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Distribution of the particles of a critical branching Wiener process

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Consider a critical branching Wiener process in $\mathbb{R}^d$. Let $\{F_T(x), T=1,2,\ldots, x\in\mathbb{R}^d\}$ be the distribution of the particles living at time T. The main result of this paper tells us that any given absolutely continuous function $F(x)$ will be well approximated by $F_T(x)$ with positive probability if T is big enough and the process does not die out up to T.

Keywords: critical branching Wiener process, empirical distribution of particles, limit theorems, measure-valued process

1. Introduction

Consider the following model:

(i) A particle starts from position $\mathbf{0} \in \mathbb{R}^d$ and executes a Wiener process $W(t) \in \mathbb{R}^d$.
(ii) Arriving at time $t = 1$ at the new location $W(1)$, it dies.
(iii) At death it is replaced by Y off-springs where

$$\mathbf{P}\{Y = \ell\} = p_\ell \qquad (\ell = 0, 1, 2, \ldots)$$

and

$$p_\ell \geq 0, \qquad \sum_{\ell=0}^{\infty} p_\ell = 1.$$

(iv) Each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps and so on. All Wiener processes and offspring numbers are assumed independent of one another.

A more formal definition is given in Révész (1994, p. 91).

Let $A \subset \mathbb{R}^d$ be a Borel set and let $\lambda(A, t)$ $(t = 0, 1, 2, \ldots)$ be the number of particles located in A at time t. Then

$$B(t) = \lambda(\mathbb{R}^d, t)$$

is the number of particles living at t and $\{B(t), t = 0, 1, 2, \ldots\}$ is a branching process. From now on we assume that

$$1 \leq m = \sum_{k=0}^{\infty} k p_k < \infty$$

and

$$0 < \sigma^2 = \sum_{k=0}^{\infty}(k-m)^2 p_k < \infty.$$

It is well known (cf. Athreya and Ney 1972) that

$$\lim_{t\to\infty}\frac{B(t)}{m^t} = B \qquad \text{a.s.}$$

where

$$P\{B=0\} = \begin{cases} 1 & \text{if} \quad m=1, \\ q & \text{if} \quad m>1 \end{cases}$$

and $q < 1$ depends on the distribution $\{p_k\}$.

On the limit properties of $\lambda(A,t)$, $(t\to\infty)$ in the case $m>1$ we have the following theorem:

Theorem A. (Révész 1994, Theorem 6.4, p. 107). *For any $x \in \mathbb{R}^d$ and $\epsilon > 0$*

$$\lim_{T\to\infty} T^{1/2-\epsilon}\left|\frac{\lambda(\{y : y \le xT^{1/2}\},T)}{m^T} - B\Phi(x)\right| = 0 \qquad \text{a.s.} \tag{1.1}$$

where

$$\Phi(x) = \Phi(x_1,x_2,\dots,x_d) = \frac{1}{(2\pi)^{d/2}}\int_{-\infty}^{x_d}\cdots\int_{-\infty}^{x_1}\exp\left(-\frac{y_1^2+y_2^2+\dots+y_d^2}{2}\right)\mathrm{d}y_1\mathrm{d}y_2\dots\mathrm{d}y_d.$$

Further, for any fixed $x \in \mathbb{R}^d$ and $0 < \epsilon < 1$ we have

$$\lim_{T\to\infty} T^{1-\epsilon}\left|(2\pi T)^{d/2}\frac{\lambda(C(x),T)}{m^T} - B\right| = 0 \qquad \text{a.s.} \tag{1.2}$$

where

$$C(x) = \{y : \|y-x\| \le r_d\}$$

and

$$r_d = \begin{cases} 2^{-1} & \textit{if} \quad d=1, \\ \pi^{-1/2} & \textit{if} \quad d=2, \\ \pi^{-1/2}(\Gamma(d/2+1))^{1/d} & \textit{if} \quad d\ge 3, \end{cases}$$

i.e. $C(x)$ is the ball in $\mathbb{R}^d$ around x of volume 1.

Theorem A tells us that in the case $m>1$ the particles at time T (if T is big enough) are distributed according to the normal law.

In the present paper we investigate the case $m=1$. Hence from now on we assume that $m=1$. Since if $m=1$ then $B=0$ a.s., we study the limit behaviour of $\lambda(A,T)$ as $T\to\infty$, under the condition $\{B(T) > 0\}$. In this case it turns out that the distribution of the particles may agree to any given smooth enough distribution with positive probability. In fact we prove the following theorem:

Theorem 1. *Let $F(x)$ $(x \in \mathbb{R}^d)$ be an arbitrary, given, absolutely continuous distribution function with bounded density. Then for any $\epsilon > 0$ there exist a $\delta = \delta(\epsilon) > 0$ and a $T_0 = T_0(\epsilon) > 0$ such that*

$$P\{\sup_{x \in \mathbb{R}^d} |F_T(x) - F(x)| \le \epsilon | B(T) > 0\} \ge \delta$$

if $T \ge T_0$, where

$$F_T(x) = \frac{\lambda(\{y : y \le xT^{1/2}\}, T)}{B(T)}.$$

We are also interested in studying the expectation vector of the empirical distribution $F_T(x) = F_T(x_1, x_2, \dots, x_d)$. Let

$$F_T^{(i)}(x_i) = F_T(\infty, \dots, \infty, x_i, \infty, \dots, \infty) = \frac{\lambda(\{y = (y_1, \dots, y_d) : y_i \le x_i T^{1/2}\}, T)}{B(T)},$$

$$m_i = m_i(T) = \int_{-\infty}^{+\infty} x_i \mathrm{d}F_T^{(i)}(x_i), \qquad (i = 1, 2, \dots, d)$$

and

$$m = m(T) = (m_1, m_2, \dots, m_d).$$

Then we have the following theorem:

Theorem 2.

