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New large-deviation local theorems for sums of independent and identically distributed random vectors when the limit distribution is α-stable

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A class of absolutely continuous distributions in $\mathbb{R}^d$ is considered. Each distribution belongs to the domain of normal attraction of an α-stable law. The limit law is characterized by a spectral measure which is absolutely continuous with respect to the spherical Lebesgue measure. The large-deviation problem for sums of independent and identically distributed random vectors when the underlying distribution belongs to that class is studied. At the focus of attention are the deviations in the directions, where the spectral density equals zero. The main conclusion is that the deviation in such a direction is explained by two abnormally large summands.

Keywords: normal domain of attraction; spectral measure; strictly α-stable density

1. Introduction

Let $\xi, \xi^{(1)}, \xi^{(2)}, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^d$, $d > 1$. Assume that the distribution of ξ belongs to the *domain of attraction* of a non-Gaussian α-stable law S. This means that there exist sequences $a^{(n)} \in \mathbb{R}^d$ and $b_n \in \mathbb{R}^1_+$ such that the sequence of random vectors $b_n^{-1}(\xi^{(1)} + \ldots + \xi^{(n)} - a^{(n)})$ as $n \to \infty$ converges in distribution to a random vector ζ having the distribution S. Denote by $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^d$, that is, $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$ for $x = (x_1, \ldots, x_d)$. It is well known that the convergence mentioned takes place if and only if the tail function $P(|\xi| > t)$ is of regular variation as $t \to \infty$ with the exponent lying in $(-2, 0)$ while the measure

$$\mu_t(E) = P(|\xi|^{-1}\xi \in E \,|\, |\xi| > t),$$

defined on the σ-algebra $\mathcal{B}_{\mathbb{S}^{d-1}}$ of the Borel subsets of the unit sphere $\mathbb{S}^{d-1}$, weakly converges as $t \to \infty$ to a probability measure μ (Theorem 4.2 in Rvacheva 1954, Corollary 6.20 in Araujo and Giné 1980). In other words,

$$P(|\xi| > t) = t^{-\alpha} l(t), \qquad \mu_t \overset{w}{\Rightarrow} \mu, \tag{1.1}$$

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where $\alpha \in (0, 2)$ and $l(t)$ slowly varies as $t \to \infty$.

If in (1.1) $\lim_{t\to\infty} l(t) = l_0$, where l_0 is a positive constant, then the distribution of ξ belongs to the domain of *normal* attraction of S and the normalizing sequence is of the form

$$b_n = b_0 n^{1/\alpha} \tag{1.2}$$

(Kalinauskaite 1974). The choice of b_0 given l_0 determines the total variation of the so-called *spectral measure* of S. Denote by $\hat{s}(y)$, $y \in \mathbb{R}^d$, the characteristic function of S. It is well known that $\hat{s}(y)$ admits the representation

$$-\ln \hat{s}(y) =$$
$$\begin{cases} \mathrm{i}\langle a, y\rangle + \int_{\mathbb{S}^{d-1}} |\langle y, e\rangle|^\alpha (1 - \mathrm{i}\,\mathrm{sign}(\langle y, e\rangle)\tan(\pi\alpha/2))\nu(\mathrm{d}e), & \text{if } \alpha \in (0,1)\cup(1,2), \\ \mathrm{i}\langle a, y\rangle + \int_{\mathbb{S}^{d-1}} |\langle y, e\rangle|(1 + \mathrm{i}(2/\pi)\,\mathrm{sign}(\langle y, e\rangle)\ln|\langle y, e\rangle|)\nu(\mathrm{d}e), & \text{if } \alpha = 1, \end{cases}$$

where $a \in \mathbb{R}^d$ is a location, while ν is a bounded measure defined on the $\mathcal{B}_{\mathbb{S}^{d-1}}$ (Theorem 2.3.1 in Samorodnitsky and Taqqu 1994). It is ν that is called the *spectral measure* of S. It is easy to see that

$$\mu(E) = \frac{\nu(E)}{\nu(\mathbb{S}^{d-1})}, \qquad E \in \mathcal{B}_{\mathbb{S}^{d-1}}.$$

Choosing $b_0 = (l_0 \alpha c_\alpha)^{1/\alpha}$ with

$$c_\alpha = \int_0^\infty \frac{1 - \cos u}{u^{1+\alpha}}\,\mathrm{d}u = \frac{\pi}{2\Gamma(1+\alpha)\sin(\pi\alpha/2)},$$

we obtain $\nu(\mathbb{S}^{d-1}) = 1$ and, therefore, $\nu \equiv \mu$. Throughout the paper we adhere to such a choice of b_0 in (1.2).

Assume that the distribution of ξ is absolutely continuous with a bounded density $p(x)$. Denote by $p_n(x)$ and $\tilde{p}_n(x) = b_n^d p_n(b_n x)$ the densities of $\zeta^{(n)} = \xi^{(1)} + \ldots + \xi^{(n)} - a^{(n)}$ and $b_n^{-1}\zeta^{(n)}$, $n = 1, 2, \ldots,$ respectively. By the local limit theorem of Gnedenko, as $n \to \infty$,

$$\sup_{x\in\mathbb{R}^d} |\tilde{p}_n(x) - s(x)| = o(1), \tag{1.3}$$

where $s(x)$ is the density of the limit α-stable law S.

From (1.3) it follows that $\tilde{p}_n(x) \to 0$ as $|x| \to \infty$. Theorems dealing with the asymptotic behaviour of $\tilde{p}_n(x)$ for such x are called *large-deviation local* theorems. Such theorems assume certain asymptotic regularity of the underlying density. The following regularity conditions were introduced in Zaigraev (1999).

Definition 1.1. *A density function $p(x)$ belongs to the class $\mathcal{P}$ if it is bounded and admits the representation*

$$p(x) = \frac{h(e_x) + \theta(x)\omega(|x|)}{|x|^{d+\alpha}}, \qquad |x| > 1$$

where $\alpha \in (0, 1) \cup (1, 2)$, $h(e)$ *is a continuous function on* $\mathbb{S}^{d-1}$, $e_x = |x|^{-1}x$, $|\theta(x)| \leqslant 1$, *and* $\omega(t) \to 0$ *as* $t \to \infty$.

It is easily seen that for a distribution having a density belonging to $\mathcal{P}$ the normalizing sequence b_n has the form (1.2) with

$$b_0 = (l_0 \alpha c_\alpha)^{1/\alpha}, \qquad l_0 = \alpha^{-1} \int_{\mathbb{S}^{d-1}} h(e)\sigma(\mathrm{d}e),$$

that is, such distribution belongs to the domain of normal attraction of S determined by α and

$$\nu(E) = \frac{\displaystyle\int_E h(e)\sigma(\mathrm{d}e)}{\displaystyle\int_{\mathbb{S}^{d-1}} h(\varepsilon)\sigma(\mathrm{d}\varepsilon)}, \qquad E \in \mathcal{B}_{\mathbb{S}^{d-1}},$$

where σ denotes the spherical Lebesque measure on $\mathcal{B}_{\mathbb{S}^{d-1}}$. In other words, ν is absolutely continuous with respect to σ and, furthermore,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\sigma}(e) = \frac{h(e)}{\displaystyle\int_{\mathbb{S}^{d-1}} h(\varepsilon)\sigma(\mathrm{d}\varepsilon)}, \qquad e \in \mathbb{S}^{d-1}.$$

In what follows we consider the case where S is a strictly α-stable distribution and $\alpha \neq 1$. This implies that the location $a = \mathbf{0}$ and, therefore, we may put $a^{(n)} = \mathbf{0}$.

Let ρ_n denote any sequence such that $\lim_{n\to\infty}\rho_n = \infty$. The following statement was proven in Zaigraev (1999); see his Theorem 1.

Proposition 1.1. *If* $p(x) \in \mathcal{P}$ *then, for any* $\delta > 0$,

$$\lim_{n\to\infty} \sup_{e_x \in E_\delta, |x| \geqslant n^{1/\alpha}\rho_n} \left| \frac{p_n(x)}{np(x)} - 1 \right| = 0,$$

where $E_\delta = (e \in \mathbb{S}^{d-1} : h(e) \geqslant \delta)$, *while*

$$\lim_{n\to\infty} \sup_{e_x \in E^0, |x| \geqslant n^{1/\alpha}\rho_n} n^{-1}|x|^{d+\alpha} p_n(x) = 0, \tag{1.4}$$

where $E^0 = (e \in \mathbb{S}^{d-1} : h(e) = 0)$.

