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On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process

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We prove the following probabilistic properties of a multidimensional Ornstein–Uhlenbeck process driven by a general Lévy process, under mild regularity conditions: the strong Feller property; the existence of a smooth transition density; and the exponential β-mixing property. As a class of possible invariant distributions of an Ornstein–Uhlenbeck process, we also discuss centred and non-skewed multidimensional generalized hyperbolic distributions.

Keywords: mixing bound; multidimensional generalized hyperbolic distribution; operator self-decomposability; Ornstein–Uhlenbeck process driven by a Lévy process

1. Introduction

Given a d-dimensional time-homogeneous Lévy process Z starting from the origin and a $d \times d$ matrix Q, the d-dimensional Ornstein–Uhlenbeck process X driven by Z (henceforth referred to as an OU process) is defined by

$$X_t = \mathrm{e}^{-tQ} X_0 + \int_0^t \mathrm{e}^{-(t-s)Q} \, \mathrm{d}Z_s, \qquad t \in \mathbb{R}_+, \tag{1.1}$$

where X_0 is supposed to be independent of Z. The OU process is equivalently defined as the unique strong solution of the stochastic differential equation

$$\mathrm{d}X_t = -QX_t \, \mathrm{d}t + \mathrm{d}Z_t. \tag{1.2}$$

Under some regularity conditions on Q and the Lévy measure of Z, X admits a unique invariant distribution F, and the class of all possible Fs forms the class of all Q-self-decomposable distributions: see Section 2 below and references therein; see also Wolfe (1982) and Jacod (1985) for the one-dimensional case. Needless to say, if Z is a Wiener process with a certain drift and covariance matrix, then X is a well-known Gaussian OU process.

Despite the simple structure of X, we have a wide choice of invariant distributions for various Z and Q. Several examples of one-dimensional non-Gaussian OU processes can be found in Barndorff-Nielsen and Shephard (2001b). See also the recent stimulating work by Barndorff-Nielsen and Shephard (2001a), where a class of stochastic volatility models is suggested and a positive stationary OU process describes an unobservable volatility process whose marginal distribution is, for example, the generalized inverse Gaussian (see (A.1)

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below). We shall give a concrete example of a class of multidimensional non-Gaussian OU processes in Section 5 below.

In this paper, we shall provide sets of sufficient conditions for the three properties of an OU process X: first, the transition semigroup of X is strong Feller if Z has either non-degenerate Gaussian part or absolutely continuous divergent Lévy measure; secondly, the existence of a smooth transition density follows from good behaviour of the Lévy measure of Z near the origin, even if the Gaussian covariance matrix of Z degenerates; and finally, in the strictly stationary case, X is exponentially β-mixing if the marginal distribution F admits a finite pth-order absolute moment for some $p > 0$. In particular, for general Lévy-driven OU processes, no mixing bound has yet been specified, to the author's knowledge. As a consequence of the mixing property, a very broad subclass of general OU processes may be ergodic: this is an important feature for statistical inference, although we do not go any further in this direction in this paper. We are not concerned here with any mixing bound other than exponential; however, see Remark 4.4 below.

This paper is organized as follows. Section 2 presents several previous results concerning OU processes. We shall discuss the strong Feller property and the existence of a smooth transition density in Section 3, and the exponential β-mixing bound in Section 4. In Section 5, we shall first give the explicit Lévy density of the d-dimensional generalized hyperbolic distribution (GH_d), and then validate the existence of the strictly stationary OU process whose marginal distribution is the centred and non-skewed version of GH_d. The former generalizes the previous one-dimensional result: see, for example, Prause (1998, Proposition 1.31 and Theorem 1.43). Finally a brief summary of GH_d is given in the Appendix for readers' convenience.

In this paper, we suppose that all stochastic processes are defined on a given probability space $(\Omega, \mathcal{F}, P)$, and denote by E the expectation operator under P. The following notation is used:

- $\mathcal{L}(\eta)$ stands for the distribution of a random variable η under P.
- We write $|A| = \{\mathrm{trace}(A^{\mathrm{T}}A)\}^{1/2}$ for any matrix (or vector) A, with T denoting transposition.
- $M_+(\mathbb{R}^d)$ stands for the set of all real $d \times d$ matrices such that the real parts of all eigenvalues are positive: $Q \in M_+(\mathbb{R}^d)$ if and only if $|\mathrm{e}^{-tQ}| \to 0$ as $t \to \infty$.
- For a Polish space S, $\mathcal{B}(S)$ stands for the Borel σ-field generated by S, and the space of bounded $\mathcal{B}(S)$-measurable real-valued functions (bounded continuous functions and C^k-functions whose derivatives are bounded) defined on S is denoted by $b\mathcal{B}(S)$ ($C_b(S)$ and $C_b^k(S)$). We denote by $\|\cdot\|_\infty$ the supremum norm on these spaces.
- $\varphi_\eta(\cdot)$ stands for the characteristic function of a random variable or a distribution η, and we denote by $\log \varphi_\eta(\cdot)$ the so-called distinguished logarithm of $\varphi_\eta(\cdot)$ for an infinitely divisible distribution (or random variable) η: see, for example, Sato (1999, Lemma 7.6).

2. Fundamental results on OU processes

Here we mention several important results concerning OU processes. Most of them will be used in the following sections.

2.1. Operator self-decomposability and OU process

Let d be a positive integer and $Z = (Z_t)_{t\in\mathbb{R}_+}$ a d-dimensional time-homogeneous cadlag Lévy process such that $Z_0 = 0$ almost surely. Denote by (b, C, ν) the generating triplet of Z: that is, $b = (b_k)_{k=1}^d \in \mathbb{R}^d$, $C = (C_{kl})_{k,l=1}^d$ is a $d \times d$ symmetric non-negative definite matrix and ν is a σ-finite measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min(1, |z|^2)\nu(\mathrm{d}z) < \infty$, for which

$$\varphi_{Z_t}(u) = \exp\left[t\left\{\mathrm{i}u^{\mathrm{T}}b - \frac{1}{2}u^{\mathrm{T}}Cu + \int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{i}u^{\mathrm{T}}z} - 1 - \mathrm{i}u^{\mathrm{T}}z1_U(z))\nu(\mathrm{d}z)\right\}\right], \tag{2.1}$$

for $u \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$, where $1_U(z)$ denotes the indicator function of the unit sphere $U = \{z \in \mathbb{R}^d : |z| \leqslant 1\}$. Throughout this paper, we implicitly exclude the trivial Lévy process $Z_t = at$ with some constant $a \in \mathbb{R}^d$.

Given a $d \times d$ matrix Q, let X be the d-dimensional OU process defined by (1.1). As already noted, we assume that the initial variable X_0 is independent of Z. Obviously X is a Markov process whose sample path is cadlag. The stochastic integral on the right-hand side of (1.1) is well defined as a certain limit in probability: see Sato and Yamazato (1983, Theorem 2.1) for details. Of course the Lévy–Itô decomposition of Z (see Sato 1999, Theorem 19.3) leads directly to a more concrete expression for this stochastic integral.

Writing $Q = (Q_{jk})_{j,k=1}^d$, $x = (x_k)_{k=1}^d \in \mathbb{R}^d$ and $\partial_j = \partial/\partial x_j$, the infinitesimal generator $\mathcal{A}$ of X is given by

$$\mathcal{A}f(x) = -\sum_{k,l=1}^{d} Q_{kl}x_l\partial_k f(x) + \sum_{k=1}^{d} b_k\partial_k f(x) + \frac{1}{2}\sum_{k,l=1}^{d} C_{kl}\partial_k\partial_l f(x)$$
$$+ \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - \sum_{k=1}^{d} y_k\partial_k f(x)1_U(y)\right)\nu(\mathrm{d}y). \tag{2.2}$$

This $\mathcal{A}$ acts on the set of all real-valued $C^2(\mathbb{R}^d)$ functions with compact support: see Sato and Yamazato (1984, Theorem 3.1). The last three terms on the right-hand side of (2.2) correspond to the infinitesimal generator of Z: see Sato (1999, Theorem 31.5).