(i) *$m_1, m_2, \dots, m_d$ are i.i.d. random variables.*
(ii) $\mathrm{E}m_i = 0\,(i = 1, 2, \dots, d)$.
(iii) *$m_1, m_2, \dots, m_d$ are normally distributed.*
(iv) $\lim_{T\to\infty} \mathrm{E}m_i^2 = \sigma^2/2$.

Theorem 1 is an analogue of (1.1). We are also interested in finding an analogue of (1.2), i.e. we intend to study the limit properties of $\lambda(C, T)$ where $C = C(0)$.

First we mention the following trivial facts:

$$\mathrm{E}(\lambda(C, T) | B(T)) \sim B(T)(2\pi T)^{-d/2}$$

and

$$\mathrm{E}(\lambda(C, T) | B(T) > 0) \sim \frac{\sigma^2 T}{2}(2\pi T)^{-d/2}.$$

Now we formulate our next theorem.

Theorem 3. (i) *In the case $d = 1$*

$$\lim_{T\to\infty} T^{-1/2} \mathrm{E}(\lambda(C, T) | \lambda(C, T) > 0) = \mathcal{L}_1$$

where

$$\mathcal{L}_1 = \frac{\sigma^2}{2(2\pi)^{1/2}} \sum_{k=1}^{\infty} \int_0^1 \cdots \int_0^1 \left(\frac{x_1 \dots x_k}{2 - x_1 \dots x_k} \right)^{1/2} dx_1 \dots dx_k = \frac{\sigma^2}{4} \left(\frac{\pi}{2} \right)^{1/2}.$$

(ii) *In the case $d = 2$, for any T big enough,*

$$\ell_2 \log T \leq \mathrm{E}(\lambda(\mathcal{C}, T) | \lambda(\mathcal{C}, T) > 0) \leq L_2 \log T$$

where

$$\ell_2 = \frac{\sigma^2}{8\pi}, \qquad L_2 = \frac{\sigma^2}{4\pi}.$$

(iii) *In the case $d = 3$, for any T big enough,*

$$\ell_3 \leq \mathrm{E}(\lambda(\mathcal{C}, T) | \lambda(\mathcal{C}, T) > 0) \leq L_3$$

where

$$\ell_3 = \frac{\sigma^2}{\pi^{3/2} 2^5}, \qquad L_3 = \frac{\sigma^2}{\pi^{3/2} 2^{7/2}}.$$

(iv) *In the case $d \geq 4$, for any T big enough,*

$$\ell_d \leq \mathrm{E}(\lambda(\mathcal{C}, T) | \lambda(\mathcal{C}, T) > 0) \leq L_d$$

where the values of L_d and ℓ_d are not given exactly.

Since

$$P\{\lambda(\mathcal{C}, T) > 0 | B(T) > 0\} = \frac{\mathrm{E}(\lambda(\mathcal{C}, T) | B(T) > 0)}{\mathrm{E}(\lambda(\mathcal{C}, T) | \lambda(\mathcal{C}, T) > 0)},$$

Theorem 3 clearly implies the following result:

Theorem 4. *In the case $d = 1$*

$$\lim_{T \to \infty} P\{\lambda(\mathcal{C}, T) > 0 | B(T) > 0\} = \frac{\sigma^2}{2\mathcal{L}_1 (2\pi)^{1/2}} = \frac{2}{\pi}.$$

In the case $d \geq 2$, for any T big enough,

$$P\{\lambda(\mathcal{C}, T) > 0 | B(T) > 0\} \leq \begin{cases} \dfrac{\sigma^2}{4\pi\ell_2 \log T} = \dfrac{2}{\log T} & \text{if} \quad d = 2, \\ \dfrac{\sigma^2 T}{2\ell_3} (2\pi T)^{-3/2} = 2^{5/2} T^{-1/2} & \text{if} \quad d = 3, \\ \dfrac{\sigma^2 T}{2\ell_d} (2\pi T)^{-d/2} & \text{if} \quad d \geq 4 \end{cases}$$

and

$$P\{(\mathcal{C},T)>0|B(T)>0\}\geq\begin{cases}\dfrac{\sigma^2}{4\pi L_2\log T}=\dfrac{1}{\log T} & \text{if} \quad d=2,\\ \dfrac{\sigma^2 T}{2L_3}(2\pi T)^{-3/2}=2T^{-1/2} & \text{if} \quad d=3,\\ \dfrac{\sigma^2 T}{2L_d}(2\pi T)^{-d/2} & \text{if} \quad d\geq 4.\end{cases}$$

It is also easy to see by Theorem 3 that the following result holds:

Theorem 5. *Uniformly in T*

$$\lim_{K\to\infty} P\{\lambda(\mathcal{C},T)>KT^{1/2}|\lambda(\mathcal{C},T)>0\}=0 \qquad \text{if} \quad d=1,$$
$$\lim_{K\to\infty} P\{\lambda(\mathcal{C},T)>K\log T|\lambda(\mathcal{C},T)>0\}=0 \qquad \text{if} \quad d=2,$$
$$\lim_{K\to\infty} P\{\lambda(\mathcal{C},T)>K|\lambda(\mathcal{C},T)>0\}=0 \qquad \text{if} \quad d\geq 3.$$

Remark. Theorems 3–5, in the case $d=1$ ($d=2$) are closely related to Theorem 3.11 (2.11) of Fleischman (1978).