From (1.3) it also follows that for any fixed $e \in \mathbb{S}^{d-1}$ there exists a sequence $t_n(e) \to \infty$ such that

$$\lim_{n\to\infty} \sup_{0 \leqslant t \leqslant t_n(e)} \left| \frac{\tilde{p}_n(te)}{s(te)} - 1 \right| = 0. \tag{1.5}$$

We call the interval $[0, t_n(e)]$ the *zone of local attraction in the direction* e (cf. Ibragimov and Linnik 1971, Chapter 9).

How wide could the zones of local attraction be? Proposition 1.1, together with Corollary 2 of Arkhipov (1989), enables us to answer this question at least in the case $e \in E_\delta$. Let

$p(x) \in \mathcal{P}$. Assume, additionally, that all derivatives of $h(e)$, up to order $2 + d/2$, are bounded. For more about the derivatives of functions defined on $\mathbb{S}^{d-1}$ see, for example, Groemer (1996, Section 1.2). Then, by Corollary 2 of Arkhipov (1989), there exists $c > 0$ such that, for $|x| \neq 0$,

$$s(x) = \frac{h(e_x)}{b_0^\alpha |x|^{d+\alpha}} + \frac{c\theta(x)}{|x|^{d+2\alpha}}, \tag{1.6}$$

where $|\theta(x)| \leqslant 1$. So, for $e \in E_\delta$ as $t \to \infty$,

$$s(te) = \frac{h(e)}{b_0^\alpha t^{d+\alpha}}(1 + c\theta(te)t^{-\alpha}).$$

Together with Proposition 1.1, this implies that, for any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{x \in K_\delta} \left| \frac{\tilde{p}_n(x)}{s(x)} - 1 \right| = 0,$$

where

$$K_\delta = (x \in \mathbb{R}^d : e_x \in E_\delta).$$

In other words, for $e \in E_\delta$ we have $t_n(e) \equiv \infty$. In particular, if $E_\delta = \mathbb{S}^{d-1}$ then $K_\delta = \mathbb{R}^d$ and we obtain the strong form of the Gnedenko local limit theorem,

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}^d} \left| \frac{\tilde{p}_n(x)}{s(x)} - 1 \right| = 0.$$

Let us call the directions $e \in E^0$ *singular*. Up to now the asymptotic behaviour of $\tilde{p}_n(te)$ in the singular directions remains unknown. The problem is that the regular directions $e \in E_\delta$ are studied using methods that were worked out for one-dimensional heavy-tailed distributions (Nagaev 1969; Tkachuk 1973). Unfortunately, those methods have proved to be insufficient for analysing singular directions and require further development.

It is the basic goal of the paper to outline an approach to the problem of large deviations in the singular directions. The problem seems to be extremely difficult, even in the simplest case considered below.

The paper is organized as follows. In Section 2 we state and comment on the main results. Auxiliary facts are given in Section 3. Sections 4 and 5 are devoted to the proof of the main results.

2. Main results

Recall that we confine ourselves to the case $\alpha \in (0, 1) \cup (1, 2)$ and assume that there is a finite number of singular directions. The following definition specifies the case.

Definition 2.1. *A density function $p(x)$ belongs to the class $\mathcal{P}_0$ if it is bounded and satisfies the following conditions:*

(i) $\#(E^0) < \infty$ *(there is a finite number of singular directions).*
(ii) *For* $|x| > 1$, *the density* $p(x)$ *admits the representation*

$$p(x) = \frac{h(e_x)}{|x|^{d+\alpha}} + \frac{d(e_x) + \theta(x)\omega(|x|)}{|x|^{d+\beta}}, \tag{2.1}$$

where $\alpha \in (0, 1) \cup (1, 2)$, $\beta > \alpha$, *and* $d(e)$ *is a continuous function on* $\mathbb{S}^{d-1}$.
(iii) *For all* $e^* \in E^0$, *there exists a positive constant* h_0 *such that*

$$h(e) \leqslant h_0|e - e^*|^2, \qquad e \in \mathbb{S}^{d-1}. \tag{2.2}$$

Obviously, $\mathcal{P}_0 \subset \mathcal{P}$. It is worth recalling that the function $h(e)$ coincides, up to a constant multiplier, with the spectral density of the limit α-stable law.

Under the additional restriction on the smoothness of $h(e)$, one can compare the asymptotic behaviour of $p(x)$ and $s(x)$ in a direction $e^* \in E^0$. Suppose that $p(x) \in \mathcal{P}_0$ and all derivatives of $h(e)$, up to order $4 + d/2$, are bounded. From Corollary 2 of Arkhipov (1989) it follows that the limit α-stable density $s(x)$ admits the expansion (cf. (1.6))

$$s(x) = \frac{h(e_x)}{b_0^\alpha |x|^{d+\alpha}} + \frac{b(e_x)}{|x|^{d+2\alpha}} + \frac{c\theta(x)}{|x|^{d+3\alpha}}, \qquad |x| \neq 0, \tag{2.3}$$

where $b(e)$ is a continuous function on $\mathbb{S}^{d-1}$. Since $h(e^*) = 0$ we conclude that, for $t \to \infty$,

$$s(te^*) = b(e^*)t^{-(d+2\alpha)}(1 + O(t^{-\alpha})), \qquad p(te^*) = d(e^*)t^{-(d+\beta)}(1 + o(1)),$$

provided $d(e^*) > 0$.

Below we prove that $b(e^*) > 0$. Thus, the relation between $p(te^*)$ and $s(te^*)$ depends on that between β and 2α.

Denote

$$\pi(x) = h(e_x)|x|^{-(d+\alpha)}, \qquad |x| \neq 0. \tag{2.4}$$

This function is not integrable, but

$$\pi_2(e^*) = \int_{\mathbb{R}^d} \pi(x)\pi(e^* - x)\mathrm{d}x < \infty$$

(see Lemma 3.7 below). The following theorem contains our main result.

Theorem 2.1. *Let* $p(x) \in \mathcal{P}_0$, $e^* \in E^0$, *and* $d(e^*) > 0$. *Then as* $n \to \infty$, $t \to \infty$,

$$\tilde{p}_n(te^*) = \frac{\pi_2(e^*)}{2b_0^{2\alpha}} t^{-(d+2\alpha)}(1 + \omega_1(n, t)) + \frac{d(e^*)}{n^{\beta/\alpha-1} b_0^\beta} t^{-(d+\beta)}(1 + \omega_2(n, t)),$$

where

$$\lim_{n\to\infty} \sup_{t \geqslant \rho_n} |\omega_i(n, t)| = 0, \qquad i = 1, 2.$$

Now we are able to compare the asymptotics of $\tilde{p}_n(te^*)$ and $s(te^*)$.

Corollary 2.2. *Suppose that the conditions of Theorem 2.1 are fulfilled and all derivatives of $h(e)$, up to order $4 + d/2$, are bounded. If $\beta \geqslant 2\alpha$, then*

$$\lim_{n\to\infty} \sup_{t\geqslant 0} \left| \frac{\tilde{p}_n(te^*)}{s(te^*)} - 1 \right| = 0.$$

If $\beta < 2\alpha$, then

$$\lim_{n\to\infty} \sup_{t\in[0,n^\kappa/\rho_n]} \left| \frac{\tilde{p}_n(te^*)}{s(te^*)} - 1 \right| = 0,$$

where $\kappa = (\beta/\alpha - 1)/(2\alpha - \beta)$. But if $t \geqslant n^\kappa \rho_n$, then, as $n \to \infty$,

$$\frac{\tilde{p}_n(te^*)}{s(te^*)} \to \infty.$$

In particular, if $\beta \geqslant 2\alpha$, that is, $p(te^*) = O(s(te^*))$, then the zone of local attraction in the direction e^* is infinite.

In the proof of Theorem 2.1 an unexpected phenomenon is utilized. To the best of our knowledge this phenomenon has been never discussed. Usually, large deviations of a sum of i.i.d. random vectors having a heavy-tailed distribution arise when exactly *one* of the summands is abnormally large (Nagaev and Zaigraev 1998; Zaigraev 1999). In particular, it is true for the regular directions $e \in E_\delta$. The following statement was proven in Zaigraev (1999); see his Theorem 2 therein.

Proposition 2.3. *If $p(x) \in \mathcal{P}$ then, for any $\delta > 0$ and any $A \in \mathcal{B}_{\mathbb{R}^d}$,*

$$\lim_{n\to\infty} \sup_{|x|\geqslant \rho_n b_n} \sup_{e_x \in E_\delta} \left| P\big(b_n^{-1}(\zeta^{(n)} - \xi') \in A \big| \zeta^{(n)} = x\big) - S(A) \right| = 0,$$

where ξ' denotes the summand in the sum $\zeta^{(n)}$ with largest absolute value.

