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The scaling limit behaviour of periodic stable-like processes

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We prove a functional non-central limit theorem for scaled Markov processes generated by pseudo-differential operators of periodic variable order. Two different situations occur. If the measure of the set where the order function attains its minimum α_o is positive with respect to the invariant measure, the limit turns out to be an α_o-stable Lévy process. In the other case the scaled sequence converges in probability to the zero function. The large deviation for this convergence is typical of processes having heavy-tail increments. It turns out that only a finite number of large jumps can be recovered on large scales. We also apply the results in order to understand the recurrence and transience of periodic stable-like processes.

Keywords: functional non-central limit theorem; homogenization; stable-like process; large deviations; heavy-tail increment

1. Introduction

Let Λ be a lattice in $\mathbb{R}^d$ such that $\mathbb{R}^d/\Lambda$ is compact, and let $\alpha \in C^1(\mathbb{R}^d)$ be Λ-periodic with $0 < \alpha < 2$. We denote by $\mathcal{M}_s(S^{d-1})$ the space of Borel measures on S^{d-1} satisfying $\mu(A) = \mu(-A)$ for all measurable $A \subset S^{d-1}$. Let $\mu : \mathbb{R}^d \to \mathcal{M}_s(S^{d-1})$ be Λ-periodic and continuously differentiable. Furthermore, we will assume that there exist $c_1, c_2 \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}^d$ and $p \in S^{d-1}$ one has

$$c_1 \leqslant \int_{S^{d-1}} |\langle p, \phi \rangle|^{\alpha(x)} \mu(x, \mathrm{d}\phi) \leqslant c_2.$$

Using the notation $\hat{y} = y/|y| \in S^{d-1}$, we define the family of measures

$$\eta(x, \mathrm{d}\xi) = |\xi|^{-1-\alpha(x)} \mathrm{d}|\xi| \mu(x, \mathrm{d}\hat{\xi})$$

on $\mathbb{R}^+ \times S^{d-1} = \mathbb{R}^d \backslash \{0\}$. The pseudo-differential operator with variable order

$$Lu(x) := \int_{\mathbb{R}^d} \left(u(x+\xi) - u(x) - \frac{\langle \xi, \nabla u(x) \rangle}{1 + |\xi|^2} \mathbb{1}_{B_1(0)}(\xi) \right) \eta(x, \mathrm{d}\xi)$$

generates a Feller semigroup $(T_t)_{t \geqslant 0}$ on the space of bounded continuous function $C_b(\mathbb{R}^d)$ (see Kolokoltsov 2000; Jacob and Leopold 1993; Kikuchi and Negoro 1997). The associated cadlag process is a Markov process X with transition probability densities given by the fundamental solutions of the parabolic problem $Lu = \partial_t u$. This process is called stable-like in the literature, since it behaves locally like a stable process with position-dependent scaling exponent.

Let $N^X(\omega, \mathrm{d}y, \mathrm{d}t)$ be the random measure associated with the jump process $\Delta X_t := X_t - X_{t-}$. Its compensator is $\nu(\omega, \mathrm{d}y, \mathrm{d}t) := \eta(X_{t-}(\omega), \mathrm{d}y)\mathrm{d}t$ (see Schilling 1998: 582). The following pathwise description of the stable-like process L can be given by

$$X_t(\omega) = X_0(\omega) + \int_0^t \int_{B_1(0)^c} yN^X(\omega, \mathrm{d}y, \mathrm{d}s)$$

$$+ \int_0^t \int_{B_1(0)} y\big(N^X(\omega, \mathrm{d}y, \mathrm{d}s) - \nu(\omega, \mathrm{d}y, \mathrm{d}s)\big).$$

For a modern introduction to the stochastic analysis related to Lévy processes, see Applebaum (2004). Let $\Pi_\Lambda : \mathbb{R}^d \to \mathbb{R}^d/\Lambda$ be the covering map. For all Λ-periodic functions f there exists a unique $f_\Lambda : \mathbb{R}^d/\Lambda \to \mathbb{R}$ such that $f = f_\Lambda \circ \Pi_\Lambda$. We denote by X^Λ the process on $\mathbb{R}^d/\Lambda$ obtained by projection of X with respect to Π_Λ. It follows from the Λ-periodicity of η and α that X^Λ is a Markov process on $\mathbb{R}^d/\Lambda$. The associated semigroup on $C(\mathbb{R}^d/\Lambda)$ will be denoted by T^Λ. The following proposition is a version of Doeblin's theorem (see Doob 1953: 197).

Lemma 1. *There exists an invariant measure π for X^Λ on $\mathbb{R}^d/\Lambda$ with $\pi(\mathbb{R}^d/\Lambda) < \infty$ and constants C, $\lambda > 0$ such that, for all Λ-periodic $f \in C(\mathbb{R}^d)$ and $t \geqslant 0$,*

$$\int_{\mathbb{R}^d/\Lambda} f_\Lambda \, \mathrm{d}\pi = 0 \text{ implies } \|T_t f\|_{\sup} \leqslant C\mathrm{e}^{-\lambda t}\|f\|_{\sup}.$$

Proof. The existence of transition probability densities for X^Λ follows from the existence of transition probability densities for X (see Kolokoltsov 2000: 759). Moreover, one has

$$p_\Lambda(t, x, y) = \sum_{l \in \Lambda} p(t, x_0, l + y_0),$$

where x_0 and y_0 are arbitrary points in $\Pi^{-1}(\{x\})$ and $\Pi^{-1}(\{y\})$, respectively. It follows from the positivity of the transition probability densities of X given by Kolokoltsov (2000: 761) that $(x, y) \mapsto p_\Lambda(t, x, y)$ is bounded away from zero. The lemma then follows from a result in Bensoussan *et al.* (1978: 365). □

We denote by $\mathbb{D}_x([0, T])$ the space of cadlag paths starting in x and by $\rho : \mathbb{D}_x([0, T]) \times \mathbb{D}_x([0, T]) \to [0, \infty[$ the Skorokhod metric on $\mathbb{D}_x([0, T])$ (see Jacod and Shiryaev 1987: 288). We will show that for $\alpha_o := \inf \alpha$ the scaled processes

$$X_t^{(n)} := n^{-1/\alpha_o}(X_{nt} - X_0)$$

converge in distribution with respect to the Skorokhod topology to a suitable stable Lévy process if $\pi(\{\alpha = \alpha_o\}) > 0$. For diffusions without jumps having a periodic generator, similar results can be found in Bensoussan *et al.* (1978) and Bhattacharya (1985).