2. Lemmas on the critical branching process

Lemma A. *For any* $t\geq 0$

$$\mathrm{E}B(t)=1, \tag{2.1}$$

and as $t\to\infty$

$$P\{B(t)>0\}\sim\frac{2}{\sigma^2 t}, \tag{2.2}$$

$$\mathrm{E}(B(t)|B(t)>0)=\frac{1}{P\{B(t)>0\}}\sim\frac{\sigma^2 t}{2}, \tag{2.3}$$

$$\lim_{t\to\infty} P\left\{\frac{B(t)}{t}>z|B(t)>0\right\}=\exp\left(-\frac{2z}{\sigma^2}\right) \qquad (z\geq 0). \tag{2.4}$$

For any $0\leq s<t<\infty$, let $Q(s,t)$ be the number of those particles which are living at time s and which have at least one offspring living at time t. Clearly

$$B(s)\geq Q(s,t), \qquad B(t)\geq Q(s,t),$$
$$\{Q(s,t)=0\}=\{B(t)=0\} \qquad (0\leq s<t)$$

and as a function of s, $Q(s,t)$ is non-decreasing. Hence on the set $\{B(t)>0\}$ one can define a sequence $\nu_2=\nu_2(t)\leq\nu_3=\nu_3(t)\leq\ldots\leq\nu_\mu=\nu_\mu(t)<t$ and a random variable $\mu=\mu(t)$

as follows:

$$\nu_k = \inf\{s : 0 \le s \le t,\ Q(s,t) \ge k\}$$

and μ is the largest integer for which $\nu_\mu < t$.

Lemma B (Révész, 1994).

$$\mathrm{E}(Q(s,t)|B(t) > 0) = \frac{1}{P\{B(t) > 0\}} \frac{1}{\mathrm{E}(B(t-s)|B(t-s) > 0)} = \frac{t}{t-s}(1 + o(t-s)). \quad (2.5)$$

$$\mathrm{E}(Q^2(s,t)|B(t) > 0) = \frac{t(t+s)}{(t-s)^2}(1 + o(t-s)) \quad (2.6)$$

and for any fixed $k = 2, 3, \ldots,$

$$\mathrm{E}((t - \nu_k)|B(t) \ge k) \sim \frac{t}{k}, \quad (2.7)$$

$$\mathrm{E}((t - \nu_k)^2|B(t) \ge k) \sim \frac{2t^2}{k(k+1)}, \quad (2.8)$$

$$\lim_{t\to\infty} P\left\{\frac{\nu_k}{t} < x|B(t) \ge k\right\} = x^{k-1} \qquad (0 \le x \le 1). \quad (2.9)$$

Consider a fixed $[0, T]$-branch $\{P_0, P_1, \ldots, P_T\}$ of the underlying branching process, i.e. P_0 is the particle at $0, P_1$ is an offspring of $P_0, \ldots, P_i$ is an offspring of $P_{i-1} (i = 1, 2, \ldots, T)$. Clearly such a branch exists if and only if $B(T) > 0$. Let $\xi_1 = \xi_1(T)$ be the first time-point where a new $[\xi_1, T]$-branch starts which lives up to time T, i.e. ξ_1 is the smallest i for which P_i has an offspring $Q_{i+1} \ne P_{i+1}$ having at least one offspring living at time T. Let $\xi_2 = \xi_2(T)$ be the second element of the fixed $[0, T]$-branch where a new $[\xi_2, T]$-branch starts which lives up to time T, i.e. ξ_2 is the smallest j for which $j > \xi_1$ and P_j has an offspring $Q_{j+1} \ne P_{j+1}$ which has an offspring living at time T. Continuing this procedure, we get a random sequence $1 \le \xi_1 < \xi_2 < \ldots < \xi_\nu < T$ where $\nu = \nu(T)$ is the largest integer for which $\xi_\nu < T$.

Now consider a partition of the $B(T) - 1$ particles living at time T, not considering the terminal point P_T of the fixed $[0, T]$-branch. The first class $C_1 = C_1(T)$ consists of the terminal points of those $[\xi_1, T]$-branches which branch from the fixed $[0, T]$-branch at ξ_1. $C_2 = C_2(T)$ consists of the terminal points of those $[\xi_2, T]$-branches which branch from the fixed $[0, T]$-branch at ξ_2, etc.

Let $U_1, U_2, \ldots$ be a sequence of independent random variables uniformly distributed on $[0, 1]$ and introduce the following notation:

$$V_1 = 1,$$

$$V_k = \prod_{j=1}^{k-1}(1 - U_j) \qquad (k = 2, 3, \ldots),$$

$$L_K = \sum_{j=1}^{k} U_j V_j = 1 - \prod_{j=1}^{k}(1 - U_j) \stackrel{\mathcal{D}}{=} 1 - \prod_{j=1}^{k} U_j \qquad (k = 1, 2, \ldots).$$

Then we have the following lemma:

Lemma 1. *For any k fixed and $\epsilon > 0$*

$$\lim_{T\to\infty} P\left\{\frac{\xi_1}{T} < x_1, \frac{\xi_2}{T} < x_2, \ldots, \frac{\xi_k}{T} < x_k \middle| B(T) > 0\right\} = P\{L_1 < x_1, L_2 < x_2, \ldots, L_k < x_k\}, \tag{2.10}$$

$$\lim_{T\to\infty} (T-\xi_k)^{-1}\mathrm{E}(|C_k| | \xi_k, B(T) > 0) = \frac{\sigma^2}{2}, \tag{2.11}$$

$$\lim_{T\to\infty} P\left\{\left|\frac{\nu(T)}{\log T} - 1\right| \geq \epsilon \middle| B(T) > 0\right\} = 0, \tag{2.12}$$

$$\lim_{T\to\infty} \mathrm{E}\left(\frac{\nu(T)}{\log T} \middle| B(T) > 0\right) = 1 \tag{2.13}$$

and

$$\lim_{T\to\infty} T^{-1/2}\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1 - \frac{\xi_k}{T}\right)^{-1/2}\right) = 2, \tag{2.14}$$

where $|C_k|$ is the cardinality of C_k.

In order to prove Lemma 1, we first prove the two following lemmas.