It turns out that within $\mathcal{P}_0$ large deviations in a singular direction $e^* \in E^0$ are explained by *two* abnormally large summands if $\beta \geqslant 2\alpha$ or $\beta < 2\alpha$ but $\rho_n \leqslant t \leqslant n^\kappa/\rho_n$. It is the first term in the representation for $\tilde{p}_n(te^*)$ that reflects this phenomenon.

The following statement gives a more formal description of the phenomenon and should be compared with Proposition 2.3.

Theorem 2.4. *Let the conditions of Theorem 2.1 be fulfilled and $A, B, C \in \mathcal{B}_{\mathbb{R}^d}$.*
If $\beta \geqslant 2\alpha$, then

$$\lim_{n\to\infty} \sup_{t\geqslant \rho_n} \Bigg| P\big(b_n^{-1}\xi' \in tA,\ b_n^{-1}\xi'' \in tB,\ b_n^{-1}(\zeta^{(n)} - \xi' - \xi'') \in C \big| \zeta^{(n)} = b_n te^*\big)$$
$$- S(C)(\pi_2(e^*))^{-1} \int_{A\cap(e^*-B)} \pi(u)\pi(e^* - u)\mathrm{d}u \Bigg| = 0, \quad (2.5)$$

where ξ' and ξ'' denote the two summands in the sum $\zeta^{(n)}$ with largest absolute value.

If $\beta < 2\alpha$, then (2.5) *remains valid if the supremum is taken over $\rho_n \leqslant t \leqslant n^\kappa/\rho_n$. As for bigger values of t, the following relation holds:*

$$\lim_{n\to\infty} \sup_{t\geqslant \rho_n n^\kappa} \left| P\big(b_n^{-1}(\zeta^{(n)} - \xi') \in A | \zeta^{(n)} = b_n t e^*\big) - S(A) \right| = 0, \tag{2.6}$$

where ξ' denotes the summand in the sum $\zeta^{(n)}$ with largest absolute value.

Now assume that $\sigma(E^0) > 0$, that is, there is a 'hole' in the support of $h(e)$. Then the nature of the large deviation in the direction $e^* \in \operatorname{int} E^0$ is the same as in the case of finite number of singular directions. More precisely, the large deviation is explained by two abnormally large summands. However, the case seems to be much more complicated, as does that concerning the asymptotic formula for $\tilde{p}_n(te^{(n)})$ for $t \to \infty$, as $e^{(n)} \to \partial E^0$. The authors will discuss these cases elsewhere.

3. Auxiliary statements

Henceforth, $B_k(\mathbf{0}, r) = (x \in \mathbb{R}^k : |x| \leqslant r)$, for $r > 0$, and c denotes any positive constant whose concrete value is of no importance. This means that, for example $c + c = c$ and $c^2 = c$. By $\omega(t)$ we denote any non-negative function such that $\lim_{t\to\infty}\omega(t) = 0$, while θ always varies within $[-1, 1]$.

3.1. Marginal densities

Let $e^* \in E^0$ and C be an orthogonal matrix such that $Ce^* = (0, \ldots, 0, 1)^{\mathrm{T}}$. If ξ has a density from $\mathcal{P}_0$ then the density of $C\xi$ also belongs to $\mathcal{P}_0$ with $h(e)$ be replaced by $h(C^{\mathrm{T}}e)$. So, without loss of generality, we may assume that $e^* = (0, \ldots, 0, 1)^{\mathrm{T}}$. For any $x = (x_1, \ldots, x_d)$, define $\bar{x} = (x_1, \ldots, x_{d-1})$. Then $x = (\bar{x}, x_d)$.

Lemma 3.1. *If $p(x) \in \mathcal{P}_0$, then the marginal density $\bar{p}(\bar{x})$ of $\bar{\xi} = (\xi_1, \ldots, \xi_{d-1})$ is bounded and admits the representation (cf. Definition 1.1)*

$$\bar{p}(\bar{x}) = \frac{\bar{h}(e_{\bar{x}}) + \theta\omega(|\bar{x}|)}{|\bar{x}|^{d-1+\alpha}}, \qquad |\bar{x}| > 1,$$

where $\bar{h}(e)$ is a strictly positive continuous function on $\mathbb{S}^{d-2}$.

Proof. Since $\mathcal{P}_0 \subset \mathcal{P}$, we may represent $p(x)$ as

$$p(x) = \frac{h(e_x) + \theta\omega(|x|)}{|x|^{d+\alpha}}, \qquad |x| > 1.$$

Note, first, that

$$\int_{-\infty}^{\infty} \frac{\omega((|\bar{x}|^2+z^2)^{1/2})\mathrm{d}z}{(|\bar{x}|^2+z^2)^{(d+\alpha)/2}} = \theta\omega(|\bar{x}|)\int_{-\infty}^{\infty}(|\bar{x}|^2+z^2)^{-(d+\alpha)/2}\mathrm{d}z = \theta\omega(|\bar{x}|)|\bar{x}|^{-d-\alpha+1}.$$

Further,

$$\int_{-\infty}^{\infty} \frac{h(e_{(\bar{x},z)})\mathrm{d}z}{(|\bar{x}|^2+z^2)^{(d+\alpha)/2}} = |\bar{x}|^{-d-\alpha+1}\int_{-\infty}^{\infty} \frac{h(e_{(e_{\bar{x}},z)})\mathrm{d}z}{(1+z^2)^{(d+\alpha)/2}}.$$

It is evident that the function

$$\bar{h}(e_{\bar{x}}) = \int_{-\infty}^{\infty} \frac{h(e_{(e_{\bar{x}},z)})\mathrm{d}z}{(1+z^2)^{(d+\alpha)/2}}$$

is continuous and positive. The lemma is proven. □

Let $\bar{\xi}^{(1)}, \ldots, \bar{\xi}^{(n)}$ be independent copies of $\bar{\xi}$. Denote by $\bar{p}_n(\bar{x})$ the densities of $\bar{\xi}^{(1)} + \ldots + \bar{\xi}^{(n)}$, $n = 1, 2, \ldots$. Here

$$\bar{b}_n = \bar{b}_0 n^{1/\alpha}, \qquad \bar{b}_0 = (\bar{l}_0 \alpha c_\alpha)^{1/\alpha}, \qquad \bar{l}_0 = \alpha^{-1}\int_{\mathbb{S}^{d-2}} \bar{h}(e)\overline{\sigma}(\mathrm{d}e),$$

where $\overline{\sigma}$ denotes the spherical Lebesque measure on $\mathcal{B}_{\mathbb{S}^{d-2}}$. From Lemma 3.1 and Proposition 1.1 we arrive at the following statement.

Corollary 3.2. *Let $\bar{s}(\bar{x})$ be the density of the α-stable distribution determined by α and $\bar{\nu}$, where*

$$\bar{\nu}(E) = \frac{\int_E \bar{h}(e)\overline{\sigma}(\mathrm{d}e)}{\int_{\mathbb{S}^{d-2}} \bar{h}(\varepsilon)\overline{\sigma}(\mathrm{d}\varepsilon)}, \qquad E \in \mathcal{B}_{\mathbb{S}^{d-2}}.$$

Under the conditions of Lemma 3.1 the strong form of the Gnedenko local limit theorem holds, that is,

$$\lim_{n\to\infty} \sup_{\bar{x}\in\mathbb{R}^{d-1}} \left| \frac{\bar{b}_n^{d-1}\,\bar{p}_n(\bar{b}_n\bar{x})}{\bar{s}(\bar{x})} - 1 \right| = 0.$$

The proof of the next result is similar to that of Lemma 3.1.

