Marcinkiewicz integrals with rough kernel associated with Schrödinger operators and commutators on generalized vanishing local Morrey spaces

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Abstract

Let $L = -\Delta + V(x)$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential $V(x)$ belonging to the reverse Hölder class. In this paper, using the some conditions on $\varphi(x.r)$, we dwell on the boundedness of Marcinkiewicz integrals with rough kernel associated with schrödinger operators and commutators generated by these operators and local Campanato functions both on generalized local Morrey spaces and on generalized vanishing local Morrey spaces, respectively. As an application of the above results, the boundedness of parametric Marcinkiewicz integral and its commutator both on generalized local Morrey spaces and on generalized vanishing local Morrey spaces is also obtained.

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1 Introduction and main results

In this section, we will give some background material that is needed for later chapters. We assume that our readers are familiar with the foundation of real analysis. Since it is impossible to squeeze everything into just a few pages, sometimes we will refer the interested readers to some papers and references.

Notation. Let $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ . . . . etc . be points of the real n-dimensional space $\mathbb{R}^n$. Let $x.\xi = \sum_{i=1}^{n} x_i \xi_i$ stand for the usual dot product in $\mathbb{R}^n$ and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ for the Euclidean norm of x.

· By x', we always mean the unit vector corresponding to x, i.e. $x' = \frac{x}{|x|}$ for any $x \neq 0$.

· $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ represents the unit sphere and dx' is its surface measure.

· By $B(x,r)$, we always mean the open ball centered at x of radius r and by $(B(x,r))^C$, we always mean its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. We also have $CB(x,r) = B(x,Cr)$ for $C > 0$.

· $F \approx G$ means $F \gtrsim G \gtrsim F$; while $F \gtrsim G$ means $F \geq CG$ for a constant $C > 0$; and p' and s' always denote the conjugate index of any $p > 1$ and $s > 1$, that is, $\frac{1}{p'} := 1 - \frac{1}{p}$ and $\frac{1}{s'} := 1 - \frac{1}{s}$.

· C stands for a positive constant that can change its value in each statement without explicit mention.

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· The Lebesgue measure of a measurable set E is denoted as $|E|$. Roughly speaking: in one-dimension $|E|$ is the length of E, in two-dimension it is the area of E, and in three dimension (or higher) it is the "volume" of E.

· We use the notation

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\, dy.$$

· $\|\Omega\|_{L_s(S^{n-1})} := \left(\int\limits_{S^{n-1}} |\Omega(z')|^s \, d\sigma(z') \right)^{\frac{1}{s}}.$

In 1938, Morrey [23] considered regularity of the solution of elliptic partial differential equations(PDEs) in terms of the solutions themselves and their derivatives. This is a very famous work by Morrey [23]. We define Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ via the following norm.

A measurable function $f \in L_p(\mathbb{R}^n)$, $p \in (1,\infty)$, belongs to the Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ with $\lambda \in [0,n)$ if the following norm is finite:

$$\|f\|_{L_{p,\lambda}} = \left(\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{1/p}.$$

When $\lambda = 0$, $L_{p,\lambda}(\mathbb{R}^n)$ coincides with the Lebesgue space $L_p(\mathbb{R}^n)$. Recently, Chen et al. [6] gave a criterion of the boundedness of a general linear or sublinear operators with rough kernel on Morrey spaces:

Theorem 1.1. [6] Let $0 < \lambda < n$. Suppose that $\Omega \in L_q(S^{n-1})$ for $q > \frac{n}{n-\lambda}$ and T_Ω is a sublinear operator with rough kernel satisfying following

$$|T_\Omega f(x)| \le C \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| \, dy.$$

Let $1 < p < \infty$. If the operator T_Ω is bounded on $L_p(\mathbb{R}^n)$, then T_Ω is bounded on $L_{p,\lambda}(\mathbb{R}^n)$.

We also refer readers to the elegant book [1] for further information about these spaces and references on recent developments in this field associated with harmonic analysis.

On the other hand, the study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [33] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $L_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

Later in [34] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [22] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa has proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VL_{p,\lambda}(\mathbb{R}^n)$ (see [26]). For the properties and applications of vanishing Morrey spaces, see also [4, 27].

The concept of the generalized local (central) Morrey space $LM_{p,\varphi}^{\{x_0\}}$ has been introduced in [3] and studied in [12, 13, 14]. Later, motivated by [3, 12, 13] and using parabolic metric, Gürbüz [15, 16] has introduced parabolic generalized local (central) Morrey space and showed that the boundedness of a class of parabolic rough operators and their commutators on these spaces. But, this topic exceeds the scope of this paper. Thus, we omit the details here. The spaces we are interested in this paper are of the following forms:

Definition 1.2. (generalized local (central) Morrey space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))} < \infty.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r))} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and the weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LL_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLL_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

For the properties and applications of generalized local (central) Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, see also [3, 12, 13, 14].

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f;x_0,r) := \frac{|B(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))}}{\varphi(x_0,r)}$$

and

$$\mathfrak{M}_{p,\varphi}^{W}(f;x_0,r) := \frac{|B(x_0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r))}}{\varphi(x_0,r)}.$$

Extending the definitions of the vanishing Morrey spaces and vanishing generalized Morrey spaces given by Vitanza [33] and Samko [27] to the case of the generalized local (central) Morrey spaces, in [13], Gürbüz has introduced the following definitions:

Definition 1.3. [13]**(generalized vanishing local Morrey space)** The generalized vanishing local Morrey space $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \mathfrak{M}_{p,\varphi}(f;x_0,r) = 0.$$

Definition 1.4. [13] **(weak generalized vanishing local Morrey space)** The weak generalized vanishing local Morrey space $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \mathfrak{M}_{p,\varphi}^{W}(f; x_0, r) = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\varphi(x_0, r)} = 0, \tag{1.1}$$

and

$$\sup_{0<r<\infty} \frac{1}{\varphi(x_0, r)} < \infty, \tag{1.2}$$

which make the spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VLM_{p,\varphi}^{\{x_0\}}} \equiv \|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}(f; x_0, r),$$

$$\|f\|_{WVLM_{p,\varphi}^{\{x_0\}}} = \|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}^{W}(f; x_0, r),$$

respectively. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are closed subspaces of the Banach spaces $LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, respectively, which may be shown by standard means.