Lemma 2. *For any $0 < \epsilon < 1$, let*

$$A_n = A_n(\epsilon) = \{(x_1, x_2, \ldots, x_n) : 0 < x_i < 1, x_1 x_2 \ldots x_n > \epsilon\}$$

and

$$I_n = I_n(\epsilon) = \int_{A_n} (x_1 x_2 \ldots x_n)^{-1/2} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

Then we have

$$\lim_{\epsilon \searrow 0} \epsilon^{1/2} \sum_{n=1}^{\infty} I_n = 2.$$

Proof. Let

$$J_n = J_n(\epsilon) = \int_{A_n} (x_1 x_2 \ldots x_n)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

Then

$$\begin{aligned} I_n &= \int_{A_{n-1}} (x_1 x_2 \ldots x_{n-1})^{-1/2}\left(\int_{\epsilon(x_1 x_2 \ldots x_{n-1})^{-1}}^{1} x_n^{-1/2} \mathrm{d}x_n\right) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \\ &= 2\int_{A_{n-1}} (x_1 x_2 \ldots x_{n-1})^{-1/2}(1 - \epsilon^{1/2}(x_1 x_2 \ldots x_{n-1})^{-1/2}) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \\ &= 2I_{n-1} - 2\epsilon^{1/2} J_{n-1}. \end{aligned}$$

By a simple calculation we get

$$I_1 = 2(1-\epsilon^{1/2}),$$

$$J_n = \frac{\left(\log\frac{1}{\epsilon}\right)^n}{n!}.$$

Hence, by induction,

$$I_n = 2^n\left(1-\epsilon^{1/2}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\log\frac{1}{\epsilon}\right)^k}{k!}\right) = 2^n\epsilon^{1/2}\sum_{k=n}^{\infty}\frac{\left(\frac{1}{2}\log\frac{1}{\epsilon}\right)^k}{k!}.$$

Consequently

$$\sum_{n=1}^{\infty} I_n = 2\epsilon^{-1/2} - 2$$

□

and Lemma 2 is proved.

Lemma 3.

$$\mathrm{E}L_k = 1 - 2^{-k}, \tag{2.15}$$

$$\mathrm{E}\log(\mathrm{e}U_k) = 0, \tag{2.16}$$

$$\mathrm{E}(\log(\mathrm{e}U_k)^2 = 1, \tag{2.17}$$

$$-\infty < \mathrm{E}(\log \mathrm{e}U_k))^r < \infty \qquad (r = 1, 2, \ldots), \tag{2.18}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log(\mathrm{e}U_k) = 0 \qquad \text{a.s.}, \tag{2.19}$$

$$\lim_{n\to\infty} P\left\{n^{-1/2}\sum_{k=1}^{n}\log(\mathrm{e}U_k) < x\right\} = \Phi(x), \tag{2.20}$$

$$\lim_{\epsilon\searrow 0}\frac{\mu(\epsilon)}{\log(1/\epsilon)} = 1 \qquad \text{a.s.}, \tag{2.21}$$

$$\lim_{\epsilon\searrow 0}\mathrm{E}\left(\frac{\mu(\epsilon)}{\log(1/\epsilon)}\right) = 1, \tag{2.22}$$

$$\lim_{\epsilon\searrow 0}\epsilon^{1/2}\mathrm{E}\left(\sum_{k=1}^{\mu}(U_1U_2\ldots U_k)^{-1/2}\right) = 2 \tag{2.23}$$

where

$$\mu(\epsilon) = \min\{k : L_k \geq 1-\epsilon\}.$$

Proof. Expressions (2.15)–(2.18) can be obtained by trivial calculations. Equations (2.19) and (2.20) are, respectively, consequences of (2.16) and (2.17). In order to prove (2.22), observe that

$$\begin{aligned} P\{\mu > k) &= P\{U_1 U_2 \dots U_k > \epsilon\} \\ &= P\left\{\frac{\log(\mathrm{e}U_1) + \log(\mathrm{e}U_2) + \dots + \log(\mathrm{e}U_k)}{k} > 1 - \frac{\log(1/\epsilon)}{k}\right\}, \end{aligned}$$

and for any $0 < \delta < 1$ we have

$$\begin{aligned} \mathrm{E}\mu &= \sum_{k=1}^{\infty} P\{\mu \geq k\} \\ &= \sum_{k=1}^{(1-\delta)\log(1/\epsilon)} P\{\mu \geq k\} + \sum_{k=(1-\delta)\log(1/\epsilon)}^{(1+\delta)\log(1/\epsilon)} P\{\mu \geq k\} + \sum_{k=(1+\delta)\log(1/\epsilon)}^{\infty} P\{\mu \geq k\}. \end{aligned}$$

Clearly

$$\lim_{\epsilon \searrow 0} P\{\mu \geq k\} = 1 \text{ uniformly in } k \leq (1-\delta)\log(1/\epsilon),$$

$$\sum_{k=(1-\delta)\log(1/\epsilon)}^{(1+\delta)\log(1/\epsilon)} P\{\mu \geq k\} \leq 2\delta \log(1/\epsilon),$$

$$\begin{aligned} &\lim_{\epsilon \searrow 0} \sum_{k=(1+\delta)\log(1/\epsilon)}^{\infty} P\{\mu \geq k\} \\ &\leq \lim_{\epsilon \searrow 0} \sum_{k=(1+\delta)\log(1/\epsilon)}^{\infty} P\left\{\frac{\log(\mathrm{e}U_1) + \log(\mathrm{e}U_2) + \dots + \log(\mathrm{e}U_k)}{k} > \frac{\delta}{1+\delta}\right\} = 0. \end{aligned}$$

The last three relations clearly imply (2.22).