Denote by $C_0(\mathbb{R}^d)$ the set of all real-valued continuous functions on $\mathbb{R}^d$ vanishing as $|x| \to \infty$, let $\|\cdot\|_0$ stand for the operator norm and write $\mathrm{e}^{sQ}E = \{y \in \mathbb{R}^d : y = \mathrm{e}^{sQ}x, x \in E\}$ for $E \in \mathcal{B}(\mathbb{R}^d)$ and $s \in \mathbb{R}_+$. The next proposition specifies the characteristic function of the transition probability of X:

Proposition 2.1. (Sato and Yamazato 1984, Theorem 3.1). *The smallest closed extension of $\mathcal{A}$ is the infinitesimal generator of a strongly continuous non-negative semigroup $\{P_t : t \geqslant 0\}$ such that*

$$P_t f(x) = \int_{\mathbb{R}^d} f(y)P(t, x, \mathrm{d}y) \tag{2.3}$$

for the transition probability $P(t, x, \cdot)$ of X, $\|P_t\|_0 = 1$, and that $f \in C_0(\mathbb{R}^d)$ implies that

$P_t f(x) \in C_0(\mathbb{R}^d)$ for each $t \in \mathbb{R}_+$. Moreover, for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, $P(t, x, \cdot)$ is an infinitely divisible distribution such that

$$\varphi_{P(t,x,\cdot)}(u) = \exp\left\{ \mathrm{i}x^{\mathrm{T}} \mathrm{e}^{-tQ^{\mathrm{T}}} u + \int_0^t \log \varphi_{Z_1}(\mathrm{e}^{-sQ^{\mathrm{T}}} u) \mathrm{d}s \right\}. \tag{2.4}$$

In particular, the generating triplet of $P(t, x, \cdot)$ is given by $(b_{t,x}, C_t, \nu_t)$, where

$$b_{t,x} = \mathrm{e}^{-tQ} x + \int_0^t \mathrm{e}^{-sQ} b \, \mathrm{d}s + \int_{\mathbb{R}^d} \int_0^t \mathrm{e}^{-sQ} z \left\{ 1_U(\mathrm{e}^{-sQ} z) - 1_U(z) \right\} \mathrm{d}s \nu(\mathrm{d}z),$$

$$C_t = \int_0^t \mathrm{e}^{-sQ} C e^{-sQ^{\mathrm{T}}} \mathrm{d}s,$$

$$\nu_t(E) = \int_0^t \nu(\mathrm{e}^{sQ} E) \mathrm{d}s, \qquad E \in \mathcal{B}(\mathbb{R}^d).$$

Now let us recall the *operator self-decomposability* of a distribution on $\mathbb{R}^d$. Let $Q \in M_+(\mathbb{R}^d)$. Then an infinitely divisible distribution F on $\mathbb{R}^d$ is called *Q-self-decomposable* if there exists a random variable $\eta_{t,Q}$ such that, for each $t \in \mathbb{R}_+$,

$$\varphi_F(u) = \varphi_F(\mathrm{e}^{-tQ^{\mathrm{T}}} u) \varphi_{\eta_{t,Q}}(u), \qquad u \in \mathbb{R}^d. \tag{2.5}$$

An infinitely divisible distribution F on $\mathbb{R}^d$ which is Q-self-decomposable for some $Q \in M_+(\mathbb{R}^d)$ is called operator self-decomposable. If $d = 1$, then the operator self-decomposability of F means that F is Q-self-decomposable for any $Q > 0$, and hence simply called *self-decomposable* in this case. According to, for example, Jurek and Mason (1993, Theorem 3.3.5), $\mathcal{L}(\eta_{t,Q})$ is infinitely divisible: for a strictly stationary OU process, $\mathcal{L}(\eta_{t,Q})$ indeed corresponds to the distribution of the second term on the right-hand side of (1.1). We should note that the support of any Q-self-decomposable distribution (more generally, any infinitely divisible distribution) is unbounded except for delta distributions: see Sato (1999, Corollary 24.4).

Now assume that

$$\int_{|z|>1} \log|z| \nu(\mathrm{d}z) < \infty, \tag{2.6}$$

or, equivalently, $\mathrm{E}[\log\{\max(1, |Z_1|)\}] < \infty$. It is known that the class of all possible invariant distributions of X forms the class of all Q-self-decomposable distributions:

Proposition 2.2. (Sato and Yamazato 1984, Theorem 4.1 and 4.2). *The following two statements hold true.*

(a) *Let $Q \in M_+(\mathbb{R}^d)$. If* (2.6) *holds, there exists a limit distribution F such that*

$$P(t, x, A) \to F(A), \qquad \text{as } t \to \infty, \tag{2.7}$$

for any $x \in \mathbb{R}^d$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$. *This* F *is* Q*-self-decomposable and the unique invariant distribution of* X. *Moreover, the characteristic function of* F *is given by*

$$\varphi_F(u) = \exp\left\{\int_0^\infty \log \varphi_{Z_1}(\mathrm{e}^{-sQ^{\mathrm{T}}} u)\mathrm{d}s\right\}. \tag{2.8}$$

In particular, the generating triplet of F *is given by* $(b_\infty, C_\infty, \nu_\infty)$, *where*

$$b_\infty = Q^{-1}b + \int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-sQ}z\left\{1_U(\mathrm{e}^{-sQ}z) - 1_U(z)\right\}\mathrm{d}s\nu(\mathrm{d}z),$$

$$C_\infty = \int_0^\infty \mathrm{e}^{-sQ}C\mathrm{e}^{-sQ^{\mathrm{T}}}\mathrm{d}s,$$

$$\nu_\infty(E) = \int_0^\infty \nu(\mathrm{e}^{sQ}E)\mathrm{d}s, \qquad E \in \mathcal{B}(\mathbb{R}^d).$$

Conversely, every Q*-self-decomposable distribution appears in this way. The correspondence between* $\mathcal{A}$ *and* F *is one-to-one.*

(b) *Let* Q *be any* $d \times d$ *matrix. If* (2.6) *fails to hold, then* X *has no invariant distribution, and moreover, for any* $x \in \mathbb{R}^d$, $P(t, x, \cdot)$ *does not converge to any probability measure as* $t \to \infty$.

Proposition 2.2 explicitly relates (b, C, ν) and $(b_\infty, C_\infty, \nu_\infty)$.

Remark 2.1. We note that, contrary to the first statement in Proposition 2.2, the assumptions $Q \in M_+(\mathbb{R}^d)$ and (2.6) are not required in Proposition 2.1. This fact will be implicitly used in Section 3.

Recall that a measure F on $\mathbb{R}^d$ is called *non-degenerate* if $F(a + V) < 1$ for any constant $a \in \mathbb{R}^d$ and subspace $V \subset \mathbb{R}^d$ such that $\dim(V) \leqslant d - 1$. The following general result is known:

Proposition 2.3. (Yamazato 1983). *Assume that an operator self-decomposable distribution* F *on* $\mathbb{R}^d$ *is non-degenerate. Then* F *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$.

There is a useful criterion for non-degeneracy of F:

Proposition 2.4. (Sato 1999, Proposition 24.19). *A probability measure* F *on* $\mathbb{R}^d$ *is non-degenerate if and only if there exist positive constants* c_1 *and* c_2 *such that* $|\varphi_F(u)| \leqslant 1 - c_1|u|^2$ *for any* $|u| < c_2$.

2.2. A useful formula

Under (2.6), X is strictly stationary if $\mathcal{L}(X_0)$ is the invariant distribution of X. In this case we can specify the characteristic function of Z_1 for each given F:

Lemma 2.5. (Barndorff-Nielsen *et al.* 1998, Lemma 5.1). *Let F be Q-self-decomposable for some $Q \in M_+(\mathbb{R}^d)$. If $\varphi_F(u)$ is differentiable for $u \neq 0$, and the real-valued function $u \mapsto \{\nabla_u \log \varphi_F(u)\} Q^{\mathrm{T}} u$ can be defined at $u = 0$ by continuity, then there exists a strictly stationary OU process with the marginal distribution F, and in this case Z is determined by*

$$\varphi_{Z_1}(u) = \exp\left[\{\nabla_u \log \varphi_F(u)\} Q^{\mathrm{T}} u\right], \tag{2.9}$$

where ∇_u denotes the gradient operator.

Remark 2.2. In general, the Q-self-decomposability of a distribution η on $\mathbb{R}^d$ does not necessarily imply that η itself depends on Q, and hence F may be independent of Q in the setting of Lemma 2.5; that is to say, the effect of the drift matrix Q on F may be cancelled out by the choice of Z determined by (2.9).

Remark 2.3. When $d = 1$, a more refined statement can be given under some regularity conditions. Suppose that F admits a differentiable Lévy density $g_F(x)$ which does not depend on $Q > 0$, and that the Lévy measure of Z admits a density $g_Z(x)$. Then we have

$$g_Z(x) = -Q\left\{ g_F(x) + x \frac{\mathrm{d}}{\mathrm{d}x} g_F(x) \right\}. \tag{2.10}$$

See, for example, Barndorff-Nielsen and Shephard (2001a, Section 2.2). The relation (2.10) enables us to determine Z for each given F, while such a simple expression is not available for $d \geqslant 2$.

Remark 2.4. According to Proposition 2.2 and Lemma 2.5, the following two practical constructions of a strictly stationary OU process X are possible, as pointed out by Barndorff-Nielsen and Shephard (2001a, Sections 2.3 and 2.4). First, for each given $Q \in M_+(\mathbb{R}^d)$ and Z satisfying (2.6), the marginal distribution F (depending on Q in general) can be determined by (2.8). Secondly, for each given marginal Q-self-decomposable distribution F which does not depend on Q, Z can be determined by (2.9) with any Q for which F is Q-self-decomposable, and this Q is the drift matrix appearing in (1.2). In the latter case, we should note that F may restrict the form of $Q \in M_+(\mathbb{R}^d)$; we shall treat such an example in Section 5.

