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Constructing tests for normal order-restricted inference

ARTHUR COHEN,[1]* HAROLD B. SACKROWITZ[1] and ESTER SAMUEL-CAHN[2]

[1]*The State University of New Jersey Rutgers, Department of Statistics, Faculty of Arts and Sciences, Hill Center for the Mathematical Sciences, Busch Campus, Piscataway, NJ 08855, USA*
[2]*Department of Statistics, Hebrew University, Jerusalem 91905, Israel*

For normal models we consider the problem of testing a null hypothesis against an order-restricted alternative. The alternative always consists of a cone minus the null space. We offer sufficient conditions for a class of tests to be complete and for unbiasedness of tests. Both sets of sufficient conditions are expressed in terms of the notion of cone order monotonicity. A method of constructing tests that are unbiased and in the complete class is given. The method yields new tests of value to many problems. Detailed applications and a simulation study are offered for testing homogeneity of means against the simple order alternative and for testing homogeneity against the matrix order alternative.

Keywords: Bayes-type tests; complete class; cone order monotonicity; cone ordering; convexity; dual cone; likelihood ratio test; matrix order alternative; unbiased tests

1. Introduction and summary

The most typical form of a multiparameter one-sided alternative encountered in hypothesis testing is that in which the alternative is a cone. Such models (which can be seen in a wide variety of areas such as analysis of variance and contingency tables) are especially interesting and difficult in the multiparameter case. A common aspect of these problems is the non-existence of classically optimal tests (uniformly most powerful invariant, uniformly most powerful unbiased, etc.). Hence one finds the likelihood ratio test (LRT) (when feasible) and many *ad hoc* procedures being used. Because no formal methodology exists to handle such problems it is not unusual to find *ad hoc* procedures developed in the literature which are inadmissible and which can be substantially improved upon (see, for example, Cohen and Sackrowitz 1992a; 1992b; 1994).

Using the notion of cone order monotonicity, we are able to present a framework in which such problems can be approached. The goal is to ensure that tests which are developed have, at least, the properties of belonging to a non-trivial complete class and of unbiasedness. In the important case of normal random variables, we present a method of developing good tests for the unknown variance case, beginning with a good test in the known variance case. This presents a valuable tool as, for many such cone alternatives, the common statistical technique of studentizing leads to inadmissible procedures (see Cohen and Sackrowitz 1993).

* To whom correspondence should be addressed.

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We will see that approaching these problems from the point of view of functions which are monotone with respect to certain cone orderings is particularly natural for exponential family distributions. One indication of this stems from studying Eaton's (1970) theorem. It will be shown that Eaton's theorem implies that under certain conditions the cone order monotone tests form a complete class.

The notion of cone orderings supplies a common thread running through much of the existing literature relating to admissibility and unbiasedness of tests in such problems. Many of the quantities which arise in these problems are most easily expressed and studied from this viewpoint. Furthermore, it allows us to deal with models for which previous methods are insufficient.

In this paper our attention will be confined to normal models. That is, we let $X_{ij} \sim \mathrm{N}(\mu_i, \sigma^2)$ for $i = 1, \ldots, k$; $j = 1, \ldots, n$; $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^{\mathrm{T}}$ and σ are unknown. Also let $\bar{X}_i = \sum_{j=1}^{n} X_{ij}/n$, $\bar{\boldsymbol{X}} = (\bar{X}_1, \ldots, \bar{X}_k)^{\mathrm{T}}$, $T = \sum_{i=1}^{k} \sum_{j=1}^{n} X_{ij}^2$, $S^2 = T - n\sum_{i=1}^{k} \bar{X}_i^2$. We wish to test $H_0 : \boldsymbol{\mu} \in \Omega_0$ against $H_1 : \boldsymbol{\mu} \in \Omega - \Omega_0$ where Ω is a closed convex cone and Ω_0 is a closed linear space of dimension r.

We need to define cone order monotonicity. Towards this end, let $\boldsymbol{x} \in \mathbb{R}^k$ and let $\mathscr{K}$ be a closed convex cone in $\mathbb{R}^k$. A partial order, called the cone order with respect to $\mathscr{K}$, is defined as follows: $\boldsymbol{x} \le [\mathscr{K}]\boldsymbol{y}$, if $\boldsymbol{y} - \boldsymbol{x} \in \mathscr{K}$. A function f is cone order monotone (non-decreasing) with respect to $\mathscr{K}$ if $f(\boldsymbol{x}) \le f(\boldsymbol{y})$ whenever $\boldsymbol{x} \le [\mathscr{K}]\boldsymbol{y}$. We use COM[$\mathscr{K}$] to describe this.

The dual cone $\mathscr{K}^*$, of $\mathscr{K}$, is defined as $\mathscr{K}^* = \{\boldsymbol{v} \in \mathbb{R}^k : \boldsymbol{v}^{\mathrm{T}}\boldsymbol{\theta} \ge 0$, for every $\boldsymbol{\theta} \in \mathscr{K}\}$. In terms of results for this paper we offer the following:

(1) Restatements of two previous results in terms of cone order monotonicity. The two results are essentially stated in Cohen *et al.* (1993) for the case where Ω is a polyhedral cone and m, the minimum number of generators of Ω^*, is bounded above by $k - r$. No such restriction is required here. The two results pertain to
  (a) Eaton's (1970) complete class theorem in terms of test functions which are COM[Ω^*];
  (b) a sufficient condition for unbiasedness in terms of test functions which are COM[$\Omega \cap \Omega_0^{\perp}$], where $\Omega_0^{\perp}$ is the orthocomplement of Ω_0 within $\mathbb{R}^k$.

(2) A general method of constructing unbiased tests which lie in a non-trivial complete class. It represents a sensible approach relative to previously used *ad hoc* procedures.