In order to prove (2.21), we note that by (2.19) we have

$$U_1 U_2 \dots U_n = \exp(-n(1+o(1)) \qquad \text{a.s.},$$

$$U_1 U_2 \dots U_\mu = \exp(-\mu(1+o(1)) \leq \epsilon, \tag{2.24}$$

$$U_1 U_2 \dots U_{\mu-1} = \exp(-(\mu-1)(1+o(1))) > \epsilon; \tag{2.25}$$

in the last two equations $o(1)$ is a function of μ converging to 0 a.s. as $\mu \to \infty$, i.e. as $\epsilon \to 0$. Clearly (2.24) implies (2.25).

Let

$$B_k = \{U_1 U_2 \dots U_k > \epsilon\}.$$

Then

$$\mathrm{E}\left(\sum_{k=1}^{\mu} (U_1 U_2 \dots U_k)^{-1/2}\right) = \sum_{k=1}^{\infty} \int_{B_k} (U_1 U_2 \dots U_k)^{-1/2} \mathrm{d}P = \sum_{k=1}^{\infty} I_k.$$

Hence we have (2.23) by Lemma 2 and Lemma 3 is proved.

Proof of Lemma 1. Since $\xi_1 = \nu_2$, we have (2.10) for $k = 1$ by (2.9). When $k = 2$, observe that

$$\frac{\xi_2}{T} = \frac{\xi_2 - \xi_1}{T - \xi_1}\left(1 - \frac{\xi_1}{T}\right) + \frac{\xi_1}{T},$$

where

$$\lim_{T\to\infty} P\left\{\frac{\xi_2 - \xi_1}{T - \xi_1} < x_1,\ \frac{\xi_1}{T} < x_2 \middle| B(T) \geq 2\right\} = x_1 x_2 \qquad (0 \leq x_1, x_2 \leq 1),$$

and we have (2.10) for $k = 2$. Continuing this procedure, we complete the proof of (2.10) by induction.

(2.11) is a simple consequence of (2.3). Equations (2.12), (2.13) and (2.14) follow from Lemma 3. Lemma 1 is proved. □

3. Proofs of Theorems 1 and 2

We introduce the following notation:

(i) $\mathscr{P}(s,t) = \{P_1^{(s)}, P_2^{(s)}, \ldots, P_{Q(s,t)}^{(s)}\} \subset \mathbb{R}^d$ is the set of locations of those particles at time s which have an offspring living at time t.

(ii) $\mathscr{P}_i(s,u,t) = \{P_{1i}^{(s,u)}, P_{2i}^{(s,u)}, \ldots\} \subseteq \mathbb{R}^d$ is the set of the locations of those offspring of $P_i^{(s)}$ at time $u (0 \leq su \leq t)$ which have an offspring living at time t.

(iii) $\mathscr{P}(s,u,t) = \bigcup_{i=1}^{Q(s,t)} \mathscr{P}_i(s,u,t) = \mathscr{P}(u,t)$.

(iv) $X_i^{s,u} = \max_j \|P_i^{(s)} - P_{i,j}^{(s,u)}\|$.

(v) $D(s,u) = \max_{1\leq i\leq Q(s,t)} X_i^{(s,u)}$.

Clearly $D(s,u)$ tells us how far from their ancestors the particles go during the time interval (s,u). Note that the number of elements of $\mathscr{P}(s,u,t)$ is $Q(u,t)$.

Lemma 4. *Let* $0 \leq s < u \leq t < \infty$. *Then*

$$P\{D(s,u) \geq K(u-s)^{1/2} | Q(u,t)\} \leq (1 + o(u-s))Q(u,t)\exp\left(-\frac{K^2}{2}\right).$$

Proof. The probability that a particle moves at least $K(u-s)^{1/2}$ during a time interval (s,u) is

$$P\{W(u-s) \geq K(u-s)^{1/2}\} \leq (1 + o(u-s))\exp\left(-\frac{K^2}{2}\right).$$

Since we have to take into consideration $Q(u,t)$ particles, we get Lemma 4. □

Lemma 5. *For any $\epsilon > 0$ there exists a $0 < \delta = \delta(\epsilon) < 1$ such that*

$$P\left\{D\left(\frac{t}{2}, t\right) \geq \epsilon t^{1/2}\right\} \leq \delta \tag{3.1}$$

for any $t > 0$.

Proof. Observe that

$$D\left(\frac{t}{2}, t\right) \leq \sum_{k=1}^{\infty} D\left(t - \frac{t}{2^k}, t - \frac{t}{2^{k+1}}\right),$$

and applying Lemma 4 with

$$s = s_k = t - \frac{t}{2^k}, \qquad u = u_k = t - \frac{t}{2^{k+1}}, \qquad K = \frac{\epsilon}{5} 2^{(k+1)/4},$$

we get

$$P\left\{D\left(t - \frac{t}{2^k}, t - \frac{t}{2^{k+1}}\right) \geq K\left(\frac{t}{2^{k+1}}\right)^{1/2} \Big| Q\left(t - \frac{t}{2^{k+1}}, t\right)\right\}$$
$$\leq (1 + o(2^{-k}t)) Q\left(t - \frac{t}{2^{k+1}}, t\right) \exp\left(-\frac{K^2}{2}\right).$$

Hence

$$P\left\{D\left(t - \frac{t}{2k}, t - \frac{t}{2^{k+1}}\right) \geq \frac{\epsilon}{5} 2^{-(k+1)/4} t^{1/2} \Big| Q\left(t - \frac{t}{2^{k+1}}, t\right), k = 1, 2, \ldots\right\}$$
$$\geq (1 + o(2^{-k}t)) Q\left(t - \frac{t}{2^{k+1}}, t\right) \exp\left(-\frac{\epsilon^2}{50} 2^{(k+1)/2}\right).$$

Since

$$\frac{\epsilon}{5} t^{1/2} \sum_{k=1}^{\infty} 2^{-(k+1)/4} \leq \epsilon t^{1/2},$$

and by (2.5)

$$\sum_{k=1}^{\infty} Q\left(t - \frac{t}{2^{k+1}}, t\right) \exp\left(-\frac{\epsilon^2}{50} 2^{(k+1)/2}\right) < \frac{1}{2}$$

with positive probability, we have (3.1). □

For any $x \in \mathbb{R}^d$ let

$$D_x\left(\frac{t}{2}\right) = \max_{1 \leq i \leq Q(t/2, t)} \| P_i^{(t/2)} - x t^{1/2} \|.$$

Clearly $D_x(t/2)$ tells us how close to $xt^{1/2}$ at time $t/2$ were located those particles which have an offspring living at time t.