2.3. Autocorrelation structure of OU processes and their finite superposition

Let $d = 1$ and assume that an OU process X of (1.1) is second-order stationary with a marginal distribution F. Then it is well known that the autocorrelation function of X is given by $\phi_X(t) = \exp(-Qt)$, which does not depend on F. More flexible correlation structures can be obtained by introducing a finite sum $Y_t = \sum_{j=1}^{m} X_t^{(j)}$ of m independent OU processes $X^{(j)}$, $j = 1, 2, \ldots, m$. Since each $X^{(j)}$ can possess any self-decomposable marginal distribution, so can Y (clearly the self-decomposability is preserved under convolutions). If each $X^{(j)}$ is second-order stationary and satisfies

$$\mathrm{d}X_t^{(j)} = -Q^{(j)} X_t^{(j)}\,\mathrm{d}t + \mathrm{d}Z_t^{(j)}, \tag{2.11}$$

where $Q^{(j)}$, $j = 1, 2, \ldots, m$, are positive constants and $Z^{(j)}$, $j = 1, 2, \ldots, m$, are independent one-dimensional Lévy processes, then the resulting autocorrelation function of Y is given by

$$\phi_Y(t) = \sum_{j=1}^{m} \frac{v_j}{v_1 + \ldots + v_m} \exp(-Q^{(j)} t), \tag{2.12}$$

where $v_j = \mathrm{var}[X_0^{(j)}]$. See Barndorff-Nielsen and Shephard (2001a, Section 3) or Barndorff-Nielsen *et al.* (1998, Section 5) for details. See also Barndorff-Nielsen (2001) for a more general construction of long-memory autocorrelation structure based on an independently scattered random measure, which is a generalization of the Poisson random measure.

An explicit expression for the d-dimensional analogue of (2.12) is also available. Let $X^{(j)}$, $j = 1, 2, \ldots, m$, be d-dimensional OU processes of the form (2.11) while now $Q^{(j)} \in M_+(\mathbb{R}^d)$ and $Z^{(j)}$ are also d-dimensional ($j = 1, 2, \ldots, m$). Define Y as above except that each $X^{(j)}$, and hence Y, may not be strictly stationary: that is, here $\mathcal{L}(Y_0)$ is any non-degenerate distribution with finite second-order absolute moment. In this case, denoting by $V_u = [V_{u,ij}]_{i,j=1}^{d}$ the covariance matrix of $\mathcal{L}(Y_u)$, $V_{u,ii}$ is strictly positive due to the independence between Y_0 and $Z^{(j)}$, $j = 1, 2, \ldots, m$. For $s, t \in \mathbb{R}_+$ such that $s \leqslant t$, define the $d \times d$ matrix $\Gamma_Y(s, t) = [\gamma_{ij}^Y(s, t)]_{i,j=1}^{d}$ by

$$\Gamma_Y(t) = \mathrm{E}[Y_t Y_s^{\mathrm{T}}] - m_t m_s^{\mathrm{T}},$$

where m_u denotes the mean vector of Y_u for each $u \in \mathbb{R}_+$. Then the autocorrelation function of Y is defined by $\Phi(s, t) = [\rho_{ij}^Y(s, t)]_{i,j=1}^{d}$, where

$$\rho_{ij}^Y(s, t) = (V_{t,ii} V_{s,jj})^{-1/2} \gamma_{ij}^Y(s, t), \qquad i, j = 1, \ldots, d. \tag{2.13}$$

Let $\Xi_{ik}^{Q^{(l)}}(u)$ and $X^{(l),k}$ denote the (i, k)th entry of $\mathrm{e}^{-uQ^{(l)}}$ and the kth component of $X^{(l)}$, respectively.

Proposition 2.6. *Suppose that* $\mathcal{L}(Y_0)$ *is non-degenerate and that* $\mathrm{E}[|Y_t|^2] < \infty$ *for each* $t \in \mathbb{R}_+$. *Then we have*

$$\rho_{ij}^Y(s, t) = (V_{t,ii} V_{s,jj})^{-1/2} \sum_{k=1}^{d} \sum_{l=1}^{m} \sum_{l'=1}^{m} \Xi_{ik}^{Q^{(l)}}(t - s)\mathrm{cov}[X_s^{(l),k}, X_s^{(l'),j}] \tag{2.14}$$

for $i, j = 1, \ldots, d$ and $s, t \in \mathbb{R}_+$ such that $s \leqslant t$.

The proof is similar to the one-dimensional case, and thus is omitted.

Remark 2.5. Suppose that $m = 1$ and that Y is second-order stationary with a marginal distribution F. Then Proposition 2.6 says that, in the multidimensional case, the autocorrelation function of Y does indeed depend on the covariance structure of F, unlike the one-dimensional case. In particular, if Q is diagonal with (j, j)th entries $q_j > 0$, then (2.14) becomes

$$\rho_{ij}(t) = \mathrm{e}^{-q_i t} \frac{V_{0,ij}}{(V_{0,ii} V_{0,jj})^{1/2}} \tag{2.15}$$

for each i and j. Up to a multiplicative constant depending on F, (2.15) is the same form as the one-dimensional case.

3. The strong Feller property and smoothness of the transition density

Let X be given by (1.1) for some Q and Z, and denote by $(S, \mathcal{B}(S))$, $S \subset \mathbb{R}^d$, the state space of X, where S is open and convex. In this section, we require neither (2.6) nor $Q \in M_+(\mathbb{R}^d)$, so that X may have no invariant distribution. Recall that (b, C, ν) denotes the generating triplet of Z.

3.1. The strong Feller property

Recall that the Markov transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ of (2.3) is called *strong Feller* if $P_t f \in C_b(S)$ for any $f \in b\mathcal{B}(S)$ and $t > 0$. This property in a certain sense connects with the absolute continuity of the transition probability $P(t, x, \cdot)$; see Remark 3.2 below. A general exposition of the strong Feller property can be found in Girsanov (1960).

Theorem 3.1. *Suppose that either of the following conditions holds true:*

(a) $\mathrm{rank}(C) = d$;
(b) $\nu(\mathbb{R}^d) = \infty$ *and* ν *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$.

Then $(P_t)_{t \in \mathbb{R}_+}$ *is strong Feller.*

Proof. Fix $t > 0$ and $f \in b\mathcal{B}(S)$ arbitrarily. Obviously P_t maps $b\mathcal{B}(S)$ to $b\mathcal{B}(S)$. Write $Y_t = \int_0^t \mathrm{e}^{-(t-s)Q} \mathrm{d}Z_s$, where from Proposition 2.1 we know that $\mathcal{L}(Y_t)$ is infinitely divisible with Gaussian covariance matrix and Lévy measure given by $C_t = \int_0^t \mathrm{e}^{sQ} C \mathrm{e}^{-sQ^{\mathrm{T}}} \mathrm{d}s$ and $\nu_t(\mathrm{d}y) = \int_0^t \nu(\mathrm{e}^{sQ}\,\mathrm{d}y)\mathrm{d}s$, respectively.

Let us prove the continuity of $x \mapsto P_t f(x)$. Since the assumption ensures the absolute continuity of $\mathcal{L}(Z_u)$ for any $u > 0$, the above expressions for C_t and $\nu_t(\mathrm{d}y)$ lead to the absolute continuity of $\mathcal{L}(Y_t)$. Denote by $g(z)$ the density of $\mathcal{L}(Y_t)$. For arbitrary $\varepsilon > 0$, we can find an $h \in C_c^\infty(S)$ such that $\|g - h\|_{L^1(\mathrm{d}x)} < \varepsilon$, where $C_c^\infty(S)$ stands for the set of all smooth functions with compact support. Then using Taylor's expansion for h, we see that, for any $x, y \in S$,

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= \left| \int f(\mathrm{e}^{-tQ}x + z)g(z)\mathrm{d}z - \int f(\mathrm{e}^{-tQ}y + z)g(z)\mathrm{d}z \right| \\
&= \left| \int f(\mathrm{e}^{-tQ}x + z)(g(z) - h(z))\mathrm{d}z \right. \\
&\quad \left. + \int f(\mathrm{e}^{-tQ}x + z)\{h(z) - g(z + \mathrm{e}^{-tQ}(x - y))\}\mathrm{d}z \right| \\
&\leqslant \|f\|_\infty \left(\varepsilon + \int |h(z) - g(z + \mathrm{e}^{-tQ}(x - y))|\mathrm{d}z \right) \\
&\leqslant \|f\|_\infty \big(2\varepsilon + \|\nabla h\|_\infty |\mathrm{e}^{-tQ}\| x - y| \big).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we are done. □

Remark 3.1. The previous result of Kwon and Lee (1999) for the strong Feller property of general one-dimensional diffusions with jumps requires certain moment conditions on ν and also $C > 0$. Hence, Theorem 3.1 asserts that we can considerably weaken their sufficient conditions for OU processes.

Remark 3.2. Suppose that X admits an invariant distribution F. The strong Feller property ensures the existence of a density $q_t(x, y)$ of $P(t, x, \cdot)$ with respect to $F(\cdot)$; see Lin (1989, Theorem 2.1). Since Proposition 2.4 ensures the existence of a function f such that $F(\mathrm{d}y) = f(y)\mathrm{d}y$, we see that, whenever F exists, $P(t, x, \cdot)$ admits a density $p_t(x, y) = q_t(x, y)f(y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ for each $t \in \mathbb{R}_+$. However, this argument does not generally guarantee the smoothness of $p_t(x, y)$ in y since we do not know whether $q_t(x, y)$ is smooth or not.

3.2. Smoothness of the transition probabilities

Recall that, for any OU process, the characteristic function of $P(t, x, \cdot)$ for each $t \in \mathbb{R}_+$ and $x \in S$ is explicitly given by (2.4). This is advantageous to investigation of the existence of a C_b^∞ transition density.

Theorem 3.2. *The following statements hold true for each* $t \in \mathbb{R}_+$.