(3) Application of the results in (1) and (2) to the problem of testing homogeneity against a simple order alternative and the problem of testing homogeneity against the matrix order alternative. See Robertson *et al.* (1988, pp. 12, 26, 32, and 394) for a discussion of such an alternative. Rosenbaum (1991) gives an additional data set appropriate for matrix order analysis. Included in these applications are suggested new tests and simulation studies comparing the power of the new tests to the power of the LRT. The new tests are sometimes computationally easier to carry out than the LRT. Furthermore, some of the new tests are considerably more powerful than the LRT for certain alternatives of interest.

As previously mentioned, we do not require that Ω be a polyhedral cone. When Ω is polyhedral, we do not require that the minimum number of generators m of Ω^* be less than $k - r$. In fact for testing homogeneity of the components of $\boldsymbol{\mu}$ against the matrix order alternative, $m > k - r = k - 1$. The problem of testing $\boldsymbol{\mu} = \boldsymbol{0}$ against a circular cone alternative is another example where $m \geqslant k - r$.

In Section 2 we deal with the preliminaries. In Sections 3, 4 and 5 we obtain the results in (1), (2) and (3), respectively.

2. Preliminaries

We make an additional assumption about Ω_0. Namely, we take Ω_0 to be orthogonal to Ω^*. This assumption still enables us to accommodate a large number of common applications. See Robertson *et al.* (1988) for many such applications.

Now for the model described in Section 1, $(\bar{X}, T)$ or $(\bar{X}, S^2)$ qualify as complete sufficient statistics. Let the matrix $\boldsymbol{B}^{r\times k}$ be composed of rows that qualify as a basis for the linear space Ω_0. Then under H_0, $(\boldsymbol{B}\bar{X}, T)$ is a complete sufficient statistic. (See equation (3.4) which offers the joint density of $\bar{X}$, T under Ω_0.) When Ω_0 is a boundary of $\Omega - \Omega_0$, we will use a conditional (on $\boldsymbol{B}\bar{X}, T$) approach based on Neyman structure. Note that in this instance there is no equivalent version of the sufficient statistic in terms of $(\boldsymbol{B}\bar{X}, S^2)$.

We need results concerning projections. If $\mathscr{C}$ is a closed convex cone, denote the projection of $\boldsymbol{x}$ onto $\mathscr{C}$ as $\boldsymbol{P}(\boldsymbol{x}|\mathscr{C})$. The projection of $\boldsymbol{x}$ onto $\mathscr{C}$ is defined as the unique point in $\mathscr{C}$ which is closest to $\boldsymbol{x}$ in terms of Euclidean distance. By virtue of Robertson *et al.* (1988, Theorem 8.2.7), we have

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{x}|\mathscr{C}) = \|\boldsymbol{P}(\boldsymbol{x}|\mathscr{C})\|^2, \tag{2.1}$$

where $\|\cdot\|$ is Euclidean norm.

Next note that by examining the likelihood function for the normal model of Section 1, we find that the maximum likelihood estimator of $\boldsymbol{\mu}$, when $\boldsymbol{\mu} \in \Omega$, is $\boldsymbol{P}(\bar{X}|\Omega)$. The maximum likelihood estimator of σ^2, say $\hat{\sigma}^2_\Omega$, when $\boldsymbol{\mu} \in \Omega$, is a multiple of $T - n\|\boldsymbol{P}(\bar{X}|\Omega)\|^2$.

Before proceeding, we need to clarify some notation and elaborate on the distinction between $(\bar{X}, T)$, $(\bar{X}, S^2)$, and $(\boldsymbol{B}\bar{X}, T)$. Since the sample space of the sufficient statistics can be expressed either in terms of $(\bar{X}, T)$ or $(\bar{X}, S^2)$, test functions are denoted by $\phi(\bar{x}, t)$ which can also be written as $\bar{\phi}(\bar{x}, s^2)$. An important distinction to keep in mind is when we condition on $S^2 = s^2$, or when we condition on $(\boldsymbol{B}\bar{X} = z,\ T = t)$. When we say that a function $g(\bar{x}, t)$ is COM$[\Omega^*]$ given $(\boldsymbol{B}\bar{X} = z,$ $T = t)$ we mean that $g(\bar{x} + \lambda, t) \geq g(\bar{x}, t)$ for any $\lambda \in \Omega^*$, and $\bar{x}$ such that $\boldsymbol{B}\bar{x} = z$, $T = t$ for any z, t. Note $\boldsymbol{B}\lambda = \boldsymbol{0}$ since $\lambda \in \Omega^*$, $\mathbf{B}$ is a basis for Ω_0, and Ω_0 is orthogonal to Ω^*. Hence $\boldsymbol{B}(\bar{x} + \lambda) = z$. When we say a function $h(\bar{x}, s^2)$ is COM$[\Omega]$, we mean that $h(\bar{x} + \lambda, s^2) \geq h(\bar{x}, s^2)$ for any $\lambda \in \Omega$ and any fixed s^2.

We conclude with two cone order monotonicity properties for $T - n\|\boldsymbol{P}(\bar{X}|\Omega)\|^2$, proved in the Appendix.

Lemma 2.1 Under the above assumptions

(i) $-[T - n\|\boldsymbol{P}(\bar{X}|\Omega)\|^2]$ is COM$[\Omega]$ for fixed $S^2 = T - n\|\bar{X}\|^2 = s^2$;
(ii) $-[T - n\|\boldsymbol{P}(\bar{X}|\Omega)\|^2]$ is COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{X} = z$, $T = t$.

