Logarithmic-Sheffer polynomials of the second kind

Paolo Emilio Ricci

International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39, 00186 - Roma, Italia
E-mail: paoloemilioricci@gmail.com

Abstract

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them have been studied. In this article new sets of logarithmic-Sheffer polynomials are introduced. Connection with Bell numbers are shown.

2010 Mathematics Subject Classification. **33C99**. 05A10, 11P81
Keywords. Sheffer polynomials, Generating functions, Monomiality principle, Shift operators, Combinatorial analysis.

1 Introduction

In recent articles [9, 21], new sets of Sheffer [23] and Brenke [8] polynomials, based on higher order Bell numbers [5, 17, 19, 20, 14, 21], have been studied. Furthermore, several integer sequences associated [24] with the considered polynomials sets both of exponential [1, 2] and logarithmic type have been introduced [9].
It is worth to note that exponential and logarithmic polynomials have been recently studied in the multidimensional case [14, 15, 16].
In this article new sets of logarithmic-Sheffer polynomials are introduced.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [23] by means of the exponential generating function [25] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\ , \tag{2.1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\ , \qquad (a_0 \neq 0)\,, \\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\ , \qquad (h_0 = 0)\,. \end{aligned} \tag{2.2}$$

According to a different characterization (see [22, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a

Tbilisi Mathematical Journal 11(3) (2018), pp. 95–106.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 02 July 2018.
Accepted for publication: 18 July 2018.

delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} , \qquad (g_0 \neq 0) ,$$
$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} , \qquad (f_0 = 0,\ f_1 \neq 0) . \tag{2.3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} , \tag{2.4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]} , \qquad H(t) = f^{-1}(t) . \tag{2.5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!} . \tag{2.6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [7].

3 New Logarithmic-Sheffer polynomial sets

We introduce, for shortness, the following compact notation.
Put, by definition:

$E_0(t) := \exp(t) - 1$
$E_1(t) := E_0(E_0(t)) = \exp(\exp(t) - 1) - 1$
. . . .
$E_r(t) := E_0(E_{r-1}(t)) = \exp(\ldots \exp(\exp(t) - 1) - 1) \cdots - 1 , \qquad [(r+1) -\text{ times } \exp] ,$

and in a similar way:

$\Lambda_0(t) := \log(t+1)$
$\Lambda_1(t) := \Lambda_0(\Lambda_0(t)) = \log(\log(t+1) + 1)$
. . . .
$\Lambda_r(t) := \Lambda_0(\Lambda_{r-1}(t)) = \log\left(\log\left(\ldots(\log(t+1) + 1)\ldots\right) + 1\right) , \qquad [(r+1) -\text{ times } \log] .$

Remark 3.1. Note that, for every integers r, k, h,

$$E_r(\Lambda_r(t)) = t\,, \qquad \Lambda_r(E_r(t)) = t\,,$$

$$(\mathit{if}\ \ k > h) \qquad E_k(\Lambda_h(t)) = E_{k-h-1}(t)\,, \quad E_h(\Lambda_k(t)) = \Lambda_{k-h-1}(t)\,,$$

$$(\mathit{if}\ \ k > h) \qquad \Lambda_k(E_h(t)) = \Lambda_{k-h-1}(t)\,, \quad \Lambda_h(E_k(t)) = E_{k-h-1}(t)\,,$$

$$e^{E_r(t)} = E_{r+1}(t) + 1\,, \qquad e^{\Lambda_r(t)} = \Lambda_{r-1}(t) + 1\,.$$

Remark 3.2. Note that the coefficients of the Taylor expansion of $H_1(t)$ are given by the Bell numbers $b_n = b_n^{[1]}$

$$E_1(t) = \sum_{n=1}^{\infty} b_n^{[1]}\,\frac{t^n}{n!}\,,$$

and, in general the coefficients of the Taylor expansion of $H_r(t)$ are given by the higher order Bell numbers $b_n^{[r]}$

$$E_r(t) = \sum_{n=1}^{\infty} b_n^{[r]}\,\frac{t^n}{n!}\,.$$

The higher order Bell numbers, also known as higher order exponential numbers, have been considered in *[17, 18, 19]*, and used in *[21]* in the framework of Brenke and Sheffer polynomials.