Theorem 1. *If $\pi(\alpha = \alpha_o) > 0$ the processes $X^{(n)}$ converge in distribution with respect to the Skorokhod topology to an α_o-stable Lévy process Y with compensator given by*

$$\nu_o(\mathrm{d}y) := \int_{\mathbb{R}^d/\Lambda} \pi(\mathrm{d}x) \mathbb{1}_{\{\alpha=\alpha_o\}}(x)\eta(x, \mathrm{d}y).$$

Proof. According to Jacod and Shiryaev (1987) we have to prove that the characteristics of the sequence $X^{(n)}$ converge to the characteristics of Y. Since the measure μ is symmetric, the process X is a local martingale. We denote by $N^{(n)}(\omega, \mathrm{d}y, \mathrm{d}t)$ the random measure associated with the jump process

$$\Delta X_t^{(n)} := n^{-1/\alpha_o}(X_{nt} - X_{nt-}).$$

Let

$$K^{(n)}(x, A) := \int_{\mathbb{R}^d} \mathbb{1}_{A\setminus\{0\}}(n^{-1/\alpha_o} y)\eta(x, \mathrm{d}y).$$

Then $\nu^{(n)}(\omega, \mathrm{d}y, \mathrm{d}t) := nK^{(n)}(X_{nt-}(\omega), \mathrm{d}y)\mathrm{d}t$ is the compensator of the random measure $N^{(n)}(\omega, \mathrm{d}y, \mathrm{d}t)$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a uniformly continuous bounded function vanishing in a neighbourhood of zero. Using the definition of the $*$-product from Jacod and Shiryaev (1987: 66), we can compute

$$\begin{aligned}
g * \nu_t^{(n)} &= \int_0^{nt}\int_{\mathbb{R}^d} g(n^{-1/\alpha_o} y)\eta(X_{s-}, \mathrm{d}y)\mathrm{d}s \\
&= n\int_0^{t}\int_{\mathbb{R}^d} g(n^{-1/\alpha_o} y)\eta(X_{ns-}, \mathrm{d}y)\mathrm{d}s \\
&= \int_0^{t}\int_{\mathbb{R}^d} g(n^{1/\alpha(X_{ns-})-1/\alpha_o} y)\eta(X_{ns-}, \mathrm{d}y)\mathrm{d}s \\
&= \int_0^{t}\int_{\mathbb{R}^d} \mathbb{1}_{\{\alpha>\alpha_o\}}(X_{ns-}) g(n^{1/\alpha(X_{ns-})-1/\alpha_o} y)\eta(X_{ns-}, \mathrm{d}y)\mathrm{d}s \\
&\quad + \int_0^{t}\int_{\mathbb{R}^d} \mathbb{1}_{\{\alpha=\alpha_o\}}(X_{ns-}) g(y)\eta(X_{ns-}, \mathrm{d}y)\mathrm{d}s \\
&\simeq \int_0^{t}\int_{\mathbb{R}^d} \mathbb{1}_{\{\alpha=\alpha_o\}}(X_{ns-}) g(y)\eta(X_{ns-}, \mathrm{d}y)\mathrm{d}s,
\end{aligned}$$

since g is equal to zero in a neighbourhood of zero and

$$\mathbb{1}_{\{\alpha>\alpha_o\}}(X_{ns-}) n^{1/\alpha(X_{ns-})-1/\alpha_o} y \to 0 \quad \mathbb{P}\text{-a.s.} \qquad \text{for } n \to \infty.$$

We now prove that for $n \to \infty$ the sequence $g * \nu_t^{(n)}$ converges in $L^2(\Omega, \mathbb{P})$ to

$$g * \nu_0 = t\int_{\mathbb{R}^d/\Lambda}\int_{\mathbb{R}^d} \mathbb{1}_{\{\alpha=\alpha_o\}}(x) g(y)\eta(x, \mathrm{d}y)\pi(\mathrm{d}x).$$

Moreover, we define

$$U(z) := \int_{\mathbb{R}^d} g(y)\left(\mathbb{1}_{\{\alpha=\alpha_o\}}(z)\eta(z, \mathrm{d}y) - \int_{\mathbb{R}^d/\Lambda} \mathbb{1}_{\{\alpha=\alpha_o\}}(x)\eta(x, \mathrm{d}y)\pi(\mathrm{d}x) \right).$$

Therefore,

$$\mathbb{E}\left[\left(\int_0^t U(X_{ns-})\mathrm{d}s\right)^2\right] = 2\int_0^t \int_0^s \mathbb{E}[U(X_{ns-})U(X_{nr-})]\mathrm{d}r\,\mathrm{d}s.$$

The Markov property of X and Lemma 1 imply that

$$\begin{aligned}
&2\int_0^t \int_0^s \mathbb{E}[U(X_{ns-})U(X_{nr-})]\mathrm{d}r\,\mathrm{d}s \\
&= 2\int_0^t \int_0^s \mathbb{E}[\mathbb{E}[U(X_{ns-})|\mathcal{F}_{nr}]U(X_{nr-})]\mathrm{d}r\,\mathrm{d}s \\
&= 2\int_0^t \int_0^s \mathbb{E}\big[(T_{n(s-r)}U)(X_{nr-})U(X_{nr-})\big]\mathrm{d}r\,\mathrm{d}s \\
&\leqslant 2\int_0^t \int_0^s C\mathrm{e}^{n(s-r)\lambda}\|U\|_\infty^2 \mathrm{d}r\,\mathrm{d}s \\
&= \frac{2C\|U\|_\infty^2}{n\lambda}\int_0^t (1-\mathrm{e}^{-ns\lambda})\mathrm{d}s \leqslant \frac{2C\|U\|_\infty^2 t}{n\lambda} \to 0.
\end{aligned}$$

Since the martingale $Y^{(n)}$ has no continuous part and no fixed time of discontinuity (see Jacod and Shiryaev 1987: 101 and 70), the modified second characteristic with respect to a given truncation function h is given by (see Jacod and Shiryaev 1987: 79)

$$\tilde{C}_t^{(n)} = (hh^{\mathrm{T}}) * \nu_t^{(n)} = \int_0^{nt} \mathbb{1}_{\{\alpha=\alpha_o\}}(X_{s-})\int_{\mathbb{R}^d} hh^{\mathrm{T}}(n^{-1/\alpha_o}y)\eta(X_{s-}, \mathrm{d}y)\mathrm{d}s,$$

where h^{T} denotes the transpose of the vector-valued function h. Similarly, one can prove that $\tilde{C}_t^{(n)}$ converges in $L^2(\Omega, \mathbb{P})$ to the second modified characteristic of Z given by

$$\tilde{C}_t = (hh^{\mathrm{T}}) * \bar{\nu}_t = t\int_{\mathbb{R}^d/\Lambda} \mathbb{1}_{\{\alpha=\alpha_o\}}(x)\int_{\mathbb{R}^d} hh^{\mathrm{T}}(y)\eta(x, \mathrm{d}y)\pi(\mathrm{d}x).$$

□

2. Applications

The transience and recurrence of Lévy processes have been extensively studied (see, for example, Sato 1999). For processes generated by more general pseudo-differential operators with non-homogeneous symbols much less is known. In Kolokoltsov *et al.* (2002) the transience of Newtonian systems driven by stable processes was analysed. We now wish to use the limit theorem of the previous section to link the recurrence and transience of X to those of the limiting Lévy process Y. This will give criteria for the recurrence and transience of the stable-like process X.

For this section we assume that $\pi(\{\alpha_o = \alpha\}) > 0$. For a given Borel set $U \subset \mathbb{R}^d$ we define the set of recurrent paths

$$R(U) := \{\omega \in \mathbb{D}_0([0, \infty[); \forall m \in \mathbb{N}, \exists t \geqslant m \text{ such that } \omega(t) \in U\}.$$

We note that X is recurrent if and only if $\mathbb{P}_X(R(U)) = 1$ for all open sets $U \subset \mathbb{R}^d$. For a Borel set $U \subset \mathbb{R}^d$ we define the set of transient paths

$$T(U) := \{\omega \in \mathbb{D}_0([0, \infty[); \exists s \geqslant 0 \text{ such that } \omega(t) \notin U, \forall t \geqslant s\},$$

and note that X is transient if and only if $\mathbb{P}_X(T(U)) = 1$ for all open bounded sets $U \subset \mathbb{R}^d$. The set $T(U)$ is the complement of the set $R(U)$ in $\mathbb{D}_0([0, \infty[)$. Furthermore, the set $R(U)$ is open in $\mathbb{D}_0([0, \infty[)$ if U is open in $\mathbb{R}^d$. In order to apply Theorem 1, we need the following lemma.