3. A complete class and unbiased tests

To start, we define an acceptance section of a non-randomized test $\phi(\bar{x}, t)$. The acceptance region of the test $\phi(\bar{x}, t)$ is

$$A = \{(\bar{x}, t) : \phi(\bar{x}, t) = 0\}.$$

The acceptance section of the test $\phi(\bar{x}, t)$ for fixed $B\bar{x} = z$, $T = t$ is

$$A \cap \{(\bar{x}, t) : B\bar{x} = z,\ T = t\}.$$

A complete class theorem based on Eaton (1970) is as follows:

Theorem 3.1 Let $\mathscr{D}$ be a class of test functions $\phi(\bar{x}, t)$ such that

(a) $\phi(\bar{x}, t)$ is COM[Ω^*] given $B\bar{x} = z$, $T = t$;
(b) for any given (z, t), the acceptance sections of the test are convex.

Then for testing $H_0 : \mu \in \Omega_0$ vs $H_1 : \mu \in \Omega - \Omega_0$, the class of tests $\mathscr{D}$ is a complete class.

Proof
The proof is essentially given in Eaton (1970). To aid in seeing the connection between the statement here and the proof of Eaton's theorem, one should examine the density of $(\bar{X}, T)$ under the null and the alternative hypotheses. The density of $(\bar{X}, T)$ expressed in exponential family form is

$$f_{\bar{X},T}(\bar{x}, t) = \beta(\mu, \sigma^2) \exp(-t/2\sigma^2) \exp(n x^{\mathrm{T}} \mu/\sigma^2) I_{(0,t]}(n\|\bar{x}\|^2). \tag{3.1}$$

Under H_0, $\mu \in \Omega_0$. Since the rows of B are a basis for Ω_0, μ can be written as $\mu = B^{\mathrm{T}}\rho$ for some r-dimensional vector ρ. Hence when $\mu \in \Omega_0$, (3.1) can be written as

$$f_{\bar{X},T}(\bar{x}, t) = \beta^*(\rho, \sigma^2) \exp(-t/2\sigma^2) \exp\{n(B\bar{x})^{\mathrm{T}}\rho/\sigma^2\} I_{(0,t]}(n\|\bar{x}\|^2). \tag{3.2}$$

When $\mu \in \Omega$, observe that the ratio of (3.1) to (3.2) is COM[Ω^*] for fixed $B\bar{x} = z$, $T = t$. To see this, note that $(\bar{x} + \lambda)^{\mathrm{T}}\mu \geq \bar{x}^{\mathrm{T}}\mu$ since $\lambda \in \Omega^*$, $\mu \in \Omega$. The properties of the densities of $(\bar{X}, T)$ under $\mu \in \Omega$ and under $\mu \in \Omega_0$ ensure by virtue of Eaton's proof that Bayes tests and weak limits of Bayes tests have properties (a) and (b). Whereas Eaton's proof yields an essentially complete class, the fact that $(\bar{X}, T)$ is a complete statistic implies that the essentially complete class becomes a complete class. □

The following theorem provides a sufficient condition for a test to be unbiased. It is a restatement of Theorem 3.2 of Cohen *et al.* (1993).

Theorem 3.2 Let $\phi(\bar{x}, s^2)$ be a size α test which is COM[$\Omega \cap \Omega_0^{\perp}$]. Then $\phi(\bar{x}, s^2)$ is unbiased.

4. Construction of unbiased tests in the complete class

An intuitive method of constructing tests is as follows. Assume σ^2 is known (say $\sigma^2 = 1$ without loss of generality). Consider a test

$$\phi(\bar{x}) = \begin{cases} 1 & \text{if } U(\bar{x}) > K_\alpha \\ 0 & \text{otherwise,} \end{cases}$$

where $U(\bar{x})$ is COM[Ω^*] for fixed $B\bar{x} = z$ and COM[$\Omega \cap \Omega_0^{\perp}$], and K_α is chosen so the test has size α. Note that $U(\bar{X})$ depends on $\bar{X}$ only (not on T). Such a test is unbiased, and if its acceptance sections are convex, it is in the complete class where $\sigma^2 = 1$. Next suppose σ^2 is unknown and make the further assumption that $U(\bar{x})$ be a convex function of $\bar{x}$. Base the test on $[U(\bar{x})/s]$, or, what usually amounts to the same test, on $U(\bar{x}/s)$. Such a test will remain unbiased when σ^2 is unknown, but, as shown in Cohen and Sackrowitz (1993), it will be inadmissible for $m \geq 2$.

Another method which retains the unbiasedness property of the test but yields a test in the complete class is as follows: Let

$$\psi(\bar{x}, t) = \begin{cases} 1 & \text{if } [U(\bar{x})/\hat{\sigma}_\Omega] > K'_\alpha \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where K'_α is chosen so that the test has size α. (This can be done, for example, when the distribution of $U(\bar{X})/\sigma$ is parameter-free under H_0.)

Theorem 4.1 Let $X_{ij} \sim N(\mu_i, \sigma^2)$, for $i = 1, 2, \ldots, k$ and $j = 1, \ldots, n$, where $\boldsymbol{\mu}$ and σ^2 are unknown. Test $H_0 : \boldsymbol{\mu} \in \Omega_0$ against $H_1 : \boldsymbol{\mu} \in \Omega - \Omega_0$, where Ω is a closed convex cone and Ω_0 is a closed linear space of dimension r that is orthogonal to Ω^*. Let $U(\bar{x})$ be COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$ and COM$[\Omega \cap \Omega_0^\perp]$. Let $\hat{\sigma}^2_\Omega$ be the maximum likelihood estimator for σ^2 when $\boldsymbol{\mu} \in \Omega$. Then the test defined in (4.1) is unbiased and lies in the complete class of Theorem 3.1.