(a) *If* $\mathrm{rank}(C) = d$*, then* $P(t, x, \cdot)$ *admits a* C_b^∞ *density.*
(b) *If there exist constants* $\alpha \in (0, 2)$ *and* $c > 0$ *such that*

$$\int_{\{z:|v^\mathrm{T} z| \leqslant 1\}} |v^\mathrm{T} z|^2 \nu(\mathrm{d}z) \geqslant c|v|^{2-\alpha} \tag{3.1}$$

for any $v \in \mathbb{R}^d$ *satisfying* $|v| \geqslant 1$*, then* $P(t, x, \cdot)$ *admits a* C_b^∞ *density.*

Proof. As is well known, it suffices to show that $\int |u|^k |\varphi_{P(t,x,\cdot)}(u)| du < \infty$ for any non-negative integer k; see, for example, Sato (1999, Proposition 28.1).

It follows from (2.1) and (2.4) that, for all $u \in \mathbb{R}^d$,

$$|\varphi_{P(t,x,\cdot)}(u)| \leqslant \left| \exp\left\{ \int_0^t \log \varphi_{Z_1}(\mathrm{e}^{-sQ^{\mathrm{T}}} u) \mathrm{d}s \right\} \right|$$

$$\leqslant \exp\left\{ -\frac{1}{2} u^{\mathrm{T}} \left(\int_0^t \mathrm{e}^{-sQ} C \mathrm{e}^{-sQ^{\mathrm{T}}} \mathrm{d}s \right) u \right\} |J_t(u)|,$$

where

$$J_t(u) = \exp\left\{ \int_0^t \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}u^{\mathrm{T}}\mathrm{e}^{-sQ}z} - 1 - \mathrm{i}u^{\mathrm{T}}\mathrm{e}^{-sQ} z 1_U(z)) \nu(\mathrm{d}z) \mathrm{d}s \right\}, \tag{3.2}$$

so that, whenever $\mathrm{rank}(C) = d$, assertion (a) follows since $|J_t(u)| \leqslant 1$ for any t and u.

Turning to (b), find two positive finite constants m_t^Q and M_t^Q such that $m_t^Q \leqslant |\mathrm{e}^{-sQ^{\mathrm{T}}}| \leqslant M_t^Q$ for any $s \in [0, t]$: such constants do indeed exist since $s \mapsto |\mathrm{e}^{-sQ^{\mathrm{T}}}|$ is continuous and zero-free. Since C is non-negative definite, we have $|\varphi_{P(t,x,\cdot)}(u)| \leqslant |J_t(u)|$. Fix u large enough so that $|\mathrm{e}^{-sQ^{\mathrm{T}}} u| \geqslant 1$. Then, using the inequality $1 - \cos(x) \geqslant 2(x/\pi)^2$ for $|x| \leqslant \pi$ and assumption (3.1), we have

$$|\varphi_{P(t,x,\cdot)}(u)| \leqslant \exp\left\{ -\tilde{c} \int_0^t |\mathrm{e}^{-sQ^{\mathrm{T}}} u|^{2-\alpha} \mathrm{d}s \right\}$$

for some constant $\tilde{c} > 0$. It is obvious that there exists a constant $c_t^Q > 0$ such that

$$\tilde{c} \int_0^t |\mathrm{e}^{-sQ^{\mathrm{T}}} u|^{2-\alpha} \mathrm{d}s \geqslant c_t^Q |u|^{2-\alpha},$$

and we thus obtain $|\varphi_{P(t,x,\cdot)}(u)| \leqslant \exp\{-c_t^Q |u|^{2-\alpha}\}$ for sufficiently large u. The proof is complete. □

Remark 3.3. For the solution of a stochastic differential equation driven by a pure-jump Lévy process with infinitely many small jumps, Picard (1996) obtained sufficient conditions for the existence of a C_b^∞ transition density; one can readily apply his Corollary 4.4 for OU processes without Gaussian component. However, Theorem 3.2 says that, for OU processes, we do not need the infinite-dimensional stochastic calculus (the so-called Malliavin calculus) which is essential in his work.

4. Exponential β-mixing bounds for a strictly stationary OU process

In this section, we shall obtain the exponential β-mixing bound for X of (1.1) under strict stationarity. Here we suppose that $Q \in M_+(\mathbb{R}^d)$ and that the marginal Q-self-decomposable distribution F is given. Recall that in this case Z can be determined by (2.9): see Remark 2.2. Due to the Markov nature of X, the β-mixing coefficient $\beta_X(t)$ of X is given by

$$\beta_X(t) = \int \|P(t, x, \cdot) - F(\cdot)\|_{\mathrm{TV}} F(\mathrm{d}x), \tag{4.1}$$

where $\|\cdot\|_{\mathrm{TV}}$ stands for the total variation norm; see, for example, Doukhan (1994, Section 2.4). To attain our goal, we shall utilize the results of Tuominen and Tweedie (1979) and Nummelin and Tuominen (1982). Denote the state space of X by $(S_F, \mathcal{B}(S_F))$, where $S_F \subset \mathbb{R}^d$ is an open set.

Write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and denote by $X^{(\Delta)} = (X_n^{(\Delta)})_{n \in \mathbb{N}_0}$ the discrete-time Markov chain regularly sampled from X at the time points $0, \Delta, 2\Delta, \ldots$ for a constant $\Delta > 0$. As usual, we call this $X^{(\Delta)}$ the Δ-*skeleton chain* (associated with X). Obviously the m-step transition semigroup of $X^{(\Delta)}$ is equal to $P_{m\Delta}$. Following Tuominen and Tweedie (1979), we call the transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ *simultaneously* φ*-irreducible* (for some σ-finite measure φ) if any Δ-skeleton chain $X^{(\Delta)}$ is φ-irreducible, that is, $\sum_{n=1}^{\infty} P(n\Delta, x, A) > 0$ for any $x \in S_F$ and $A \in \mathcal{B}(S_F)$ such that $\varphi(A) > 0$.

Lemma 4.1. (Tuominen and Tweedie 1979, Proposition 1.2). *If* $(P_t)_{t \in \mathbb{R}_+}$ *is simultaneously* φ*-irreducible, then any* Δ*-skeleton chain is aperiodic.*

It is easy to see that any OU process is weak Feller, that is, $P_t f \in C_b(S)$ for any $f \in C_b(S)$ and $t > 0$. Therefore any compact set $K \in \mathcal{B}(S_F)$ such that $F(K) > 0$ is *small*: see, for example, Meyn and Tweedie (1992, Section 3). The following general result (the so-called Foster–Lyapunov criterion) will be used in order to derive the geometric ergodicity of $X^{(\Delta)}$:

Proposition 4.2. (Nummelin and Tuominen 1982, Theorem 2.1 and 3.1). *Let* $x = (x_n)_{n \in \mathbb{N}_0}$ *be a* φ*-irreducible aperiodic Markov chain with an* n*-step transition probability* $P^n(x, \mathrm{d}y)$ *(the superscript* $n \in \mathbb{N}$ *is suppressed when* $n = 1$*), and denote the state space of* x *by* $(S, \mathcal{B}(S))$*, where* $\mathcal{B}(S)$ *is countably generated. Assume that there exist a measurable function* $g : S \to \mathbb{R}_+$*, a small set* $K \in \mathcal{B}(S)$ *and constants* $c_1 \in (0, 1)$ *and* $c_2 > 0$ *such that*

$$\sup_{z \in K} \int_{K^c} g(y) P(z, \mathrm{d}y) < \infty, \tag{4.2}$$

where K^c *stands for the complement of* K*, and that*

$$\int g(y) P(z, \mathrm{d}y) \leqslant c_1 g(z) - c_2 \tag{4.3}$$

for any $z \in K^c$*. Then* x *is geometrically ergodic, that is, there exists a constant* $\rho \in (0, 1)$ *such that*

$$\int \|P^n(z, \cdot) - F\|_{\mathrm{TV}} F(\mathrm{d}z) = O(\rho^n), \qquad \text{as } n \to \infty. \tag{4.4}$$

The claim in this section is the following:

Theorem 4.3. *Let $Q \in M_+(\mathbb{R}^d)$ and X be the strictly stationary OU process given by* (1.1) *with a Q-self-decomposable marginal distribution F. If we have*

$$\int |x|^p F(\mathrm{d}x) < \infty \tag{4.5}$$

for some $p > 0$, then there exists a constant $a > 0$ such that $\beta_X(t) = O(\mathrm{e}^{-at})$ as $t \to \infty$. In particular, X is ergodic.

Proof. For each Δ and $n \in \mathbb{N}$, we have

$$X_n^{(\Delta)} = \mathrm{e}^{-\Delta Q} X_{n-1}^{(\Delta)} + \xi_n,$$

where $\xi = (\xi_n)_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables such that $\mathcal{L}(\xi_1) = \mathcal{L}(\int_0^\Delta \mathrm{e}^{-(\Delta - s)Q}\, \mathrm{d}Z_s)$. Obviously $X^{(\Delta)}$ is strictly stationary with the marginal distribution F.

Step 1. First, we show that $X^{(\Delta)}$ is geometrically ergodic. Note that the simultaneous F-irreducibility of $X^{(\Delta)}$ follows directly from (2.7) since $\lim_{t\to\infty} P(t, x, A) = \lim_{n\to\infty} P(n\Delta, x, A)$ for any $\Delta > 0$ and $A \in \mathcal{B}(S_F)$. Hence, for any Δ, the aperiodicity of $X^{(\Delta)}$ is implied by Lemma 4.1.

