subseteq (A \setminus B)^{*r}$.

Q.E.D.

Theorem 3.8. Let $(X$,RO$(X,\tau), I)$ be an regular ideal space and A, B subsets of X. Then, for regular local functions, the following properties hold:

(i) If $I_0 \in I$, then $(A \setminus I_0)^{*r} = A^{*r} = (A \cup I_0)^{*r}$,

(ii) If $U \subseteq X$, then $U \cap (U \cap A)^{*r} \subseteq U \cap A^{*r}$,

(iii) If $A \subseteq X$ and $U \in$ RO(X,τ), then $U \cap A \in I \Longrightarrow U \cap A^{*r} = \varnothing$,

(iv) If $A \subseteq X$, then $(A \cap A^{*r})^{*r} \subseteq A^{*r}$.

Proof. (i) Since $I_0 \in I$, by theorem 3.2 (iii) $I_0^{*r} = \varnothing$. By theorem 3.7 (v), $A^{*r} = (A \setminus I_0)^{*r}$ and by theorem 3.7 (iii), $(A \cup I_0)^{*r} = A^{*r} \cup I_0^{*r} = \varnothing \cup A^{*r} = A^{*r}$.

(ii) Since $U \cap A \subseteq A$, by theorem 3.6 (i), $(U \cap A)^{*r} \subseteq A^{*r}$ and hence $U \bigcap (U \cap A)^{*r} \subseteq U \cap A^{*r}$.

(iii) Let $U \cap A \in I$, then for every $x \in U, x \notin A^{*r}$ because of $U \in \mathrm{RO}(X, \tau)$. So $U \cap A^{*r} = \varnothing$.

(iv) By theorem 3.6 (i) $(A \cap A^{*r})^{*r} \subseteq (A^{*r})^{*r}$. On the other hand, from theorem 3.7 (ii) we have $(A \cap A^{*r})^{*r} \subseteq (A^{*r})^{*r} \subseteq A^{*r}$.

Q.E.D.

In literature [4] for ideal topological spaces we will obtain $cl^*(A) = A \cup A^*$ Kuratowski Closure operator. But in [1], [7] and [11] we are not able to define a Kuratowski Closure operator with the help of (regular semi, semi and pre) local function. Because that functions do not provide a theorem 3.7 (iii) given above for $()^{*r}$-Operator. However, in [7] Khan and Noiri say that $(A \cup B)_* = A_* \cup B_*$. But the following example, we see that this is not true:

Example 3.9. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ with $I = \{\varnothing, \{c\}\}$. So semi open sets are SO$=\{\varnothing, X, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$. Let $A = \{a\}$ and $B = \{d\}$ then Let $A \cup B = \{a, d\}$. Thus we have $A_* = \{a, c\}$ and $B_* = \{d\}$. So $(A \cup B)_* = (\{a, d\})_* = X$ but $A_* \cup B_* = \{a, c, d\}$. Hence $(A \cup B)_* \neq A_* \cup B_*$.

We are able to define a closure operator with the help of regular local function. Because the $()^{*r}$ operator satisfy the conditions of Theorem 3.2 (iv), Theorem 3.7 (ii) and Theorem 3.7 (iii). And thus $\mathrm{Cl}_I^{*r} : \wp(X) \longrightarrow \wp(X)$, $\mathrm{Cl}_I^{*r} = A \cup A^{*r}, \forall A \in \wp(X)$ is a Kuratowski closure operator. Hence it generates a τ^{*r} topology:

$$\tau^{*r}(I) = \{A \in \wp(X) : \mathrm{Cl}_I^{*r}(X \setminus A) = X \setminus A\}$$

4 ψ_r-operator

In topological space $clA = X \setminus int(X \setminus A)$ [6] is remarkable result. Many useful result have been proved with the help of this result. This relation is the motivation of defining the operator ψ_r.

Definition 4.1. Let $(X, \mathrm{RO}(X, \tau), I)$ be a regular ideal space. An operator $\psi_r : \wp(X) \longrightarrow \tau$ is defined as: $\psi_r(A) = \{x \in X | \exists U_x \in \mathrm{RO}(X, x) : U_x \setminus A \in I\}$, for every $A \in \wp(X)$.

We observe that $\psi_r(A) = X \setminus (X \setminus A)^{*r}$.

Theorem 4.2. Let $(X, \mathrm{RO}(X, \tau), I)$ be a regular ideal space and $A, B \in \wp(X)$.