Remark 3.3. Note that the coefficients of the Taylor expansion of $\Lambda_0(t)$ are given by the logarithmic numbers $l_n^{[1]} = (-1)^{n-1}(n-1)!$

$$\Lambda_0(t) = \sum_{n=1}^{\infty} l_n^{[1]}\,\frac{t^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\,\frac{t^n}{n!}\,,$$

and, in general the coefficients of the Taylor expansion of $\Lambda_{r-1}(t)$ are given by the higher order logarithmic numbers $l_n^{[r]}$

$$\Lambda_{r-1}(t) = \sum_{n=1}^{\infty} l_n^{[r]}\,\frac{t^n}{n!}\,.$$

The higher order logarithmic numbers, which are the counterpart of the higher order Bell (exponential) numbers, have been considered in *[9]*, and used there in the framework of new sets of Sheffer polynomials.

3.1 The polynomials $\mathcal{M}_k^{(0)}(x)$

Therefore, we consider the Sheffer polynomials, defined through their generating function, by putting

$$A(t) = e^t \,, \qquad H(t) = \Lambda_0(t) \,,$$

$$G(t,x) = \exp\left[t + x\,\Lambda_0(x)\right] = e^t\,(t+1)^x = \sum_{k=0}^{\infty} \mathcal{M}_k^{(0)}(x)\,\frac{t^k}{k!} \,, \tag{3.1}$$

3.2 A differential identity

Theorem 3.4. - For any $k \geq 0$, the polynomials $\mathcal{M}_k^{(0)}(x)$ satisfy the differential identity:

$$[\mathcal{M}_k^{(0)}(x)]' = \sum_{h=0}^{k-1} \frac{(-1)^h\,k!}{(h+1)\,(k-h-1)!}\;\mathcal{M}_{k-h-1}^{(0)}(x) \;. \tag{3.2}$$

Proof. - Differentiating $G(t,x)$ with respect to x, we have

$$\frac{\partial G(t,x)}{\partial x} = G(t,x)\,\log(t+1)\,, \tag{3.3}$$

i.e.

$$\sum_{k=1}^{\infty} [\mathcal{M}_k^{(0)}(x)]'\,\frac{t^k}{k!} = \sum_{k=0}^{\infty} \mathcal{M}_k^{(0)}(x)\,\frac{t^k}{k!}\,\sum_{k=0}^{\infty}(-1)^k\,\frac{t^{k+1}}{k+1} = \sum_{k=0}^{\infty}(-1)^{k+1}\,\frac{t^{k+1}}{(k+1)} =$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{k}(-1)^h \mathcal{M}_{k-h}^{(0)}(x)\,\frac{t^{k+1}}{(h+1)\,(k-h)!}$$

and therefore:

$$\sum_{k=1}^{\infty} [\mathcal{M}_k^{(0)}(x)]'\,\frac{t^k}{k!} = \sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h \mathcal{M}_{k-h-1}^{(0)}(x)\,\frac{k!}{(h+1)\,(k-h-1)!}\,\frac{t^k}{k!}\,,$$

so that our result follows by shifting the indexes in the last equation.

3.3 Recurrence relation for the $\mathcal{M}_k^{(0)}(x)$

Theorem 3.5. - For any $k \geq 0$, the polynomials $\mathcal{M}_k^{(0)}(x)$ satisfy the following recurrence relation:

$$\mathcal{M}_{k+1}^{(0)}(x) = \mathcal{M}_k^{(0)}(x) + x\sum_{h=0}^{k} \frac{(-1)^{k-h}\,k!}{h!}\,\mathcal{M}_h^{(0)}(x)\,. \tag{3.4}$$

Proof. - Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t} = G(t,x)\,\left[1+\frac{x}{t+1}\right], \tag{3.5}$$

and therefore

$$\sum_{k=0}^{\infty}\mathcal{M}_{k+1}^{(0)}(x)\,\frac{t^k}{k!} = \sum_{k=0}^{\infty}\mathcal{M}_{k}^{(0)}(x)\,\frac{t^k}{k!} + x\,\sum_{k=0}^{\infty}\mathcal{M}_{k}^{(0)}(x)\,\frac{t^k}{k!}\,\sum_{k=0}^{\infty}(-1)^k\,k!\,\frac{t^k}{k!}\,,$$

i.e.

$$\sum_{k=0}^{\infty}\mathcal{M}_{k+1}^{(0)}(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\mathcal{M}_{k}^{(0)}(x)\,\frac{t^k}{k!} + x\,\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}\mathcal{M}_{h}^{(0)}(x)(-1)^{k-h}\,(k-h)!\,,$$

so that the recurrence relation (3.4) follows.

3.4 Generating function's PDE

Theorem 3.6. The generating function *(3.1)*$_2$ satisfies the homogeneous linear PDE

$$(t+1+x)\,\frac{\partial G(t,x)}{\partial t} = (t+1)\,\log(t+1)\,\frac{\partial G(t,x)}{\partial x}\,. \tag{3.6}$$

Proof. - By taking the ratio between the members of equations (3.3) and (3.5), we find equation (3.6).