Lemma 6. *For any* $\epsilon > 0, \mathrm{K} > 0$ *there exists a* $\delta = \delta(\epsilon, \mathrm{K}) > 0$ *such that*

$$P\left\{D_{Kx}\left(\frac{t}{2}\right) \leq \epsilon t^{1/2}\right\} \geq \delta,$$

provided that $||x|| \leq 1$.

Proof. Essentially the same as that of Lemma 4. □

Lemma 7. *For any* $d = 1, 2, \ldots, \delta > 0, K > 0$ *there exists an* $\epsilon = \epsilon(\delta, K) > 0$ *such that*

$$P\{\lambda(\bar{C}(x, \delta t^{1/2}), t) = 0 | B(t) > 0\} \geq \epsilon,$$

where

$$C(a, r) = \{x : x \in \mathbb{R}^d, ||x - a|| \leq r\},$$

$$||x|| \leq Kt^{1/2},$$

and

$$\bar{C}(\cdot, \cdot) = \mathbb{R}^d - C(\cdot, \cdot).$$

Lemma 7 tells us that for any $x \in \mathbb{R}^d$ with $||x|| \leq Kt^{1/2}$, all particles living at time t will be located in a ball around x of radius $\delta t^{1/2}$ with positive probability.

Proof. The lemma is a trivial consequence of Lemmas 5 and 6. □

Proof of Theorem 1. Clearly for an $\epsilon > 0$ there exist a $1 < \ell = \ell(\epsilon) < \infty$, a $C = C(\epsilon) > 0$ and a partition $A_1, A_2, \ldots, A_\ell$ of $\mathbb{R}^d$ such that

$$\int_{A_i} \mathrm{d}F(x) \leq \epsilon \qquad (i - 1, 2, \ldots, \ell)$$

and an A_i contains a ball of radius $C\epsilon$.

Let

$$\nu_\ell = \inf\{s : 0 \leq s \leq t, Q(s, t) = \ell = \ell(\epsilon)\}$$

and let the locations of these ℓ particles at time ν_ℓ be

$$Q_1, Q_2, \ldots, Q_\ell.$$

Observe that

$$P\{\max_{1 \leq i \leq \ell} ||Q_i|| \leq t^{1/2}\} \geq \delta$$

and that by (2.9)

$$P\{t - \nu_\ell \geq \epsilon t\} \geq \delta$$

for some $\delta = \delta(\epsilon) > 0$ if ℓ is big enough, say $\ell > 1/\epsilon$.

Let $B_i(t - \nu_\ell)$ be the number of offspring of the particle located in Q_i at time ν_ℓ. Then

$$P\{\max_{1 \leq i \leq j \leq \ell} |B_i(t - \nu_\ell) - B_j(t - \nu_\ell)| \leq \epsilon^2 t\} \geq \delta.$$

Then by Lemma 7 all $B_i(t-\nu_\ell)$ offspring of the particle located in Q_i at time ν_ℓ will be located in A_i with positive probability. Hence we have Theorem 1. □

In order to prove Theorem 2 we first prove the following:

Lemma 8. *Let $d=1$ and $P_1, P_2, \ldots, P_{B(T)}$ be the locations of the particles at time T. Then we have*

$$\lim_{T\to\infty} T^{-3}\mathrm{E}((P_1+P_2+\ldots+P_{B(T)})^2 | B(T)>0) = \frac{\sigma^2}{2}.$$

Proof. Let P_1 be fixed and consider a point $P_j \in C_k$ (cf. Lemma 1). Then

$$\mathrm{E}(P_1P_j|\xi_k) = \xi_k.$$

Hence, by (2.11),

$$\mathrm{E}\left(\sum_{j\in C_k} P_1P_j \middle| \xi_k\right) \sim \frac{\sigma^2}{2}\xi_k(T-\xi_k)$$

and

$$\begin{aligned}\mathrm{E}\left(\sum_{j\in C_k} P_1P_j \middle| B(T)>0\right) &\sim \frac{\sigma^2}{2}T^2\mathrm{E}\left(\frac{\xi_k}{T}\left(1-\frac{\xi_k}{T}\right)\right)\\ &\sim \frac{\sigma^2}{2}T^2\mathrm{E}((1-U_1U_2\ldots U_k)U_1U_2\ldots U_k)\\ &= \frac{\sigma^2}{2}T^2\left(\frac{1}{2^k}-\frac{1}{3^k}\right).\end{aligned}$$

Consequently

$$\mathrm{E}\left(\sum_{j=2}^{B(T)} P_1P_j \middle| B(T)>0\right) \sim \frac{\sigma^2}{4}T^2$$

which implies Lemma 8. □

Proof of Theorem 2. (i)–(iii) are trivial. Hence it is enough to prove (iv) in the case $d=1$, which is a straight consequence of Lemma 8. □

4. Proofs of Theorems 3–5

Let

$$b(t) = (b_1(t), b_2(t), \ldots, b_d(t)) \qquad (0 \le t \le 1)$$

where $b_1(t), b_2(t), \ldots, b_d(t)$ are independent Brownian bridges. Let $\{W(t) \in \mathbb{R}^d, t \ge 0\}$ be a Wiener process. Assume that $b(\cdot)$ and $W(\cdot)$ are independent.