Put $\delta = |\mathrm{e}^{-\Delta Q}|$. To prove (4.2) and (4.3), we note that there exists a (sufficiently large) Δ such that $\delta < 1$ since $Q \in M_+(\mathbb{R}^d)$. Fix Δ thus in the rest of this proof.

Suppose that $p \in (0, 1]$ without loss of generality. Since $(a+b)^p \leqslant a^p + b^p$ for any non-negative a and b, it follows from the strict stationarity of X that

$$\begin{aligned} \mathrm{E}[|\xi_1|^p] &= \mathrm{E}[|X_1^{(\Delta)} - \mathrm{e}^{-\Delta Q} X_0^{(\Delta)}|^p] \\ &\leqslant \mathrm{E}[(|X_1^{(\Delta)}| + \delta |X_0^{(\Delta)}|)^p] \\ &\leqslant (1 + \delta^p)\mathrm{E}[|X_0|^p] < \infty \end{aligned}$$

under (4.5). Put $C_\eta = \{x \in S_F : |x| \leqslant \eta\}$ for some constant $\eta > 0$; then C_η is a small set by virtue of its compactness. Since ξ_1 and $X_0^{(\Delta)}$ are independent, we have

$$\begin{aligned} \int_{C_\eta^c} |y|^p P(\Delta, x_0, \mathrm{d}y) &\leqslant \mathrm{E}[|\mathrm{e}^{-\Delta Q} x_0 + \xi_1|^p] \\ &\leqslant \delta^p \eta^p + \mathrm{E}[|\xi_1|^p] < \infty \end{aligned}$$

for any $x_0 \in C_\eta$. Since the upper bound does not depend on the choice of x_0, we obtain (4.2). On the other hand, for $x_1 \in C_\eta^c$, we similarly obtain

$$\int |y|^p P(\Delta, x_1, \mathrm{d}y) \leqslant c_1 |x_1|^p - c_2, \tag{4.6}$$

where c_1 is a constant such that $\delta^p < c_1 < 1$, and $c_2 = (c_1 - \delta^p)|x_1|^p - \mathrm{E}[|\xi_1|^p]$. The set C_η^c is not empty for any η since the support of F is unbounded, and therefore we can choose η

large enough so that $c_2 > 0$. Thus (4.3) also follows, and hence Proposition 4.2 yields the geometric ergodicity of $X^{(\Delta)}$ with $g(x) = |x|^p$.

Step 2. By step 1, there exists a constant ρ such that $\rho \in (0, 1)$ and that

$$\int \sup_{|f| \leqslant 1} |P_{n\Delta} f(x) - F(f)| F(\mathrm{d}x) = O(\rho^n), \qquad \text{as } n \to \infty, \tag{4.7}$$

where $F(f) = \int f(y) F(\mathrm{d}y)$. Denote by $[t]$ the integer part of $t \in \mathbb{R}_+$, and put $t_\Delta = [t/\Delta]\Delta$ and $f_t = P_{t_\Delta} f \in b\mathcal{B}(S_F)$. Then (4.1), a property of the semigroup, the invariance of F and (4.7) yield that

$$\begin{aligned}
\beta_X(t) &= \int \sup_{|f| \leqslant 1} |P_t f(x) - F(f)| F(\mathrm{d}x) \\
&= \int \sup_{|f| \leqslant 1} |[P_{t_\Delta} P_{t - t_\Delta} f](x) - F(f)| F(\mathrm{d}x) \\
&= \int \sup_{|f| \leqslant 1} |[P_{t_\Delta} P_{t - t_\Delta} f](x) - F(P_{t - t_\Delta}(f))| F(\mathrm{d}x) \\
&\leqslant \int_{\mathbb{R}^d} \sup_{|f_t| \leqslant 1} |P_{t_\Delta} f_t(x) - F(f_t)| F(\mathrm{d}x) \\
&= O(\rho^{t_\Delta/\Delta})
\end{aligned}$$

as $t \to \infty$, so taking $a = -(\log \rho)/\Delta$ completes the proof (recall that strict stationarity and β-mixing property imply ergodicity). □

Remark 4.1. Condition (4.5) results from the general Lévy-driven setting. This condition is slightly stronger than (2.6), which is the necessary and sufficient condition for the existence of F; however, (4.5) would be satisfactory for applications.

Remark 4.2. The class of all operator-stable distributions (see Section 5.2) is contained in the class of all operator self-decomposable distributions. Since any operator-stable distribution on $\mathbb{R}^d$ satisifes (4.5) (see Jurek and Mason 1993, Theorem 4.12.6), any OU process with an operator-stable marginal distribution is exponentially β-mixing.

Remark 4.3. Let f be any F-integrable function defined on S_F and denote by $P^X_{F_0}$ the image measure of an OU process X such that $\mathcal{L}(X_0) = F_0$ for some F_0. Since $\lim_{t\to\infty} \|P(t, x, \cdot) - F\|_{\mathrm{TV}} = 0$ for any $x \in S_F$ under the assumptions in Theorem 4.1, we have

$$\frac{1}{t}\int_0^t f(X_s)\mathrm{d}s \to F(f) \qquad \text{as } t \to \infty \tag{4.8}$$

almost surely under $P^X_{F_0}$ for any F_0; see Bhattacharya (1982, pp. 193–194) for an illustration. The ergodic theorem (4.8) is often crucial for parameter estimation.

Theorem 4.3 implies that an OU process may be ergodic even if the distribution of its driving Lévy process is singular, as illustrated by the following example. Define a distribution F on $\mathbb{R}_+$ by

$$\varphi_F(u) = \exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}uy} - 1)y^{-1}\delta_c([y, \infty))\mathrm{d}y\right\}, \tag{4.9}$$

where δ_c stands for the delta measure at $c > 0$. Then the Lévy density of F is $\delta_c([y, \infty))y^{-1}$, which is discontinuous at $y = c$. Also, let q be a positive constant and define a Lévy measure ν on $\mathbb{R}_+$ by $\nu(\mathrm{d}z) = q\delta_c(\mathrm{d}z)$. Now consider the one-dimensional OU process given by

$$\mathrm{d}X_t = -qX_t\,\mathrm{d}t + \mathrm{d}Z_t, \tag{4.10}$$

where $\mathcal{L}(X_0) = F$ and Z is a subordinator with the Lévy measure ν, that is, Z is a compound Poisson process with intensity q and the degenerate jump distribution δ_c. Then X is strictly stationary with the marginal distribution F, and also exponentially β-mixing. Indeed, we have

$$\begin{aligned}\mathrm{E}[\mathrm{e}^{\mathrm{i}uX_t}|X_0 = x] &= \exp\left\{\mathrm{i}u\mathrm{e}^{-qt}x + \int_0^t \log\varphi_{Z_1}(\mathrm{e}^{-qs}u)\mathrm{d}s\right\}\\ &= \exp\left\{\mathrm{i}u\mathrm{e}^{-qt}x + \int_{ue^{-qt}}^u (\mathrm{e}^{\mathrm{i}vc} - 1)v^{-1}\,\mathrm{d}v\right\}\\ &\to \varphi_F(u), \qquad \text{as } t\to\infty,\end{aligned}$$

by applying (2.4), and it is also not difficult to see that (4.5) holds (apply, for example, Jurek and Mason 1993, Proposition 1.8.13, with the submultiplicative function $\max(1, |x|)$). Hence, according to Theorem 4.3, X is exponentially β-mixing and ergodic.

Since $\beta_Y(t) \leqslant \sum_{j=1}^m \beta_{X^{(j)}}(t)$ for the β-mixing coefficients of Y and $(X^{(j)})_{j=1}^m$ (see, for example, Doukhan 1994, p. 4, Theorem 1), we can also establish the exponential mixing bound for a finitely superposed OU process $Y = \sum_{j=1}^m X^{(j)}$ (recall the notation in Section 2.3) as a simple corollary of Theorem 4.3:

Corollary 4.4. *Let $X^{(j)}$, $j = 1, 2, \ldots, m$, be independent strictly stationary OU processes with marginal distributions $F^{(j)}$ such that $\int |x|^p F^{(j)}(\mathrm{d}x) < \infty$, $j = 1, 2, \ldots, m$, for some $p > 0$. Then there exists a constant $a > 0$ such that $\beta_Y(t) = O(\mathrm{e}^{-at})$ as $t \to \infty$.*

Remark 4.4. One may ask if it is possible to establish some mixing bound other than exponential under strict stationarity. By Theorem 4.3, we must choose an F satisfying $\int |x|^p F(\mathrm{d}x) = \infty$ for any $p > 0$. Such a case rarely appears, at least in practice. For several types of Markov chain or process, many researchers worked on the derivation of non-exponential bounds such as subgeometric and polynomial; see, for example, Tweedie (1983), Tuominen and Tweedie (1994), Veretennikov (1997, 1999) and Fort and Moulines (2000). However, for a strictly stationary OU process, we observe that most of the sufficient conditions in such works yield the exponential rate. For example, Tuominen and Tweedie

(1994) gave a set of sufficient conditions for the polynomial mixing bound for a first-order Markov chain $x = (x_n)_{n\in\mathbb{N}_0}$ of the form

$$x_n = G(x_{n-1}) + \eta_n, \tag{4.11}$$

where $\eta = (\eta_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables and the function G defined on the state space of x satisfies a certain regularity condition. Any Δ-skeleton chain sampled from an OU process corresponds to $G(x) = \mathrm{e}^{-\Delta Q}x$, which satisfies the condition of Tuominen and Tweedie (1994). However, they also require that $\mathrm{E}[|\eta_1|^m] < \infty$ for some $m \geqslant 2$, which yields $\int |x|^2 F(\mathrm{d}x) < \infty$ in the case of stationary OU processes, and hence we can derive the exponential β-mixing bound from Theorem 4.3; in other words, the simple linear structure of G here does not suffice to obtain the desired mixing bound within their framework. By the same token, the sufficient conditions of Veretennikov (1999) for the polynomial β-mixing property of a class of Markov chains also yield the exponential mixing bound for OU processes. We do not pursue this topic in this paper.