Proof

Note that $U(\bar{x})$ is COM$[\Omega \cap \Omega_0^\perp]$ for fixed s^2 (it does not depend on S^2) and Lemma 2.1(i) implies $-\hat{\sigma}_\Omega$ is COM$[\Omega]$ for fixed s^2 and so it is COM$[\Omega \cap \Omega_0^\perp]$ for fixed s^2. Thus by Theorem 3.2 the test is unbiased. Next note that $U(\bar{x})$ is COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$, $T = t$, and Lemma 2.1 implies $-\hat{\sigma}_\Omega$ is COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$, $T = t$. Furthermore, it can be shown, using Lemma A in the Appendix, that acceptance sections of the test are convex. Thus by Theorem 3.1, the test $\psi(\bar{x}, t)$ of (4.1) is in the complete class. □

Corollary 4.2 Consider the model of Theorem 4.1. Suppose $\Omega^* \oplus \Omega_0 \supset \Omega$. Then if $U(\bar{x})$ is COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$, the test in (4.1) is unbiased and in the complete class.

Proof

It suffices to show that if $\Omega^* \oplus \Omega_0 \supset \Omega$, then if $U(\bar{x})$ is COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$, then it is COM$[\Omega \cap \Omega_0^\perp]$. Now $\Omega^* \oplus \Omega_0 \supset \Omega = \Omega \cap \Omega_0^\perp \oplus \Omega_0$, so $\Omega^* \supset \Omega \cap \Omega_0^\perp$. If $\omega \in \Omega \cap \Omega_0^\perp$, then $\omega \in \Omega^*$ and $U(\bar{x} + \omega) \geq U(\bar{x})$ provided $\boldsymbol{B}\bar{\boldsymbol{x}} = \boldsymbol{B}(\bar{x} + \omega) = z$. The last statement follows since $\boldsymbol{B}\omega = \boldsymbol{0}$.

Remark 4.1

The successful use of $\hat{\sigma}_\Omega$ in the denominator of the statistic in (4.1) answers a query raised in Berk and Marcus (1995).

In continuing the construction of tests we seek statistics $U(\bar{X})$ that are COM$[\Omega^*]$ for fixed $\boldsymbol{B}\bar{\boldsymbol{x}} = z$. Towards this end, express $\Omega = \Omega'' \oplus \Omega_0$ where Ω'' is orthogonal to Ω_0. Let $\boldsymbol{a}_\nu$ denote the generators of Ω'' and let $Y_\nu = \boldsymbol{a}_\nu^{\mathrm{T}}\bar{\boldsymbol{X}}/\sqrt{\boldsymbol{a}_\nu^{\mathrm{T}}\boldsymbol{a}_\nu}$, $\nu = 1, \ldots, p$, where p is the number of generators of Ω''. Note that the Y_ν are COM$[\Omega^*]$ given $\boldsymbol{B}\bar{\boldsymbol{X}} = z$. It follows that functions $U(\bar{X}) = U^*(Y_1, \ldots, Y_p) = U^*(\boldsymbol{Y})$ which are non-decreasing functions of $\boldsymbol{Y}$ are COM$[\Omega^*]$.

In this phase of the construction it is convenient to choose $[U(\bar{X})/\hat{\sigma}_\Omega]$ to be location- and scale-invariant whenever the problem is location- and scale-invariant so that the ensuing test is similar. In recommending tests other than the LRT we consider tests such that

$$U^*(Y) = \max\left[\max_{1\le\nu\le p} \gamma_\nu Y_\nu, \gamma_{p+1}\bar{Y}\right], \tag{4.2}$$

where $\bar{Y} = \sum_{\nu=1}^{m} Y_\nu/p$ and γ_ν are non-negative weights assigned. Tests based on (4.2), with $\gamma_{p+1} = 0$, are Bayes-type tests that are designed to perform well against certain alternatives by choosing higher weights for the Y_νs that correspond to those alternatives. Furthermore, $U^*(Y)$ in (4.2) is COM[Ω^*] and is such that acceptance sections of the resulting test are convex in deference to Theorem 3.1.

We note that an important step in the above construction is the determination of $\boldsymbol{a}_\nu$, $\nu = 1, \dots, p$. Consider situations where Ω is a finite polyhedral cone, i.e.

$$\Omega = \{\boldsymbol{\mu} : \boldsymbol{\mu} \in \mathbb{R}^k,\ \Gamma\boldsymbol{\mu} \ge \mathbf{0}\},$$

where Γ is an $m \times k$ matrix. The null space is $\Omega_0 = \{\boldsymbol{\mu} : \boldsymbol{\mu} \in \mathbb{R}^k, \Gamma\boldsymbol{\mu} = \mathbf{0}\}$. The rows of Γ are the generators of Ω^*. When $m \le k - r$ the generators $\boldsymbol{a}_\nu$ are the columns of the matrix $\Gamma^{\mathrm{T}}(\Gamma\Gamma^{\mathrm{T}})^{-1}$ (see Cohen *et al.* 1993, p. 144). When $m > k - r$ and the rows of Γ are contrasts with two non-zero elements, i.e. rows with $+1$ and one -1, one can use results of Berk and Marcus (1995, Theorem 3.13) to determine the $\boldsymbol{a}_\nu$. Examples are offered in the next section.

5. Applications

5.1. Simple order alternative

For the simple order alternative we determine unbiased tests that lie in a non-trivial complete class using the notion of cone order monotonicity. From Cohen *et al.* (1993), it follows that $\Omega^* \oplus \Omega_0 \supset \Omega$, thus enabling an application of Corollary 4.2. We offer tests that illustrate the construction of Section 4 and a simulation study of the power of the constructed tests compared with the power of the LRT. The model is as in Section 1. Let $\Omega_0 = \{\boldsymbol{\mu} : \mu_1 = \mu_2 = \cdots = \mu_k\}$ and $\Omega = \{\boldsymbol{\mu} : \mu_1 \ge \mu_2 \ge \cdots \ge \mu_k\}$. Test $H_0 : \boldsymbol{\mu} \in \Omega_0$ against $H_1 : \boldsymbol{\mu} \in \Omega - \Omega_0$. The alternative H_1 is called the simple order alternative. The matrix Γ of Section 4 which determines Ω is

$$\Gamma = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 1 & -1 \end{pmatrix}. \tag{5.1}$$

In turn Γ determines $\boldsymbol{a}_\nu$, $\nu = 1, \dots, k-1$, as the columns of $\Gamma^{\mathrm{T}}(\Gamma\Gamma^{\mathrm{T}})^{-1}$.