For any $0 \le s < 1$ and $T > 0$, define the process

$$\Gamma(t,s,T) = \Gamma(t) = \begin{cases} T^{1/2} b(tT^{-1}) & \text{if } 0 \le t \le sT, \\ W(t - sT) + T^{1/2} b(s) & \text{if } sT \le t \le T. \end{cases}$$

Lemma 9. *The density function $\gamma(x)$ of $\Gamma(T)$ is*

$$\gamma(x) = (2\pi(1-s^2)T)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1-s^2)T}\right) \qquad (x \in \mathbb{R}^d).$$

Proof. Consider the case $d = 1$. Then the density function of $T^{1/2}b(s)$ is

$$(2\pi s(1-s)T)^{-1/2} \exp\left(-\frac{x^2}{2s(1-s)T}\right)$$

and the conditional density function of $\Gamma(T)$ given $T^{1/2}b(s) = m$ is

$$(2\pi(1-s)T)^{-1/2} \exp\left(-\frac{(x-m)^2}{2(1-s)T}\right).$$

Hence

$$\begin{aligned} \gamma(x) &= \psi \frac{s^{-1/2}}{\pi} \int_{-\infty}^{+\infty} \exp\left(-\psi\left(\frac{m^2}{s} + (x-m)^2\right)\right) \mathrm{d}m \\ &= \psi \frac{s^{-1/2}}{\pi} \left(\frac{s(1-s)T}{1+s}\right)^{1/2} \exp\left(-\frac{x^2}{2(1-s^2)T}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y \\ &= (2\pi(1-s^2)T)^{-1/2} \exp\left(-\frac{x^2}{2(1-s^2)T}\right), \end{aligned}$$

where

$$\psi = (2(1-s)T)^{-1}.$$

This proves the lemma for the case $d = 1$. The general case follows immediately from the case $d = 1$. □

Lemma 10. *Assume that* $0 \le s = s(T) < 1$ *and*

$$\lim_{T \to \infty} (1-s^2)T = \infty.$$

Then

$$P\{\Gamma(T) \in C\} \sim (2\pi(1-s^2)T)^{-d/2}$$

as $T \to \infty$.

Proof. This is a trivial consequence of Lemma 9. □

Proof of Theorem 3. Assume that $\lambda(\mathcal{C},T)>0$. Then there exists a $[0,T]$-branch of the underlying branching Wiener process having a terminal point in $\mathcal{C}$ at time T. Fix this branch. Consider the time-points $0<\xi_1<\xi_2<\ldots<\xi_\nu<T$ and the sets $C_1,C_2,\ldots,C_\nu$ of terminal points with respect to the fixed $[0,T]$-branch. Then, by Lemma 10,

$$P\{\text{ an element of } C_k \text{ belongs to } \mathcal{C}|\xi_k\}\sim\left(2\pi\left(1-\left(\frac{\xi_k}{T}\right)^2\right)T\right)^{-d/2}$$

and, by (2.11),

$$\mathrm{E}(\#\text{ of those elements of } C_k \text{ which belong to } \mathcal{C}|\xi_k)$$

$$\sim\frac{\sigma^2(T-\xi_k)}{2\left(2\pi\left(1-\left(\frac{\xi_k}{T}\right)^2\right)T\right)^{d/2}}=\frac{\sigma^2}{2(2\pi)^{d/2}}\frac{T^{1-d/2}}{\left(1-\frac{\xi_k}{T}\right)^{d/2-1}\left(1+\frac{\xi_k}{T}\right)^{d/2}}.$$

Hence

$$\mathrm{E}(\lambda(\mathcal{C},T)|\lambda(\mathcal{C},T)>0)\sim\frac{\sigma^2T^{1-d/2}}{2(2\pi)^{d/2}}\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1-\frac{\xi_k}{T}\right)^{1-d/2}\left(1+\frac{\xi_k}{T}\right)^{-d/2}\right). \tag{4.1}$$

Since $0<\xi_k/T<1$, we have

$$\frac{\sigma^2T^{1-d/2}}{\pi^{d/2}2^{d+1}}\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1-\frac{\xi_k}{T}\right)^{1-d/2}\right)$$

$$\leq\mathrm{E}(\lambda(\mathcal{C},T)|\lambda(\mathcal{C},T)>0)\leq\frac{\sigma^2T^{1-d/2}}{2(2\pi)^{d/2}}\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1-\frac{\xi_k}{T}\right)^{1-d/2}\right). \tag{4.2}$$

In the case $d=1$, by (4.1) and Lemma 1,

$$\mathrm{E}(\lambda(\mathcal{C},T)|\lambda(\mathcal{C},T)>0)\sim\frac{\sigma^2T^{1/2}}{2(2\pi)^{1/2}}\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1-\frac{\xi_k}{T}\right)^{1/2}\left(1+\frac{\xi_k}{T}\right)^{-1/2}\right)$$

$$\sim\frac{\sigma^2T^{1/2}}{2(2\pi)^{1/2}}\sum_{k=1}^{\infty}\int_0^1\cdots\int_0^1\left(\frac{x_1x_2\ldots x_k}{2-x_1x_2\ldots x_k}\right)^{1/2}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_k.$$

Now we prove the following identity.

$$\sum_{k=1}^{\infty}\int_0^1\cdots\int_0^1\left(\frac{x_1x_2\ldots x_k}{2-x_1x_2\ldots x_k}\right)^{1/2}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_k=\frac{\pi}{2}. \tag{4.3}$$

Since

$$(1-x)^{-1/2}=\sum_{i=0}^{\infty}a_ix^i\qquad(0<x<1)$$

where

$$a_i = \binom{-1/2}{i},$$

we have

$$\left(\frac{x}{2-x}\right)^{1/2} = \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\frac{a_i}{2^i}\,x^{i+1/2}.$$