5. GH_d distribution and OU process

In this section, we shall first obtain the explicit Lévy density of the general $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$ distribution with respect to the Lebesgue measure on $\mathbb{R}^d$, and then formulate the strictly stationary OU process whose marginal distribution is the centred and non-skewed GH_d, which does not depend on Q. We should note that GH_d can be defined as a normal variance–mean mixture associated with the generalized inverse Gaussian (GIG) distribution (see (A.1)), which is self-decomposable. See the Appendix for a brief review of GH_d.

5.1. Explicit Lévy densities of GH_d

It is known that the Lévy measure of GH_1 admits an explicit density with respect to the Lebesgue measure, as stated in the Introduction. Here we give its d-dimensional version, using the general theory of the subordination of Lévy processes.

Before the statement, let us recall that the $GIG(\lambda, \delta, \gamma)$ distribution admits the explicit Lévy density $f(x)$ given by

$$f(x) = \frac{\mathrm{e}^{-\gamma^2 x/2}}{x}\left\{\max(0, \lambda) + \frac{1}{2}\int_0^\infty \exp\left(-\frac{xy}{2\delta^2}\right) g_\lambda(y)\mathrm{d}y\right\} \tag{5.1}$$

with

$$g_\lambda(y) = \frac{2}{y\pi^2\left\{J^2_{|\lambda|}(y^{1/2}) + Y^2_{|\lambda|}(y^{1/2})\right\}}$$

where $J_{|\lambda|}$ ($Y_{|\lambda|}$) stands for the Bessel functions of first (second) kind; see Barndorff-Nielsen and Shephard (2001a, Theorem 2).

Theorem 5.1. *The Lévy measure of $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$ admits the density given by*

$$h(x) = \frac{2e^{\beta^T x}}{\{2\pi(x^T\Lambda^{-1}x)^{1/2}\}^{d/2}\{\det(\Lambda)\}^{1/2}}\Bigg\{\max(0, \lambda)\alpha^{d/2}K_{d/2}(\alpha(x^T\Lambda^{-1}x)^{1/2})$$

$$+\int_0^\infty \frac{(\alpha^2+2y)^{d/4}K_{d/2}(\{(\alpha^2+2y)(x^T\Lambda^{-1}x)\}^{1/2})}{\pi^2 y\Big\{J^2_{|\lambda|}(\delta(2y)^{1/2})+Y^2_{|\lambda|}(\delta(2y)^{1/2})\Big\}}\,dy\Bigg\}. \tag{5.2}$$

Proof. First of all, we note that the Lévy measure ν of $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$ is given by

$$\nu(B) = \int_0^\infty\int_B \frac{f(s)}{(2s\pi)^{d/2}\{\det(\Lambda)\}^{1/2}}\exp\bigg\{-\frac{1}{2}(x-s\Lambda\beta)^T\Lambda^{-1}(x-s\Lambda\beta)\bigg\}dx\,ds \tag{5.3}$$

for $B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$, where $f(s)$ is given by (5.1) and $\gamma^2 = \alpha^2 - \beta^T\Lambda\beta \geqslant 0$ (see (A.5)): (5.3) follows from (A.3) and Sato (1999, Theorem 30.1). By (5.1) and interchanging the order of integration, (5.3) can be rewritten as

$$\begin{aligned}\nu(B) &= \int_B\int_0^\infty \frac{s^{-d/2-1}e^{\beta^T x-\gamma^2 s/2}}{2(2\pi)^{d/2}\{\det(\Lambda)\}^{1/2}}\exp\bigg\{-\frac{1}{2}\bigg(\frac{x^T\Lambda^{-1}x}{s}+s\beta^T\Lambda\beta\bigg)\bigg\}\\ &\quad\times\bigg(\int_0^\infty e^{-sy/(2\delta^2)}g_\lambda(y)dy\bigg)ds\,dx\\ &\quad+\int_B\int_0^\infty \max(0,\lambda)\frac{s^{-d/2-1}e^{\beta^T x-\gamma^2 s/2}}{2(2\pi)^{d/2}\{\det(\Lambda)\}^{1/2}}\\ &\quad\times\exp\bigg\{-\frac{1}{2}\bigg(\frac{x^T\Lambda^{-1}x}{s}+s\beta^T\Lambda\beta\bigg)\bigg\}ds\,dx\\ &= \int_B h_1(x)dx + \int_B h_2(x)dx,\end{aligned}$$

say, so that $h(x) = h_1(x) + h_2(x)$.

For $h_1(x)$, by interchanging the order of integrations and then using the expression (A.1) to eliminate the factor integrating to 1, we have

$$\begin{aligned}h_1(x) &= \frac{e^{\beta^T x}}{(2\pi)^{d/2}\{\det(\Lambda)\}^{1/2}}\int_0^\infty g_\lambda(y)\int_0^\infty \frac{s^{-d/2-1}}{2}\exp\bigg[-\frac{1}{2}\bigg\{\frac{x^T\Lambda^{-1}x}{s}+s\bigg(\alpha^2+\frac{y}{\delta^2}\bigg)\bigg\}\bigg]ds\,dy\\ &= \frac{e^{\beta^T x}}{(2\pi)^{d/2}\{\det(\Lambda)\}^{1/2}}\int_0^\infty g_\lambda(y)\bigg(\frac{\alpha^2+y/\delta^2}{x^T\Lambda^{-1}x}\bigg)^{d/4}K_{d/2}\big(\{(\alpha^2+y/\delta^2)(x^T\Lambda^{-1}x)\}^{1/2}\big)dy.\end{aligned}$$

Here we have used the property $K_\lambda(\cdot) = K_{-\lambda}(\cdot)$. Then the change of variable $y = 2\delta^2 z$ and the definition of $g_\lambda(\cdot)$ give

$$h_1(x) = \frac{2e^{\beta^T x}}{\{2\pi(x^T\Lambda^{-1}x)^{1/2}\}^{d/2}\{\det(\Lambda)\}^{1/2}}$$

$$\times \int_0^\infty \frac{(\alpha^2+2z)^{d/4} K_{d/2}\big(\{(\alpha^2+2z)(x^T\Lambda^{-1}x)\}^{1/2}\big)}{\pi^2 z\Big\{J^2_{|\lambda|}(\delta(2z)^{1/2}) + Y^2_{|\lambda|}(\delta(2z)^{1/2})\Big\}}\, dz. \tag{5.4}$$

For $h_2(x)$, analogous calculations yield

$$h_2(x) = \frac{2\alpha^{d/2}e^{\beta^T x}\max(0, \lambda)}{\{2\pi(x^T\Lambda^{-1}x)^{1/2}\}^{d/2}\{\det(\Lambda)\}^{1/2}} K_{d/2}\big(\alpha(x^T\Lambda^{-1}x)^{1/2}\big). \tag{5.5}$$

Combining (5.4) and (5.5) concludes the proof. □

By Lemma A.1, each one-dimensional marginal distribution of $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$ belongs to the GH_1 family, hence the Lévy measure of $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$ obviously has infinite mass around the origin.

Let us consider a special case of $\lambda = -\frac{1}{2}$, namely, the so-called d-dimensional normal inverse Gaussian distribution denoted by $NIG_d(\alpha, \beta, \delta, \mu, \Lambda)$. In the one-dimensional case, its Lévy density is given by $(\pi|x|)^{-1}\alpha\delta e^{\beta x}K_1(\alpha|x|)$; see, for example, Barndorff-Nielsen (1998).

