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On L^α-convergence $(1 \leqslant \alpha \leqslant 2)$ for a bisexual branching process with population-size dependent mating

MANUEL MOLINA*, MANUEL MOTA** and ALFONSO RAMOS†

Department of Mathematics, University of Extremadura, 06071 Badajoz, Spain.
*E-mail: *mmolina@unex.es; **mota@unex.es; †aramos@unex.es*

We investigate the L^α-convergence, $1 \leqslant \alpha \leqslant 2$, of the class of bisexual branching processes with population-size dependent mating, suitably normalized, to a finite limit W such that $P(W > 0) > 0$. Through different probabilistic approaches, we provide some necessary and sufficient conditions for such convergence. In particular we establish, by analogy with the classical Kesten and Stigum result for Bienaymé–Galton–Watson processes, a logarithmic criterion for L^1-convergence.

Keywords: bisexual processes; branching processes; limiting behaviour; martingales; population-size dependent processes

1. Introduction

In order to describe the probabilistic evolution of populations by sexual reproduction, Daley (1968) introduced the *bisexual Galton–Watson process* as a discrete-time branching model. In Daley's model, a deterministic function establishes the number of couples (female–male mating units) produced. The model has received considerable attention in the literature. In particular, its extinction probability is studied in Daley (1968), Hull (1982, 1984), Bruss (1984), Daley *et al.* (1986) and Alsmeyer and Rösler (1996, 2002); its limiting behaviour is investigated in Bagley (1986) and González and Molina (1996, 1997); and some inferential problems are considered in González Fragoso (1995), Molina *et al.* (1998) and González *et al.* (2001a). In an attempt to improve the mathematical modelling, certain classes of discrete-time bisexual branching processes have recently been introduced and some theoretical results presented: see, for example, González *et al.* (2000, 2001b), Molina *et al.* (2002, 2003, 2004a, 2004b) or Xing and Wang (2005). Hull (2003) provides a survey of the literature associated with discrete-time bisexual processes. A theory of continuous-time bisexual branching processes has yet to be sufficiently developed, and indeed only Asmussen (1980) and more recently Molina and Yanev (2003) have considered this possibility.

This paper deals with the *bisexual process with population-size dependent mating* introduced in Molina *et al.* (2002) as a branching model $\{(F_n, M_n)\}_{n\geqslant 1}$ initiated with $Z_0 = N \geqslant 1$ couples and defined in recursive form for $n = 0, 1, \ldots$ by

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$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n}(f_{ni}, m_{ni}), \qquad Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1}),$$

where the empty sum is taken as (0, 0), $\{(f_{ni}, m_{ni}) : i = 1, 2, \ldots; n = 0, 1, \ldots\}$ is a sequence of independent and identically distributed, non-negative, integer-valued random variables, and $\{L_k\}_{k\geqslant 0}$ is a sequence of non-negative real functions on $\mathbb{R}^+ \times \mathbb{R}^+$ assumed to be integer-valued on the integers and such that $L_k(x, y) \leqslant xy$, $x, y \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, with $\mathbb{R}^+$ and $\mathbb{Z}^+$ denoting the non-negative real and integer numbers, respectively. Intuitively, (f_{ni}, m_{ni}) represents the number of females and males descending from the ith couple of the nth generation. It follows that (F_{n+1}, M_{n+1}) is the number of females and males in the $(n+1)$th generation, which form Z_{n+1} couples according to the mating function L_{Z_n}. These couples reproduce independently through the same offspring probability distribution for each generation. Notice that, in each generation, the function governing the mating changes depending on the number of couples in the previous generation. Indeed, the motivation behind this stochastic process was the interest in developing bisexual models to describe the probabilistic evolution of two-sex populations in which, because of environmental, social or other factors, matings between females and males could be influenced by the number of their progenitor couples.

It can be directly verified that $\{(Z_{n-1}, F_n, M_n)\}_{n\geqslant 1}$ and $\{Z_n\}_{n\geqslant 0}$ are homogeneous Markov chains, with 0 being an absorbing state for $\{Z_n\}_{n\geqslant 0}$. For each $k = 1, 2, \ldots,$ we introduce the mean growth rate per couple,

$$r_k := \mathrm{E}[Z_{n+1}Z_n^{-1}|Z_n = k] = k^{-1}\mathrm{E}\left[L_k\left(\sum_{i=1}^{k} f_{ni}, \sum_{i=1}^{k} m_{ni}\right)\right].$$

Note that r_k can be interpreted intuitively as the expected growth rate per couple when, in a certain generation, there are k couples. These expected values will play a major role in this work. In order to investigate some results concerning limiting evolution in this class of bisexual processes, we shall begin by making the following working assumptions:

Assumption 1. *$L^* : \mathbb{Z}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined as $L^*(k, x, y) = L_k(x, y)$, $x, y \in \mathbb{R}^+$, $k = 0, 1, \ldots,$ is a superadditive function, that is,*

$$L^*(k_1 + k_2, x_1 + x_2, y_1 + y_2) \geqslant L^*(k_1, x_1, y_1) + L^*(k_2, x_2, y_2).$$

Assumption 2. *$r := \lim_{k\to\infty} r_k > 1$ and $P(Z_n \to \infty | Z_0 = N) > 0$.*

Remark 1. Assumption 1 extends the classical superadditivity condition usually imposed on the mating function in the Daley bisexual process literature; see Hull (1982). It expresses the following intuitive concept: if the females and males descending from $k_1 + k_2$ couples live together then there will be more matings than if the females and males descending from k_1 couples and those descending from k_2 couples live separately.