(i) $\psi_r(A) \supseteq \mathrm{rint}(A)$,

(ii) $\psi_r(A)$ is open,

(iii) If $A \subseteq B$, than $\psi_r(A) \subseteq \psi_r(B)$,

(iv) $\psi_r(A) \cup \psi_r(B) \subseteq \psi_r(A \cup B)$,

(v) $\psi_r(A \cap B) = \psi_r(A) \cap \psi_r(B)$,

(vi) $\psi_r(A) \subseteq \psi(A)$.

Proof. (i) $\psi_r(A) = X \setminus (X \setminus A)^{*r} \supseteq X \setminus rcl(X \setminus A)$ by theorem 3.7 (i). So $\psi_r(A) \supseteq rint(A)$.

(ii) Since A^{*r} is a closed, then $(X \setminus A)^{*r}$ is closed. So $X \setminus (X \setminus A)^{*r} = \psi_r(A)$ is a open set.

(iii)

$$\begin{aligned} A \subseteq B \Rightarrow X \setminus A \supseteq X \setminus B &\Rightarrow (X \setminus A)^{*r} \supseteq (X \setminus B)^{*r} \\ &\Rightarrow X \setminus (X \setminus A)^{*r} \subseteq X \setminus (X \setminus B)^{*r} \\ &\Rightarrow \psi_r(A) \subseteq \psi_r(B) \end{aligned}$$

(iv) Proof is obvious from theorem 4.2 (iii).

(v)

$$\begin{aligned} \psi_r(A \cap B) &= X \setminus [X \setminus (A \cap B)]^{*r} \\ &= X \setminus [(X \setminus A) \cup (X \setminus B)]^{*r} \\ &= X \setminus [(X \setminus A)^{*r} \cup (X \setminus B)^{*r}] \\ &= [X \setminus (X \setminus A)^{*r}] \cap [X \setminus (X \setminus B)^{*r}] \\ &= \psi_r(A) \cap \psi_r(B) \end{aligned}$$

(vi) From theorem 3.2 (i), we have that

$$\begin{aligned} (X \setminus A)^* \subseteq (X \setminus A)^{*r} &\Rightarrow X \setminus (X \setminus A)^{*r} \subseteq X \setminus (X \setminus A)^* \\ &\Rightarrow \psi_r(A) \subseteq \psi(A) \end{aligned}$$

Q.E.D.

Theorem 4.3. Let $(X,\mathrm{RO}(X,\tau),I)$ be a regular ideal space and $A, B \in \wp(X)$.

(i) $\psi_r(A) = \psi_r(\psi_r(A))$ if and only if $(X \setminus A)^{*r} = [(X \setminus A)^{*r}]^{*r}$,

(ii) If $I_0 \in I$, then $\psi_r(A \setminus I_0) = \psi_r(A)$,

(iii) If $I_0 \in I$, then $\psi_r(A \cup I_0) = \psi_r(A)$,

(iv) If $(A \setminus B) \cup (B \setminus A) \in I$, then $\psi_r(A) = \psi_r(B)$,

(v) If $A \in RO(X,\tau)$, then $A \subseteq \psi_r(A)$.

Proof. (i) Proof is obvious from definition of $\psi_r(A)$ and the fact: $\psi_r(\psi_r(A)) = X \setminus [X \setminus (X \setminus (X \setminus A)^{*r})]^{*r} = X \setminus [(X \setminus A)^{*r}]^{*r}$.

(ii) By theorem 3.8 (i),we have

$$\begin{aligned} \psi_r(A \setminus I_0) &= X \setminus [X \setminus (A \setminus I_0)]^{*r} \\ &= X \setminus [(X \setminus A) \cup I_0]^{*r} \\ &= X \setminus (X \setminus A)^{*r} \\ &= \psi_r(A) \end{aligned}$$

(iii) By theorem 3.8 (i),we have

$$\begin{aligned}\psi_r(A \cup I_0) &= X \setminus [X \setminus (A \cup I_0)]^{*r} \\ &= X \setminus [(X \setminus A) \setminus I_0]^{*r} \\ &= X \setminus (X \setminus A)^{*r} \\ &= \psi_r(A)\end{aligned}$$

(iv) Assume $(A \backslash B) \cup (B \backslash A) \in I$. Let $A \backslash B = I_1$ and $B \backslash A = I_2$. Observe that by heredity $I_1, I_2 \in I$. Also observe that $B = (A \setminus I_1) \cup I_2$. Thus $\psi_r(A) = \psi_r(A \setminus I_1) = \psi[(A \setminus I_1) \cup I_2] = \psi_r(B)$ by theorem 4.3 (ii) and (iii).