Let $U_1, U_2, \dots$ be i.i.d. random variables uniformly distributed on $[0,1]$. Then

$$\int_0^1 \cdots \int_0^1 \left(\frac{x_1\dots x_k}{2-x_1\dots x_k}\right)^{1/2} \mathrm{d}x_1\dots\mathrm{d}x_k = \mathrm{E}\left(\frac{U_1\dots U_k}{2-U_1\dots U_k}\right)^{1/2}$$
$$= \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\frac{a_i}{2^i}\mathrm{E}(U_1\dots U_k)^{i+1/2} = \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\frac{a_i}{2^i}\left(\frac{1}{i+3/2}\right)^k$$

and

$$\sum_{k=1}^{\infty}\mathrm{E}\left(\frac{U_1\dots U_k}{2-U_1\dots U_k}\right)^{1/2} = \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\frac{a_i}{2^i}\sum_{k=1}^{\infty}\left(\frac{1}{i+3/2}\right)^k = \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\frac{a_i}{2^i}\,\frac{2}{2i+1}$$
$$= \sqrt{2}\int_0^{\infty}\mathrm{e}^{-x}\sum_{i=0}^{\infty}a_i\left(\frac{\mathrm{e}^{-2x}}{2}\right)^i\mathrm{d}x = \sqrt{2}\int_0^{\infty}\mathrm{e}^{-x}\left(1-\frac{\mathrm{e}^{-2x}}{2}\right)^{-1/2}\mathrm{d}x$$
$$= \sqrt{2}\int_0^1\left(1-\frac{y^2}{2}\right)^{-1/2}\mathrm{d}y = 2\int_0^1\frac{1}{\sqrt{2-y^2}}\mathrm{d}y$$
$$= 2\arcsin\left(\frac{y}{\sqrt{2}}\right)\Big|_0^1 = \frac{\pi}{2}.$$

Hence we have (4.3) as well as Theorem 3 for $d=1$.

In the case $d=2$

$$\mathrm{E}\left(\sum_{k=1}^{\nu}\left(1-\frac{\xi_k}{T}\right)^{1-d/2}\right) = \mathrm{E}\nu \sim \log T.$$

Hence by (4.2) we have

$$\frac{\sigma^2}{8\pi}\log T \le \mathrm{E}(\lambda(C,T)\,|\,\lambda(C,T)>0) \le \frac{\sigma^2}{4\pi}\log T$$

as stated in part (ii).

In the case $d=3$ by (2.14) we have our statement (iii).

The general case can be treated similarly and, in turn, we have Theorem 3. □

As we mentioned in the Introduction, Theorems 4 and 5 easily follow from Theorem 3.

5. Questions

Question 1. $F_T(x)$ $(x \in \mathbb{R}^d, T = 1, 2, \ldots)$ of Theorem 1 for any T is a random, empirical distribution function. Hence we have a probability measure P_T on the space of the distributions on $\mathbb{R}^d$. Very likely P_T, as $T \to \infty$, converges weakly to a limit measure. Prove the existence of this limit measure and characterize it. This question was proposed by O. Barndorff-Nielsen. $m(T)$ of Theorem 2 is the random expectation of $F_T(x)$ according to the law of P_T. Theorem 2 came about as a I tried to answer this question.

Question 2. Investigate the limit properties of

$$\max_{x \in \mathbb{R}^d} \lambda(\mathcal{C}(x), T) \qquad (d = 1, 2, \ldots)$$

as $T \to \infty$ on the set $\{B(T) > 0\}$.

Question 3. Let $d = 1$ and let

$$\lambda^-(T) = \max\{x : x < 0, \lambda(\mathcal{C}(x), T) = 0\},$$
$$\lambda^+(T) = \min\{x : x > 0, \lambda(\mathcal{C}(x), T) = 0\}.$$

Investigate the limit properties of

$$\lambda^+(T) - \lambda^-(T) \qquad (T \to \infty)$$

on the set $\{\lambda(\mathcal{C}, T) > 0\}$.

6. A secret

Let $\{W_{ij}(t), t \geq 0\}$ $(i, j = 1, 2, \ldots)$ be an array of independent $\mathbb{R}^d$-valued Wiener processes. Consider a system of non-independent Wiener processes.

$$w(t) = w(i(1), i(2), \ldots, i([\lg T]), T, t)$$
$$= \begin{cases} W_{11}(t) & \text{if} \quad 0 \leq t \leq \alpha_1 T, \\ w(\alpha_1 T) + W_{2,i(2)}(t - \alpha_1 T) & \text{if} \quad \alpha_1 T \leq t \leq \alpha_2 T, \\ \ldots & \\ w(\alpha_k T) + W_{k+1, i(k+1)}(t - \alpha_k T) & \text{if} \quad \alpha_k T \leq t \leq \alpha_{k+1} T, \end{cases}$$

where

$$\alpha_k = 1 - 2^{-k}, \qquad k = 1, 2, \ldots, [\lg T] - 1,$$
$$i(1) = 1, \qquad i(2) = 1, 2 \qquad \ldots \qquad i(k) = 1, 2, \ldots, 2^{k-1},$$
$$0 \leq t \leq (1 - 2^{-[\lg T]})T, \qquad T \geq 2.$$

It is easy to see that the process $w(\cdot)$ is very similar to a critical branching Wiener process. Hence my secret, which I share with you, is the following: I try to solve any question regarding a critical branching Wiener process by replacing it by the above model. If I succeed in doing so, I try to prove the same for the branching model by Lemma 1 which essentially claims that the two models are close to each other. In fact I followed this method in proving the above theorems. I have to confess that I could not answer the three questions in Section 5 even in this simple situation.

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