At this point we let $k = 6$ and propose two tests designed to have good power against slippage alternatives. The tests are:

$$\phi_1 : \text{reject } H_0 \text{ if } \max\left[\max_{2\le\nu\le 5} Y_\nu, 3Y_1\right] > 4.18\tilde{\sigma}_\Omega$$

and

$$\phi_2 : \text{reject } H_0 \text{ if } \max\left[\max_{3\le\nu\le5} Y_\nu, 2Y_1, 2Y_2\right] > 3.25\hat{\sigma}_\Omega.$$

In the above tests the special generators $\boldsymbol{a}_1 = (5, -1, -1, -1, -1, -1)/\sqrt{30}$ and $\boldsymbol{a}_2 = (2, 2, -1, -1, -1, -1)/2\sqrt{3}$ receive special attention. They reflect the two largest slippage alternatives. The test ϕ_1 gives the generator $\boldsymbol{a}_1$ three times the weight of any other generator, while ϕ_2 gives each of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ twice the weight of any other generator. The numbers 4.18 in test ϕ_1 and 3.25 in test ϕ_2 are chosen so that each test has size 0.05 when $n = 10$ observations are taken for each population. Table 5.1 contains the simulated power functions for the LRT, ϕ_1 and ϕ_2. Each value of the power function is based on 50 000 replications. The power of all tests depends on μ/σ. Alternative points were chosen so that some indicate slippage while others do not.

The simulation indicates that for slippage-type alternatives the tests ϕ_1 and ϕ_2 are definitely preferable to the LRT. For non-slippage-type alternatives the LRT is preferred. The test based on ϕ_2 is a compromise between ϕ_1 and the LRT. The simulation clearly demonstrates that it will make a significant difference which of several natural tests is applied.

Should k be different from 6, tests analogous to ϕ_1 and ϕ_2 could easily be determined.

Remark 5.1
Theorem 4.1 is true with only mild modifications if the sample size in each population is n_i instead of n (see Cohen *et al.* (1993, Remark 4.4).

5.2. Matrix order alternatives

For the matrix order alternative we determine unbiased tests that lie in a complete class using the notions of cone order monotonicity. We also show that $\Omega^* \oplus \Omega_0 \supset \Omega$, thus enabling an application of Corollary 4.2. We offer tests that illustrate the construction of Section 4 and finish

Table 5.1. Simulated power functions: simple order alternative

μ/σ						*LRT*	ϕ_1	ϕ_2
0.0	0.0	0.0	0.0	0.0	0.0	0.050	0.050	0.050
2.0	0.0	0.0	0.0	0.0	0.0	0.373	0.549	0.484
2.0	2.0	0.0	0.0	0.0	0.0	0.580	0.435	0.666
2.0	2.0	1.0	1.0	0.0	0.0	0.547	0.324	0.493
3.0	1.0	1.0	0.0	0.0	0.0	0.727	0.766	0.753
3.0	3.0	1.0	1.0	1.0	0.0	0.767	0.525	0.778
4.0	2.0	1.0	0.0	0.0	0.0	0.941	0.931	0.952
3.0	3.0	2.0	1.0	0.0	0.0	0.873	0.547	0.794
4.0	1.0	1.0	1.0	0.0	0.0	0.882	0.953	0.903
4.0	1.0	0.0	0.0	0.0	0.0	0.916	0.061	0.954
4.0	0.0	0.0	0.0	0.0	0.0	0.897	0.970	0.949
3.0	3.0	0.0	0.0	0.0	0.0	0.881	0.694	0.932
3.0	0.0	0.0	0.0	0.0	0.0	0.679	0.843	0.784

up with a simulation study of the power of the constructed tests compared with the power of the LRT.

To start, let X_{ijk} be independent $N(\mu_{ij}, \sigma^2)$, for $i = 1, \ldots, I$; $j = 1, \ldots, J$; $k = 1, \ldots, n$. Let $\bar{X}_{ij} = \sum_k X_{ijk}/n$, let $\bar{X}_{...} = \sum_{i,j,k} X_{ijk}/IJn$, and let $\bar{\boldsymbol{X}} = (\bar{X}_{11}, \ldots, \bar{X}_{1J}, \ \bar{X}_{21}, \ldots, \bar{X}_{1J})$. Define $T = \sum_{i,j,k} X_{ijk}^2$ and $S^2 = T - n\sum_{i,j} \bar{X}_{ij}^2$. Let

$$\Omega_0 = \{\boldsymbol{\mu} : \mu_{ij} \text{ equal for } i = 1, \ldots, I;\ j = 1, \ldots, J\},\ I \text{ or } J \geq 2,$$

and

$$\Omega = \{\boldsymbol{\mu} : \mu_{ij} \leq \mu_{i(j+1)}; \mu_{ij} \leq \mu_{(i+1)j}, \text{ for all } i, j\}.$$

Test $H_0 : \boldsymbol{\mu} \in \Omega_0$ against $H_1 : \boldsymbol{\mu} \in \Omega - \Omega_0$. The alternative H_1 is called the matrix order alternative.

The LRT is unbiased and lies in the complete class of Theorem 3.1. Whereas the LRT can be expected to have favourable power properties, it is not always easy to carry it out computationally. We proceed to construct other tests described in Section 4.