(v) Since $A \in$ RO(X, τ), $(X \setminus A) \in$RC(X, τ). So $(X \setminus A) = rcl(X \setminus A)$. From Theorem 3.7 (i) we have

$$\begin{aligned}(X \setminus A)^{*r} \subseteq rcl(X \setminus A) = X \setminus A &\Longrightarrow (X \setminus A)^{*r} \subseteq X \setminus A \\ &\Longrightarrow A \subseteq X \setminus (X \setminus A)^{*r} \\ &\Longrightarrow A \subseteq \psi_r(A)\end{aligned}$$

Q.E.D.

Example 4.4. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{b\}, \{c, d\}, \{b, c, d\}\}$ with $I = \{\varnothing, \{d\}\}$. Then for $A = \{c\}$, we have $\psi_r(A) = \{c, d\} \supseteq A$ but $A = \{c\}$ is not a regular open set.

Theorem 4.5. Let $(X$,RO$(X, \tau), I)$ be a regular ideal space and $A \subseteq X$.

(i) $\psi_r(A) = \bigcup\{U \in$RO$(X, \tau) : U \setminus A \in I\}$,

(ii) $\psi_r(A) \supseteq \bigcup\{U \in$RO$(X, \tau) : (U \setminus A) \cup (A \setminus U) \in I\}$.

Proof. (i) Proof is obvious from definition of $\psi_r(A)$.

(ii) Since I is heredity, we have
$\bigcup\{U \in$RO$(X, \tau) : (U \setminus A) \cup (A \setminus U) \in I\} \subseteq \bigcup\{U \in$RO$(X, \tau) : U \setminus A \in I\} = \psi_r(A)$

Q.E.D.

5 RO-codense deal

Definition 5.1. Let (X, τ, I) be an ideal topological spaces and $A \subseteq X$. If $\tau \cap I = \{\varnothing\}$ then we say the I is codense ideal [5].

Definition 5.2. Let $(X$,RO$(X, \tau), I)$ be a regular ideal spaces and $A \subseteq X$. If RO$(X, \tau) \cap I = \{\varnothing\}$ then we say the I is RO-codense ideal.

Theorem 5.3. Let $(X$,RO$(X, \tau), I)$ be a regular ideal spaces. If I is RO-codense ideal with RO(X, τ), then $X = X^{*r}$.

Proof. It is obvious that $X^{*r} \subseteq X$. Let $x \notin X^{*r}$, for $x \in X$. Then there is at least one $U_x \in$RO(X, τ) that provide $U_x \cap X \in I$. Hence $U_x \cap X = U_x \in I$. But RO$(X, \tau) \cap I = \{\varnothing\}$. It is a contradiction. So $X = X^{*r}$. Q.E.D.

Theorem 5.4. The followings are equivalent for $(X,\mathrm{RO}(X,\tau),I)$ regular ideal space.

(i) $\mathrm{RO}(X,\tau)\cap I=\{\varnothing\}$,

(ii) $\psi_r(\varnothing)=\varnothing$,

(iii) If $I_0\in I, \psi_r(I_0)=\varnothing$.

Proof. i⇒ii: Let $\mathrm{RO}(X,\tau)\cap I=\{\varnothing\}$. From definition of ψ_r operator and Theorem 5.3 (5.3), we have $\psi_r(\varnothing)=X\setminus(X\setminus\varnothing)^{*r}=X\setminus X^{*r}=\varnothing$.

ii⇒iii: Let $I_0\in I$ and $\psi_r(\varnothing)=\varnothing$. Also because of Theorem 3.8 (i), we have obtained $(X\setminus I_0)^{*r}=X^{*r}$. So we have

$$\psi_r(I_0)=X\setminus(X\setminus I_0)^{*r}=X\setminus X^{*r}=\psi_r(\varnothing)=\varnothing.$$

iii⇒i: Let $A\in\mathrm{RO}(X,\tau)\cap I$. Then because of $A\in I$ and Theorem 5.4 (iii), we have $\psi_r(A)=\varnothing$. Also $A\subseteq\psi_r(A)=\varnothing$ since $A\in\mathrm{RO}(X,\tau)$ and $A\subseteq\psi_r(A)$. And so $A=\varnothing$. Hence we have $\mathrm{RO}(X,\tau)\cap I=\{\varnothing\}$. Q.E.D.