3.5 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\,(p_n(x)) = np_{n-1}(x)\,, \qquad \hat{M}\,(p_n(x)) = p_{n+1}(x)\,, \qquad (n=1,2,\dots). \tag{3.7}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.
This definition traces back to a paper by J.F. Steffensen [26], recently improved by G. Dattoli [11] and widely used in several applications.
Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 3.7. Let $(p_n(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty}p_n(x)\,\frac{t^n}{n!}\,, \tag{3.8}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \ , \qquad (a_0 \neq 0)\,,$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n \ , \qquad (\gamma_n \neq 0 \quad \forall n)\,, \tag{3.9}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} h_n\, t^{n+1} \ , \qquad (h_0 \neq 0)\,. \tag{3.10}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \qquad \sigma(x^n) = \frac{\gamma_{n-1}}{\gamma_n}\; x^{n-1}\,, \qquad (n = 1, 2, \dots). \tag{3.11}$$

Put

$$\sigma^{-1}(x^n) = \frac{\gamma_{n+1}}{\gamma_n}\; x^{n+1} \quad (n = 0, 1, 2, \dots). \tag{3.12}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma)\,, \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + x D_x H'[f(\sigma)]\sigma^{-1}\,, \tag{3.13}$$

where *prime* denotes the ordinary derivatives with respect to t.

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (3.10) simply reduces to the derivative operator D_x. Furthermore, we have:

$$f(t) = H^{-1}(t) = e^t - 1 = E_0(t)\,,$$

$$\frac{A'(t)}{A(t)} = 1\,, \qquad H'(t) = \frac{1}{t+1}\,,$$

and consequently

$$f(\sigma) = E_0(D_x)\,,$$

$$H'[f(\sigma)] = H'[E_0(D_x)] = \frac{1}{E_0(D_x) + 1} = e^{-D_x}\,.$$

Theorem 3.8. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(0)}(x)\}$ are quasi-monomial under the action of the operators

$$\begin{aligned} \hat{P} &= f(D_x) = E_0(D_x) = \sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!}\,, \\ \hat{M} &= 1 + x\, e^{-D_x} = 1 + x \sum_{k=0}^{\infty} \frac{(-1)^k\, D_x^k}{k!}\,. \end{aligned} \tag{3.14}$$

3.6 Differential equation for the $\mathcal{M}_k^{(0)}(x)$

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x)\,. \tag{3.15}$$

In the present case, recalling equations (3.15), we have

Theorem 3.9. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(0)}(x)\}$ satisfy the differential equation

$$\sum_{k=0}^{n-1} \frac{D_x^{k+1}}{(k+1)!} + x \sum_{k=0}^{n-1} \sum_{h=0}^{k} (-1)^{k-h} \frac{D_x^{k+1}}{(h+1)!\,(k-h)!}\; \mathcal{M}_n^{(0)}(x) = n\, \mathcal{M}_n^{(0)}(x)\,. \tag{3.16}$$

Proof - Equation (3.15), by using equations (3.14), becomes

$$\sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!} + x \sum_{k=0}^{\infty} \sum_{h=0}^{k} (-1)^{k-h} \frac{D_x^{k+1}}{(h+1)!\,(k-h)!}\; \mathcal{M}_n^{(0)}(x) = n\, \mathcal{M}_n^{(0)}(x)$$

and furthermore, for any fixed n, the last series expansion reduces to a finite sum, with upper limit $n-1$, when it is applied to a polynomial of degree n.

Remark 3.10. – Here we show the first few values of the Bell-Sheffer polynomials $\mathcal{M}_k^{(0)}(x)$, defined by the generating function $(3.1)_2$

$$\begin{aligned} &\mathcal{M}_0^{(0)}(x) = 1, \\ &\mathcal{M}_1^{(0)}(x) = x + 1, \\ &\mathcal{M}_2^{(0)}(x) = x^2 + x + 1, \\ &\mathcal{M}_3^{(0)}(x) = x^3 + 2x + 1, \\ &\mathcal{M}_4^{(0)}(x) = x^4 - 2x^3 + 5x^2 + 1, \\ &\mathcal{M}_5^{(0)}(x) = x^5 - 5x^4 + 15x^3 - 15x^2 + 9x + 1. \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

4 Iterated Logarithmic-Sheffer polynomial sets

Here we iterate the procedure introduced in Sect. 3, by considering the Sheffer polynomial sets defined by putting

$$\begin{aligned} &A(t) = e^t\,, \qquad H(t) = \Lambda_1(t)\,, \\ &G(t,x) = \exp\left[t + x\,\Lambda_1(x)\right] = \sum_{k=0}^{\infty} \mathcal{M}_k^{(1)}(x)\,\frac{t^k}{k!}\;. \end{aligned} \tag{4.1}$$