Corollary 5.2. *The Lévy measure of* $NIG_d(\alpha, \beta, \delta, \mu, \Lambda)$ *admits the density given by*

$$q(x) = \frac{2\delta e^{\beta^T x}}{\{\det(\Lambda)\}^{1/2}} \left\{\frac{\alpha}{2\pi(x^T\Lambda^{-1}x)^{1/2}}\right\}^{(d+1)/2} K_{(d+1)/2}\big(\alpha(x^T\Lambda^{-1}x)^{1/2}\big). \tag{5.6}$$

Proof. Taking $\lambda = -\frac{1}{2}$ reduces $GIG(\lambda, \delta, \gamma)$ to the inverse Gaussian distribution with the Lévy density given by

$$\tilde{f}(x) = \frac{\delta}{(2\pi)^{1/2}} x^{-3/2} \exp\left(-\frac{\gamma^2 x}{2}\right).$$

Replacing $f(s)$ by this $\tilde{f}(s)$ in the proof of Theorem 5.1 yields (5.6). □

5.2. A relation between centred and non-skewed GH_d and the OU process

Now we consider the strictly stationary OU process whose marginal distribution is the centred and non-skewed $GH_d(\lambda, \alpha, \delta, \Lambda)$ with density

$$pGH_d(x; \lambda, \alpha, \delta, \Lambda) = \frac{\alpha^{d/2}K_\lambda(\delta\alpha)K_{\lambda-d/2}\big(\alpha(\delta^2 + x^T\Lambda^{-1}x)^{1/2}\big)}{(2\pi)^{d/2}\delta^\lambda(\delta^2 + x^T\Lambda^{-1}x)^{d/4-\lambda/2}}. \tag{5.7}$$

Here we assume that $\det(\Lambda) = 1$ for simplicity, and denote $GH_d(\lambda, \alpha, 0, \delta, 0, \Lambda)$ simply by

$GH_d(\lambda, \alpha, \delta, \Lambda)$. Put $\Lambda = [\Lambda_{ij}]_{i,j=1}^d$. Let $\Xi_{ij}^Q(t)$ denote the (i, j)th entry of the matrix e^{-tQ} with some $Q \in \mathbb{R}^d \otimes \mathbb{R}^d$, and let I_d stand for the $d \times d$ identity matrix.

Proposition 5.3. *Let $Q \in \mathbb{R}^d \otimes \mathbb{R}^d$ be of the form*

$$Q = rI_d/2 + \Lambda^{1/2} S \Lambda^{-1/2}, \tag{5.8}$$

where $r > 0$ is a constant and $S \in \mathbb{R}^d \otimes \mathbb{R}^d$ is a skew-symmetric matrix. Then any $GH_d(\lambda, \alpha, \delta, \Lambda)$ is Q-self-decomposable. The associated strictly stationary OU process X is strong Feller and exponentially β-mixing, with the driving Lévy process Z given by

$$\varphi_{Z_1}(u) = \begin{cases} \exp\left(-\dfrac{2\lambda u^{\mathrm{T}} \Lambda Q^{\mathrm{T}} u}{\alpha^2 + u^{\mathrm{T}} \Lambda u} \right), & \textit{for } \delta = 0 (\alpha > 0) \textit{ and } \lambda > 0, \\ \exp\left[- \left\{ \dfrac{K_{\lambda+1}\left(\delta(\alpha^2 + u^{\mathrm{T}} \Lambda u)^{1/2} \right)}{K_\lambda \left(\delta(\alpha^2 + u^{\mathrm{T}} \Lambda u)^{1/2} \right)} \right\} \dfrac{\delta u^{\mathrm{T}} \Lambda Q^{\mathrm{T}} u}{(\alpha^2 + u^{\mathrm{T}} \Lambda u)^{1/2}} \right], & \textit{otherwise}. \end{cases} \tag{5.9}$$

Moreover, the autocorrelation matrix function $\Phi(t) = [\rho_{ij}(t)]_{i,j=1}^d$ of X is given by

$$\rho_{ij}(t) = (\Lambda_{ii} \Lambda_{jj})^{-1/2} \sum_{k=1}^d \Lambda_{jk} \Xi_{ik}^Q(t). \tag{5.10}$$

Proof. First, let us prove the Q-self-decomposability of $GH_d(\lambda, \alpha, \delta, \Lambda)$. To this end, we introduce *strict operator stability.*

Denote by $M_I(\mathbb{R}^d)$ the class of all $d \times d$ matrices whose eigenvalues have real parts in $I \subset \mathbb{R}$ and by F^{*t} the tth convolution of a distribution F on $\mathbb{R}^d$. Put

$$t^Q = \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} Q^k \tag{5.11}$$

for $t > 0$ and $d \times d$ matrix Q. Now suppose that $Q \in M_{[1/2,\infty)}(\mathbb{R}^d)$. Then a distribution F on $\mathbb{R}^d$ is called *Q-stable* if and only if there exists a deterministic function $b : (0, \infty) \to \mathbb{R}^d$ such that, for every $t > 0$,

$$\varphi_{F^{*t}}(u) = \varphi_F(t^{Q^{\mathrm{T}}} u) \mathrm{e}^{\mathrm{i} b(t)^{\mathrm{T}} u}. \tag{5.12}$$

The characterization (5.12) of Q-stability goes back to Sharpe (1969). When $b(t) \equiv 0$ in (5.12), F is called *strictly Q-stable*. See Sato (1987) for a detailed analysis of strict Q-stability. A Lévy process L is called (strictly) Q-stable if $\mathcal{L}(L_1)$ is (strictly) Q-stable. We shall utilize the following result:

Proposition 5.4. *Let Z be a (one-dimensional) self-decomposable subordinator and Y a d-dimensional strictly Q-stable Lévy process with some $Q \in M_{[1/2,\infty)}(\mathbb{R}^d)$. Then, the subordinated Lévy process $L_t = Y_{Z_t}$ is $\tilde{Q}$-self-decomposable for any $\tilde{Q}$ of the form $\tilde{Q} = rQ \in M_+(\mathbb{R}^d)$ with some $r > 0$.*

Proposition 5.4 is a reduced version of Theorem 6.1 of Barndorff-Nielsen *et al.* (2001), where more general multivariate subordinators are taken into consideration.

Remark 5.1. The reason for setting $\beta = \mu = 0$ in this section is that the author presently does not know whether it is possible to replace strict Q-stability of Y in Proposition 5.4 with Q-stability (Sato (2001) gave an affirmative answer when $d = 1$ and Y is a Wiener process with non-zero drift): if this is possible, then β and μ may be non-zero due to the construction of general GH_d. We should remark that, if $d \geqslant 2$ and $Q = cI_d$ for some constant $c > 0$ (i.e. $S \equiv 0$), then there exists a non-Q-self-decomposable GH_d with $\beta \neq 0$; see Shanbhag and Sreehari (1979, p. 24).

Any Wiener process Y with no drift and a non-singular covariance matrix Λ is strictly Q-stable for some Q. Indeed, by Jurek and Mason (1993, Theorem 4.6.10), we know that the exponent Q of Y is of the form

$$Q = \frac{1}{2} I_d + \Lambda^{1/2} S \Lambda^{-1/2}, \tag{5.13}$$

where S is a $d \times d$ skew-symmetric matrix. For any $t > 0$ and Q as in (5.13), we have

$$\begin{aligned} t^{Q^{\mathrm{T}}} &= t^{I_d/2} t^{\Lambda^{-1/2} S^{\mathrm{T}} \Lambda^{1/2}} \\ &= t^{I_d/2} \Lambda^{-1/2} \left\{ \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} (S^{\mathrm{T}})^k \right\} \Lambda^{1/2} \\ &= t^{I_d/2} \Lambda^{-1/2} t^{S^{\mathrm{T}}} \Lambda^{1/2}, \end{aligned}$$

and hence

$$H(u) := u^{\mathrm{T}} t^{Q} \Lambda t^{Q^{\mathrm{T}}} u - t u^{\mathrm{T}} \Lambda u = 0. \tag{5.14}$$

Then (5.14) ensures strict Q-stability of Y since, for $F = N_d(\xi, \Lambda)$,

$$\begin{aligned} \varphi_{F^{*t}}(u) &= \varphi_F(t^{Q^{\mathrm{T}}} u) \exp\{\mathrm{i}\xi^{\mathrm{T}}(tI_d - t^{Q^{\mathrm{T}}})u\} \exp\{H(u)/2\} \\ &= \varphi_F(t^{Q^{\mathrm{T}}} u) \exp\{\mathrm{i}\xi^{\mathrm{T}}(tI_d - t^{Q^{\mathrm{T}}})u\}, \end{aligned}$$

so that the convolution $F * \delta_{(-\xi)} = N_d(0, \Lambda)$ is strictly Q-stable for Q of (5.13). Thus Proposition 5.4 guarantees the existence of the $GH_d(\lambda, \alpha, \delta, \Lambda)$ OU process with any Q of the form (5.8).

The strong Feller property of any $GH_d(\lambda, \alpha, \delta, \Lambda)$ OU process automatically follows from Theorems 3.1 and 5.1.

Since the absolute moments of any order exist when $\alpha > 0$ (allowing $\delta = 0$), we discuss only the case of $\alpha = 0$ for the exponential β-mixing property. In this case, the associated $GH_d(\lambda, \alpha, \delta, \Lambda)$ becomes the d-dimensional centred and non-skewed Student t whose density is given by

$$p_{t_d}(x; \psi, \delta, \Lambda) = \frac{\delta^{\psi}\Gamma((\psi + d)/2)}{\pi^{d/2}\Gamma(\psi/2)}(\delta^2 + x^{\mathrm{T}}\Lambda^{-1}x)^{-(\psi+d)/2}, \tag{5.15}$$

where $\psi = -2\lambda > 0$ corresponds to the degrees of freedom (each marginal is one-dimensional Student t with the same degrees of freedom ψ; see Lemma A.1). In this case, only rth-order ($r < d - 1 + \psi$) absolute moments exist; however, the condition (4.5) still holds for $p \in (0, r]$, and hence it is concluded that any OU process whose marginal distribution is $GH_d(\lambda, \alpha, \delta, \Lambda)$ is exponentially β-mixing.