Under Assumption 1, the existence of the asymptotic rate r was proved in Molina *et al.*

(2002). Under Assumptions 1 and 2, it was verified in Molina *et al.* (2004b) that $\{W_n\}_{n\geqslant 0}$, where $W_n := r^{-n}Z_n$, is a supermartingale with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$, where $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$, and, with some additional requirements, its almost sure convergence to a finite limit W such that $P(W > 0) > 0$ (i.e. W is non-degenerate at zero) was established. Some necessary and sufficient conditions for both almost sure and L^1-convergence of the sequences $\{W_n\}_{n\geqslant 0}$, $\{\overline{F}_n\}_{n\geqslant 1}$ and $\{\overline{M}_n\}_{n\geqslant 1}$, where $(\overline{F}_n, \overline{M}_n) := r^{-n}(F_n, M_n)$, were also derived in Molina *et al.* (2004b). In particular, the following results were established:

Theorem A. *The L^1-convergence of $\{W_n\}_{n\geqslant 0}$ to W is equivalent to the L^1-convergence of $\{\overline{F}_n\}_{n\geqslant 1}$ to $r^{-1}\mu_1 W$ and of $\{\overline{M}_n\}_{n\geqslant 1}$ to $r^{-1}\mu_2 W$, where $\mu_1 := \mathrm{E}[f_{01}]$ and $\mu_2 := \mathrm{E}[m_{01}]$.*

Theorem B. *Given a non-increasing sequence $\{\varepsilon_k\}_{k\geqslant 1}$, $\varepsilon_k := r - r_k$, if $\sum_{k=1}^{\infty} k^{-1}\varepsilon_k < \infty$, then $\lim_{n\to\infty} \mathrm{E}[W_n | Z_0 = N] > 0$ for N such that $P(Z_n \to \infty | Z_0 = N) > 0$.*

In the usual methodology of branching process theory research, the natural next step is to study the L^α-convergence, $1 \leqslant \alpha \leqslant 2$, of the sequences $\{W_n\}_{n\geqslant 0}$, $\{\overline{F}_n\}_{n\geqslant 1}$ and $\{\overline{M}_n\}_{n\geqslant 1}$. In Section 2, under a general methodological framework, sufficient conditions are given for L^α-convergence to non-degenerate limits, with the main result extending that obtained in Molina *et al.* (2004b) for $\alpha = 1$. In Section 3, in a more specific methodological framework similar to that used in Klebaner (1984, 1985) for asexual population-size dependent branching processes, some necessary and sufficient conditions for L^2-convergence are determined. These results are based on weaker requirements than those imposed in Section 2 for $\alpha = 2$, and generalize those proved in González and Molina (1997) for Daley's model. Finally, by analogy with the classical Kesten and Stigum result for Bienaymé–Galton–Watson processes, a logarithmic criterion for L^1-convergence is established in Section 4.

2. L^α-convergence

In this section a sufficient condition is provided for the L^α-convergence, $1 \leqslant \alpha \leqslant 2$, of $\{W_n\}_{n\geqslant 0}$ to a non-degenerate limit. In the proof of the main result of this section we shall use some techniques which extend those used in Klebaner (1984, 1985) for asexual population-size dependent branching processes.

First, we establish a strong relationship between the L^α-convergence of the sequences $\{\overline{F}_n\}_{n\geqslant 1}$ and $\{\overline{M}_n\}_{n\geqslant 1}$ and the L^α-convergence of $\{W_n\}_{n\geqslant 0}$.

Theorem 1. *Suppose that $\mathrm{E}[f_{01}^\alpha] < \infty$ and $\mathrm{E}[m_{01}^\alpha] < \infty$. The following statements are equivalent:*

(i) *$\{W_n\}_{n\geqslant 0}$ is L^α-convergent to W.*
(ii) *$\{\overline{F}_n\}_{n\geqslant 1}$ is L^α-convergent to $r^{-1}\mu_1 W$.*
(iii) *$\{\overline{M}_n\}_{n\geqslant 1}$ is L^α-convergent to $r^{-1}\mu_2 W$.*

Proof. It is sufficient to prove that (i) and (ii) are equivalent. Assume that (i) holds. It is clear that

$$\|\overline{F}_{n+1} - r^{-1}\mu_1 W\|_\alpha \leqslant \|\overline{F}_{n+1} - r^{-1}\mu_1 W_n\|_\alpha + r^{-1}\mu_1\|W_n - W\|_\alpha.$$

By hypothesis, $\lim_{n\to\infty}\|W_n - W\|_\alpha = 0$. Also

$$E[|\overline{F}_{n+1} - r^{-1}\mu_1 W_n|^\alpha] = r^{-(n+1)\alpha}E[E[|F_{n+1} - \mu_1 Z_n|^\alpha | Z_n]]. \tag{1}$$