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

In [30], Stein has defined the Marcinkiewicz integral for higher dimensions. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \quad \text{for any} \quad \mu > 0, x \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.4}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

(c) $\Omega \in Lip_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma \quad \text{for any} \quad x', y' \in S^{n-1}.$$

(d) $\Omega \in L_1(S^{n-1})$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [21, 31, 32].

Remark 1.5. We easily see that the Marcinkiewicz integral operator of higher dimension μ_Ω can be regarded as a generalized version of the classical Marcinkiewicz integral in the one dimension case. Also, it is easy to see that μ_Ω is a special case of the Littlewood-Paley g-function if we take

$$g(x) = \Omega(x')\,|x|^{-n+1}\,\chi_{|x|\le 1}(|x|).$$

When Ω satisfies some size conditions, the kernel of the operator μ_Ω has no regularity, and so the operator μ_Ω is called rough Marcinkiewicz integral operator. The theory of Operators with rough kernel is a well studied area on some kinds of function spaces (see [3, 5, 6, 7, 12, 13, 15, 16, 17, 18, 21] for example).

Now we give the definition of the commutator generalized by μ_Ω and b by

$$\mu_{\Omega,b}(f)(x) = \left(\int_0^\infty |F_{\Omega,t,b}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t,b}(f)(x) = \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f(y)dy.$$

On the other hand, in this paper we consider the Schrödinger operator

$$L = -\Delta + V(x) \text{ on } \mathbb{R}^n, \qquad n \ge 3$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q, for some exponent $q \ge \frac{n}{2}$; that is, there exists a constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{\frac{1}{q}} \le \frac{C}{|B|}\int_B V(x)\,dx, \tag{1.5}$$

holds for every ball $B \subset \mathbb{R}^n$; see [28].

We introduce the definition of the reverse Hölder index of V as $q_0 = \sup\{q : V \in RH_q\}$. It is worth pointing out that the RH_q class is that, if $V \in RH_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (1.5), such that $V \in RH_{q+\varepsilon}$. Therefore, under the assumption $V \in RH_{\frac{n}{2}}$, we may conclude $q_0 > \frac{n}{2}$. Throughout this paper, we always assume that $0 \ne V \in RH_n$. In particular, Shen [28] has considered L_p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \cdots, n$.

Similar to the Marcinkiewicz integral operator with rough kernel μ_Ω, we define the Marcinkiewicz integral operator with rough kernel $\mu_{j,\Omega}^L$ associated with the Schrödinger operator L by

$$\mu_{j,\Omega}^L f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| \, K_j^L(x,y) f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)\,|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial_{x_j}} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when $V = 0$, $K_j^\Delta(x,y) = \widetilde{K_j^\Delta}(x,y)\,|x-y| = ((x_j - y_j)/|x-y|)/|x-y|^{n-1}$ and $\widetilde{K_j^\Delta}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial_{x_j}} \Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K_j^\Delta(x,y)$ and $\mu_{j,\Omega} = \mu_{j,\Omega}^\Delta$ and so, $\mu_{j,\Omega}^\Delta$ is defined by

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| \, K_j(x,y) f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Obviously, μ_j are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the properties of $\mu_{j,\Omega}^L$.

On the other hand, for $b \in L_1^{loc}(\mathbb{R}^n)$, denote by B the multiplication operator defined by $\mathrm{B}f(x) = b(x) f(x)$ for any measurable function f. If $\mu_{j,\Omega}^L$ is a linear operator on some measurable function space, then the commutator formed by B and $\mu_{j,\Omega}^L$ is defined by

$$\mu_{j,\Omega,b}^L f(x) = [b, \mu_{j,\Omega}^L] f(x) := \left(\mathrm{B}\mu_{j,\Omega}^L - \mu_{j,\Omega}^L \mathrm{B}\right) f(x) = b(x)\, \mu_{j,\Omega}^L f(x) - \mu_{j,\Omega}^L(bf)(x).$$

The commutators we are interested in here are of the form

$$\mu_{j,\Omega,b}^L f(x) = [b, \mu_{j,\Omega}^L] f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| \, K_j^L(x,y) \left[b(x) - b(y)\right] f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where Ω satisfies both (1.3) and (1.4). It is worth noting that for a constant C, if $\mu_{j,\Omega}^L$ is linear we have,

$$\begin{aligned} [b + C, \mu_{j,\Omega}^L] f &= (b + C)\, \mu_{j,\Omega}^L f - \mu_{j,\Omega}^L((b + C) f) \\ &= b\mu_{j,\Omega}^L f + C\mu_{j,\Omega}^L f - \mu_{j,\Omega}^L(bf) - C\mu_{j,\Omega}^L f \\ &= [b, \mu_{j,\Omega}^L] f. \end{aligned}$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in BMO$ (bounded mean oscillation space) or $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space) has had the most historical significance.

Gao and Tang [10] have shown that Marcinkiewicz integral μ_j^L is bounded on $L_p(\mathbb{R}^n)$, for $1 < p < \infty$, and are bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Later, Akbulut and Kuzu [2] have shown that the Marcinkiewicz integral operators with rough kernel $\mu_{j,\Omega}^L$, $j = 1, \dots, n$, associated with the Schrödinger operator L are bounded on $L_p(\mathbb{R}^n)$, for $1 < p < \infty$, and are bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ that we need. Their results can be formulated as follows.