First recognize that Ω is a closed convex cone. To see this, let

$$\boldsymbol{\mu}^{IJ \times 1} = (\mu_{11}, \ldots, \mu_{1J},\ \mu_{21}, \ldots, \mu_{IJ})^{\mathrm{T}},$$

$$\mathbf{A}^{(J-1)\times J} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix};$$

$\mathbf{I}_J^{J\times J}$ is the identity matrix of order J, $\mathbf{0}_1^{(J-1)\times J}$ and $\mathbf{0}_2^{J\times J}$ are matrices of zeros, and

$$\mathbf{Q}^{[I(J-1)+(I-1)J]\times IJ} = \begin{pmatrix} \mathbf{A} & \mathbf{0}_1 & \cdots & \cdots & \cdots & \mathbf{0}_1 \\ \mathbf{0}_1 & \mathbf{A} & \mathbf{0}_1 & \cdots & \cdots & \mathbf{0}_1 \\ \vdots & & & & & \\ \mathbf{0}_1 & \mathbf{0}_1 & \cdots & \cdots & \cdots & \mathbf{A} \\ -\mathbf{I}_J & I_J & \mathbf{0}_2 & \cdots & \cdots & \mathbf{0}_2 \\ \mathbf{0}_2 & -I_J & \mathbf{I}_J & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \\ \vdots & & & & & \\ \mathbf{0}_2 & \mathbf{0}_2 & \cdots & \cdots & -\mathbf{I}_J & \mathbf{I}_J \end{pmatrix}.$$

Then it follows that Ω can be expressed as $\mathbf{Q}\boldsymbol{\mu} \geq \mathbf{0}$ or as $\Omega_{\mathbf{Q}} = \{\boldsymbol{\mu} : \mathbf{Q}\boldsymbol{\mu} \geq \mathbf{0}\}$ so that $\Omega_{\mathbf{Q}}$ is a polyhedral cone. Let $\Omega_{\mathbf{Q}}^*$ be the dual of $\Omega_{\mathbf{Q}}$ so that the rows of $\mathbf{Q}$ are the generators of $\Omega_{\mathbf{Q}}^*$ and note that Ω_0 is orthogonal to the rows of $\mathbf{Q}$.

In seeking results we prove

Lemma 5.1 The cone $\Omega_{\mathbf{Q}}$ is such that

$$\Omega_{\mathbf{Q}}^* \oplus \Omega_0 \supseteq \Omega_{\mathbf{Q}}. \tag{5.2}$$

The proof is given in the Appendix.

Remark 5.2
Property (5.2), along with Theorem 4.2 of Cohen *et al.* (1994), immediately yields that those tests which are COM[Ω^*] for fixed $B\bar{x} = z$ are unbiased when σ^2 is known.

We now construct tests. We evaluate the tests by comparing their simulated power functions with that of the simulated power of the LRT.

The first step in determining these tests is to find the generators of Ω. Writing $\Omega = \tilde{\Omega} \oplus \Omega_0$ and using a method given by Berk and Marcus (1995, Section 3), we find that the generators of $\tilde{\Omega}$ are constructed by using the upper sets principle as follows.

Consider the matrix

$$\mathscr{M} = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1J} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2J} \\ \vdots & & & \\ \mu_{I1} & \cdots & \cdots & \mu_{IJ} \end{pmatrix}.$$

An upper set U is a subset of the elements of $\mathscr{M}$ such that whenever $\mu_{ij} \in U$, $\mu_{i^*j^*}$ lies in U for $i \leq i^*$, $j \leq j^*$. Each upper set U determines a vector $\boldsymbol{c}$ of order $IJ \times 1$ such that $\boldsymbol{c} = (c_{11}, c_{12}, \ldots, c_{1J}, c_{21}, \ldots, c_{IJ})^{\mathrm{T}}$, with $c_{ij} = 1$ if $\mu_{ij} \in U$ and $c_{ij} = 0$ if $\mu_{ij} \notin U$. The generators of $\tilde{\Omega}$ are all vectors $\boldsymbol{c}$ that correspond to upper sets U, except $\boldsymbol{c} = \mathbf{1}$. There are a total of $m = |(I + J/J) - 2|$ generators of $\tilde{\Omega}$.

Next express $\Omega = \Omega'' \oplus \Omega_0$, where Ω'' is orthogonal to Ω_0, and note that the generators of Ω'' are $\boldsymbol{a}_\nu$, where

$$\boldsymbol{a}_\nu = \mathbf{1}^{\mathrm{T}}\mathbf{1}\mathbf{c}_\nu - (\mathbf{1}^{\mathrm{T}}\mathbf{c}_\nu)\mathbf{1}, \qquad \nu = 1, 2, \ldots, m,$$

where $\boldsymbol{c}_\nu$ are the generators of $\tilde{\Omega}$. Now note that $Y_\nu = \boldsymbol{a}_\nu^{\mathrm{T}}\bar{\boldsymbol{X}}/\sqrt{\boldsymbol{a}_\nu^{\mathrm{T}}\mathbf{a}_\nu}$ are COM[Ω^*] given $\bar{X}_{...} = x_{...}$.

At this point we let $I = 2$ and $J = 3$, so that $m = 8$. We propose two tests. The first test is designed to have power properties similar to that of the LRT. The advantage of the proposed test is that it is sometimes computationally easier to carry out than the LRT. The first test is

$$\phi_1 : \text{Reject } H_0 \text{ if } \max\left[\max_{1 \leq \nu \leq 8} Y_\nu,\ (5/3)\bar{Y}\right] > 2.15\hat{\sigma}_\Omega.$$

The second test was designed to have favourable properties against alternatives exhibiting increasing column effects. This test is

$$\phi_2 : \text{Reject } H_0 \text{ if } \max\left[\max_{3 \leq \nu \leq 8} Y_\nu,\ 2Y_1,\ 2Y_2\right] 3.36\hat{\sigma}_\Omega,$$

where the generators receiving extra weight are

$$\boldsymbol{a}_1 = (-2, 1, 1, -2, 1, 1)/2\sqrt{3}, \qquad \boldsymbol{a}_2 = (-1, -1, 2, -1, -1, 2)/2\sqrt{3}.$$