Lemma 2. *Let Y be a strong Markov process with cadlag paths on $\mathbb{R}^d$ and let U, $V \subset \mathbb{R}^d$ be such that*

$$\delta := \inf_{x \in U} \mathbb{P}(Y_{1/2} \in V | Y_0 = x) > 0.$$

Then $\mathbb{P}_Y(R(U) \backslash R(V)) = 0$.

Proof. Let $\tau(0) = 0$ and $\tau(n) := \inf\{t \geqslant max(n, \tau(n-1) + 1); Y_t \in U\}$. Let $t \mapsto T_t$ be the semigroup associated with Y on $L^\infty(\mathbb{R}^d)$. Then by the strong Markov property for all $l \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\left[\prod_{m=l}^{k} \mathbb{1}_{V^c}(Y_{\tau(m)+1/2})\right] \\
&\quad = \mathbb{E}\left[\prod_{m=l}^{k-1} \mathbb{1}_{V^c}(Y_{\tau(m)+1/2}) \mathbb{E}\left[\mathbb{1}_{V^c}(Y_{\tau(k)+1/2}) | \mathcal{F}_{\tau(k)}\right]\right] \\
&\quad = \mathbb{E}\left[\prod_{m=l}^{k-1} \mathbb{1}_{V^c}(Y_{\tau(m)+1/2}) T_{1/2} \mathbb{1}_{V^c}(Y_{\tau(k)})\right] \\
&\quad \leqslant (1-\delta) \mathbb{E}\left[\prod_{m=l}^{k-1} \mathbb{1}_{V^c}(Y_{\tau(m)+1/2})\right] \\
&\quad \leqslant (1-\delta)^{k-l} \to 0 \qquad \text{if } k \to \infty.
\end{aligned}$$

With the set $C_m := \{Y_{\tau(m)+1/2} \in V\}$ we see that for all $l \in \mathbb{N}$,

$$\mathbb{P}_Y\left(R(U)\cap\bigcap_{m=l}^{k} C_m^c\right) \leqslant \mathbb{E}\left[\prod_{m=l}^{k} \mathbb{1}_{V^c}(Y_{\tau(m)+1/2})\right] \to 0 \qquad \text{if } k\to\infty.$$

Since $\limsup C_m \subset R(V)$, it follows that

$$\mathbb{P}_Y(R(U)\cap R(V)^c) \leqslant \mathbb{P}_Y\left(R(U)\cap\bigcup_{l\in\mathbb{N}}\bigcap_{m\geqslant l} C_m^c\right) = 0.$$

□

Lemma 3. *For all bounded open sets* $U\subset\mathbb{R}^d$, $\mathbb{P}_Y(\partial R(U)) = 0$.

Proof. We have $\partial R(U)\subset R(U_1)$, where $U_1 := \{x\in\mathbb{R}^d;\ \mathrm{dist}(x, U) < 1\}$. Further, Y is a strong Markov process on $\mathbb{R}^d$ and

$$\delta := \inf_{x\in U_1} \mathbb{P}(Y_{1/2}\in U | Y_0 = x) > 0.$$

Therefore, one can apply Lemma 2. Since $R(U)$ is open, it follows that

$$\mathbb{P}_Y(\partial R(U)) = \mathbb{P}_Y(\partial R(U)\backslash R(U)) \leqslant \mathbb{P}_Y(R(U_1)\backslash R(U)) = 0.$$

Proposition 1. *X is recurrent if and only if Y is recurrent.*

Proof. By Lemma 3, $\mathbb{P}_Y(\partial R(B)) = 0$ for all open balls B in $\mathbb{R}^d$. It follows from Theorem 1 that

$$\lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(R(B)) = \mathbb{P}_Y(R(B)).$$

Now recurrence of X implies that $\mathbb{P}_X(R(U)) = 1$ for all open sets U. Therefore, $\mathbb{P}_{X^{(n)}}(R(U)) = 1$ for all $n\in\mathbb{N}$. This implies that $\mathbb{P}_Y(R(U)) = 1$ for all open $U\subset\mathbb{R}^d$ and thus the recurrence of Y.

Now assume that Y is recurrent. Thus

$$\lim_{n\to\infty} \mathbb{P}_X(R(B_{n^{1/\alpha}}(0))) = \lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(R(B_1(0))) = \mathbb{P}_Y(R(B_1(0))) = 1.$$

For all $\epsilon > 0$ there exists $r\in\mathbb{N}$ such that $\mathbb{P}_X(R(B_r(0))) \geqslant 1-\epsilon$. Since the fundamental solutions are positive (see Kolokoltsov 2000: 759), for a given open set $U\subset\mathbb{R}^d$, one has

$$\delta := \inf_{x\in B_r(0)} \int_U p(1/2, x, y)\mathrm{d}y > 0,$$

where $p(t, x, y)$ is the density of the transition probability of X with respect to the Lebesgue measure. Furthermore, X satisfies the strong Markov property. By Lemma 2 we have $\mathbb{P}_X(R(B_r(0))\backslash R(U)) = 0$. Thus, we have $\mathbb{P}_X(R(U)) \geqslant \mathbb{P}_X(R(B_r(0))) \geqslant 1-\epsilon$. Letting ϵ go to zero proves the proposition. □

Proposition 2. *X is transient if and only if Y is transient.*

Proof. Since $\partial T(U) = \partial R(U)$, we know that $\mathbb{P}_Y(\partial T(U)) = 0$. Therefore, Theorem 1 implies

$$\lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(T(U)) = \mathbb{P}_Y(T(U)).$$

Now if X is transient, then $\mathbb{P}_X(T(U)) = 1$ for all bounded $U \subset \mathbb{R}^d$. From this it follows that $\mathbb{P}_{X^{(n)}}(T(U)) = 1$. This implies $\mathbb{P}_Y(T(U)) = 1$ for all bounded $U \subset \mathbb{R}^d$ and therefore the transience of Y.

Assume that Y is transient, thus

$$\mathbb{P}_X(T(B_r(0))) \geqslant \mathbb{P}_{X^{(n)}}(T(B_r(0))) \to \mathbb{P}_Y(T(B_r(0))) = 1.$$

Therefore, $\mathbb{P}_X(T(B_r(0))) = 1$ for all $r > 0$. This means that X is transient. □

The following theorem concludes the recurrence and transience properties of the process X in the case where $\pi(\{\alpha_o = \alpha\}) > 0$.

Theorem 2. *Under the assumption that* $\pi(\{\alpha_o = \alpha\}) > 0$ *we have the following two cases:*

(i) *If* $d = 1$ *the process* X *is transient if* $0 < \alpha_o < 1$ *and recurrent if* $1 \leqslant \alpha_o \leqslant 2$.
(ii) *If* $d \geqslant 2$ *then* X *is transient.*

Proof. This follows from the two previous propositions and classical results on the recurrence and transience of stable processes (see Sato 1999: 260). □

3. Large deviations

In this section we wish to treat the case where $\pi(\{\alpha = \alpha_o\}) = 0$. For this we assume that there exists a $\mu \in \mathcal{M}_s(S^{d-1})$ such that $\mu(x, \mathrm{d}\hat{y}) = \mu(\mathrm{d}\hat{y})$ for all $x \in \mathbb{R}^d$. It will turn out that in this situation the sequence $X^{(n)}$ converges in probability to the constant zero function. It is therefore interesting to understand the large deviations from this convergence. Large deviations for Markov processes with increments possessing heavy tails were investigated in Wentzell (1990). As a first step towards the large deviations of $X^{(n)}$ we analyse the Lévy process Z with Lévy measure given by

$$\eta(\mathrm{d}y) = \mu(\mathrm{d}\hat{y}) \int_{\mathbb{R}^d/\Lambda} |y|^{-1-\alpha(x)} \pi(\mathrm{d}x) \mathrm{d}|y| = \mu(d\hat{y}) \int_{\alpha_o}^{2} |y|^{-1-q} F_\alpha(\mathrm{d}q) \mathrm{d}|y|,$$

where $F_\alpha = \pi \circ \alpha^{-1}$ is the image measure of π with respect to α. We will soon see that the scaled processes

$$Z^{(n)} = n^{-1/\alpha_o}(Z_{nt} - Z_0)$$

satisfies conditions (A), (B), (C) and (D) in Wentzell (1990: 141). This gives us insight into the deviations of $Z^{(n)}$ from paths performing only a finite number of jumps and being constant between those jumps.