Theorem 1.6. (see [2]) Let $1 < p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ satisfies (1.3) and $V \in RH_n$. Then, for every $q' < p < \infty$ or $1 < p < q$, there is a constant C independent of f such that

$$\left\|\mu_{j,\Omega}^L f\right\|_{L_p} \le C \left\|f\right\|_{L_p},$$

and for $p = 1$

$$\left\|\mu_{j,\Omega}^L f\right\|_{WL_1} \le C \left\|f\right\|_{L_1}.$$

On the other hand, there are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [3, 7, 12, 13, 15, 16, 17, 18, 20, 25]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [8, 9, 11, 24, 29]).

In this paper, we assume that b is in the local Campanato spaces $LC_{p,\lambda}^{\{x_0\}}$ and consider its boundedness properties on generalized local (central) Morrey space.

Let us recall the defination of the space of $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space).

Definition 1.7. (see [3, 12, 13]) Let $1 \le p < \infty$ and $0 \le \lambda < \frac{1}{n}$. A function $f \in L_p^{loc}(\mathbb{R}^n)$ is said to belong to the $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, if

$$\|f\|_{LC_{p,\lambda}^{\{x_0\}}} = \sup_{r>0} \left(\frac{1}{|B(x_0, r)|^{1+\lambda p}} \int_{B(x_0,r)} \left|f(y) - f_{B(x_0,r)}\right|^p dy \right)^{\frac{1}{p}} < \infty. \tag{1.6}$$

Define

$$LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \left\{ f \in L_p^{loc}(\mathbb{R}^n) : \|f\|_{LC_{p,\lambda}^{\{x_0\}}} < \infty \right\}.$$

Remark 1.8. If two functions which differ by a constant are regarded as a function in the space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, then $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ becomes a Banach space. The space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ when $\lambda = 0$ is just the $LC_p^{\{x_0\}}(\mathbb{R}^n)$. Apparently, (1.6) is equivalent to the following condition:

$$\sup_{r>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B(x_0, r)|^{1+\lambda p}} \int_{B(x_0,r)} |f(y) - c|^p dy \right)^{\frac{1}{p}} < \infty.$$

For local Campanato space's historical development and its backgrounds, see also [3, 12, 13, 14].

The following lemma plays a key role in the proof of following Lemma 2.2.

Lemma 1.9. (see [12, 13, 14]) Let b be function in $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{n}$ and r_1, $r_2 > 0$. Then

$$\left(\frac{1}{|B(x_0, r_1)|^{1+\lambda p}} \int\limits_{B(x_0,r_1)} \left| b(y) - b_{B(x_0,r_2)} \right|^p dy \right)^{\frac{1}{p}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{LC_{p,\lambda}^{\{x_0\}}}, \tag{1.7}$$

where $C > 0$ is independent of b, r_1 and r_2.

From this inequality (1.7), we have

$$\left| b_{B(x_0,r_1)} - b_{B(x_0,r_2)} \right| \le C \left(1 + \ln \frac{r_1}{r_2} \right) |B(x_0, r_1)|^{\lambda} \|b\|_{LC_{p,\lambda}^{\{x_0\}}}, \tag{1.8}$$

and it is easy to see that

$$\left\| b - (b)_{B(x_0,r)} \right\|_{L_p(B(x_0,r))} \le C r^{\frac{n}{p} + n\lambda} \|b\|_{LC_{p,\lambda}^{\{x_0\}}}. \tag{1.9}$$

Our main results can be formulated as follows.

Theorem 1.10. Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ satisfies (1.3) and $V \in RH_n$.

Let also, for $q' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies the condition

$$\int\limits_r^{\infty} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C \varphi_2(x_0, r), \tag{1.10}$$

and for $1 < p < q$ the pair (φ_1, φ_2) satisfies the condition

$$\int\limits_r^{\infty} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{q}+1}} dt \le C \, \varphi_2(x_0, r) r^{\frac{n}{q}},$$

where C does not depend on r.

Then the operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ are bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$ for $p = 1$. Moreover, for $p > 1$

$$\left\| \mu_{j,\Omega}^L f \right\|_{LM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p = 1$

$$\left\| \mu_{j,\Omega}^L f \right\|_{WLM_{1,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{1,\varphi_1}^{\{x_0\}}}.$$

Now using above Theorem 1.10, we get the boundedness of the operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ on the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$.

Theorem 1.11. Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.3) and $V \in RH_n$. Let also, for $q' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (1.1)-(1.2) and

$$c_\delta := \int_\delta^\infty \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt < \infty \tag{1.11}$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C_0 \varphi_2(x_0, r), \tag{1.12}$$

and for $1 < p < q$ the pair (φ_1, φ_2) satisfies conditions (1.1)-(1.2) and also

$$c_{\delta'} := \int_{\delta'}^\infty \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{q}+1}} dt < \infty$$

for every $\delta' > 0$, and

$$\int_r^\infty \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{q}+1}} dt \leq C_0\, \varphi_2(x_0, r) r^{\frac{n}{q}},$$

where C_0 does not depend on $r > 0$.

Then the operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ are bounded from $VLM_{p,\varphi_1}^{\{x_0\}}$ to $VLM_{p,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $VLM_{1,\varphi_1}^{\{x_0\}}$ to $WVLM_{1,\varphi_2}^{\{x_0\}}$ for $p = 1$. Moreover, we have for $p > 1$

$$\left\| \mu_{j,\Omega}^L f \right\|_{VLM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{VLM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p = 1$

$$\left\| \mu_{j,\Omega}^L f \right\|_{WVLM_{1,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{VLM_{1,\varphi_1}^{\{x_0\}}}.$$

Remark 1.12. Akbulut and Kuzu [2] have shown that the boundedness of the operator $\mu_{j,\Omega}^L$ from generalized Morrey spaces in itself by considering the following conditions (1.13) and (1.14) according to conditions (1.11) and (1.12)

$$c_\delta := \int_\delta^\infty \varphi_1(x, t) \frac{dt}{t^{\frac{n}{p}+1}} < \infty \tag{1.13}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi_1(x, t)}{t^{\frac{n}{p}+1}} dt \leq C_0 \frac{\varphi_2(x, r)}{r^{\frac{n}{p}}}, \tag{1.14}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. In fact, this difference stems from the definitions of spaces. Because, in this paper the definitions of generalized local (central) and generalized vanishing local Morrey spaces are given with the concept of normalized norm, but these definitions are not given in this [2]. In other words, if we use the definition of vanishing generalized Morrey space in [2], then we use conditions (1.13) and (1.14) instead of conditions (1.11) and (1.12). Also, we would like to remark that the main method employed in this paper is a combination of ideas and arguments from [2].