Table 5.2 contains simulated powers for the LRT, ϕ_1 and ϕ_2. Samples of size $n = 10$ were taken for each population. The simulation is based on 50 000 replications. The power of all tests depends on $\boldsymbol{\mu}/\sigma$. The parameter points, written in the original matrix form, were chosen so that some of them reflect increasing column effects. However, other alternatives are considered so that a balanced presentation is made. The simulation indicates that test ϕ_1 has a power function similar to that of the

Table 5.2. Simulated power functions: matrix order alternative

μ/σ			*LRT*	ϕ_I	ϕ_2
0.0	0.0	0.0	0.050	0.050	0.050
0.0	0.0	0.0			
0.0	0.0	0.0	0.373	0.369	0.276
0.0	0.0	2.0			
0.0	0.0	1.0	0.197	0.196	0.257
0.0	2.0	1.0			
0.0	0.0	2.0	0.519	0.517	0.651
0.0	0.0	2.0			
0.0	0.0	2.0	0.561	0.561	0.618
0.0	1.0	2.0			
0.0	1.0	2.0	0.483	0.473	0.588
0.0	1.0	2.0			
0.0	2.0	2.0	0.520	0.515	0.648
0.0	2.0	2.0			
0.0	0.0	2.0	0.828	0.832	0.784
0.0	0.0	3.0			
0.0	2.0	3.0	0.717	0.704	0.769
1.0	2.0	3.0			
0.0	0.0	3.0	0.843	0.840	0.927
0.0	0.0	3.0			
0.0	0.0	3.0	0.854	0.851	0.908
0.0	1.0	3.0			
0.0	1.0	3.0	0.799	0.788	0.881
0.0	1.0	3.0			
0.0	2.0	3.0	0.651	0.636	0.607
2.0	2.0	3.0			

LRT. Test ϕ_2 is clearly the best for alternatives with increasing column effects. The simulation clearly indicates another instance where it makes a significant difference which of several natural tests is applied.

Remark 5.3

In the above development of the matrix order alternative we assumed that the number of observations in each cell was a constant n. If we allow the number of observations in the (i, j)th cell to be n_{ij}, then in general Lemma 5.1 will not hold. If for each $i = 1, \ldots, I$, $n_{i1} = \cdots = n_{iJ}$, it is easily shown that Lemma 5.1 does hold.

In general, whether unbiasedness of a test can be established by the method here, when sample sizes for each population are unequal, depends on Ω. If Ω represents the simple order cone, we remarked earlier that these methods work. On the other hand, if Ω represents the matrix order alternative then the methods work only in special cases. Other cones would require additional examination.

Remark 5.4
Unbiasedness of the LRT for the matrix order alternative has been established by Hu and Wright (1994).

Appendix

Proof of Lemma 2.1

(i) Write $\|\bar{X}\|^2 = \|\bar{X} - P(\bar{X}|\Omega) + P(\bar{X}|\Omega)\|^2$. Expand and use (2.1) to find

$$\|P(\bar{X}|\Omega)\|^2 = \|\bar{X}\|^2 - \|\bar{X} - P(\bar{X}|\Omega)\|^2. \tag{A.1}$$

Furthermore, by Robertson *et al.* (1988, p. 102, (2.6.8)), we have that $-\|\bar{X} - P(\bar{X}|\Omega)\|^2$ is COM[Ω] with $T - n\|\bar{X}\|^2$ fixed. Hence (i) follows.

(ii) Consider

$$\begin{aligned}\|P(\bar{X}|\Omega)\|^2 &= \|P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0) + P(\bar{X}|\Omega_0)\|^2 \\ &= \|P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0)\|^2 + \|P(\bar{X}|\Omega_0)\|^2 \\ &\quad + 2\{P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0)\}^{\mathrm{T}} P(\bar{X}|\Omega_0).\end{aligned} \tag{A.2}$$

By Robertson *et al.* (1988, p. 46, Theorem 1.7.2), we may write

$$\{P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0)\}^{\mathrm{T}} P(\bar{X}|\Omega_0) = \{\bar{X} - P(\bar{X}|\Omega_0) - P(\bar{X}|\Omega^*)\}^{\mathrm{T}} P(\bar{X}|\Omega_0)$$

using (2.1) and the fact that Ω_0 is orthogonal to Ω^*.

Thus

$$\|P(\bar{X}|\Omega)\|^2 = \|P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0)\|^2 + \|P(\bar{X}|\Omega_0)\|^2.$$

The argument leading to Robertson *et al.* (1988, p. 102, (2.6.9)) will imply that $\|P(\bar{X}|\Omega) - P(\bar{X}|\Omega_0)\|^2$ is COM[Ω^*]. The proof of (ii) is complete once we recognize that as the maximum likelihood estimator of $\theta \in \Omega_0$, $P(\bar{X}|\Omega_0)$ must be a function only of the sufficient statistic $B\bar{X}$. Thus if $B\bar{X}$ is fixed, $P(\bar{X}|\Omega_0)$ is fixed.