By Lemma 1 it is natural to expect the large deviations of $X^{(n)}$ to be similar to those of

$Z^{(n)}$. However, $X^{(n)}$ does not satisfy the conditions in Wentzell (1990). We therefore have to modify the proofs in Wentzell (1990) and use the asymptotic equality of $X^{(n)}$ and $Z^{(n)}$ given by Lemma 1 in appropriate places. Following Wentzell's method, we first define a family of measures Q^k with support on the set of paths in $\mathbb{D}_0([0, T])$ which perform k jumps and are constant between those jumps. Those paths can be represented in

$$\mathbb{E}^k := \{(x_1, \ldots, x_k, t_1, \ldots, t_k); 0 < t_1 < \ldots < t_k \leqslant T, x_i \neq 0\}$$

in terms of the following map:

$$\Gamma_k : \mathbb{E}^k \to \mathbb{D}_0([0, T]); (t_1, x_1, \ldots, t_k, x_k) \mapsto \sum_{i=1}^{k} x_i \mathbb{1}_{[t_i, t_{i+1}[} + x_k \mathbb{1}_{[t_k, T]}.$$

On $\mathbb{E}^k$ we define the measure

$$R^k(\mathrm{d}x_1, \ldots, \mathrm{d}t_k) := \bar{\eta}(\mathrm{d}x_1)\bar{\eta}(\mathrm{d}x_2 - \mathrm{d}x_1) \ldots \bar{\eta}(\mathrm{d}x_k - \mathrm{d}x_{k-1})\lambda(\mathrm{d}t_1) \ldots \lambda(\mathrm{d}t_k),$$

with $\bar{\eta}(\mathrm{d}x) := |x|^{-1-\alpha_o}\mathrm{d}|x|\mu(\mathrm{d}\hat{x})$. The image measure $Q^k := R^k \circ \Gamma_k^{-1}$ on $\mathbb{D}_0([0, T])$ describes the limiting behaviour of the scaled processes. In the following we will denote the sup-norm distance of two elements γ_1 and γ_2 from $\mathbb{D}_0([0, T])$ by $d_T(\gamma_1, \gamma_2)$. We will also denote the jump measure of $Z^{(n)}$ by $\eta^{(n)}(\mathrm{d}y)$.

Proposition 3. *For all $\tilde{f} \in C_b(\mathbb{R}^d)$ vanishing in a neighbourhood of zero and $n \in \mathbb{N}$ large, the following statements hold:*

$$\text{(i)} \quad g(n)^{-1} \int_{\mathbb{R}^d} \tilde{f}(y)\eta(\mathrm{d}y) \to \int_{\mathbb{R}^d} \tilde{f}(y)\bar{\eta}(\mathrm{d}y),$$

$$\text{(ii)} \quad \eta^{(n)}(y; |y| > \delta) \leqslant K(\delta)g(n) \text{ for all } \delta > 0,$$

$$\text{(iii)} \quad \int_{R^d} (1 \wedge y^2)\eta^{(n)}(\mathrm{d}y) \leqslant Kg(n),$$

with suitable constants $K(\delta)$, $K > 0$ and

$$g(n) := \int_{\alpha_o}^{2} n^{1-q/\alpha_o} F_\alpha(\mathrm{d}q).$$

Proof. A Taylor expansion of $q \mapsto |y|^{-1-q}$ gives

$$\begin{aligned}
&\int_{\alpha_o}^{2} n^{1-q/\alpha_o}|y|^{-1-q} F_\alpha(\mathrm{d}q) \\
&= |y|^{-1-\alpha_o} \int_{\alpha_o}^{2} n^{1-q/\alpha_o} F_\alpha(\mathrm{d}q) + \int_{\alpha_o}^{2} n^{1-q/\alpha_o}|y|^{-1-\xi(q)}\log(|y|)F_\alpha(\mathrm{d}q) \\
&=: g(n)|y|^{-1-\alpha_o}\mu(\mathrm{d}\hat{y}) + h(n, y)
\end{aligned}$$

for suitable $\xi(q) \in [\alpha_o, q]$. Therefore, for all $\tilde{f} \in C_b(\mathbb{R}^d)$ vanishing in a neighbourhood of zero and $n \to \infty$,

$$g(n)^{-1} \int_{\mathbb{R}^d} \tilde{f}(y) \eta(\mathrm{d}y) \to \int_{\mathbb{R}^d} \tilde{f}(y) \bar{\eta}(\mathrm{d}y)$$

with $\bar{\eta}(\mathrm{d}y) = |y|^{-1-\alpha_o}\, \mathrm{d}|y| \mu(\mathrm{d}\hat{y})$. This proves statement (i).

Furthermore, for large $n \in \mathbb{N}$,

$$\begin{aligned}
\eta^{(n)}(y;\, |y| > \delta) &= \int_{S^{d-1}} \int_\delta^\infty \int_{\alpha_o}^2 n^{1-q/\alpha_o} |y|^{-1-q} F_\alpha(\mathrm{d}q) \mathrm{d}|y| \mu(\mathrm{d}\hat{y}) \\
&= g(n) \int_{S^{d-1}} \int_\delta^\infty \left(|y|^{-1-\alpha_o} + \frac{h(n, y)}{g(n)} \right) \mathrm{d}|y| \mu(\mathrm{d}\hat{y}) \\
&\leqslant g(n)(\bar{\eta}(y;\, |y| > \delta) + 1).
\end{aligned}$$

This proves (ii) with $K(\delta) := \bar{\eta}(y;\, |y| > \delta) + 1$.

Finally, for $n \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned}
&\int_{R^d} (1 \wedge y^2) \eta^{(n)}(\mathrm{d}y) \\
&\quad = g(n) \int_{S^{d-1}} \int_0^\infty (1 \wedge y^2) \left(|y|^{-1-\alpha_o} + \frac{h(n, y)}{g(n)} \right) \mathrm{d}|y| \mu(\mathrm{d}\hat{y}) \\
&\quad \leqslant g(n) \left(\int_{R^d} (1 \wedge y^2) \bar{\eta}(\mathrm{d}y) + 1 \right).
\end{aligned}$$

Therefore, (iii) follows with

$$K := \int_{R^d} (1 \wedge y^2) \bar{\eta}(\mathrm{d}y) + 1.$$

□

Corollary 1. *One has for all* $A \subset \mathbb{D}_0([0, T])$ *satisfying* $d_T(A, \Gamma_{k-1}(\mathbb{E}^k)) > 0$ *and* $\lim_{\delta \downarrow 0} Q^k(A_{+\delta} \backslash A_{-\delta}) = 0$*, that*

$$\mathbb{P}(Z^{(n)} \in A) = (g(n))^k Q^k(A) + o((g(n))^k).$$

In particular, $Z^{(n)}$ *converges in probability to the zero function.*

Proof. By Proposition 3, conditions (A), (B), (C) and (D) in Wentzell (1990) are satisfied. Therefore, the theorem presented in Wentzell (1990: 155) proves our claim. □