We find:

$$f(t) = H^{-1}(t) = E_1(t)\,, \qquad \frac{A'(t)}{A(t)} = 1\,,$$

$$H'(t) = \frac{1}{\left[\Lambda_0(t)+1\right](t+1)}\,,$$

and consequently

$$f(\sigma) = E_1(D_x)\,,$$

$$H'[f(\sigma)] = H'[E_1(D_x)] = \frac{1}{\left[\Lambda_0(E_1(D_x))+1\right]\left[E_1(D_x)+1\right]} =$$

$$= \frac{1}{\left[E_0(D_x)+1\right]\left[E_1(D_x)+1\right]} = \frac{e^{-D_x}}{E_1(D_x)+1}\,.$$

Theorem 4.1. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(1)}(x)\}$ are quasi-monomial under the action of the operators

$$\begin{aligned} &\hat{P} = E_1(D_x) = \sum_{k=0}^{\infty} b_k^{[1]}\,\frac{D_x^{k+1}}{(k+1)!}\,, \\ &\hat{M} = 1 + x\,\frac{e^{-D_x}}{E_1(D_x)+1}\,. \end{aligned} \tag{4.2}$$

4.1 Differential equation for the $\mathcal{M}_k^{(1)}(x)$

According to the results of monomiality principle [11, 12], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{4.3}$$

In the present case, recalling equations (3.13), we have

Theorem 4.2. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(1)}(x)\}$ satisfy the differential equation

$$\left[1 + x\,\frac{e^{-D_x}}{E_1(D_x)+1}\right] E_1(D_x)\,\mathcal{M}_n^{(1)}(x) = n\,\mathcal{M}_n^{(1)}(x)\,. \tag{4.4}$$

Remark 4.3. – Here we show the first few values of the Bell-Sheffer polynomials $\mathcal{M}_k^{(1)}(x)$, defined by the generating function $(4.1)_2$

$$\begin{aligned}
&\mathcal{M}_0^{(1)}(x) = 1,\\
&\mathcal{M}_1^{(1)}(x) = x+1,\\
&\mathcal{M}_2^{(1)}(x) = x^2+1,\\
&\mathcal{M}_3^{(1)}(x) = x^3-3x^2+4x+1,\\
&\mathcal{M}_4^{(1)}(x)) = x^4-8x^3+22x^2-15x+1,\\
&\mathcal{M}_5^{(1)}(x) = x^5-15x^4+80x^3-165x^2+108x+1.
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

5 The general case

In general, by putting

$$\begin{aligned}
&A(t) = e^t, \qquad H(t) = \Lambda_r(t)\,,\\
&G(t,x) = \exp\left[t + x\,\Lambda_r(t)\right] = \sum_{k=0}^{\infty} \mathcal{M}_k^{(r)}(x)\,\frac{t^k}{k!}\ ,
\end{aligned} \tag{5.1}$$

we find:

$$f(t) = H^{-1}(t) = E_r(t)\,,$$

$$\frac{A'(t)}{A(t)} = 1\,, \qquad H'(t) = \left[\prod_{\ell=0}^{r-1}[\Lambda_\ell(t)+1]\,(t+1)\right]^{-1}\ ,$$

and consequently

$$f(\sigma) = E_r(D_x)\,,$$

$$H'[f(\sigma)] = H'[E_r(D_x)] = \left[\prod_{\ell=0}^{r-1}\left[\Lambda_\ell(E_r(D_x))+1\right][E_r(D_x)+1]\right]^{-1}\ .$$

Recalling Remark 3.1, we find

$$\Lambda_\ell(E_r(D_x)) = E_{r-\ell-1}(D_x)\,,$$

$$\prod_{\ell=0}^{r-1}\left[E_{r-\ell-1}(D_x)+1\right][E_r(D_x)+1] = \prod_{\ell=0}^{r}[E_\ell(D_x)+1]\,,$$

so that we have the theorem:

Theorem 5.1. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(r)}(x)\}$ are quasi-monomial under the action of the operators

$$
\begin{aligned}
\hat{P} &= E_r(D_x)\,, \\
\hat{M} &= 1 + x\left[\prod_{\ell=0}^{r}[E_\ell(D_x)+1]\right]^{-1}.
\end{aligned}
\tag{5.2}
$$

5.1 Differential equation for the $\mathcal{M}_k^{(r)}(x)$

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$
\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{5.3}
$$

In the present case, recalling equations (5.2), we have

Theorem 5.2. The Bell-Sheffer polynomials $\{\mathcal{M}_k^{(r)}(x)\}$ satisfy the differential equation

$$
\left\{1 + x\left[\prod_{\ell=0}^{r}[E_\ell(D_x)+1]\right]^{-1}\right\} E_r(D_x)\ \mathcal{M}_n^{(r)}(x) = n\,\mathcal{M}_n^{(r)}(x)\,. \tag{5.4}
$$

References

[1] E.T. Bell, Exponential polynomials, *Annals of Mathematics*, **35** (1934), 258–277.