Now, applying the von Bahr–Esseen inequality (von Bahr and Esseen 1965), one deduces that

$$\begin{aligned} E[|F_{n+1} - \mu_1 Z_n|^\alpha | Z_n = k] &= E\left[\left|\sum_{i=1}^{k}(f_{ni} - \mu_1)\right|^\alpha\right] \\ &\leqslant 2\sum_{i=1}^{k} E[|f_{ni} - \mu_1|^\alpha] = 2kE[|f_{01} - \mu_1|^\alpha], \end{aligned}$$

and then from (1) one obtains

$$\begin{aligned} E[|\overline{F}_{n+1} - r^{-1}\mu_1 W_n|^\alpha] &\leqslant 2E[|f_{01} - \mu_1|^\alpha]r^{-(n+1)\alpha}E[Z_n] \\ &= 2E[|f_{01} - \mu_1|^\alpha]r^{-(n+1)\alpha}r^n E[W_n] \\ &\leqslant 2E[|f_{01} - \mu_1|^\alpha]r^{-1}r^{n(1-\alpha)}N. \end{aligned}$$

Therefore, there exists a constant C_α such that

$$\|\overline{F}_{n+1} - r^{-1}\mu_1 W_n\|_\alpha \leqslant C_\alpha\|f_{01} - \mu_1\|_\alpha r^{n(1-\alpha)/\alpha},$$

and the right-hand side converges to 0 because $\|f_{01}\|_\alpha < \infty$ and $r > 1$.

Conversely, if one assumes that $\{\overline{F}_n\}_{n\geqslant 1}$ is L^α-convergent to $r^{-1}\mu_1 W$, then

$$\|W_n - W\|_\alpha \leqslant r\mu_1^{-1}\big(\|r^{-1}\mu_1 W_n - \overline{F}_{n+1}\|_\alpha + \|\overline{F}_{n+1} - r^{-1}\mu_1 W\|_\alpha\big).$$

The second term on the right-hand side converges to 0 by hypothesis, and, using reasoning similar to that considered for (1), the first term also converges to 0, so that the proof is concluded. □

For all $1 \leqslant \alpha \leqslant 2$, let us introduce the sequence $\{R_{\alpha,k}\}_{k\geqslant 1}$ where

$$R_{\alpha,k} := k^{-1}E[|Z_{n+1} - rZ_n|^\alpha | Z_n = k]^{1/\alpha}, \qquad k = 1, 2, \ldots.$$

Note that the expected value per couple $R_{\alpha,k}$ provides information about the L^α-distance between the variable $k^{-1}L_k(\sum_{i=1}^k f_{ni}, \sum_{i=1}^k m_{ni})$ and the asymptotic growth rate r. Its definition involves both the reproduction law and the mating function, that is to say, the main features describing the behaviour of the process. We show that the order of magnitude of these rates will determine the L^α-convergence of the sequence $\{W_n\}_{n\geqslant 0}$, and consequently, by Theorem 1, of $\{\overline{F}_n\}_{n\geqslant 1}$ and $\{\overline{M}_n\}_{n\geqslant 1}$.

Proposition 1. *Given* $1 \leqslant \alpha \leqslant 2$, *then* $|\varepsilon_k| \leqslant R_{\alpha,k}$, $k = 1, 2, \ldots$.

Proof. The proof is immediate by the application of Jensen's inequality. Indeed, given $1 \leqslant \alpha \leqslant 2$ and $k = 1, 2, \ldots,$

$$\left| k^{-1}\mathrm{E}\left[L_k\left(\sum_{i=1}^{k} f_{ni}, \sum_{i=1}^{k} m_{ni} \right) \right] - r \right| \leqslant k^{-1}\mathrm{E}\left[\left| L_k\left(\sum_{i=1}^{k} f_{ni}, \sum_{i=1}^{k} m_{ni} \right) - kr \right|^{\alpha} \right]^{1/\alpha}.$$

□

Theorem 2. *If, for some* $1 \leqslant \alpha \leqslant 2$, $\{R_{\alpha,k}\}_{k\geqslant 1}$ *is a non-increasing sequence and* $\sum_{k=1}^{\infty} k^{-1} R_{\alpha,k} < \infty$, *then* $\{W_n\}_{n\geqslant 0}$ *is almost surely and* L^{α}*-convergent to a finite and non-degenerate limit.*

Proof. Almost sure convergence to a finite non-negative random variable is derived from the fact that $\{W_n\}_{n\geqslant 0}$ is a supermartingale with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$. By Proposition 1, one deduces that $\sum_{k=1}^{\infty} k^{-1}\varepsilon_k < \infty$. Then, applying Thorem B, one obtains $0 < \lim_{n\to\infty}\mathrm{E}[W_n] < \infty$. Therefore it suffices to verify that $\{W_n\}_{n\geqslant 0}$ is a Cauchy sequence in L^{α}. For $n = 0, 1, \ldots,$

$$\|W_{n+1} - W_n\|_{\alpha} = r^{-n-1}\mathrm{E}[\mathrm{E}[|Z_{n+1} - rZ_n|^{\alpha}|Z_n]]^{1/\alpha} = r^{-n-1}\mathrm{E}[(Z_n R_{\alpha, Z_n})^{\alpha}]^{1/\alpha}. \tag{2}$$