We now return to the large deviations of $X^{(n)}$. Its jump measure is given by $\eta^{(n)}(x, \mathrm{d}\xi) := n\eta(n^{1/\alpha_o} x, \mathrm{d}\xi)$. We will denote by $\mathbb{P}_x^{(n)}$ the distribution of the Markov process generated by

$$L^{(n)}f(x) := \int_{\mathbb{R}^d}\left(f(x+\xi) - f(x) - \frac{\langle \xi, \nabla f(x)\rangle}{1+|\xi|^2}\mathbb{1}_{B_1(0)}(\xi)\right)\eta^{(n)}(x, \mathrm{d}\xi)$$

on $\mathbb{D}_x([0, T])$ and by $\mathbb{E}_x^{(n)}[F]$ the corresponding expectation of a random variable $F : \mathbb{D}_x([0, T]) \to \mathbb{R}$. We also define a sequence of stopping times on $\mathbb{D}_0([0, T])$ by $\tau_0^\epsilon := 0$, $\tau_k^\epsilon := \tau^\epsilon(\tau_{k-1}^\epsilon)$ with

$$\gamma \mapsto \tau^\epsilon(s) := \inf\{t > s;\ |\gamma_t - \gamma_{t-}| \geqslant \epsilon\}.$$

When there is no danger of ambiguity we will often write τ^ϵ for τ_1^ϵ. Finally, we define

$$\nu^\epsilon := \mathrm{card}\{k > 0;\ \tau^\epsilon \leqslant T\}.$$

In the proof of the following theorem we will need several results concerning those stopping times. These results, Lemmas 4–8, are straightforward modifications of lemmas presented in Wentzell (1990: 145–148) and are given in Section 4 for the reader's convenience.

Theorem 3. *For all* $A \subset \mathbb{D}_0([0, T])$ *satisfying* $d_T(A, 0) > 0$ *and* $\lim_{\delta\downarrow 0} Q^1(A_{+\delta}\backslash A_{-\delta}) = 0$,

$$\mathbb{P}^{(n)}(A) = g(n)Q^1(A) + o(g(n)).$$

In particular, $X^{(n)}$ *converges in probability to the zero function.*

Proof. In order to prove the statement we prove that for arbitrary $\kappa > 0$ and for large $n \in \mathbb{N}$,

$$g(n)(Q^1(A) - \kappa) \leqslant \mathbb{P}_0^{(n)}(A) \leqslant g(n)(Q^1(A) + \kappa).$$

There exists a $\delta_0 > 0$ such that $\delta_0 \leqslant d_T(A, 0)/6$ and $Q^1(A_{2\delta_0}\backslash A_{-2\delta_0}) < \kappa/2$. Define

$$\chi_-(t, x, \xi) := H(d_T(\Gamma(t, x+\xi), A^c)/\delta_0 - 1),$$

$$\chi_+(t, x, \xi) := 1 - H(d_T(\Gamma(t, x+\xi), A)/\delta_0 - 1),$$

where

$$H(z) := \begin{cases} 1, & \text{for } z \geqslant 1, \\ z, & \text{for } 0 \leqslant z \leqslant 1, \\ 0, & \text{for } z \leqslant 0. \end{cases}$$

It therefore follows that $\chi_\pm(t, x, \xi) = 0$ for $|x + \xi| \leqslant 3\delta_0$. Furthermore, we define

$$V_\pm(t, x, \xi) := \chi_\pm(t, x, \xi)(1 - H(|x|/\delta_0 - 1)).$$

and, for $0 < \delta < \delta_0$,

$$\overline{\chi}_+(t, \xi) := \sup_{|x|<\delta} \chi_+(t, x, \xi)$$

$$\underline{\chi}_-(t, \xi) := \inf_{|x|<\delta} \chi_-(t, x, \xi).$$

We then have $V_\pm(t, x, \xi) = 0$ for $|\xi| < \delta_0$ and $\overline{\chi}_+(t, \xi) = 0 = \underline{\chi}_-(t, \xi)$ for $|\xi| < 2\delta_0$. If δ in the definition of $\overline{\chi}_+$ and $\underline{\chi}_-$ is chosen small enough, we obtain

$$\left|\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \bar{\eta}(\mathrm{d}\xi)\chi_+(t,0,\xi) - \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \bar{\eta}(\mathrm{d}\xi)\overline{\chi}_+(t,\xi)\right| \leqslant \kappa/4$$

and

$$\left|\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \bar{\eta}(\mathrm{d}\xi)\chi_-(t,0,\xi) - \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \bar{\eta}(\mathrm{d}\xi)\underline{\chi}_-(t,\xi)\right| \leqslant \kappa/4.$$

Finally we choose $\epsilon := \delta/(2m-1)$ with $m \geqslant 2$.

We have by Lemma 6 and $d_T(A, 0) > 0$ that

$$\mathbb{P}_0^{(n)}(A, \nu^\epsilon = 0) \leqslant \mathbb{P}_0^{(n)}\big(\nu^\epsilon = 0, d_T(\gamma, 0) > \delta\big) = O\big((g(n))^2\big).$$

Further, by Lemma 7 and the strong Markov property we have

$$\begin{aligned}
&\mathbb{P}_0^{(n)}(A, \nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) \geqslant \delta)\\
&\leqslant \mathbb{P}_0^{(n)}(\nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) \geqslant \delta)\\
&\leqslant \mathbb{P}_0^{(n)}\left(\sup_{t\in[0,\tau^\epsilon[} |\gamma_t - \gamma_0| \geqslant \delta\right) + \mathbb{P}_0^{(n)}\left(\sup_{t\in[\tau^\epsilon,\tau_2^\epsilon[} |\gamma_t - \gamma_{\tau^\epsilon}| \geqslant \delta\right)\\
&\leqslant \mathbb{P}_0^{(n)}\left(\sup_{t\in[0,\tau^\epsilon[} |\gamma_t - \gamma_0| \geqslant \delta\right) + \mathbb{E}_0^{(n)}\left[\mathbb{P}_{\gamma_{\tau^\epsilon}}^{(n)}\left(\sup_{t\in[0,\tau^\epsilon[} |\gamma_t - \gamma_0| \geqslant \delta\right)\right]\\
&= \mathrm{O}\big((g(n)^2)\big).
\end{aligned}$$

From Lemma 5 it follows that $\mathbb{P}_0^{(n)}(A, \nu^{(n)} \geqslant 2) = O((g(n))^2)$. Those considerations imply that

$$\begin{aligned}
\mathbb{P}_0^{(n)}(A) &= \mathbb{P}_0^{(n)}(A, \nu^\epsilon = 0) + \mathbb{P}_{x_0}^{(n)}(A, \nu^\epsilon \geqslant 2)\\
&\quad + \mathbb{P}_0^{(n)}(A, \nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) < \delta)\\
&\quad + \mathbb{P}_0^{(n)}(A, \nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) \geqslant \delta)\\
&= \mathbb{P}_0^{(n)}(A, \nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) \geqslant \delta) + O\big((g(n))^2\big).
\end{aligned}$$