Lemma 3.3. *Let $\xi = (\xi_1, \ldots, \xi_d)$, $d \geqslant 2$. If $p(x) \in \mathcal{P}_0$ then the marginal density $p^*(t)$ of the random variable $\xi_d \in \mathbb{R}^1$ is bounded and admits the representation*

$$p^*(t) = \frac{h^*(t) + \theta\omega(|t|)}{|t|^{1+\alpha}}, \qquad |t| > 1,$$

where

$$h^*(t) = \int_{\mathbb{R}^{d-1}} \frac{h(e_{(\bar{x},\mathrm{sign}(t))})\mathrm{d}\bar{x}}{(1+|\bar{x}|^2)^{(d+\alpha)/2}} > 0.$$

3.2. Truncated densities

Consider a random vector $\eta = \eta_r$ such that

$$P(\eta \in A) = P(\xi \in A \mid |\xi| \leqslant r), \qquad A \in \mathcal{B}_{\mathbb{R}^d}.$$

Obviously, the distribution of η is absolutely continuous and its density $q(x) = q_r(x)$ is of the form

$$q(x) = (P(|\xi| \leqslant r))^{-1} p(x) \mathbf{1}_{B_d(\mathbf{0},r)}(x). \tag{3.1}$$

Let $q_n(x)$ be the densities of $\eta^{(1)} + \ldots + \eta^{(n)}$, $n = 1, 2, \ldots,$ where $\eta^{(j)}$, $j = 1, 2, \ldots,$ are independent copies of η.

Lemma 3.4. *Let the distribution of ξ be from the domain of normal attraction of S. If $\sup_{x\in\mathbb{R}^d} p(x) < \infty$, then*

$$\lim_{n\to\infty} \sup_{r\geqslant \rho_n b_n} \sup_{x\in\mathbb{R}^d} |b_n^d q_n(b_n x) - s(x)| = 0. \tag{3.2}$$

Proof. Let $r \geqslant \rho_n b_n$. Denote by $\psi(y)$ and $\varphi_r(y)$ the characteristic functions corresponding to $p(x)$ and $q(x) = q_r(x)$, respectively. We should verify the following relations:

$$\lim_{n\to\infty} \sup_{|y|\leqslant Y} |\varphi_r^n(b_n^{-1} y) - \hat{s}(y)| = 0;$$

$$\sup_{r\geqslant\rho_n b_n} |\varphi_r(y)| \leqslant |\psi(y)|(1 + O(n^{-1})), \qquad |y| \leqslant Y;$$

$$\sup_{r\geqslant\rho_n b_n} \sup_{x\in\mathbb{R}^d} q(x) < \infty.$$

Here Y is any fixed positive constant.

The first two relations follow from the representation

$$\varphi_r(y) = \psi(y) + O(r^{-\alpha}).$$

The third one is quite evident. The lemma is proven. □

Let $\xi = (\xi_1, \ldots, \xi_d) = (\bar{\xi}, \xi_d)$. Consider the random vector $\bar{\eta} \in \mathbb{R}^{d-1}$ and the random variable $\eta_d \in \mathbb{R}^1$ such that

$$P(\bar{\eta} \in A) = P(\bar{\xi} \in A \mid |\bar{\xi}| \leqslant r), \qquad A \in \mathcal{B}_{\mathbb{R}^{d-1}},$$

and

$$P(\eta_d \leqslant t) = P(\xi_d \leqslant t \mid |\xi_d| \leqslant r), \qquad t \in \mathbb{R}^1.$$

Obviously, both distributions are absolutely continuous, with densities

$$\bar{q}(\bar{x}) = (P(|\bar{\xi}| \leqslant r))^{-1} \bar{p}(\bar{x}) \mathbf{1}_{B_{d-1}(\mathbf{0},r)}(\bar{x}),$$

where $\bar{p}$ is the density of $\bar{\xi}$, and

$$q^*(t) = (P(|\xi_d| \leqslant r))^{-1} p^*(t) \mathbf{1}_{[-r,r]}(t),$$

respectively.

Let $\bar{q}_n$ and q_n^*, $n > 1$, stand for the nth convolutions of $\bar{q}$ and q^*, respectively.

Lemma 3.5. *If* $p(x) \in \mathcal{P}_0$ *then, as* $n \to \infty$, $zn^{-1/\alpha} \to \infty$,

$$\int_{|\bar{x}|>z} \bar{q}_n(\bar{x}) \mathrm{d}\bar{x} = o(nz^{-\alpha}), \qquad \int_{|t|>z} q_n^*(t) \mathrm{d}t = o(nz^{-\alpha}),$$

provided $rn^{-1/\alpha} \to \infty$, $zr^{-1} \to \infty$.

Proof. First, note that by Lemma 3.1 $\bar{p}(\bar{x})$ satisfies the conditions of Theorem 1 of Zaigraev (1999). So, in order to prove the first statement it suffices to alter slightly the proof of Lemma 4 therein, where, in contrast to the case considered here, the projection of $\bar{\xi}$ onto the direction of the large deviation is truncated. As to the second statement, it follows from Lemma 3.3 and Tkachuk (1973). $\square$

3.3. More about the tail properties of $s(x)$

Here we give two facts concerning the asymptotic properties of the limit α-stable density. The first is a direct corollary of Theorem 1 of Fristedt (1972).

Lemma 3.6. *If a random vector* ζ *has the distribution* S *then, as* $r \to \infty$,

$$\mathrm{E}\big(|\zeta|^2 \mathbf{1}_{B_d(\mathbf{0},r)}(\zeta)\big) = cr^{2-\alpha}(1 + o(1)).$$

The next fact is very important, but not quite obvious.

Lemma 3.7. *Let* $p(x) \in \mathcal{P}_0$, *and* $\pi(x)$ *is given by* (2.4). *Then*

$$\pi_2(e^*) = \int_{\mathbb{R}^d} \pi(x)\pi(e^* - x) \mathrm{d}x < \infty.$$

Proof. Represent $\pi_2(e^*)$ as follows:

$$\pi_2(e^*) = \int_{|x| \leqslant 1/2} + \int_{|x-e^*| \leqslant 1/2} + \int_{(x:|x|>1/2, |x-e^*|>1/2)} = I_1 + I_2 + I_3.$$

Obviously, $I_1 = I_2$. From (2.2) and (2.4), it follows that

$$I_1 \leqslant c \int_{|x| \leqslant 1/2} h(e_{e^*-x}) |x|^{-d-\alpha} \mathrm{d}x \leqslant c \int_{|x| \leqslant 1/2} |e_{e^*-x} - e^*|^2 |x|^{-d-\alpha} \,\mathrm{d}x.$$

For $|x| \leqslant 1/2$ we have $|e^* - x| \geqslant 1/2$ and, therefore,

$$|e_{e^*-x} - e^*|^2 \leqslant \frac{2|\bar{x}|^2}{|e^* - x|^2} \leqslant 8|x|^2. \tag{3.3}$$

Hence,

$$I_1 \leqslant c\int_{|x|\leqslant 1/2} |x|^{-d-\alpha+2}\,\mathrm{d}x < \infty.$$

Further,

$$I_3 \leqslant c\int_{|x|>1/2} |x|^{-d-\alpha}\mathrm{d}x < \infty.$$

The lemma is proven. □

4. Proof of Theorem 2.1

Put $z = b_n t$ and let $t \to \infty$. Recall that $b_n = b_0 n^{1/\alpha}$.

Consider the events

$$A_{n,0} = \{|\xi^{(j)}| \leqslant \gamma z,\ j = 1, \dots, n\},$$

$$A_{n,1} = \{|\xi^{(j)}| \leqslant \gamma z,\ j = 1, \dots, n-1,\ |\xi^{(n)}| > \gamma z\},$$

$$A_{n,2} = \{|\xi^{(j)}| \leqslant \gamma z,\ j = 1, \dots, n-2,\ |\xi^{(n-1)}| > \gamma z,\ |\xi^{(n)}| > \gamma z\};$$

$A_{n,3}$ is the event that at least three variables among $|\xi^{(1)}|, \dots, |\xi^{(n)}|$ are greater than γz, where the constant $\gamma \in (0, 1)$ is to be specified later.

Then $p_n(ze^*)$ can be represented as

$$p_n(ze^*) = p_{n,0}(ze^*) + np_{n,1}(ze^*) + \frac{n(n-1)}{2}\,p_{n,2}(ze^*) + p_{n,3}(ze^*), \tag{4.1}$$

where

$$p_{n,j}(ze^*) = \lim_{|\Delta|\to 0} |\Delta|^{-d} P\big(ze^* \prec \zeta^{(n)} \leqslant ze^* + \Delta,\ A_{n,j}\big), \qquad j = 0, 1, 2, 3$$

and $\prec$ ($\leqslant$) denotes componentwise ordering.