Theorem 1.13. Suppose that $x_0 \in \mathbb{R}^n$, $1<p<\infty$, $\Omega \in L_q(S^{n-1})$,$1<q\leq\infty$ satisfies (1.3) and $V \in RH_n$. Let $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $0\leq\lambda<\frac{1}{n}$.

Let also, for $q'\leq p$ the pair (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\,\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt \leq C\varphi_2(x_0,r), \tag{1.15}$$

and for $p_1<q$ the pair (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\,\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{q}+1-n\lambda}}dt \leq C\varphi_2(x_0,r)\,r^{\frac{n}{q}},$$

where C does not depend on r.
Then, the operators $\mu_{j,\Omega,b}^L$, $j=1,\ldots,n$ are bounded from $LM_{p_1,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$. Moreover,

$$\left\|\mu_{j,\Omega,b}^L f\right\|_{LM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}.$$

Now using above Theorem 1.13, in the following theorem we also obtain the boundedness of the operators $\mu_{j,\Omega,b}^L$, $j=1,\ldots,n$ on the vanishing generalized local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$.

Theorem 1.14. Suppose that $x_0 \in \mathbb{R}^n$, $1<p<\infty$, $\Omega \in L_q(S^{n-1})$,$1<q\leq\infty$ satisfies (1.3) and $V \in RH_n$. Let $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $0\leq\lambda<\frac{1}{n}$. Let also, for $q'\leq p$, the pair (φ_1,φ_2) satisfies conditions (1.1)-(1.2) and

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x_0,t)\frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt \leq C_0\varphi_2(x_0,r), \tag{1.16}$$

where C_0 does not depend on $r>0$,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\varphi_2(x_0,r)}=0 \tag{1.17}$$

and

$$c_\delta := \int_\delta^\infty (1+\ln|t|)\,\varphi_1(x_0,t)\,\frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt<\infty \tag{1.18}$$

for every $\delta>0$, and for $p_1<q$ the pair (φ_1,φ_2) satisfies conditions (1.1)-(1.2) and also

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x_0,t)\frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{q}+1-n\lambda}}dt \leq C_0\varphi_2(x_0,r)r^{\frac{n}{q}},$$

where C_0 does not depend on $r > 0$,

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\varphi_2(x_0, r)} = 0$$

and

$$c_{\delta'} := \int_{\delta'}^{\infty} (1 + \ln|t|)\, \varphi_1(x_0, t)\, \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{p_1} - \frac{n}{q} + 1 - n\lambda}} dt < \infty$$

for every $\delta' > 0$.

Then the operators $\mu^L_{j,\Omega,b}$, $j = 1, \ldots, n$ are bounded from $VLM^{\{x_0\}}_{p_1,\varphi_1}$ to $VLM^{\{x_0\}}_{p,\varphi_2}$. Moreover,

$$\left\|\mu^L_{j,\Omega,b} f\right\|_{VLM^{\{x_0\}}_{p,\varphi_2}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \|f\|_{VLM^{\{x_0\}}_{p_1,\varphi_1}}.$$

Now, we give following Lemma 1.15 that we use in this paper.

Lemma 1.15. (see [35] page 143) Let f be a real-valued nonnegative function and measurable on E. Then

$$\left(\operatorname*{essinf}_{x\in E} f(x)\right)^{-1} = \operatorname*{esssup}_{x\in E} \frac{1}{f(x)}. \tag{1.19}$$

2 Proof of theorems

To prove the theorems (Theorems 1.10, 1.11, 1.13 and 1.14), we need the following lemmas.

Lemma 2.1. (see [2]) Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ satisfies (1.3) and $V \in RH_n$.

If $p > 1$ and $q' \le p$, then the inequality

$$\left\|\mu^L_{j,\Omega} f\right\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))}\, dt \tag{2.1}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

If $p > 1$ and $p < q$, then the inequality

$$\left\|\mu^L_{j,\Omega} f\right\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} t^{\frac{n}{q}-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))}\, dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

Moreover, for $q > 1$ the inequality

$$\left\|\mu^L_{j,\Omega} f\right\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))}\, dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_1(\mathbb{R}^n)$.

We end this part by presenting the following lemma, which is the heart of the proofs of Theorems 1.13 and 1.14.

Lemma 2.2. Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.3) and $V \in RH_n$. Let also $b \in LC^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 \leq \lambda < \frac{1}{n}$.

Then, for $q' \leq p$, the inequality

$$\|\mu^L_{j,\Omega,b} f\|_{L_p(B(x_0,r))} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt \tag{2.2}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_{p_1}(\mathbb{R}^n)$.

Also, for $p_1 < q$, the inequality

$$\|\mu^L_{j,\Omega,b} f\|_{L_p(B(x_0,r))} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} + \frac{n}{q} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_{p_1}(\mathbb{R}^n)$.