For $\gamma \in \{\nu^\epsilon = 1, d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) < \delta\}$ the following two statements are valid:

$$\gamma \in A \text{ implies } \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon}) \in A_{+\delta} \text{ implies } \chi_+(\tau^\epsilon, \gamma_{\tau^\epsilon-}, \Delta\gamma_{\tau^\epsilon}) = 1,$$

$$\chi_-(\tau^\epsilon, \gamma_{\tau^\epsilon-}, \Delta\gamma_{\tau^\epsilon}) > 0 \text{ implies } \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon}) \in A_{-\delta} \text{ implies } \gamma \in A.$$

Furthermore, it follows from $d_T(\gamma, \Gamma(\tau^\epsilon, \gamma_{\tau^\epsilon})) < \delta$ that $|\gamma_{\tau^\epsilon-}| \leqslant \delta$, which implies that

$$\chi_\pm(\tau^\epsilon, \gamma_{\tau^\epsilon-}, \Delta\gamma_{\tau^\epsilon}) = V_\pm(\tau^\epsilon, \gamma_{\tau^\epsilon-}, \Delta\gamma_{\tau^\epsilon}).$$

Therefore, we obtain the following upper and lower bounds:

$$\begin{aligned}
&\mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon=1,\mathrm{d}_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}V_-(\tau^\epsilon,\gamma_{\tau^\epsilon-},\Delta\gamma_{\tau^\epsilon})\big]\\
&\quad\leqslant \mathbb{P}_0^{(n)}(A,\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta)\\
&\quad\leqslant \mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}V_+(\tau^\epsilon,\gamma_{\tau^\epsilon-},\Delta\gamma_{\tau^\epsilon})\big].
\end{aligned}$$

As shown above, we have by Lemma 7 and the strong Markov property that

$$\begin{aligned}
&\mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))\geqslant\delta\}}V_\pm(\tau^\epsilon,\gamma_{\tau^\epsilon-},\Delta\gamma_{\tau^\epsilon})\big]\\
&\quad\leqslant \mathbb{P}_0^{(n)}(\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))\geqslant\delta)=O\big((g(n))^2\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}V_\pm(\tau^\epsilon,\gamma_{\tau^\epsilon-},\Delta\gamma_{\tau^\epsilon})\big]\\
&\quad= \mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon=1\}}V_\pm(\tau^\epsilon,\gamma_{\tau^\epsilon-},\Delta\gamma_{\tau^\epsilon})\big]+O\big((g(n))^2\big)\\
&\quad= \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1\}}\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]+O\big((g(n))^2\big).
\end{aligned}$$

However, Lemma 5 implies that

$$\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon\geqslant2\}}\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]\\
&\quad\leqslant \mathbb{E}_0^{(n)}\big[\mathbb{1}_{\{\nu^\epsilon\geqslant2\}}\nu^\epsilon\big]\leqslant \mathbb{E}_0^{(n)}[\nu^\epsilon(\nu^\epsilon-1)]=O\big((g(n))^2\big).
\end{aligned}$$

Since $\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)=0$ for $\nu^\epsilon=0$, we obtain from Lemma 4 that

$$\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1\}}\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]\\
&\quad= \mathbb{E}_0^{(n)}\left[\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]-\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon\geqslant2\}}\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]\\
&\quad= \mathbb{E}_0^{(n)}\left[\sum_{t\in[0,T]}V_\pm(t,\gamma_{t-},\Delta\gamma_t)\right]+O\big((g(n))^2\big)\\
&\quad= \mathbb{E}_0^{(n)}\left[\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]+O\big((g(n))^2\big).
\end{aligned}$$

It follows from Lemma 7 that

$$
\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0, d_{\mathrm{T}}(\gamma,0)\geqslant\delta\}}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\leqslant \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0, d_{\mathrm{T}}(\gamma,0)\geqslant\delta\}}\int_0^T \mathrm{d}t\eta^{(n)}(\gamma_t, B_\delta(0)^c)\right]\\
&\leqslant C\mathbb{P}_0^{(n)}(\nu^\epsilon=0, d_{\mathrm{T}}(\gamma,0)\geqslant\delta) = \mathrm{O}\big((g(n))^2\big).
\end{aligned}
$$

Here we remark on one main difference from the proof in Wentzell (1990). Since we cannot prove that the probability of $\{\nu^\epsilon = 1\}$ is $O((g(n))^2)$, we have to split up the time integration on the event $\{\nu^\epsilon = 1\}$ according to the stopping time τ^ϵ. We have by Lemmas 5 and 7 that

$$
\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon\geqslant 1\}}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&= \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon\geqslant 2\}}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad + \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))\geqslant\delta\}}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad + \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&= \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_0^{\tau^\epsilon} \mathrm{d}t \int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\chi_\pm(t,\gamma_t,\xi)\right]\\
&\quad + \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_{\tau^\epsilon}^T \mathrm{d}t \int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad + O\big((g(n))^2\big).
\end{aligned}
$$

To the second term we can apply the strong Markov property. From Lemmas 1 and 5 and (i) in Proposition 3 it then follows that

$$
\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_{\tau^\epsilon}^T \mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad\leqslant \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\tau^\epsilon\leqslant T\}}\int_{\tau^\epsilon}^{\tau_2^\epsilon}\mathrm{d}t\int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\right]\\
&\quad\leqslant \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\tau^\epsilon\leqslant T\}}\mathbb{E}_{\gamma_{\tau^\epsilon}}^{(n)}\left[\int_0^{\tau^\epsilon}\mathrm{d}t\int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\right]\right]\\
&\quad\leqslant \int_0^T \mathrm{d}t\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\tau^\epsilon\leqslant T\}}\left(\int_{B_\delta(0)^c}\eta^{(n)}(\mathrm{d}\xi)+nC\mathrm{e}^{-\lambda\mathrm{nt}}\right)\right]\\
&\quad\leqslant \mathbb{P}_0^{(n)}(\nu^\epsilon\geqslant 1)\int_0^T\mathrm{d}t\big(\eta^{(n)}(\mathrm{B}_\delta(0)^c)+nK\mathrm{e}^{-\lambda\mathrm{nt}}\big)\\
&\quad= \mathrm{O}(g(n))O(g(n)).
\end{aligned}
$$

This proves that

$$
\begin{aligned}
&\mathbb{E}_0^{(n)}\left[\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad= \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0,d_T(\gamma,0)\geqslant\delta\}}\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\qquad+ \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0,d_T(\gamma,0)<\delta\}}\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\qquad+ \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon\geqslant 1\}}\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)V_\pm(t,\gamma_t,\xi)\right]\\
&\quad= \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0,d_T(\gamma,0)<\delta\}}\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\chi_\pm(t,\gamma_t,\xi)\right]\\
&\qquad+ \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1,d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_0^{\tau^\epsilon}\mathrm{d}t\int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\chi_\pm(t,\gamma_t,\xi)\right]\\
&\qquad+ O\big((g(n))^2\big).
\end{aligned}
$$

We have so far seen that

$$\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_0^{\tau^\epsilon}\mathrm{d}t\int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\underline{\chi}_-(t,\xi)\right]$$

$$+\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0, d_T(\gamma,0)<\delta\}}\int_0^{T}\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\underline{\chi}_-(t,\xi)\right]$$

$$\leqslant \mathbb{P}_0^{(n)}(A)+O\big((g(n))^2\big)$$

$$\leqslant \mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=1, d_T(\gamma,\Gamma(\tau^\epsilon,\gamma_{\tau^\epsilon}))<\delta\}}\int_0^{\tau^\epsilon}\mathrm{d}t\int_{B_\delta(0)^c}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\overline{\chi}_+(t,\xi)\right]$$

$$+\mathbb{E}_0^{(n)}\left[\mathbb{1}_{\{\nu^\epsilon=0, d_T(\gamma,0)<\delta\}}\int_0^{T}\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\overline{\chi}_+(t,\xi)\right].$$