Theorem 5.5. Let $(X,\mathrm{RO}(X,\tau),I)$ be a regular ideal spaces. If I is RO-codense ideal with $\mathrm{RO}(X,\tau)$, then $\psi_r(A)\subseteq A^{*r}$ for every $A\subseteq X$.

Proof. Let $x\in\psi_r(A)$ and $x\notin A^{*r}$ for a least one $x\in X$. Then we obtain

$$x\notin A^{*r}\Longrightarrow\exists T_x\in\mathrm{RO}(X,x):T_x\cap A\in I$$

Since $x\in\psi_r(A)$, we have $x\in\bigcup\{U\in\mathrm{RO}(X,\tau):U\setminus A\in I\}$ from Theorem 4.5 (i). Hence there is $V\in\mathrm{RO}(X,\tau)$ which satisfy $x\in V$ and $V\setminus A\in I$. Since $x\in T_x\cap V$ is a regular open set, we obtain $(T_x\cap V)\cap A\in I$ and $(T_x\cap V)\setminus A\in I$ from heredity of I. Also since I is finite additivity, we obtain

$$T_x\cap V=[(T_x\cap V)\cap A]\bigcup[(T_x\cap V)\setminus A]\in I$$

Since $T_x\cap V\neq\varnothing$ is a regular open set, $I\cap\mathrm{RO}(X,\tau)\neq\varnothing$. But it contradict with the fact I is RO-codense. So $x\in A^{*r}$.

Q.E.D.

Remark 5.6. Let $(X,\mathrm{RO}(X,\tau),I)$ be a regular ideal spaces and $A\subseteq X$. If I is RO-codense ideal, then $\psi_r(A)\subseteq\mathrm{rcl}(A)$.

6 Regular compatibility topology with an ideal

Definition 6.1. Let (X,τ,I) be an ideal topological spaces. We say the τ is compatible with the ideal I, denoted $\tau\backsim I$, if the following holds for every $A\subseteq X$, if for every $x\in A$ there exists $U\in\tau(x)$ such that $U\cap A\in I$, then $A\in I$ [10].

Definition 6.2. Let $(X,\mathrm{RO}(X,\tau),I)$ be an regular ideal spaces. We say the τ is regular compatible with the ideal I, denoted $\tau\backsim_r I$, if the following holds for every $A\subseteq X$, if for every $x\in A$ there exists $U\in\mathrm{RO}(X,x)$ such that $U\cap A\in I$, then $A\in I$.

Theorem 6.3. Let $(X,\mathrm{RO}(X,\tau),I)$ be a regular ideal space. $\tau \backsim_r I$ if and only if $\psi_r(A)\setminus A \in I$, for every $A \subseteq X$.

Proof. Let $\tau \backsim_r I$ and $\psi_r(A)\setminus A \in I$, for every $A \subseteq X$. Then we have

$$\begin{aligned} x \in \psi_r(A), x \notin A &\Longrightarrow x \in X \setminus (X \setminus A)^{*r}, x \notin A \\ &\Longrightarrow x \notin (X \setminus A)^{*r}, x \notin A \\ &\Longrightarrow \exists U \in RO(X,x); U \cap (X \setminus A) \in I, x \notin A \\ &\Longrightarrow X \setminus A \in I, x \in X \setminus A \end{aligned}$$

So $\psi_r(A)\setminus A \subseteq X \setminus A \in I$.
Conversely, let $\psi_r(A)\setminus A \in I$ for every $A \subseteq X$. Also there is $U \in \mathrm{RO}(X,x)$ which $U \cap A \in I$ for every $x \in A$. Then
$x \notin A^{*r} \Longrightarrow x \in X \setminus A^{*r} \Longrightarrow A \subseteq X \setminus A^{*r}$
Hence because of the following equation and the fact $A \subseteq X \setminus A^{*r}$ we have $\psi_r(X\setminus A)\setminus(X\setminus A) = A$.

$$\psi_r(X \setminus A) \setminus (X \setminus A) = [X \setminus (X \setminus (X \setminus A))^{*r}] \setminus (X \setminus A) = (X \setminus A^{*r}) \cap A$$

Since $\psi_r(A)\setminus A \in I$ for every $A \subseteq X$, $\psi_r(X \setminus A)\setminus (X\setminus A) = A \in I$. Q.E.D.

From the above theorem we will give the following remark.

Remark 6.4. Let $(X,\mathrm{RO}(X,\tau),I)$ be a regular ideal space and $\tau \backsim_r I$. Then $\psi_r(\psi_r(A)) = \psi_r(A)$, for every $A \subseteq X$.

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