Proof. Note that $t > 2r$ and $|x - x_0| < r$, we have $t + |x - x_0| < t + r < \frac{3}{2}t < 2t$. Moreover, for $x \in B(x_0, t)$, notice that $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.3). Then, we obtain

$$\begin{aligned}
\left(\int_{B(x_0,t)} |\Omega(x-y)|^q \, dy\right)^{\frac{1}{q}} &= \left(\int_{B(x-x_0,t)} |\Omega(z)|^q \, dz\right)^{\frac{1}{q}} \\
&\leq \left(\int_{B(0,t+|x-x_0|)} |\Omega(z)|^q \, dz\right)^{\frac{1}{q}} \\
&\leq \left(\int_{B(0,2t)} |\Omega(z)|^q \, dz\right)^{\frac{1}{q}} \\
&= \left(\int_0^{2t} \int_{S^{n-1}} |\Omega(z')|^q \, d\sigma(z') \, r^{n-1} dr\right)^{\frac{1}{q}} \\
&= C \|\Omega\|_{L_q(S^{n-1})} |B(x_0, 2t)|^{\frac{1}{q}}.
\end{aligned} \tag{2.3}$$

Let $1 < p < \infty$, $b \in LC^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{(2B)^C}(y), \qquad r > 0,$$

and have

$$\mu_{j,\Omega,b}^{L} f(x) = (b(x) - b_B)\, \mu_{j,\Omega}^{L} f_1(x) - \mu_{j,\Omega}^{L}\left((b(\cdot) - b_B) f_1\right)(x)$$
$$+ (b(x) - b_B)\, \mu_{j,\Omega}^{L} f_2(x) - \mu_{j,\Omega}^{L}\left((b(\cdot) - b_B) f_2\right)(x)$$
$$\equiv J_1 + J_2 + J_3 + J_4.$$

Hence we get

$$\left\| \mu_{j,\Omega,b}^{L} f \right\|_{L_p(B)} \le \|J_1\|_{L_p(B)} + \|J_2\|_{L_p(B)} + \|J_3\|_{L_p(B)} + \|J_4\|_{L_p(B)}.$$

By the Hölder's inequality, the boundedness of $\mu_{j,\Omega}^{L}$ on L_{p_1} (see Theorem 1.6) and (1.9) it follows that:

$$\begin{aligned}
\|J_1\|_{L_p(B)} &\le \left\| (b(\cdot) - b_B)\, \mu_{j,\Omega}^{L} f_1(\cdot) \right\|_{L_p(B)} \\
&\le \|(b(\cdot) - b_B)\|_{L_{p_2}(B)} \left\| \mu_{j,\Omega}^{L} f_1(\cdot) \right\|_{L_{p_1}(B)} \\
&\le C \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2} + n\lambda} \|f_1\|_{L_{p_1}(B)} \\
&= C \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2} + \frac{n}{p_1} + n\lambda} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{p_1}}\, dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt,
\end{aligned}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Using the boundedness of $\mu_{j,\Omega}^{L}$ on L_p (see Theorem 1.6), by the Hölder's inequality and (1.9) for J_2 we have

$$\begin{aligned}
\|J_2\|_{L_p(B)} &\le \left\| \mu_{j,\Omega}^{L} (b(\cdot) - b_B) f_1 \right\|_{L_p(B)} \\
&\lesssim \|(b(\cdot) - b_B) f_1\|_{L_p(B)} \\
&\lesssim \|b(\cdot) - b_B\|_{L_{p_2}(B)} \|f_1\|_{L_{p_1}(B)} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2} + \frac{n}{p_1} + n\lambda} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{p_1}}\, dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt,
\end{aligned}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

For J_3, it is known that $x \in B$, $y \in (2B)^C$, which implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$.

When $q' \leq p_1$, by the Fubini's theorem, the Hölder's inequality and (2.3) we have

$$\begin{aligned}
\left|\mu_{j,\Omega}^{L} f_2(x)\right| &\leq c_0 \int\limits_{(2B)^C} |\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n} dy \\
&\approx \int\limits_{2r}^{\infty} \int\limits_{2r<|x_0-y|<t} |\Omega(x-y)|\,|f(y)|\, dy t^{-1-n} dt \\
&\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |\Omega(x-y)|\,|f(y)|\, dy t^{-1-n} dt \\
&\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{q}} t^{-1-n} dt \\
&\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-\frac{n}{p_1}-1} dt. \qquad (2.4)
\end{aligned}$$

Hence, by the Hölder's inequality, (2.4) and (1.9) we get

$$\begin{aligned}
\|J_3\|_{L_p(B)} &= \left\|(b(\cdot)-b_B)\,\mu_{j,\Omega}^{L} f_2(\cdot)\right\|_{L_p(B)} \\
&= \|(b(\cdot)-b_B)\|_{L_{p_2}(B)} \left\|\mu_{j,\Omega}^{L} f_2(\cdot)\right\|_{L_{p_1}(B)} \\
&\leq \|(b(\cdot)-b_B)\|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{p_1}} \int\limits_{2r}^{\infty} t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}+n\lambda} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} dt,
\end{aligned}$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

Let $1<p<q$. Similarly to (2.3), when $y \in B(x_0,t)$, notice that

$$\left(\int\limits_{B(x_0,r)} |\Omega(x-y)|^q dy\right)^{\frac{1}{q}} \leq C \|\Omega\|_{L_q(S^{n-1})} \left|B\left(x_0,\frac{3}{2}t\right)\right|^{\frac{1}{q}}. \qquad (2.5)$$

When $p_1 < q$, by the Fubini's theorem, the Minkowski inequality and from (1.9), (2.5), we get

$$\|J_3\|_{L_p(B)} \leq \left(\int\limits_B \left| \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, |b(x) - b_B| \, |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|(b(\cdot) - b_B) \, \Omega(\cdot - y)\|_{L_p(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\leq \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|b(\cdot) - b_B\|_{L_{p_2}(B)} \, \|\Omega(\cdot - y)\|_{L_{p_1}(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+n\lambda} |B|^{\frac{1}{p_1}-\frac{1}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{q}+n\lambda} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{q}+n\lambda} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{p_1}-\frac{n}{q}+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}+\frac{n}{q}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt.$$