We begin with the contribution of the largest summand. Substituting $r = \gamma z$ in (3.1), we obtain

$$(P(|\xi| \leqslant \gamma z))^{n-1} = 1 + o(1)$$

and, therefore,

$$p_{n,1}(ze^*) = I_n(z)(1 + o(1)), \tag{4.2}$$

where, in view of (2.1),

$$I_n(z) = \int_{|u|>\gamma z} p(u)q_{n-1}(ze^* - u)\mathrm{d}u = I + J, \tag{4.3}$$

$$I = \int_{|u|>\gamma z} \frac{h(e_u)}{|u|^{d+\alpha}} q_{n-1}(ze^* - u)\mathrm{d}u, \qquad J = \int_{|u|>\gamma z} \frac{d(e_u) + \theta\omega(|u|)}{|u|^{d+\beta}} q_{n-1}(ze^* - u)\mathrm{d}u.$$

First, consider I. It is easily seen that

$$I = \iint_{|\bar{u}|^2+u_d^2>\gamma^2 z^2} \frac{h(e_{(\bar{u},u_d)})}{(|\bar{u}|^2 + u_d^2)^{(d+\alpha)/2}} q_{n-1}(-\bar{u}, z - u_d)\mathrm{d}\bar{u}\mathrm{d}u_d \leqslant (\gamma z)^{-d-\alpha} I', \tag{4.4}$$

where

$$I' = \iint_{|\bar{u}|^2+(z-u_d)^2>\gamma^2 z^2} h(e_{(\bar{u},z-u_d)}) q_{n-1}(-\bar{u}, u_d)\mathrm{d}\bar{u}\,\mathrm{d}u_d.$$

Let I_1, I_2 and I_3 be the parts of I' corresponding to the sets

$$A_1 = (u \in \mathbb{R}^d : |\bar{u}|^2 + (z - u_d)^2 > \gamma^2 z^2, |\bar{u}| \leqslant \delta z, |u_d| \leqslant \delta z),$$

$$A_2 = (u \in \mathbb{R}^d : |\bar{u}|^2 + (z - u_d)^2 > \gamma^2 z^2, |\bar{u}| \leqslant \delta z, |u_d| > \delta z),$$

$$A_3 = (u \in \mathbb{R}^d : |\bar{u}|^2 + (z - u_d)^2 > \gamma^2 z^2, |\bar{u}| > \delta z),$$

respectively, where $0 < \delta < 1 - \gamma$ is a fixed small number. Then

$$I' = I_1 + I_2 + I_3. \tag{4.5}$$

First, we estimate I_1. By (2.2),

$$h(e_{(\bar{u},z-u_d)}) \leqslant c|e_{(\bar{u},z-u_d)} - e^*|^2.$$

Since (cf. (3.3))

$$|e_{(\bar{u},z-u_d)} - e^*|^2 \leqslant \frac{2|\bar{u}|^2}{|\bar{u}|^2 + (z - u_d)^2},$$

we obtain

$$I_1 \leqslant cz^{-2} \iint_{(u:|\bar{u}|\leqslant\delta z)} |\bar{u}|^2 q_{n-1}(-\bar{u}, u_d)\mathrm{d}\bar{u}\,\mathrm{d}u_d.$$

Obviously, for all sufficiently large n,

$$q_{n-1}(u) \leqslant \frac{p_{n-1}(u)}{(P(|\xi| \leqslant \gamma z))^{n-1}} \leqslant 2p_{n-1}(u), \qquad u \in \mathbb{R}^d.$$

Then

$$I_1 \leqslant cz^{-2} \int_{|\bar{u}|\leqslant\delta z} |\bar{u}|^2 \bar{p}_{n-1}(\bar{u})\mathrm{d}\bar{u},$$

where $\bar{p}_{n-1}(\bar{u})$ is as in Lemma 3.1. By Corollary 3.2,

$$\int_{|\bar{u}|\leqslant\delta z} |\bar{u}|^2 \bar{p}_{n-1}(\bar{u})\mathrm{d}\bar{u} = \bar{b}_n^2 \int_{|\bar{u}|\leqslant\delta z\bar{b}_n^{-1}} |\bar{u}|^2 \bar{s}(\bar{u})\mathrm{d}\bar{u}(1 + o(1)).$$

From Lemma 3.6 it follows that, as $\bar{t} = z\bar{b}_n^{-1} \to \infty$,

$$\int_{|\bar{u}|\leqslant\delta\bar{t}} |\bar{u}|^2 \bar{s}(\bar{u})\mathrm{d}\bar{u} = c\delta^{2-\alpha}\bar{t}^{2-\alpha}(1+o(1)).$$

Thus, for all sufficiently large n,

$$\int_{|\bar{u}|\leqslant\delta z} |\bar{u}|^2\,\bar{p}_{n-1}(\bar{u})\mathrm{d}\bar{u} \leqslant c\delta^{2-\alpha}z^{2-\alpha}\bar{b}_n^{\alpha}$$

and, therefore,

$$I_1 \leqslant c\delta^{2-\alpha}nz^{-\alpha}. \tag{4.6}$$

The term I_2 is simpler. Here, for all sufficiently large n,

$$I_2 \leqslant c\int_{A_2} q_{n-1}(-\bar{u},\, u_d)\mathrm{d}\bar{u}\mathrm{d}u_d \leqslant c\int_{|u_d|>\delta z} q_{n-1}^{*}(u_d)\mathrm{d}u_d,$$

where q_{n-1}^{*} is defined as in Section 3.2. In view of Lemma 3.5, we have

$$I_2 = o(nz^{-\alpha}). \tag{4.7}$$

As to I_3, it is evident that, for all sufficiently large n,

$$I_3 \leqslant c\int_{|\bar{u}|>\delta z} \bar{q}_{n-1}(\bar{u})\mathrm{d}\bar{u},$$

where $\bar{q}_{n-1}$ is defined as in Section 3.2. By virtue of Lemma 3.5, we have

$$I_3 = o(nz^{-\alpha}). \tag{4.8}$$

Combining (4.4)–(4.8) yields

$$I \leqslant c\delta^{2-\alpha}nz^{-d-2\alpha}, \tag{4.9}$$

provided n is sufficiently large.

We now estimate J in (4.3). First, note that

$$J = \int_{|u|>\gamma z} \frac{d(e_u)q_{n-1}(ze^{*}-u)\mathrm{d}u}{|u|^{d+\beta}}(1+o(1)) = z^{-d-\beta}J'(1+o(1)), \tag{4.10}$$

where

$$J' = \int\!\!\int_{|\bar{u}|^2+(z-u_d)^2>\gamma^2z^2} \frac{d(e_{(\bar{u},z-u_d)})q_{n-1}(-\bar{u},\, u_d)\mathrm{d}\bar{u}\,\mathrm{d}u_d}{(|\bar{u}|^2/z^2+(1-u_d/z)^2)^{(d+\beta)/2}}.$$

Denote by J_1, J_2 and J_3 the parts of J' corresponding, respectively, to the sets A_1, A_2 and A_3, that is,

$$J' = J_1 + J_2 + J_3. \tag{4.11}$$

If $u \in A_1$ then $d(e_{(\bar{u},z-u_d)}) = d(e^{*}) + c\theta\delta$, and recall that $d(e^{*}) > 0$. If $\delta > 0$ is sufficiently small then

$$A_1 = (u \in \mathbb{R}^d : |\bar{u}| \leqslant \delta z, |u_d| \leqslant \delta z)$$

and

$$|\bar{u}|^2/z^2 + (1 - u_d/z)^2 = 1 + c\theta\delta.$$

Therefore,

$$J_1 = (d(e^*) + c\theta\delta)\int_{|u_d|\leqslant\delta z}\int_{|\bar{u}|\leqslant\delta z} q_{n-1}(\bar{u}, u_d)\mathrm{d}\bar{u}\,\mathrm{d}u_d$$

and, by Lemma 3.4,

$$J_1 = d(e^*) + c\theta\delta, \tag{4.12}$$

provided n is sufficiently large.

As to J_2, again for all sufficiently large n, by virtue of Lemma 3.5 we have

$$J_2 \leqslant c\int_{|u_d|>\delta z} q^*_{n-1}(u_d)\mathrm{d}u_d = o(nz^{-\alpha}). \tag{4.13}$$

J_3 is estimated in the same way as I_3. We thus obtain

$$J_3 = o(nz^{-\alpha}). \tag{4.14}$$

From (4.10)–(4.14) it follows that, for all sufficiently large n, we have

$$J' = d(e^*) + c\theta\delta$$

and, therefore,

$$J = (d(e^*) + c\theta\delta)z^{-d-\beta}. \tag{4.15}$$

Since $\delta > 0$ can be arbitrarily small, in view of (4.2), (4.3), (4.9) and (4.15) we arrive at the following statement.