We now concentrate on the lower bound. The upper bound can be treated analogously. Applying the arguments for $V_\pm$ above to $\underline{\chi}_-$, we see that the lower bound is equal to

$$\mathbb{E}_0^{(n)}\left[\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\underline{\chi}_-(t,\xi)\right]+O\big((g(n))^2\big).$$

Lemma 1 and Proposition 3 imply that

$$\mathbb{E}_0^{(n)}\left[\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\eta^{(n)}(\gamma_t,\mathrm{d}\xi)\underline{\chi}_-(t,\xi)\right]$$

$$\geqslant \int_0^T\mathrm{d}t\left(\int_{\mathbb{R}^d}\eta^{(n)}(\mathrm{d}\xi)\underline{\chi}_-(t,\xi)-nC\mathrm{e}^{-\lambda nt}\right)+O\big((g(n))^2\big)$$

$$\geqslant g(n)\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\bar{\eta}(\mathrm{d}\xi)\underline{\chi}_-(t,\xi)+O\big((g(n))^2\big)$$

$$\geqslant g(n)\left(\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\bar{\eta}(\mathrm{d}\xi)\chi_-(t,0,\xi)-\kappa/4\right)+O\big((g(n))^2\big)$$

$$\geqslant g(n)\left(\int_0^T\mathrm{d}t\int_{\mathbb{R}^d}\bar{\eta}(\mathrm{d}\xi)\mathbb{1}_{\{\Gamma(t,\xi)\in A_{-\delta_0}\}}-\kappa/4\right)+O\big((g(n))^2\big)$$

$$\geqslant g(n)\big(Q^1(A_{-\delta})-\kappa/2\big)+O\big((g(n))^2\big)$$

$$\geqslant g(n)\big(Q^1(A)-\kappa\big).$$

Using the same arguments for the upper bound proves the theorem. □

A refinement of the proof of the theorem gives rise to the following result.

Theorem 4. *For all* $A \subset \mathbb{D}_0([0, T])$ *with* $d_T(A, \Gamma_{k-1}(\mathbb{E}^k)) > 0$ *and* $\lim_{\delta \downarrow 0} Q^k(A_{+\delta} \backslash A_{-\delta}) = 0$*,*

$$\mathbb{P}(X^{(n)} \in A) = (g(n))^k Q^k(A) + o((g(n))^k).$$

4. Auxiliary results

The following results are slight modifications of results from Wentzell (1990).

Lemma 4. *Let* $V(t_1, x_1, \xi_1, \ldots, t_k, x_k, \xi_k)$ *be bounded and continuous such that there exists a* $\delta > 0$ *such that* $V(t_1, x_1, \xi_1, \ldots, t_k, x_k, \xi_k) = 0$ *if* $|\xi_i| < \delta$ *for* $1 \leqslant i \leqslant k$*. Then*

$$\mathbb{E}_x\left[\sum_{0 \leqslant t_1 < \ldots < t_k \leqslant T} V(t_1, \gamma_{t_1-}, \Delta\gamma_{t_1}, \ldots, t_k, \gamma_{t_k-}, \Delta\gamma_{t_k})\right]$$

$$= \mathbb{E}_x\left[\left[\int_0^T \mathrm{d}t_1 \int_{\mathbb{R}^d} \eta(\gamma_{t_1}, \mathrm{d}\xi_1) \mathbb{E}_{\gamma_{t_1}+\xi_1}\left[\left[\int_{t_1}^T \mathrm{d}t_2 \eta(\gamma_{t_2}, \mathrm{d}\xi_k) \times \ldots\right.\right.\right.\right.$$

$$\times\, \mathbb{E}_{\gamma_t} k - 1_{+\xi_{k-1}}\left[\left[\int_{t_{k-1}}^T \mathrm{d}t_k \int_{\mathbb{R}^d} \eta(\gamma_{t_k}, \mathrm{d}\xi_k) V(t_1, \gamma_{t_1}, \xi_1, \ldots, t_k, \gamma_{t_k}, \xi_k)\right] \ldots\right]\right],$$

where the expectation of γ_{t_i} *is with respect to* $\mathbb{P}_{\gamma_{t_{i-1}}+\xi_{i-1}}$ *for* $2 \leqslant i \leqslant k$*.*

Proof. See Wentzell (1990: 23). □

Lemma 5. *We have, uniformly with respect to* $x \in \mathbb{R}^d$*,*

$$\mathbb{E}_x[\nu^\epsilon(\nu^\epsilon - 1) \cdots (\nu^\epsilon - k + 1)] = O((g(n))^k)$$

and

$$\mathbb{P}_x(\nu^\epsilon \geqslant k) = O((g(n))^k).$$

Proof. We have, by Lemma 1,

$$\begin{aligned}\mathbb{E}_x\left[\eta^{(n)}(X_t^{(n)}, B_\epsilon(0)^c)\right] &= \mathbb{E}_x\left[n\eta(X_{nt}, n^{1/\alpha_o} B_\epsilon(0)^c)\right] \\ &\leqslant n\eta(n^{1/\alpha_o} B_\epsilon(0)^c) + nC\mathrm{e}^{-\lambda nt} = \eta^{(n)}(B_\epsilon(0)^c) + nC\mathrm{e}^{-\lambda nt}.\end{aligned}$$

Define $V(y_1, \ldots, y_k) := \mathbb{1}_{B_\epsilon(0)^c}(y_1) \cdots \mathbb{1}_{B_\epsilon(0)^c}(y_k)$. From Lemma 4 and (ii) in Proposition 3 we see, for large $n \in \mathbb{N}$, that

$$
\begin{aligned}
&\mathbb{E}_x^{(n)}[\nu^\epsilon(\nu^\epsilon-1)\cdots(\nu^\epsilon-k+1)]\\
&= k!\mathbb{E}_x^{(n)}\left[\sum_{0<t_1<\ldots<t_k\leqslant T} V(\Delta\gamma_{t_1},\ldots,\Delta\gamma_{t_k})\right]\\
&= k!\mathbb{E}_x^{(n)}\left[\int_0^T \mathrm{d}t_1\int_{B_\epsilon(0)^c}\eta^{(n)}(\gamma_{t_1},\mathrm{d}\xi_1)\mathbb{E}^{(n)}_{\gamma_{t_1}+\xi_1}\left[\int_{t_1}^T\mathrm{d}t_2\times\ldots\right.\right.\\
&\quad\left.\left.\times\,\mathbb{E}^{(n)}_{\gamma_{t_{k-1}}+\xi_{k-1}}\left[\int_{t_{k-1}}^T\mathrm{d}t_k\int_{B_\epsilon(0)^c}\eta^{(n)}(\gamma_{t_k},\mathrm{d}\xi_k)\right]\ldots\right]\right]\\
&\leqslant k!\int_0^T\mathrm{d}t_1\mathbb{E}_x^{(n)}\left[\int_{B_\epsilon(0)^c}\eta^{(n)}(\gamma_{t_1},\mathrm{d}\xi_1)\int_{t_1}^T\mathrm{d}t_2\mathbb{E}^{(n)}_{\gamma_{t_1}+\xi_1}[\ldots\right.\\
&\quad\left.\times\int_{t_{k-1}}^T\mathrm{d}t_k\left(\eta^{(n)}(\mathrm{B}_\epsilon(0))+Cn\mathrm{e}^{-\lambda n t_k}\right)\ldots\right]\Bigg]\\
&\leqslant k!\int_0^T\mathrm{d}t_1\int_{t_1}^T\mathrm{d}t_2\ldots\int_{t_{k-1}}^T\mathrm{d}t_k\left(\eta^{(n)}(\mathrm{B}_\epsilon(0)^c)\right)^{\mathrm{k}}+o\left((g(n)^k)\right)\\
&\leqslant k!2\mathrm{T}^k(K(\epsilon))^k(g(n))^k.
\end{aligned}
$$