Proof of Lemma 5.1

Let $\mathbf{1}_J$ and $\mathbf{0}_J$ denote $J \times 1$ vectors of 1s and 0s, respectively. Define

$$\hat{\mathbf{Q}}^{(IJ-1)\times IJ} = \begin{pmatrix} \mathbf{A} & \mathbf{0}_1 & \cdots & \cdots & \mathbf{0}_1 \\ \mathbf{0}_1 & \mathbf{A} & \mathbf{0}_1 & \cdots & \mathbf{0}_1 \\ \vdots & & & & \\ \mathbf{0}_1 & \mathbf{0}_1 & \cdots & \cdots & \mathbf{A} \\ -\mathbf{1}_J^{\mathrm{T}} & \mathbf{1}_J^{\mathrm{T}} & \mathbf{0}_J^{\mathrm{T}} & \cdots & \mathbf{0}_J^{\mathrm{T}} \\ \mathbf{0}_J^{\mathrm{T}} & -\mathbf{1}_J^{\mathrm{T}} & \mathbf{1}_J^{\mathrm{T}} & \cdots & \mathbf{0}_J^{\mathrm{T}} \\ \vdots & & & & \\ \mathbf{0}_J^{\mathrm{T}} & \mathbf{0}_J^{\mathrm{T}} & \cdots & -\mathbf{1}_J^{\mathrm{T}} & \mathbf{1}_J^{\mathrm{T}} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} I \text{ blocks} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} (I-1) \text{ rows} \end{matrix}$$

Since adding all the rows of $\mathbf{0}_2 \cdots \mathbf{0}_2 - \mathbf{I}_J\mathbf{I}_J\mathbf{0}_2 \cdots \mathbf{0}_2$ in the $\mathbf{Q}$ matrix gives $\mathbf{0}_J^{\mathrm{T}}\mathbf{0}_J^{\mathrm{T}} \cdots - \mathbf{1}_J^{\mathrm{T}}\mathbf{1}_J^{\mathrm{T}}\mathbf{0}_J^{\mathrm{T}} \cdots \mathbf{0}_J^{\mathrm{T}}$, it follows that $\Omega^*_{\hat{\mathbf{Q}}} \subseteq \Omega^*_{\mathbf{Q}}$. Hence $\Omega_{\mathbf{Q}} \subseteq \Omega_{\hat{\mathbf{Q}}}$. Furthermore, Ω_0 is orthogonal to the rows of $\hat{\mathbf{Q}}$. Thus it suffices to show that

$$\Omega^*_{\hat{\mathbf{Q}}} \oplus \Omega_0 \supseteq \Omega_{\hat{\mathbf{Q}}}, \tag{A.3}$$

since then we would have

$$\Omega^*_{\mathbf{Q}} \oplus \Omega_0 \supseteq \Omega^*_{\hat{\mathbf{Q}}} \oplus \Omega_0 \supseteq \Omega_{\hat{\mathbf{Q}}} \supseteq \Omega_{\mathbf{Q}}. \tag{A.4}$$

Now it follows from Cohen *et al.* (1993, Lemma 3.4) that (A.3) holds if and only if $(\hat{\mathbf{Q}}\hat{\mathbf{Q}}^{\mathrm{T}})^{-1}$ has all non-negative elements. Note that

$$\hat{\mathbf{Q}}\hat{\mathbf{Q}}^{\mathrm{T}} = \begin{pmatrix} \mathbf{G}_J & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_J & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{G}_J & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & J\mathbf{G}_I \end{pmatrix}, \tag{A.5}$$

where $\mathbf{G}_J = \mathbf{A}\mathbf{A}^{\mathrm{T}}$, $\mathbf{G}_I = \Gamma_I\Gamma_I^{\mathrm{T}}$,

$$\Gamma_I^{(I-1)\times I} = \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{pmatrix}.$$

From Cohen *et al.* (1993, Example 4.1) it follows that $\mathbf{G}_J^{-1}$ and $\mathbf{G}_I^{-1}$ both have all non-negative elements. Hence $(\hat{\mathbf{Q}}\hat{\mathbf{Q}}^{\mathrm{T}})^{-1}$ has all non-negative elements. This completes the proof of the lemma.

Lemma A

Lemma A The norm squared of the projection of $\bar{\boldsymbol{X}}$ onto Ω, $\|\boldsymbol{P}(\bar{\boldsymbol{X}}|\Omega)\|^2$, is a convex function of $\bar{\boldsymbol{X}}$.

Proof

Let $z = \alpha\bar{\boldsymbol{x}}_1 + (1-\alpha)\bar{\boldsymbol{x}}_2$. Then from Robertson *et al.* (1988, equation (8.2.6)) we have

$$\|\boldsymbol{P}(z|\Omega)\|^2 = z^{\mathrm{T}}\boldsymbol{P}(z|\Omega) = \{\alpha\bar{\boldsymbol{x}}_1 + (1-\alpha)\bar{\boldsymbol{x}}_2\}^{\mathrm{T}}\boldsymbol{P}(z|\Omega). \tag{A.6}$$

Use Robertson *et al.* (1988, equation (8.2.7)) so that (A.6) is less than or equal to

$$\begin{aligned} &\alpha\boldsymbol{P}(\bar{\boldsymbol{x}}_1|\Omega)^{\mathrm{T}}\boldsymbol{P}(z|\Omega) + (1-\alpha)\boldsymbol{P}(\bar{\boldsymbol{x}}_2|\Omega)^{\mathrm{T}}\boldsymbol{P}(z|\Omega) \\ &= \{\alpha\boldsymbol{P}(\bar{\boldsymbol{x}}_1|\Omega) + (1-\alpha)\boldsymbol{P}(\bar{\boldsymbol{x}}_2|\Omega)\}^{\mathrm{T}}\boldsymbol{P}(z|\Omega). \end{aligned} \tag{A.7}$$

Use the Cauchy–Schwarz inequality in (A.7) so that (A.7) is less than or equal to

$$\|\alpha\boldsymbol{P}(\bar{\boldsymbol{x}}_1|\Omega) + (1-\alpha)\boldsymbol{P}(\bar{\boldsymbol{x}}_2|\Omega)\|\|\boldsymbol{P}(z|\Omega)\|. \tag{A.8}$$

Thus we have

$$\|P(z|\Omega)\| \leq \|\alpha P(\bar{x}_1|\Omega) + (1-\alpha)P(\bar{x}_2|\Omega)\|. \tag{A.9}$$

Since an increasing convex function of a convex function is convex, the lemma follows.

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