Lemma 4.1. *Under the conditions of Theorem 2.1 as* $n \to \infty$, $zn^{-1/\alpha} \to \infty$,

$$p_{n,1}(ze^*) = d(e^*)z^{-d-\beta}(1 + o(1)) + o(nz^{-d-2\alpha}).$$

Turning now to the contribution of the two largest summands, denote

$$I = \int_{|u|>\gamma z}\int_{|v|>\gamma z} p(u)p(v)q_{n-2}(ze^* - u - v)\mathrm{d}u\,\mathrm{d}v,$$

$$I_1 = \int_{|u|>\gamma z}\int_{|v|>\gamma z} \pi(u)\pi(v)q_{n-2}(ze^* - u - v)\mathrm{d}u\,\mathrm{d}v,$$

$$I_2 = \int_{|u|>\gamma z}\int_{|v|>\gamma z} |u|^{-d-\alpha}|v|^{-d-\beta}q_{n-2}(ze^* - u - v)\mathrm{d}u\,\mathrm{d}v,$$

where $\pi(x)$ is given by (2.4). Then it is evident that

$$p_{n,2}(ze^*) = I(1 + o(1)), \tag{4.16}$$

where

$$I = I_1 + c\theta I_2. \tag{4.17}$$

First, we estimate I_1. With the change of variables $u = zx$, $v = z(e^* - x - y)$ we obtain

$$I_1 = z^{-2\alpha}\int_A \pi(x)\pi(e^* - x - y)q_{n-2}(zy)\mathrm{d}x\,\mathrm{d}y = z^{-d-2\alpha}(I_{11} + I_{12}),$$

where

$$A = \big((x,\ y) \in \mathbb{R}^{2d} : |x| > \gamma,\ |e^* - x - y| > \gamma\big).$$

Here

$$I_{1i} = z^d\int_{A_i} \pi(x)\pi(e^* - x - y)q_{n-2}(zy)\mathrm{d}x\,\mathrm{d}y, \qquad i = 1,\ 2,$$

where

$$A_1 = \big((x,\ y) \in \mathbb{R}^{2d} : |x| > \gamma,\ |e^* - x - y| > \gamma,\ |y| \leqslant \delta_n\big),$$

$$A_2 = \big((x,\ y) \in \mathbb{R}^{2d} : |x| > \gamma,\ |e^* - x - y| > \gamma,\ |y| > \delta_n\big)$$

and $\delta_n > 0$.

Choose $\delta_n \to 0$ so that $\delta_n z b_n^{-1} \to \infty$. Since the function $h(e)$ is continuous, in view of Lemma 3.4 we obtain, as $n \to \infty$,

$$I_{11} = \int_{(x:|x|>\gamma,|e^*-x|>\gamma)} \pi(x)\pi(e^* - x)\mathrm{d}x + o(1).$$

The obvious inequality $\pi(x) \leqslant c|x|^{-d-\alpha}$ yields

$$I_{12} \leqslant c\int_{|x|>\gamma} |x|^{-d-\alpha}\mathrm{d}x\int_{|y|>z\delta_n} q_{n-2}(y)\mathrm{d}y.$$

By virtue of Lemma 3.4, $I_{12} = o(1)$ as $n \to \infty$ for any $\gamma \in (0,\ 1)$. Thus,

$$I_1 = z^{-d-2\alpha}\int_{(x:|x|>\gamma,|e^*-x|>\gamma)} \pi(x)\pi(e^* - x)\mathrm{d}x + o(z^{-d-2\alpha}). \tag{4.18}$$

As to I_2, we have

$$I_2 \leqslant cz^{-d-\beta}\int_{|u|>\gamma z} |u|^{-d-\alpha}\,\mathrm{d}u\int_{\mathbb{R}^d} q_{n-2}(ze^* - u - v)\mathrm{d}v \leqslant cz^{-d-\alpha-\beta}. \tag{4.19}$$

Recall that $\beta > \alpha$. In view of (4.16)–(4.19) we arrive at the following statement.

Lemma 4.2. *Under the conditions of Theorem 2.1 as $n \to \infty$, $zn^{-1/\alpha} \to \infty$,*

$$p_{n,2}(ze^*) = z^{-d-2\alpha}\int_{(x:|x|>\gamma,|e^*-x|>\gamma)} \pi(x)\pi(e^* - x)\mathrm{d}x + o(z^{-d-2\alpha}).$$

We now consider the contribution of the ‘normal’ summands. Let $q(x) = q_r(x)$ be defined as in (3.1) with $r = \gamma z$. Consider the density $q^{(s)}(x)$, $s > 0$, associated with $q(x)$ and having the form

$$q^{(s)}(x) = (f(s))^{-1} \mathrm{e}^{sx_d} q(x),$$

where $x = (x_1, \ldots, x_d)$ and

$$f(s) = \int_{|x| \leqslant \gamma z} \mathrm{e}^{sx_d} q(x) \mathrm{d}x.$$

By Cramér's transformation we have

$$p_{n,0}(ze^*) = (P(|\xi| \leqslant \gamma z))^n (f(s))^n \mathrm{e}^{-sz} q_n^{(s)}(ze^*), \tag{4.20}$$

where $q_n^{(s)}$ is the nth convolution of $q^{(s)}$, $n > 1$. Put

$$s = \alpha \gamma^{-1} z^{-1} \ln(zn^{-1/\alpha}), \tag{4.21}$$

where $0 < \gamma < \alpha/(d + 2\alpha)$.

Lemma 4.3. *If s is given by* (4.21) *then, as $n \to \infty$, $zn^{-1/\alpha} \to \infty$,*

$$f(s) = 1 + O(n^{-1}).$$

Proof. Let $\alpha' > \alpha$ and

$$f_1(s) = \int_{|x| \leqslant \alpha'/s} \mathrm{e}^{sx_d} q(x) \mathrm{d}x, \qquad f_2(s) = \int_{\alpha'/s < |x| \leqslant \gamma z} \mathrm{e}^{sx_d} q(x) \mathrm{d}x.$$

Then

$$f(s) = f_1(s) + f_2(s). \tag{4.22}$$

If $0 < \alpha < 1$, then

$$f_1(s) = 1 - \int_{\alpha'/s < |x| \leqslant \gamma z} q(x) \mathrm{d}x + \int_{|x| \leqslant \alpha'/s} (\mathrm{e}^{sx_d} - 1) q(x) \mathrm{d}x = 1 + O(s^\alpha) + f_{11}(s).$$

By Lemma 3.3,

$$f_{11}(s) = \int_{|x| \leqslant \alpha'/s} (\mathrm{e}^{sx_d} - 1) q(x) \mathrm{d}x = O\left(s \int_{-\alpha'/s}^{\alpha'/s} |t| q^*(t) \mathrm{d}t \right) = O(s^\alpha).$$

For $1 < \alpha < 2$, we have

$$f_1(s) = 1 - \int_{\alpha'/s < |x| \leqslant \gamma z} q(x) \mathrm{d}x + s \int_{|x| \leqslant \alpha'/s} x_d q(x) \mathrm{d}x + \int_{|x| \leqslant \alpha'/s} (\mathrm{e}^{sx_d} - 1 - sx_d) q(x) \mathrm{d}x.$$

Again by virtue of Lemma 3.3,

$$\int_{|x| \leqslant \alpha'/s} (\mathrm{e}^{sx_d} - 1 - sx_d) q(x) \mathrm{d}x = O\left(s^2 \int_{-\alpha'/s}^{\alpha'/s} |t|^2 q^*(t) \mathrm{d}t \right) = O(s^\alpha).$$

Therefore, thanks to the relations $P(|\xi| \leqslant \gamma z) = 1 + O(z^{-\alpha})$ and

$$\int_{|x|\leqslant\alpha'/s} x_d\, p(x)\mathrm{d}x = -\int_{|x|>\alpha'/s} x_d\, p(x)\mathrm{d}x = O(s^{\alpha-1}),$$

we obtain

$$f_1(s) = 1 + O(s^\alpha). \tag{4.23}$$

Further,

$$f_2(s) \leqslant \int_{\alpha'/s<|x|\leqslant\gamma z} \mathrm{e}^{s|x|} q(x)\mathrm{d}x = \int_{\alpha'/s<|x|\leqslant\gamma z} |x|^{\alpha'} \exp(s|x| - \alpha' \ln|x|) q(x)\mathrm{d}x.$$