On the other hand, for J_4, when $q' \leq p$, for $x \in B$ by the Fubini's theorem, applying the Hölder's inequality and from (2.3), (1.8), (1.9) we have

$$\left|\mu^L_{j,\Omega}\left((b(\cdot) - b_B) f_2\right)(x)\right| \lesssim \int\limits_{(2B)^C} |b(y) - b_B| \, |\Omega(x-y)| \, \tfrac{|f(y)|}{|x-y|^n} dy$$

$$\lesssim \int\limits_{(2B)^C} |b(y) - b_B| \, |\Omega(x-y)| \, \tfrac{|f(y)|}{|x_0-y|^n} dy$$

$$\approx \int\limits_{2r}^{\infty} \int\limits_{2r<|x_0-y|<t} |b(y) - b_B| \, |\Omega(x-y)| \, |f(y)| \, dy \tfrac{dt}{t^{n+1}}$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} \left|b(y) - b_{B(x_0,t)}\right| |\Omega(x-y)| \, |f(y)| \, dy \tfrac{dt}{t^{n+1}}$$

$$+ \int\limits_{2r}^{\infty} \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \int\limits_{B(x_0,t)} |\Omega(x-y)| \, |f(y)| \, dy \tfrac{dt}{t^{n+1}}$$

$$\lesssim \int_{2r}^{\infty} \left\| \left(b(\cdot) - b_{B(x_0,t)} \right) f \right\|_{L_p(B(x_0,t))} \left\| \Omega(\cdot - y) \right\|_{L_q(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p}-\frac{1}{q}} \frac{dt}{t^{n+1}}$$
$$+\int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \|f\|_{L_{p_1}(B(x_0,t))} \left\| \Omega(\cdot - y) \right\|_{L_q(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p_1}-\frac{1}{q}} t^{-n-1} dt$$
$$\lesssim \int_{2r}^{\infty} \left\| \left(b(\cdot) - b_{B(x_0,t)} \right) \right\|_{L_{p_2}(B(x_0,t))} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt$$
$$+ \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \left(1 + \ln \tfrac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt$$
$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \left(1 + \ln \tfrac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt.$$

Then, we have

$$\|J_4\|_{L_p(B)} = \left\| \mu^L_{j,\Omega} \left(b(\cdot) - b_B \right) f_2 \right\|_{L_p(B)}$$
$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt.$$

When $p_1 < q$, by the Minkowski inequality, applying the Hölder's inequality and from (2.5), (1.8), (1.9) we have

$$\|J_4\|_{L_p(B)} \lesssim \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b(y) - b_{B(x_0,t)} \right| |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}$$
$$+ \left(\int_B \left| \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}$$
$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b(y) - b_{B(x_0,t)} \right| |f(y)| \left\| \Omega(\cdot - y) \right\|_{L_p(B(x_0,t))} dy \frac{dt}{t^{n+1}}$$
$$+ \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} |f(y)| \left\| \Omega(\cdot - y) \right\|_{L_p(B(x_0,t))} dy \frac{dt}{t^{n+1}}$$
$$\lesssim |B|^{\frac{1}{p}-\frac{1}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b(y) - b_{B(x_0,t)} \right| |f(y)| \left\| \Omega(\cdot - y) \right\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n+1}}$$
$$+ |B|^{\frac{1}{p}-\frac{1}{q}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} |f(y)| \left\| \Omega(\cdot - y) \right\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n+1}}$$
$$\lesssim r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} \left\| \left(b(\cdot) - b_{B(x_0,t)} \right) \right\|_{L_{p_2}(B(x_0,t))} \|f\|_{L_{p_1}(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p}} \left| B\left(x_0, \tfrac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$
$$+ r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \|f\|_{L_{p_1}(B(x_0,t))} \left| B\left(x_0, \tfrac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{\frac{n}{p_1}+1}}$$
$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \tfrac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} + \frac{n}{q} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt.$$

Now combined by all the above estimates, we end the proof of Lemma 2.2. Q.E.D.

The Proof of Theorem 1.10. Since $f \in LM^{\{x_0\}}_{p,\varphi_1}$, by (1.19) and the non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{essinf}\limits_{0<t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \leq \operatorname*{esssup}\limits_{0<t<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$
$$\leq \operatorname*{esssup}\limits_{0<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \leq \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.$$

For $q' \leq p < \infty$, since (φ_1, φ_2) satisfies (1.10), we have

$$\begin{aligned}
&\int_r^\infty \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{p}} \frac{dt}{t} \\
&\leq \int_r^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\leq C\,\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}} \int_r^\infty \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\leq C\,\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}\, \varphi_2(x_0,r).
\end{aligned}$$

Then by (2.1), we get

$$\begin{aligned}
\left\|\mu^L_{j,\Omega} f\right\|_{LM^{\{x_0\}}_{p,\varphi_2}} &= \sup_{r>0} \varphi_2\left(x_0,r\right)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \left\|\mu^L_{j,\Omega} f\right\|_{L_p(B(x_0,r))} \\
&\leq C \sup_{r>0} \varphi_2\left(x_0,r\right)^{-1} \int_r^\infty \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{p}} \frac{dt}{t} \\
&\leq C\,\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.
\end{aligned}$$

For the case of $1 \leq p < q$, we can also use the same method, so we omit the details. This completes the proof of Theorem 1.10.