In González and Molina (1997, Lemma 4.1) it was proved that there exists a non-increasing positive function $h_{\alpha}(\cdot)$ on $\mathbb{R}^{+}$ such that $R_{\alpha,k} \leqslant h_{\alpha}(k)$, $k = 1, 2, \ldots, \sum_{k=1}^{\infty} k^{-1} h_{\alpha}(k) < \infty$. Moreover, it is a matter of straightforward computation to check that $\phi_{\alpha}(x) := x h_{\alpha}^{\alpha}(x^{1/\alpha})$ is concave on $[1, \infty)$. Consequently, from (2) and Jensen's inequality,

$$\begin{aligned}\|W_{n+1} - W_n\|_{\alpha} &\leqslant r^{-(n+1)}\mathrm{E}[(Z_n^{\alpha} h_{\alpha}(Z_n))^{\alpha}]^{1/\alpha} = r^{-(n+1)}\mathrm{E}[\phi_{\alpha}(Z_n^{\alpha})]^{1/\alpha} \\ &\leqslant r^{-(n+1)}\phi_{\alpha}^{1/\alpha}\left(\mathrm{E}[Z_n^{\alpha}]\right) = r^{-1}\|W_n\|_{\alpha} h_{\alpha}(r^n \|W_n\|_{\alpha}).\end{aligned}$$

Now, since $\lim_{n\to\infty}\|W_n\|_{\alpha} > \lim_{n\to\infty}\mathrm{E}[W_n] > 0$, there exists $\delta > 0$ such that $\|W_n\|_{\alpha} > \delta$ for all n. Hence, taking into account that $h_{\alpha}(\cdot)$ is non-increasing, one deduces that

$$\|W_{n+1} - W_n\|_{\alpha} \leqslant r^{-1}\|W_n\|_{\alpha} h_{\alpha}(\delta r^n). \tag{3}$$

Making use of this inequality, one obtains

$$\|W_{n+1}\|_{\alpha} \leqslant (1 + r^{-1}h_{\alpha}(\delta r^n))\|W_n\|_{\alpha} \leqslant N \prod_{i=1}^{\infty}(1 + r^{-1}h_{\alpha}(\delta r^i)),$$

and this product is convergent because $\sum_{k=1}^{\infty} k^{-1} h_{\alpha}(k) < \infty$. Consequently there exists M such that $\|W_n\|_{\alpha} \leqslant M$ for all n, and by (3) one has

$$\|W_{n+1} - W_n\|_{\alpha} \leqslant r^{-1} M h_{\alpha}(\delta r^n).$$

Using again the fact that $\sum_{k=1}^{\infty} k^{-1} h_{\alpha}(k) < \infty$, one verifies that $\sum_{n=0}^{\infty} h_{\alpha}(\delta r^n) < \infty$, which completes the proof. □

Remark 2. Note that Theorem 2 holds if $\{R_{\alpha,k}\}_{k\geqslant 1}$ is bounded above by a non-increasing sequence $\{a_k\}_{k\geqslant 1}$ such that $\sum_{k=1}^{\infty} k^{-1}a_k < \infty$. In this case, it is not necessary for $\{R_{\alpha,k}\}_{k\geqslant 1}$ to be non-increasing.

3. L^2-convergence

In this section, by applying some specific techniques concerning L^2-bounded martingales, we provide further conditions for the L^2-convergence of $\{W_n\}_{n\geqslant 0}$, $\{\overline{F}_n\}_{n\geqslant 1}$ and $\{\overline{M}_n\}_{n\geqslant 1}$ which are weaker than those obtained in the previous section for $\alpha = 2$. First, we establish a necessary condition. Let us consider

$$\sigma_k^2 := k^{-2}\,\mathrm{var}[Z_{n+1}Z_n = k], \qquad k = 1, 2, \ldots,$$

assumed to be finite.

Proposition 2.

$$\mathrm{E}[W_{n+1}^2] = N^2 + r^{-2}\sum_{j=0}^{n} \mathrm{E}\Big[W_j^2(\sigma_{Z_j}^2 + r_{Z_j}^2 - r^2)\Big], \qquad n = 0, 1, \ldots.$$

Proof.

$$\mathrm{E}[W_{n+1}^2] = r^{-2(n+1)}\mathrm{E}[\mathrm{E}[Z_{n+1}^2|Z_n]] = r^{-2(n+1)}\mathrm{E}[Z_n^2(\sigma_{Z_n}^2 + r_{Z_n}^2)]$$
$$= \mathrm{E}[W_n^2] + r^{-2}\mathrm{E}[W_n^2(\sigma_{Z_n}^2 + r_{Z_n}^2 - r^2)], \qquad n = 0, 1, \ldots.$$

Hence, by iteration and taking into account that $Z_0 = N$, the proof is concluded. □