Since the function $st - \alpha' \ln t$ increases in $(\alpha'/s, \gamma z)$, in view of (4.21) we obtain

$$f_2(s) \leqslant (\gamma z)^{-\alpha'} \mathrm{e}^{\gamma s z} \int_{\alpha'/s<|x|\leqslant\gamma z} |x|^{\alpha'} q(x)\mathrm{d}x = O(z^{-\alpha}\mathrm{e}^{\gamma s z}) = O(n^{-1}). \tag{4.24}$$

From (4.22)–(4.24) it follows that

$$f(s) = 1 + O(s^\alpha) + O(n^{-1}).$$

It remains to note that $ns^\alpha = O(t^{-\alpha}(\ln t)^\alpha) = o(1)$. The lemma is proven. □

Lemma 4.4. *If s is defined by* (4.21) *then, as $n \to \infty$, $zn^{-1/\alpha} \to \infty$,*

$$\sup_{x\in\mathbb{R}^d} q_n^{(s)}(x) = O(n^{-d/\alpha}).$$

Proof. Let $\psi_s(y)$ be the characteristic function corresponding to $q^{(s)}(x)$. Since $q^{(s)}(x)$ is bounded, we may use the inverse formula and obtain

$$\sup_{x\in\mathbb{R}^d} q_n^{(s)}(x) \leqslant (2\pi)^{-d}(I_1 + I_2), \tag{4.25}$$

where

$$I_1 = \int_{|y|\leqslant\delta} |\psi_s(y)|^n \mathrm{d}y, \qquad I_2 = \int_{|y|>\delta} |\psi_s(y)|^n\, \mathrm{d}y$$

and $\delta > 0$ is a fixed small number.

First, we estimate I_1. We confine ourselves to the case $\alpha \in (1, 2)$ because the case $\alpha \in (0, 1)$ is much simpler. From (4.22) and (4.24) it follows that

$$\psi_s(y) = (f(s))^{-1} \int_{|x|\leqslant\gamma z} \mathrm{e}^{\mathrm{i}\langle y,x\rangle + sx_d} q(x)\mathrm{d}x = \psi_{s1}(y) + c\theta n^{-1}, \tag{4.26}$$

where

$$\psi_{s1}(y) = (f_1(s))^{-1} \int_{|x|\leqslant\alpha'/s} \mathrm{e}^{\mathrm{i}\langle y,x\rangle + sx_d} q(x)\mathrm{d}x, \qquad \alpha' > \alpha.$$

Denote

$$a(s) = (a_1(s), \ldots, a_d(s)) = (f_1(s))^{-1} \int_{|x| \leqslant \alpha'/s} x \mathrm{e}^{sx_d} q(x) \mathrm{d}x.$$

Note that $|a(s)| \to 0$ as $s \to 0$. Then

$$\psi_{s1}(y) \mathrm{e}^{-\mathrm{i}\langle a(s), y\rangle} = 1 - (f_1(s))^{-1} \int_{|x+a(s)| \leqslant \alpha'/s} (1 - \cos\langle y, x\rangle) \mathrm{e}^{s(x_d + a_d(s))} q(x + a(s)) \mathrm{d}x$$

$$+ \mathrm{i}(f_1(s))^{-1} \int_{|x+a(s)| \leqslant \alpha'/s} (\sin\langle y, x\rangle - \langle y, x\rangle) \mathrm{e}^{s(x_d + a_d(s))} q(x + a(s)) \mathrm{d}x = 1 - R + \mathrm{i}I.$$

Recall that $\alpha > 1$. By (2.1),

$$|I| \leqslant c \int_{\mathbb{R}^d} |\sin\langle y, x\rangle - \langle y, x\rangle| p(x + a(s)) \mathrm{d}x \leqslant c \int_{\mathbb{R}^d} |x + a(s)|^{-d-\alpha} |\sin\langle y, x\rangle - \langle y, x\rangle| \mathrm{d}x$$

$$\leqslant c \int_{\mathbb{R}^d} |x|^{-d-\alpha} |\sin\langle y, x\rangle - \langle y, x\rangle| \mathrm{d}x$$

$$= c|y|^\alpha \int_{\mathbb{R}^d} |x|^{-d-\alpha} |\sin\langle e_y, x\rangle - \langle e_y, x\rangle| \mathrm{d}x = c\theta |y|^\alpha,$$

while, for a sufficiently large N,

$$R \geqslant c \int_{N < |x+a(s)| \leqslant \alpha'/s, e_{x+a(s)} \in E'} (1 - \cos\langle y, x\rangle) |x + a(s)|^{-d-\alpha} \, \mathrm{d}x,$$

where

$$E' = (e \in \mathbb{S}^{d-1} : h(e) \geqslant h'/(2\sigma(\mathbb{S}^{d-1}))), \qquad h' = \int_{\mathbb{S}^{d-1}} h(e) \sigma(\mathrm{d}e).$$

So,

$$R \geqslant c|y|^\alpha \int_{N|y| < |x| \leqslant \alpha'|y|/s, e_x \in E'} (1 - \cos\langle e_y, x\rangle) |x|^{-d-\alpha} \mathrm{d}x \geqslant c \max(|y|^\alpha, s^\alpha).$$

This implies that, for all sufficiently small $\delta > 0$ (see (4.26)),

$$|\psi_s(y)| \leqslant (1 - c \max(|y|^\alpha, s^\alpha))(1 + c\theta n^{-1}).$$

Thus,

$$I_1 \leqslant cn^{-d/\alpha} + cs^d \leqslant cn^{-d/\alpha}. \tag{4.27}$$

Now we estimate I_2. It is clear that

$$I_2 \leqslant \left(\sup_{|\tau| \geqslant \delta} |\psi_s(\tau)| \right)^{n-2} \int_{\mathbb{R}^d} |\psi_s(y)|^2 \, \mathrm{d}y.$$

For a fixed $N > 0$, we have

$$\sup_{x\in\mathbb{R}^d}|q^{(s)}(x)-p(x)| = \max\left(\sup_{|x|\leqslant N}|\cdots|,\ \sup_{N<|x|\leqslant(d+\alpha)/s}|\cdots|,\ \sup_{|x|>(d+\alpha)/s}|\cdots|\right)$$

$$= \max(\Delta_1,\ \Delta_2,\ \Delta_3),$$

It is evident that as $n\to\infty$, $zn^{-1/\alpha}\to\infty$,

$$\Delta_1 = \sup_{|x|\leqslant N}|(f(s)P(|\xi|\leqslant\gamma z))^{-1}\mathrm{e}^{sx_d}p(x)-p(x)| = o(1)$$

since $s\to 0$. Further,

$$\Delta_2 \leqslant (2\mathrm{e}^{d+\alpha}+1)\sup_{|x|>N}p(x) = \omega(N).$$

Finally,

$$\Delta_3 \leqslant c\left(\sup_{(d+\alpha)/s<|x|\leqslant\gamma z}\mathrm{e}^{s|x|}|x|^{-d-\alpha}+s^{d+\alpha}\right).$$

As in the proof of Lemma 4.3, we have

$$\mathrm{e}^{s|x|-(d+\alpha)\ln|x|}\leqslant c\mathrm{e}^{\gamma sz}z^{-d-\alpha} = cn^{-1}z^{-d}.$$

Therefore, $\Delta_3 = o(1)$.

Thus, as $n\to\infty$, $zn^{-1/\alpha}\to\infty$,

$$\sup_{x\in\mathbb{R}^d}|q^{(s)}(x)-p(x)| = o(1),$$

since N can be arbitrarily large.

Now, taking into account the fact that $p(x)$ is bounded, by Parseval's equality we obtain

$$(2\pi)^{-d}\int_{\mathbb{R}^d}|\psi_s(y)|^2\,\mathrm{d}y = \int_{\mathbb{R}^d}(q^{(s)}(x))^2\,\mathrm{d}x = \int_{\mathbb{R}^d}(p(x))^2\,\mathrm{d}x + 2\theta\sup_{x\in\mathbb{R}^d}|q^{(s)}(x)-p(x)|\leqslant c.$$

It is evident that, as $s\to 0$,

$$\sup_{y\in\mathbb{R}^d}|\psi_s(y)-\psi(y)| = o(1).$$

Thus, for all sufficiently large n and sufficiently large $zn^{-1/\alpha}$, we have

$$\sup_{|y|>\delta}|\psi_s(y)|\leqslant\rho<1$$

and

$$I_2\leqslant c\rho^{n-2} = o(n^{-d/\alpha}). \tag{4.28}$$

In view of (4.25), (4.27) and (4.28), the lemma follows. □

Taking into account (4.20), (4.21) and Lemmas 4.3 and 4.4, we arrive at the following statement.