[2] E.T. Bell, The Iterated Exponential Integers, *Annals of Mathematics*, **39** (1938), 539–557.

[3] Y. Ben Cheikh, Some results on quasi-monomiality, *Appl. Math. Comput.*, **141** (2003), 63–76.

[4] A. Bernardini, P.E. Ricci, Bell polynomials and differential equations of Freud-type polynomials, *Math. Comput. Modelling*, **36** (2002), 1115–1119.

[5] A. Bernardini, P. Natalini, P.E. Ricci, Multi-dimensional Bell polynomials of higher order, *Comput. Math. Appl.*, **50** (2005), 1697–1708.

[6] R.P. Boas, R.C. Buck, Polynomials defined by generating relations, *Amer. Math. Monthly*, **63** (1958), 626–632.

[7] R.P. Boas, R.C. Buck, *Polynomial Expansions of Analytic Functions*, Springer-Verlag, Berlin, Gottingen, Heidelberg, New York, 1958.

[8] W.C. Brenke, On generating functions of polynomial systems, *Amer. Math. Monthly*, **52** (1945), 297–301.

[9] G. Bretti, P. Natalini, P.E. Ricci, A new set of Sheffer-Bell polynomials and logarithmic numbers, *Georgian Math. J.*, 2018, (to appear).

[10] G. Bretti, P. Natalini, P.E. Ricci, New sets of Euler-type polynomials, *J. Anal. Number Theory*, **52**, No. 2 (2018), 51–54.

[11] G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, in *Advanced Special Functions and Applications* (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, 9-12 May, 1999) (D. Cocolicchio, G. Dattoli and H.M. Srivastava, Editors), Aracne Editrice, Rome, 2000, pp. 147–164.

[12] G. Dattoli, P.E. Ricci, H.M. Srivastava, Editors), Advanced Special Functions and Related Topics in Probability and in Differential Equations, (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, June 24-29, 2001), in: *Applied Mathematics and Computation*, **141**, (No. 1) (2003), 1–230.

[13] G. Dattoli, B. Germano, M.R. Martinelli and P.E. Ricci, Monomiality and partial differential equations, *Math. Comput. Modelling*, **50** (2009), 1332–1337.

[14] Feng Qi, Da-Wei Niu, Dongkyu Lim, and Bai-Ni Guo, Some properties and an application of multivariate exponential polynomials, *HAL archives* (2018), available online at https://hal.archives-ouvertes.fr/hal-01745173.

[15] Feng Qi, Integral representations for multivariate logarithmic potentials, *J. Comput. Appl. Math.* **336** (2018), 54–62; available on line at https://doi.org/10.1016/j.cam.2017.11.047.

[16] Feng Qi, On multivariate logarithmic polynomials and their properties, *Indag. Math.* **336** (2018), (in press); available on line at https://doi.org/10.1016/j.indag.2018.04.002.

[17] P. Natalini, P.E. Ricci, An extension of the Bell polynomials, *Comput. Math. Appl.*, **47** (2004), 719–725.

[18] P. Natalini, P.E. Ricci, Higher order Bell polynomials and the relevant integer sequences, *Appl. Anal. Discrete Math.*, **11** (2017), 327–339.

[19] P. Natalini, P.E. Ricci, Remarks on Bell and higher order Bell polynomials and numbers, *Cogent Mathematics*, **3** (2016), 1–15.

[20] P. Natalini, P.E. Ricci, New Bell-Sheffer polynomial sets, *Axioms*, 2018, (submitted).

[21] P.E. Ricci, P. Natalini, G. Bretti, Sheffer and Brenke polynomials associated with generalized Bell numbers, *Jnanabha, Vijnana Parishad of India*, **47** (No2-2017), 337–352.

[22] S.M. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[23] I.M. Sheffer, Some properties of polynomials sets of zero type, *Duke Math. J.*, **5** (1939), 590–622.

[24] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences.* Published electronically at http://oeis.org, 2016.

[25] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[26] J.F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.*, **73** (1941), 333–366.