Theorem 3. *Suppose that there exists k_0 such that $\{\varepsilon_k\}_{k\geqslant k_0}$ and $\{\sigma_k^2\}_{k\geqslant k_0}$ are non-increasing and either $\sigma_k^2 + r_k^2 \geqslant r^2$ or $\sigma_k^2 + r_k^2 \leqslant r^2$ for all $k \geqslant k_0$. Then a necessary condition for the L^2-convergence of $\{W_n\}_{n\geqslant 0}$ to a non-degenerate random variable W is that $\sum_{k=1}^{\infty} k^{-1}\sigma_k^2 < \infty$.*

Proof. Assume without loss of generality that $k_0 = 1$. Since $\{\sigma_k^2\}_{k\geqslant 1}$ is non-increasing, there exists $\hat{\sigma}^2 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\hat{\sigma}^2(x) := \sigma_1^2\mathbf{1}_{(0,1)}(x) + \sigma_{\lfloor x\rfloor}^2\mathbf{1}_{[1,\infty)}(x)$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. It is clear that this function is also non-increasing and $\hat{\sigma}^2(k) = \sigma_k^2,\ k = 1, 2, \ldots$.

For simplicity, let $A_k := \sigma_k^2 + r_k^2 - r^2$. The L^2-convergence of $\{W_n\}_{n\geqslant 0}$ implies that $\{\mathrm{E}[W_n^2]\}_{n\geqslant 0}$ is bounded. Then, from Proposition 2,

$$\left|\sum_{n=0}^{\infty} \mathrm{E}[W_n^2 A_{Z_n}]\right| < \infty.$$

By hypothesis, the sign of A_n is the same for all n. Hence,

$$\sum_{n=0}^{\infty} \mathrm{E}[W_n^2 |A_{Z_n}|] < \infty$$

and, by the monotone convergence theorem,

$$\sum_{n=0}^{\infty} W_n^2 |A_{Z_n}| < \infty \qquad \text{a.s.}$$

Now, on $\{W > 0\}$, one has that $\{W_n\}_{n\geqslant 0}$ has a lower bound greater than 0. Consequently, taking into account that $\varepsilon_{Z_n} = r - r_{Z_n}$,

$$\left| \sum_{n=0}^{\infty} A_{Z_n} \right| = \left| \sum_{n=0}^{\infty} \left(\sigma_{Z_n}^2 - 2r\varepsilon_{Z_n} + \varepsilon_{Z_n}^2 \right) \right| < \infty \qquad \text{a.s. on } \{W > 0\}. \tag{4}$$

Since $P(W > 0) > 0$, it can be verified (see Molina *et al.* 2004b, Proposition 14), that on $\{W > 0\}$,

$$\sum_{n=0}^{\infty} \varepsilon_{Z_n} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \varepsilon_{Z_n}^2 < \infty \qquad \text{a.s.}$$

Hence, from (4), one obtains the almost sure convergence of $\sum_{n=0}^{\infty} \sigma_{Z_n}^2$ on $\{W > 0\}$.

Let $\tilde{W} := \sup_{n\geqslant 0} W_n < \infty$, which exists because $\{W_n\}_{n\geqslant 0}$ is almost surely convergent to a finite random variable. Taking into account that $\hat{\sigma}^2(\cdot)$ is non-increasing, one deduces that

$$\sum_{n=0}^{\infty} \sigma_{Z_n}^2 = \sum_{n=0}^{\infty} \hat{\sigma}^2(r^n W_n) \geqslant \sum_{n=0}^{\infty} \hat{\sigma}^2(r^n \tilde{W}) \qquad \text{a.s.}$$

Finally the convergence of $\sum_{k=1}^{\infty} k^{-1}\sigma_k^2$ is derived using reasoning similar to that in Klebaner (1984, Lemma 1). □

Theorem 4. *Suppose that $\{\varepsilon_k\}_{k\geqslant 1}$ and $\{\sigma_k^2\}_{k\geqslant 1}$ are non-increasing sequences such that $\sum_{k=1}^{\infty} k^{-1}\varepsilon_k < \infty$ and $\sum_{k=1}^{\infty} k^{-1}\sigma_k^2 < \infty$. Then $\{W_n\}_{n\geqslant 0}$ is L^2-convergent to W, where $P(W > 0) > 0$.*

Proof. Again using Lemma 4.1 of González and Molina (1997) applied to the sequence $\{\sigma_k^2\}_{k\geqslant 1}$, there exists a positive real non-increasing function $\sigma^2(\cdot)$ on $\mathbb{R}^+$ such that $\sigma_k^2 \leqslant \sigma^2(k)$ and $\sum_{k=1}^{\infty} k^{-1}\sigma^2(k) < \infty$. Moreover, it is easy to check that the function $x^2\sigma^2(x)$ is concave on $\mathbb{R}^+$. Taking into account Proposition 2 and the properties of $\sigma^2(\cdot)$, one deduces that

$$\mathrm{E}[W_{n+1}^2] \leqslant \mathrm{E}[W_n^2] + r^{-2(n+1)}\mathrm{E}[Z_n^2\sigma^2(Z_n)] \tag{5}$$

and, since $x^2\sigma^2(x)$ is concave, by Jensen's inequality,

$$\mathrm{E}[Z_n^2\sigma^2(Z_n)] \leqslant \mathrm{E}[Z_n]^2\sigma^2(\mathrm{E}[Z_n]) \leqslant \mathrm{E}[Z_n^2]\sigma^2(\mathrm{E}[Z_n])$$

so, by (5),

$$\mathrm{E}[W_n^2] \leqslant N^2 \prod_{n=0}^{\infty} \left(1 + r^{-2}\sigma^2(\mathrm{E}[Z_n])\right), \qquad n = 0, 1, \ldots .$$