Lemma 4.5. *Under the conditions of Theorem 2.1 as $n \to \infty$, $zn^{-1/\alpha} \to \infty$,*

$$p_{n,0}(ze^*) = o(n^2 z^{-d-2\alpha}),$$

provided $0 < \gamma < \alpha/(d+2\alpha)$.

In order to complete the proof of Theorem 2.1, all that remains is to estimate the term $p_{n,3}(te^*)$ in (4.1). By definition,

$$p_{n,3}(ze^*) \leqslant n^3 \int_{|u|>\gamma z} p(u)\mathrm{d}u \int_{|v|>\gamma z} p(v)\mathrm{d}v \int_{|w|>\gamma z} q_{n-3}(ze^* - u - v - w)p(w)\mathrm{d}w.$$

For all sufficiently large $|w|$, we have $p(w) \leqslant c|w|^{-d-\alpha}$. Hence,

$$\int_{|w|>\gamma z} q_{n-3}(ze^* - u - v - w)p(w)\mathrm{d}w = O\big(z^{-d-\alpha}\big)$$

and, therefore,

$$p_{n,3}(ze^*) = O\big(n^3 z^{-d-\alpha}(P(|\xi| > \gamma z))^2\big) = O\big(n^3 z^{-d-3\alpha}\big). \tag{4.29}$$

Combining (4.29) and Lemmas 4.1, 4.2 and 4.5, we conclude that, for any $0 < \gamma < \alpha/(d+2\alpha)$,

$$p_n(ze^*) =$$

$$d(e^*)nz^{-(d+\beta)}(1+o(1)) + \frac{1}{2}n^2 z^{-(d+2\alpha)} \int_{(x:|x|>\gamma, |e^*-x|>\gamma)} \pi(x)\pi(e^* - x)\mathrm{d}x + o\big(n^2 z^{-(d+2\alpha)}\big).$$

By Lemma 3.7,

$$\int_{(x:|x|>\gamma, |e^*-x|>\gamma)} \pi(x)\pi(e^* - x)\mathrm{d}x = \pi_2(e^*) + \omega(1/\gamma).$$

Since γ can be arbitrarily small and $\tilde{p}_n(te^*) = b_n^d p_n(ze^*)$, $z = b_n t$, $b_n = b_0 n^{1/\alpha}$, the theorem follows.

5. Proof of Theorem 2.4

First, let $\beta \geqslant 2\alpha$. Again put $z = b_n t$, where $b_n = b_0 n^{1/\alpha}$, and let $t \to \infty$.

Let $A_{n,i}$, $i = 0, 1, 2, 3$, be as in the proof of Theorem 2.1. Then, for any event A',

$$\lim_{|\Delta|\to 0} \frac{P(ze^* \prec \zeta^{(n)} \leqslant ze^* + \Delta, A_{n,j}, A')}{P(ze^* \prec \zeta^{(n)} \leqslant ze^* + \Delta)} \leqslant \frac{p_{n,j}(ze^*)}{p_n(ze^*)}, \qquad j = 0, 1, 3.$$

Put

$$A_R = A \cap B_d(\mathbf{0}, R), \qquad B_R = B \cap B_d(\mathbf{0}, R), \qquad C_R = C \cap B_d(\mathbf{0}, R),$$

where $R > 0$ is a fixed large number. For the sake of brevity, denote

$$P_{n,t} = P\big(b_n^{-1}\xi' \in tA_R,\, b_n^{-1}\xi'' \in tB_R,\, b_n^{-1}(\zeta^{(n)} - \xi' - \xi'') \in C_R | \zeta^{(n)} = b_n te^*\big).$$

Taking into account Lemmas 4.1, 4.2 and 4.5 as well as (4.29), we obtain

$$P_{n,t} = P'_{n,t} + o(1),$$

where

$$P'_{n,t} = \binom{n}{2} P\big(\xi' \in zA_R,\, \xi'' \in zB_R,\, \zeta^{(n)} - \xi' - \xi'' \in b_n C_R,\, A_{n,2} | \zeta^{(n)} = ze^*\big).$$

As in the proof of Theorem 2.1, where we considered the contribution of the two largest summands, we obtain

$$P'_{n,t} = z^d(\pi_2(e^*))^{-1} \int_{A^*} \pi(x)\pi(e^* - x - y) q_{n-2}(zy) \mathrm{d}x\, \mathrm{d}y + o(1),$$

where

$$A^* = \big((x, y) \in \mathbb{R}^{2d} : x \in A_R \backslash B_d(\mathbf{0}, \gamma),\, e^* - x - y \in B_R \backslash B_d(\mathbf{0}, \gamma),\, y \in z^{-1} b_n C_R\big).$$

It is clear that the set A^* is covered by A_1 defined in that proof. Further,

$$P'_{n,t} = (\pi_2(e^*))^{-1} \int_{A_1^*} \pi(x)\pi(e^* - x) \mathrm{d}x \int_{b_n C_R} q_{n-2}(y) \mathrm{d}y + o(1),$$

where

$$A_1^* = \big(x \in \mathbb{R}^d : x \in A_R \backslash B_d(\mathbf{0}, \gamma),\, e^* - x \in B_R \backslash B_d(\mathbf{0}, \gamma)\big).$$

By Lemma 3.4, as $n \to \infty$,

$$\int_{b_n C_R} q_{n-2}(y) \mathrm{d}y = S(C_R) + o(1).$$

Since γ can be arbitrarily small and R can be arbitrarily large, (2.5) follows.

The case $\beta < 2\alpha$, $\rho_n \leqslant t \leqslant n^\kappa/\rho_n$ is dealt with similarly.

Now let $\beta < 2\alpha$, $t \geqslant \rho_n n^\kappa$. For the sake of brevity, denote

$$Q_{n,t} = P\big(b_n^{-1}(\zeta^{(n)} - \xi') \in A_R | \zeta^{(n)} = b_n te^*\big).$$

Taking into account Lemmas 4.1, 4.2 and 4.5 as well as (4.29), we obtain

$$Q_{n,t} = Q'_{n,t} + o(1),$$

where

$$Q'_{n,t} = nP\big(\zeta^{(n)} - \xi' \in b_n A_R,\, A_{n,1} | \zeta^{(n)} = ze^*\big).$$

By Lemma 4.1,

$$Q'_{n,t} = \frac{z^{d+\beta}}{d(e^*)}\int_{ze^*-u\in b_nA_R} p(u)q_{n-1}(ze^*-u)\mathrm{d}u = \frac{z^{d+\beta}}{d(e^*)}\left(\int_{ze^*-u\in b_nA_R}\frac{h(e_u)}{|u|^{d+\alpha}}q_{n-1}(ze^*-u)\mathrm{d}u\right.$$

$$\left.+\int_{ze^*-u\in b_nA_R}\frac{d(e_u)+\theta\omega(|u|)}{|u|^{d+\beta}}q_{n-1}(ze^*-u)\mathrm{d}u\right) = \frac{z^{d+\beta}}{d(e^*)}(I+J).$$

For all sufficiently large n, the set $\{u\in\mathbb{R}^d : ze^*-u\in b_nA_R\}$ is covered by $\{u\in\mathbb{R}^d : |u|>\gamma z\}$. Therefore, I can be estimated as in Theorem 2.1, where we considered the contribution of the largest summand (see (4.3)–(4.9)). As to J, we have (cf. (4.10))

$$\begin{aligned} J &= \int_{ze^*-u\in b_nA_R}\frac{d(e_u)}{|u|^{d+\beta}}q_{n-1}(ze^*-u)\mathrm{d}u(1+o(1)) \\ &= z^{-d-\beta}\int_{(\bar{u},u_d)\in b_nA_R}\frac{d(e_{(-\bar{u},z-u_d)})q_{n-1}(\bar{u},u_d)\mathrm{d}\bar{u}\,\mathrm{d}u_d}{(|\bar{u}|^2/z^2+(1-u_d/z)^2)^{(d+\beta)/2}}(1+o(1)). \end{aligned}$$

For all sufficiently large n, the set b_nA_R is covered by A_1 defined in the proof of Theorem 2.1, where we considered the contribution of the largest summand (see (4.5) and (4.11)). Thus,

$$J = z^{-(d+\beta)}\left(d(e^*)\int_{b_nA_R}q_{n-1}(u)\mathrm{d}u + o(1)\right).$$

By Lemma 3.4, as $n\to\infty$,

$$\int_{b_nA_R}q_{n-1}(u)\mathrm{d}u = S(A_R)+o(1).$$

Since R can be arbitrarily large, (2.6) follows. The theorem is proven.

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