Since we have

$$\mathbb{P}_x^{(n)}(\nu^\epsilon\geqslant k)=\mathbb{P}_x^{(n)}(\nu^\epsilon(\nu^\epsilon-1)\cdots(\nu^\epsilon-k+1)\geqslant k!).$$

the second part follows from the first part together with Chebyshev's inequality. □

Lemma 6. *For* $1>\epsilon>0$ *one has, uniformly with respect to* $x\in\mathbb{R}^d$,

$$\mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[}|\gamma_t-x|\geqslant\epsilon\right)=O(g(n)).$$

Proof. We define the truncation of $\gamma\in\mathbb{D}_x([0,T])$ by

$$\gamma_t^*:=\begin{cases}\gamma_t, & \text{for } 0\leqslant t<\tau^\epsilon,\\ \gamma_{\tau^\epsilon-}, & \text{for } \tau^\epsilon\leqslant t\leqslant T.\end{cases}$$

If γ is distributed according to the distribution $\mathbb{P}_x^{(n)}$, we see that for a $f\in C^2(\mathbb{R}^d)$ the compensator of $f(\gamma^*)$ is given by the expression

$$f(x)+\int_0^{t\wedge\tau^\epsilon}L^{(n)}f(\gamma_s)\mathrm{d}s.$$

This gives, with Kolmogorov's inequality and (iii) in Proposition 3,

$$
\begin{aligned}
&\mathbb{P}_x^{(n)}(\sup|\gamma_t - x|;\ t \in [0,\ T] \cap [0,\ \tau^\epsilon[\geqslant \epsilon) \\
&= \mathbb{P}_x^{(n)}\big(\sup|\gamma_t^* - x|;\ t \in [0,\ T] \geqslant \epsilon\big) \\
&\leqslant \frac{1}{\epsilon^2} \sum_{i=1}^d \mathbb{E}_x^{(n)}\left[\int_0^{T\wedge\tau^\epsilon} \int_{B_\epsilon(0)} \xi_i^2 \eta^{(n)}(\gamma_s,\ \mathrm{d}\xi)\right] \\
&\leqslant \frac{1}{\epsilon^2} \sum_{i=1}^d \int_0^T \mathbb{E}_x\left[\int_{B_\epsilon(0)} \xi_\mathrm{i}^2 \eta^{(n)}(\mathrm{X}_s^{(n)},\ \mathrm{d}\xi)\right] \\
&\leqslant \frac{1}{\epsilon^2} \int_0^T \sum_{i=1}^d \left(\int_{B_\epsilon(0)} \xi_\mathrm{i}^2 \eta^{(n)}(\mathrm{d}\xi) + Cn\mathrm{e}^{-\lambda \mathrm{ns}}\right)\mathrm{d}s \\
&= O(g(n)).
\end{aligned}
$$

□

Lemma 7. *For* $1 > \epsilon > 0$ *one has, uniformly with respect to* $x \in \mathbb{R}^d$,

$$
\mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[} |\gamma_t - x| \geqslant (2m-1)\epsilon\right) = O((g(n))^m).
$$

Proof. We proceed by induction. For $m = 1$ the claim was proved in the previous lemma. We define a new stopping time on $\mathbb{D}_x([0,\ T])$ by

$$
\gamma \mapsto \sigma^\epsilon(x) := \tau^\epsilon \wedge \inf\{t \geqslant 0;\ |\gamma_t - x| \geqslant \epsilon\}.
$$

For $m \geqslant 2$,

$$
\begin{aligned}
A &:= \left\{\sup_{t\in[0,T]\cap[0,\tau^\epsilon[} |\gamma_t - x| \geqslant (2m-1)\epsilon\right\} \\
&\subset \{\inf\{t \geqslant 0;\ |\gamma_t - x| \geqslant \epsilon\} < \tau^\epsilon\} \\
&= \{\sigma^\epsilon(x) < \tau^\epsilon\}.
\end{aligned}
$$

This implies $|\gamma_{t-} - x| \leqslant \epsilon$ for all $\gamma \in A$ and $t \leqslant \tau^\epsilon$. Since no jump of size larger than ϵ occurs before τ^ϵ, we see that $|\gamma_t - x| \leqslant 2\epsilon$ for all $\gamma \in A$ and $t \leqslant \tau^\epsilon$. This implies that

$$
\begin{aligned}
&\left\{\sup_{t\in[0,T]\cap[0,\tau^\epsilon[} |\gamma_t - x| \geqslant (2m-1)\epsilon\right\} \\
&\subset \left\{\sigma^\epsilon(x) < \tau^\epsilon,\ \sup_{t\in[\sigma^\epsilon(x),T]\cap[\sigma^\epsilon(x),\tau^\epsilon[} |\gamma_t - x| \geqslant (2m-3)\epsilon\right\}
\end{aligned}
$$

The strong Markov property, together with Lemma 6, implies that

$$\begin{aligned}
&\mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[}|\gamma_t - x| \geqslant (2m-1)\epsilon\right)\\
&\quad\leqslant \mathbb{P}_x^{(n)}\left(\sigma^\epsilon(x) < \tau^\epsilon, \sup_{t\in[\sigma^\epsilon(x),T]\cap[\sigma^\epsilon(x),\tau^\epsilon[}|\gamma_t - x| \geqslant (2m-3)\epsilon\right)\\
&\quad\leqslant \mathbb{E}_x^{(n)}\left[\mathbb{1}_{\{\sigma^\epsilon(x)<\tau^\epsilon\}}\mathbb{P}_{\gamma_{\sigma^\epsilon(x)}}^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[}|\gamma_t - \gamma_0| \geqslant (2m-3)\epsilon\right)\right]\\
&\quad\leqslant \mathbb{P}_x^{(n)}(\sigma^\epsilon(x) < \tau^\epsilon)\sup_y \mathbb{P}_y^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[}|\gamma_t - y| \geqslant (2m-3)\epsilon\right)\\
&\quad= O(g(n))O((g(n))^{m-1}),
\end{aligned}$$

where we have used the fact that

$$\{\sigma^\epsilon(x) < \tau^\epsilon\} = \{\sup|\gamma_t - x|;\ t\in[0,\ T]\cap[0,\ \tau^\epsilon[\geqslant \epsilon\}.$$

□

Lemma 8. *For any* $k\in\mathbb{N}_0$, $m\in\mathbb{N}$, *one has, uniformly for all* $x\in\mathbb{R}^d$,

$$\mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[\tau_k^\epsilon,\tau_{k+1}^\epsilon[}|\gamma_t - \gamma_{\tau_k^\epsilon}| \geqslant (2m-1)\epsilon\right) = O((g(n))^m).$$

Proof. An induction argument using the the strong Markov property and Lemma 7 gives rise to

$$\begin{aligned}
&\mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[\tau_k^\epsilon,\tau_{k+1}^\epsilon[}|\gamma_t - \gamma_{\tau_k^\epsilon}| \geqslant (2m-1)\epsilon\right)\\
&\quad\leqslant \sup_x \mathbb{P}_x^{(n)}\left(\sup_{t\in[0,T]\cap[0,\tau^\epsilon[}|\gamma_t - x| \geqslant (2m-1)\epsilon\right)\\
&\quad= O((g(n))^m).
\end{aligned}$$

□

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