By virtue of Theorem B, $\lim_{n\to\infty} \mathrm{E}[W_n] > 0$. There therefore exists $\delta > 0$ such that $\mathrm{E}[Z_n] \geqslant \delta r^n$, $n = 0, 1, \ldots,$ and, taking into account that $\sigma^2(\cdot)$ is non-increasing,

$$\mathrm{E}[W_n^2] \leqslant N^2 \prod_{n=0}^{\infty} \left(1 + r^{-2}\sigma^2(\delta r^n)\right), \qquad n = 0, 1, \ldots .$$

Note that the product is finite if and only if $\sum_{n=1}^{\infty} \sigma^2(\delta r^n) < \infty$ or equivalently if $\sum_{k=1}^{\infty} k^{-1}\sigma^2(k) < \infty$ (see Klebaner 1984, Lemma 1), which is guaranteed by the properties of $\sigma^2(\cdot)$. Thus, we have proved that $\{W_n\}_{n=0}^{\infty}$ is L^2 bounded. Now, according to Doob's theorem, we can decompose $\{W_n\}_{n\geqslant 0}$ into $W_n = Y_n + T_n$, where $\{Y_n\}_{n\geqslant 0}$ is a martingale relative to $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ and

$$T_n = \sum_{k=0}^{n-1} (\mathrm{E}[W_{k+1}\mathcal{F}_k] - W_k) = -r^{-1} \sum_{k=0}^{n-1} W_k \varepsilon_{Z_k} \qquad \text{a.s.}$$

Since $\{W_n\}_{n\geqslant 0}$ is L^2 bounded, in order to prove that $\{Y_n\}_{n\geqslant 0}$ is an L^2 bounded martingale and therefore L^2-convergent, it only remains to prove that $\{T_n\}_{n\geqslant 0}$ is L^2-convergent.

Now,

$$\left\| \sum_{n=0}^{\infty} W_n \varepsilon_{Z_n} \right\|_2 \leqslant \sum_{n=0}^{\infty} \| W_n \varepsilon_{Z_n} \|_2 = \sum_{n=0}^{\infty} r^{-n} \mathrm{E}\left[Z_n^2 \varepsilon_{Z_n}^2\right]^{1/2}. \tag{6}$$

Using once again Lemma 4.1 of González and Molina (1997) applied to the sequence $\{\varepsilon_k\}_{k\geqslant 1}$, there exists a non-increasing positive real function $\varepsilon(\cdot)$ such that $\varepsilon_k \leqslant \varepsilon(k)$, $\sum_{k=1}^{\infty} k^{-1}\varepsilon(k) < \infty$, and the function $x\varepsilon^2(x^{1/2})$ is concave on $[1, \infty)$. Thus, from (6),

$$\begin{aligned}\left\| \sum_{n=0}^{\infty} W_n \varepsilon_{Z_n} \right\|_2 &\leqslant \sum_{n=0}^{\infty} r^{-n} \mathrm{E}\left[Z_n^2 \varepsilon^2(Z_n)\right]^{1/2} \leqslant \sum_{n=0}^{\infty} \left(\mathrm{E}\left[W_n^2\right] \varepsilon^2(\mathrm{E}[Z_n])\right)^{1/2} \\ &\leqslant \sum_{n=0}^{\infty} \| W_n \|_2 \, \varepsilon(\mathrm{E}[Z_n]) \leqslant K \sum_{n=0}^{\infty} \varepsilon(\mathrm{E}[Z_n])\end{aligned}$$

with $K > 0$, where we have used the fact that $\{W_n\}_{n\geqslant 0}$ is L^2 bounded. Finally, again using the fact that $\mathrm{E}[Z_n] \geqslant \delta r^n$ and $\varepsilon(\cdot)$ is non-increasing, one obtains

$$\left\| \sum_{n=0}^{\infty} W_n \varepsilon_{Z_n} \right\|_2 \leqslant K \sum_{n=0}^{\infty} \varepsilon(\delta r^n),$$

which is convergent by considering Lemma 1 of Klebaner (1984). □

Remark 3. Note that assumptions in Theorem 4 are weaker than those in Theorem 2 for the particular case $\alpha = 2$. In fact, if $\{R_{2,k}\}_{k\geqslant 1}$ is non-increasing and $\sum_{k=1}^{\infty} k^{-1} R_{2,k} < \infty$, then, by Proposition 1, one derives that $\sum_{k=1}^{\infty} k^{-1}\varepsilon_k < \infty$. Moreover, $\sum_{k=1}^{\infty} k^{-1} R_{2,k} < \infty$ implies

$\lim_{k\to\infty} R_{2,k} = 0$ and there exists k_0 such that $R_{2,k}^2 \leqslant R_{2,k}$, $k \geqslant k_0$. Hence, using the fact that $\sigma_k^2 \leqslant R_{2,k}^2 \leqslant R_{2,k}$, $k \geqslant k_0$, one deduces that $\sum_{k=1}^{\infty} k^{-1}\sigma_k^2 < \infty$, and consequently the assumptions in Theorem 4 hold. However, it is not possible from the conditions established in Theorem 4 to guarantee that $\sum_{k=1}^{\infty} k^{-1} R_{2,k} < \infty$.