The Proof of Theorem 1.11. The norm inequalities follow from Theorem 1.10. Thus we only have to prove that

$$\lim_{r\to 0} \mathfrak{M}_{p,\varphi_1}\left(f;x_0,r\right) = 0 \text{ implies } \lim_{r\to 0} \mathfrak{M}_{p,\varphi_2}\left(\mu^L_{j,\Omega} f;x_0,r\right) = 0 \tag{2.6}$$

and

$$\lim_{r\to 0} \mathfrak{M}_{p,\varphi_1}\left(f;x_0,r\right) = 0 \text{ implies } \lim_{r\to 0} \mathfrak{M}^W_{p,\varphi_2}\left(\mu^L_{j,\Omega} f;x_0,r\right) = 0. \tag{2.7}$$

To show that $\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} < \varepsilon$ for small r, we split the right-hand side of (2.1):

$$\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} \leq C\left[I_{\delta_0}(x_0,r) + J_{\delta_0}(x_0,r)\right],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x_0,r) := \frac{1}{\varphi_2(x_0,r)}\int_r^{\delta_0} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt,$$

and

$$J_{\delta_0}(x_0,r) := \frac{1}{\varphi_2(x_0,r)}\int_{\delta_0}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt,$$

and $r < \delta_0$ and the rest of the proof is the same as the proof of Theorem 6 in [2]. Thus, we can prove that (2.6).

The proof of (2.7) is similar to the proof of (2.6). For the case of $1 \leq p < q$, we can also use the same method, so we omit the details.

The Proof of Theorem 1.13. Since $f \in LM_{p_1,\varphi_1}^{\{x_0\}}$, by (1.19) and the non-decreasing, with respect to t, of the norm $\|f\|_{L_{p_1}(B(x_0,t))}$, we get

$$\begin{aligned}\frac{\|f\|_{L_{p_1}(B(x_0,t))}}{\operatorname*{essinf}\limits_{0<t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}} &\leq \operatorname*{esssup}_{0<t<\tau<\infty}\frac{\|f\|_{L_{p_1}(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}\\ &\leq \operatorname*{esssup}_{0<\tau<\infty}\frac{\|f\|_{L_{p_1}(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}} \leq \|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}.\end{aligned}$$

For $q' \leq p < \infty$, since (φ_1,φ_2) satisfies (1.15), we have

$$\begin{aligned}&\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}dt\\ &\leq \int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p_1}(B(x_0,t))}}{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt\\ &\leq C\|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt\\ &\leq C\|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}\varphi_2(x_0,r).\end{aligned}$$

Then by (2.2), we get

$$\begin{aligned}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{LM_{p,\varphi_2}^{\{x_0\}}} &= \sup_{r>0}\varphi_2\left(x_0,r\right)^{-1}\left|B(x_0,r)\right|^{-\frac{1}{p}}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{L_p(B(x_0,r))} \\ &\le C\left\|b\right\|_{LC_{p_2,\lambda}^{\{x_0\}}}\sup_{r>0}\varphi_2\left(x_0,r\right)^{-1}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}dt \\ &\le C\left\|b\right\|_{LC_{p_2,\lambda}^{\{x_0\}}}\left\|f\right\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}.\end{aligned}$$

For the case of $p_1 < q$, we can also use the same method, so we omit the details. This completes the proof of Theorem 1.13.

Remark 2.3. We point out that some ideas in the proofs of Theorems 1.10 and 1.13 are taken from [3, 12, 13, 15, 16]. However, the reader can find that the main techniques and non-trivial estimates used in the proofs of our conclusions are quite different from [3, 12, 13, 15, 16]. For example, using inequality about the weighted Hardy operator H_w in [3, 12, 13, 15, 16], in this paper we only used above a relationship between essential supremum and essential infimum (see Lemma 1.15).

The Proof of Theorem 1.14. The norm inequality having already been provided by Theorem 1.13, we only have to prove the implication

$$\lim_{r\to 0}\frac{r^{-\frac{n}{p_1}}\|f\|_{L_{p_1}(B(x_0,r))}}{\varphi_1(x_0,r)} = 0 \text{ implies } \lim_{r\to 0}\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} = 0. \tag{2.8}$$

To show that

$$\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} < \varepsilon \text{ for small } r,$$

we use the estimate (2.2):

$$\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} \lesssim \frac{\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}}{\varphi_2(x_0,r)}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}dt.$$

We take $r < \delta_0$, where δ_0 will be chosen small enough and split the integration:

$$\frac{r^{-\frac{n}{p}}\left\|\mu_{j,\Omega,b}^{L}f\right\|_{L_p(B(x_0,r))}}{\varphi_2(x_0,r)} \le C\left[I_{\delta_0}\left(x_0,r\right)+J_{\delta_0}\left(x_0,r\right)\right], \tag{2.9}$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}\left(x_0,r\right) := \frac{1}{\varphi_2(x_0,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\left\|f\right\|_{L_{p_1}(B(x_0,t))}dt,$$

and

$$J_{\delta_0}(x_0,r) := \frac{1}{\varphi_2(x_0,r)} \int_{\delta_0}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt$$

and $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\frac{t^{-\frac{n}{p_1}} \|f\|_{L_{p_1}(B(x_0,t))}}{\varphi_1(x_0,t)} < \frac{\varepsilon}{2CC_0}, \qquad t\le\delta_0,$$

where C and C_0 are constants from (1.16) and (2.9). This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$CI_{\delta_0}(x_0,r) < \frac{\varepsilon}{2}, \qquad 0<r<\delta_0.$$

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x_0,r) \le \frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi_2(x_0,r)} \|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}},$$

where c_{δ_0} is the constant from (1.18) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.17) we can choose small enough r such that

$$J_{\delta_0}(x_0,r) < \frac{\varepsilon}{2},$$

which completes the proof of (2.8).

For the case of $p_1<q$, we can also use the same method, so we omit the details.

Now, we give the applications of Theorem 1.10, Theorem 1.11, Theorem 1.13 and Theorem 1.14 for the parametric Marcinkiewicz integral operator.