4. L^1-convergence under a logarithmic criterion

In this section, we establish a necessary and sufficient condition, based on a logarithmic criterion, for the L^1-convergence of $\{W_n\}_{n\geqslant 0}$. Taking Theorem A into account, it will be sufficient to determine such a condition for $\{\overline{F}_n\}_{n\geqslant 1}$ or $\{\overline{M}_n\}_{n\geqslant 1}$.

Theorem 5. *If*

(i) $\{\varepsilon_k\}_{k\geqslant 1}$ *is non-increasing and such that* $\sum_{k=1}^{\infty} k^{-1}\varepsilon_k < \infty$,
(ii) $\overline{F}_{n+1} - r^{-1}\mu_1 W_{n+1} \geqslant 0$ *(*$\overline{M}_{n+1} - r^{-1}\mu_2 W_{n+1} \geqslant 0$*),* $n = 0, 1, \ldots,$

then $\mathrm{E}[f_{01}\log^+ f_{01}] < \infty$ *(*$\mathrm{E}[m_{01}\log^+ m_{01}] < \infty$*) is a necessary and sufficient condition for the* L^1*-convergence of* $\{\overline{F}_n\}_{n\geqslant 1}$ *(*$\{\overline{M}_n\}_{n\geqslant 1}$*) to a non-degenerate limit.*

Proof. First, we prove sufficiency. Write, for $n = 0, 1, \ldots,$

$$\tilde{F}_{n+1} := r^{-(n+1)} \sum_{i=1}^{Z_n} f_{ni}\mathbf{1}_{\{f_{ni}\leqslant r^n\}}$$

and

$$\tilde{D}_{n+1} := \overline{F}_{n+1} - r^{-1} W_{n+1}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}\leqslant r^{n+1}\}}].$$

If $\mathcal{F}_n := \sigma(Z_0, \ldots, Z_n)$, $n = 0, 1, \ldots,$ then

$$\sum_{n=1}^{\infty} (\tilde{F}_{n+1} - \mathrm{E}[\tilde{F}_{n+1}\mathcal{F}_n]) = \sum_{n=1}^{\infty} (\tilde{F}_{n+1} - \overline{F}_n + \tilde{D}_n) \qquad \text{a.s.}$$

Molina *et al.* (2004b) prove the L^1-convergence of $\sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \mathrm{E}[\tilde{F}_{n+1}|\mathcal{F}_n])$. Hence, if one proves that $\sum_{n=1}^{\infty}\tilde{D}_n$ is also L^1-convergent one will derive that $\sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \overline{F}_n)$ is L^1-convergent. Actually, since $\tilde{D}_{n+1} \geqslant 0$, $n = 0, 1, \ldots,$ it will be sufficient to verify that $\sum_{n=0}^{\infty}\mathrm{E}[\tilde{D}_{n+1}] < \infty$. In fact,

$$\begin{aligned}
\mathrm{E}[\tilde{D}_{n+1}] &= \mathrm{E}\big[\mathrm{E}[\overline{F}_{n+1} - r^{-1}W_{n+1}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}\leqslant r^{n+1}\}}]\mathcal{F}_n]\big] \\
&= r^{-1}\mu_1\mathrm{E}[W_n] - r^{-2}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}\leqslant r^{n+1}\}}]\mathrm{E}[W_n r_{Z_n}] \\
&= r^{-1}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}> r^{n+1}\}}]\mathrm{E}[W_n] + r^{-2}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}\leqslant r^{n+1}\}}]\mathrm{E}[W_n\varepsilon_{Z_n}] \\
&\leqslant r^{-1}\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}> r^{n+1}\}}]N + r^{-2}\mu_1\mathrm{E}[W_n\varepsilon_{Z_n}], \qquad n = 0, 1, \ldots.
\end{aligned}$$

Now, writing $F(x) := P(f_{01} \leqslant x)$ and bearing in mind that $\sum_{n=0}^{\infty} \mathbf{1}_{(r^{n+1},\infty)}(x) = O(\log^+ x)$, $x \in \mathbb{R}^+$, one has

$$\sum_{n=0}^{\infty} \mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}>r^{n+1}\}}] = \sum_{n=0}^{\infty} \int_{r^{n+1}}^{\infty} x\,\mathrm{d}F(x) = \int_0^{\infty} x \sum_{n=0}^{\infty} \mathbf{1}_{(r^{n+1},\infty)}(x)\mathrm{d}F(x)$$

$$= \int_0^{\infty} x\, O(\log^+ x)\mathrm{d}F(x) = \mathrm{E}[f_{01}\log^+ f_{01}] < \infty.$$

Since $\sum_{n=0}^{\infty}\mathrm{E}[W_n \varepsilon_{Z_n}] < \infty$ (see Molina *et al.* 2004b, Theorem 7), one concludes that $\sum_{n=0}^{\infty}\mathrm{E}[\tilde{D}_{n+1}] < \infty$.

Using the fact that there exists a finite and non-negative random variable W_F which is the almost sure limit of $\{\overline{F}_n\}_{n\geqslant 1}$, since $\tilde{F}_{n+1} \leqslant \overline{F}_{n+1}$, $n = 0, 1, \ldots,$ one deduces that

$$\mathrm{E}[W_F] \geqslant \mathrm{E}[\overline{F}_{n+1}] + \mathrm{E}\left[\sum_{k=n}^{\infty}(\tilde{F}_{k+1} - \overline{F}_k)\right], \qquad n = 0, 1, \ldots . \tag{7}$$

Taking into account that $\sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \overline{F}_n)$ converges in L^1, one deduces from (7) that $\mathrm{E}[W_F] \geqslant \limsup_{n\to\infty} \mathrm{E}[\overline{F}_n]$, and by Fatou's lemma one derives that $\mathrm{E}[W_F] \leqslant \liminf_{n\to\infty} \mathrm{E}[\overline{F}_n]$. Therefore $\mathrm{E}[W_F] = \lim_{n\to\infty} \mathrm{E}[\overline{F}_n]$ and $\{\overline{F}_n\}_{n\geqslant 1}$ is also L^1-convergent to W_F.

Finally, by Theorem A the L^1-convergence of $\{\overline{F}_n\}_{n\geqslant 1}$ implies the L^1-convergence of $\{W_n\}_{n\geqslant 0}$ to W which is almost surely equal to $r\mu_1^{-1}W_F$. Moreover, by (i) and Theorem B, one has that $\lim_{n\to\infty}\mathrm{E}[W_n] > 0$. Hence, $\mathrm{E}[W_F] > 0$ and $P(W_F > 0) > 0$.

Conversely, if $\{\overline{F}_n\}_{n\geqslant 1}$ converges in L^1 to W_F such that $P(W_F > 0) > 0$ then $P(\widehat{W} > 0) > 0$, where $\widehat{W} := \inf_{n\geqslant 0} W_n$.

It is known that

$$\sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \overline{F}_n + \tilde{D}_n) = \sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \mathrm{E}[\tilde{F}_{n+1} \mid \mathcal{F}_n]) \qquad \text{a.s.},$$

which is almost surely and L^1-convergent, by virtue of the L^2-bounded martingale convergence theorem and of the fact that $\{\overline{F}_n\}_{n\geqslant 1}$ is almost surely and L^1-convergent to W_F. Also, since $\{\tilde{F}_n\}_{n\geqslant 1}$ and $\{\overline{F}_n\}_{n\geqslant 1}$ are almost surely equal from a certain n onwards, $\sum_{n=1}^{\infty}(\tilde{F}_{n+1} - \overline{F}_n) < \infty$ almost surely. Hence, one deduces that $\sum_{n=0}^{\infty}\tilde{D}_{n+1} < \infty$ almost surely, and, taking into account that $\tilde{D}_{n+1} \geqslant 0$, $n = 0, 1, \ldots,$ and (ii),

$$\widehat{W}\sum_{n=0}^{\infty} \mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}>r^n\}}] \leqslant \sum_{n=0}^{\infty} W_n\mathrm{E}[f_{01}\mathbf{1}_{\{f_{01}>r^n\}}] < \infty \qquad \text{a.s.},$$

and one deduces that $\mathrm{E}[f_{01}\,log^+ f_{01}] < \infty$. □

Remark 4. Condition (ii) in Theorem 5 is verified if $\{L_k\}_{k\geqslant 0}$ is such that $L_k(x, y) \leqslant xr\mu_1^{-1}$ (or $L_k(x, y) \leqslant xr\mu_2^{-1}$), $k = 0, 1, \ldots$. Indeed,

$$r^{-1}\mu_1 Z_{n+1} = r^{-1}\mu_1 L_{Z_n}(F_{n+1}, M_{n+1}) \leqslant F_{n+1}$$

and therefore $\overline{F}_{n+1} - r^{-1}\mu_1 W_{n+1} \geqslant 0,\ n = 0, 1, \ldots$.

Remark 5. An interesting problem which arises from Theorem 5 is whether the set on which the limit variable W is greater than 0, equivalently $W_F > 0$, is the entire survival set of the process. This problem is partially solved in González *et al.* (2004). In this work, assuming the classical extinction–explosion duality in branching process theory, it is proved that if $Z_0 = 1$ then $\{W > 0\} = \{Z_n \to \infty\}$, and, since $W_F = r^{-1}\mu_1 W$ (see Theorem 1), $\{W_F > 0\} = \{Z_n \to \infty\}$.

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