For $0<\rho<n$, in 1960, Hörmander [19] defined the parametric Marcinkiewicz integral operator of higher dimension as

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(x)|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2},$$

where

$$F_{\Omega,t}^\rho(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\, dy.$$

Let b be a locally integrable function, the commutator generated by parametric Marcinkiewicz integral operator μ_Ω^ρ and b is defined by

$$[b,\mu_\Omega^\rho](f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} [b(x)-b(y)] f(y)\, dy \right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \qquad 0<\rho<n.$$

It is well known that the operator $\mu_\Omega^1 \equiv \mu_\Omega$ was first introduced by Stein in [30]. He proved that if Ω satisfies above condition (c), then μ_Ω is the operator of strong type (p,p) for $1<p\le 2$ and

of weak type $(1,1)$. On the other hand, in 1960, Hörmander [19] proved that if Ω satisfies above condition (c), then for $0<\rho<n$, μ_Ω^ρ is of strong type (p,p) for all $1<p<\infty$. His conclusion can be summarized as follows.

Theorem 2.4. Let $0<\rho<n$ and $f\in L_p(\mathbb{R}^n)$. If Ω satisfies above conditions (a)-(c), then μ_Ω^ρ is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. Moreover, there exists a constant $C>0$ independent of f such that

$$\|\mu_\Omega^\rho f\|_{L_p} \le C\|f\|_{L_p}.$$

Lemma 2.5. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1\le p<\infty$ and Ω satisfies above conditions (a)-(c). Then, for $1<p<\infty$ the inequality

$$\|\mu_\Omega^\rho f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt$$

holds for any ball $B(x_0,r)$ and for all $f\in L_p^{loc}(\mathbb{R}^n)$.

Moreover, for $p=1$ the inequality

$$\|\mu_\Omega^\rho f\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1}\|f\|_{L_1(B(x_0,t))}\,dt$$

holds for any ball $B(x_0,r)$ and for all $f\in L_1^{loc}(\mathbb{R}^n)$.

Proof. The proof of Lemma 2.5 is obtained in the same manner in the proof of Lemma 2.1, directly.

Q.E.D.

Lemma 2.6. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1<p<\infty$ and Ω satisfies above conditions (a)-(c). Let also $b\in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $0\le\lambda<\frac{1}{n}$.

Then, the inequality

$$\|[b,\mu_\Omega^\rho]f\|_{L_p(B(x_0,r))} \lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}dt$$

holds for any ball $B(x_0,r)$ and for all $f\in L_{p_1}^{loc}(\mathbb{R}^n)$.

Proof. The proof of Lemma 2.6 is obtained in the same manner in the proof of Lemma 2.2, directly.

Q.E.D.

Theorem 2.7. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1\le p<\infty$ and Ω satisfies above conditions (a)-(c). Let also, the pair (φ_1,φ_2) satisfies condition (1.10). Then the operator μ_Ω^ρ is $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$ for $p>1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$ for $p=1$. Moreover, for $p>1$

$$\|\mu_\Omega^\rho f\|_{LM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p=1$

$$\|\mu_\Omega^\rho f\|_{WLM_{1,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{1,\varphi_1}^{\{x_0\}}}.$$

Proof. The statement of Theorem 2.7 follows by Lemmas 2.5 and 1.15 in the same manner as in the proof of Theorem 1.10. Q.E.D.

Theorem 2.8. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1<p<\infty$ and Ω satisfies above conditions (a)-(c). Let also $b\in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $0\le\lambda<\frac{1}{n}$. Let also, the pair (φ_1,φ_2) satisfies condition (1.15). Then the operator $[b,\mu_\Omega^\rho]$ is bounded from $LM_{p_1,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$. Moreover,

$$\|[b,\mu_\Omega^\rho]f\|_{LM_{p,\varphi_2}^{\{x_0\}}}\lesssim\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}.$$

Proof. The statement of Theorem 2.8 follows by Lemmas 2.6 and 1.15 in the same manner as in the proof of Theorem 1.13. Q.E.D.

Theorem 2.9. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1\le p<\infty$ and Ω satisfies above conditions (a)-(c). Let also, the pair (φ_1,φ_2) satisfies conditions (1.1)-(1.2) and (1.11)-(1.12). Then the operator μ_Ω^ρ is bounded from $VLM_{p,\varphi_1}^{\{x_0\}}$ to $VLM_{p,\varphi_2}^{\{x_0\}}$ for $p>1$ and from $VLM_{1,\varphi_1}^{\{x_0\}}$ to $WVLM_{1,\varphi_2}^{\{x_0\}}$ for $p=1$. Moreover, we have for $p>1$

$$\|\mu_\Omega^\rho f\|_{VLM_{p,\varphi_2}^{\{x_0\}}}\lesssim\|f\|_{VLM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p=1$

$$\|\mu_\Omega^\rho f\|_{WVLM_{1,\varphi_2}^{\{x_0\}}}\lesssim\|f\|_{VLM_{1,\varphi_1}^{\{x_0\}}}.$$

Proof. The proof of Theorem 2.9 is obtained in the same manner in the proof of Theorem 1.11, directly. Q.E.D.

Theorem 2.10. Let $0<\rho<n$, $x_0\in\mathbb{R}^n$, $1<p<\infty$ and Ω satisfies above conditions (a)-(c). Let also $b\in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $0\le\lambda<\frac{1}{n}$. Let also, the pair (φ_1,φ_2) satisfies conditions (1.1)-(1.2) and (1.16)-(1.17)-(1.18). Then the operator $[b,\mu_\Omega^\rho]$ is bounded from $VLM_{p_1,\varphi_1}^{\{x_0\}}$ to $VLM_{p,\varphi_2}^{\{x_0\}}$. Moreover,

$$\|[b,\mu_\Omega^\rho]f\|_{VLM_{p,\varphi_2}^{\{x_0\}}}\lesssim\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}}.$$

Proof. The proof of Theorem 2.10 is obtained in the same manner in the proof of Theorem 1.14, directly. Q.E.D.

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