Letting $m = 1$ and $s = 0$ in (2.14) and then using (A.8), the autocorrelation matrix function turns out to be (5.10); the finitely superposed version mentioned in Section 2.3 can be computed in the same way.

A direct application of Lemma 2.5 gives (5.9) if we use the properties

$$\frac{\mathrm{d}}{\mathrm{d}z}K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z),$$

$$K_{\nu-1}(z) = K_{\nu+1}(z) - \frac{2\nu}{z}K_\nu(z),$$

together with the following expression for the characteristic function of $GH_d(\lambda, \alpha, \delta, \Lambda)$:

$$\varphi_{GH_d}(u) = \begin{cases} \left(\dfrac{\alpha^2}{\alpha^2 + u^{\mathrm{T}}\Lambda u}\right)^{\lambda/2}\dfrac{K_\lambda\left(\delta(\alpha^2 + u^{\mathrm{T}}\Lambda u)^{1/2}\right)}{K_\lambda(\delta\alpha)} & \text{for } \alpha\delta \neq 0, \\ \left(\dfrac{\alpha^2}{\alpha^2 + u^{\mathrm{T}}\Lambda u}\right)^{\lambda} & \text{for } \delta = 0,\ \alpha > 0 \text{ and } \lambda > 0, \\ \dfrac{2^{\lambda+1}K_\lambda\left(\delta(u^{\mathrm{T}}\Lambda u)^{1/2}\right)}{\Gamma(-\lambda)\delta^\lambda(u^{\mathrm{T}}\Lambda u)^{\lambda/2}} & \text{for } \alpha = 0,\ \delta > 0 \text{ and } \lambda < 0. \end{cases} \tag{5.16}$$

Here we have used the asymptotic properties (A.2) in cases where either $\delta = 0$ or $\alpha = 0$.

Summarizing the statements above now yields Proposition 5.3. □

Appendix: Multidimensional generalized hyperbolic distributions

In this appendix, we give a brief review of the general GH_d distribution originally introduced by Barndorff-Nielsen (1977) for investigation of the mass-size distribution of Aeolian sand deposits. The general GH_d depends on the parameters $(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$, where $\lambda \in \mathbb{R}$, $\alpha, \delta \geqslant 0$, $\beta, \mu \in \mathbb{R}^d$ and Λ is a $d \times d$ symmetric positive definite matrix. The interested reader may refer to Blæsild and Jensen (1981) for a systematic exposition of GH_d.

Let Y be a random variable such that $\mathcal{L}(Y) = GIG(\lambda, \delta, \gamma)$ whose density is given by

$$p_{GIG}(x; \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\}, \qquad x > 0, \tag{A.1}$$

where $K_{\lambda}(x)$, $\lambda \in \mathbb{R}$, stands for the modified Bessel function of third order with index λ; see Abramowitz and Stegun (1968). The case where $\delta = 0$ and $\lambda > 0$ ($\gamma = 0$ and $\lambda < 0$) corresponds to the gamma (inverse gamma), via the following asymptotic properties of $K_{\lambda}(\cdot)$ for $z \downarrow 0$:

$$K_{\nu}(z) \sim \begin{cases} \Gamma(\nu)2^{\nu-1}z^{-\nu}, & \text{if } \nu > 0, \\ \Gamma(-\nu)2^{-\nu-1}z^{\nu}, & \text{if } \nu < 0. \end{cases} \tag{A.2}$$

Further, when $\lambda = -\frac{1}{2}$, the *GIG* becomes the inverse Gaussian. Halgreen (1979) proved the self-decomposability of *GIG* and GH_1. See Jørgensen (1982) for a detailed exposition of *GIG*.

Let η be a d-dimensional standard normal random variable independent of Y. Then $\mathcal{L}(X) = GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$, where

$$X = \mu + Y\Lambda\beta + (Y\Lambda)^{1/2}\eta, \tag{A.3}$$

that is, GH_d is defined as the multivariate normal variance–mean mixture where the mixing distribution is *GIG*. Thus the density of GH_d with respect to the Lebesgue measure on $\mathbb{R}^d$ is given by

$$p_{GH_d}(x; \lambda, \alpha, \beta, \delta, \mu, \Lambda)$$
$$= \frac{\{(\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta)^{1/2}/\delta\}^{\lambda} K_{\lambda-d/2}(\alpha V(x; \delta, \mu, \Lambda))\exp\{\beta^{\mathrm{T}}(x-\mu)\}}{(2\pi)^{d/2}\{\det(\Lambda)\}^{1/2} K_{\lambda}\left(\delta(\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta)^{1/2}\right)\{V(x; \delta, \mu, \Lambda)/\alpha\}^{d/2-\lambda}}, \qquad x \in \mathbb{R}^d, \tag{A.4}$$

where $V(x; \delta, \mu, \Lambda) = \{\delta^2 + (x-\mu)^{\mathrm{T}}\Lambda^{-1}(x-\mu)\}^{1/2}$. Usually we assume that $\det(\Lambda) = 1$. The parameters α, β, δ and μ express heaviness of tails, degree of asymmetry, scale and location, respectively, and moreover they vary within the following region in order that (A.4) indeed defines a density:

$$\begin{aligned} &\delta \geqslant 0, \alpha > 0, \alpha^2 > \beta^{\mathrm{T}}\Lambda\beta && \text{if } \lambda > 0, \\ &\delta > 0, \alpha > 0, \alpha^2 > \beta^{\mathrm{T}}\Lambda\beta && \text{if } \lambda = 0, \\ &\delta > 0, \alpha \geqslant 0, \alpha^2 \geqslant \beta^{\mathrm{T}}\Lambda\beta && \text{if } \lambda < 0. \end{aligned} \tag{A.5}$$

The case where $\alpha = 0$ ($\delta = 0$) corresponds to a multivariate version of the symmetric Student t (normal gamma). Also, if $\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta = 0$ with $\beta \neq 0$, we obtain a multivariate version of the asymmetric Student t. These special cases result from the property $K_{\lambda}(z) = K_{-\lambda}(z)$ and (A.2). Moreover, GH_d contains a multivariate version of the hyperbolic (normal inverse Gaussian and hyperboloid), which corresponds to $\lambda = (d+1)/2$ ($\lambda = -1/2$ and $\lambda = (d-1)/2$).

The characteristic function of X is given by

$$\varphi_X(u) = \mathrm{e}^{\mathrm{i}u^{\mathrm{T}}\mu}\left\{\frac{\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta}{\alpha^2 - U(u;\beta,\Lambda)}\right\}^{\lambda/2}\frac{K_\lambda\big(\delta\{\alpha^2 - U(u;\beta,\Lambda)\}^{1/2}\big)}{K_\lambda\big(\delta(\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta)^{1/2}\big)}, \tag{A.6}$$

where $U(u;\beta,\Lambda) = (\mathrm{i}u + \beta)^{\mathrm{T}}\Lambda(\mathrm{i}u + \beta)$, hence the mean and covariance matrix are given by

$$\mathrm{E}[X] = \mu + \delta R_\lambda(\zeta)\Lambda^{1/2}\eta, \tag{A.7}$$

$$\mathrm{var}[X] = \delta^2\{\zeta^{-1}R_\lambda(\zeta)\Lambda + S_\lambda(\zeta)(\Lambda^{1/2}\eta)(\Lambda^{1/2}\eta)^{\mathrm{T}}\}, \tag{A.8}$$

where $\zeta = \delta\{\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta\}^{1/2}$, $\eta = \Lambda^{1/2}\beta\{\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta\}^{-1/2}$,

$$R_\lambda(\zeta) = \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \quad \text{and} \quad S_\lambda(\zeta) = \frac{K_{\lambda+2}(\zeta)K_\lambda(\zeta) - K_\lambda^2(\zeta)}{K_\lambda^2(\zeta)}.$$

Whenever $\alpha^2 - \beta^{\mathrm{T}}\Lambda\beta > 0$, X admits absolute moments of any order.

It is known that the GH_d family is closed under taking marginals, conditioning and regular affine transformations: see Blæsild and Jensen (1981, Theorem 1). In particular, we can specify each marginal of a given $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$:

Lemma A.1. *Let* $\mathcal{L}(X) = GH_d(\lambda, \alpha, \beta, \delta, \mu, \Lambda)$, *and write* $X = (X_1, \ldots, X_d)^{\mathrm{T}}$, $\Lambda = [\Lambda_{i,j}]_{i,j=1}^d$, $\beta = (\beta_j)_{j=1}^d$ *and* $\mu = (\mu_j)_{j=1}^d$. *Then, for each* $j = 1, 2, \ldots, d$, *we have*

$$\mathcal{L}(X_j) = GH_1(\lambda^{[j]}, \alpha^{[j]}, \beta^{[j]}, \delta^{[j]}, \mu^{[j]}, \Lambda^{[j]}), \tag{A.9}$$

where

$$\lambda^{[j]} = \lambda, \qquad \alpha^{[j]} = \Lambda_{j,j}^{-1/2}\{\alpha^2 - \hat{\beta}_j^{\mathrm{T}}(\hat{\Lambda}_{22} - \hat{\Lambda}_{21}\Lambda_{j,j}^{-1}\hat{\Lambda}_{12})\hat{\beta}_j\}^{1/2},$$

$$\beta^{[j]} = \beta_j + \Lambda_{j,j}^{-1}\hat{\Lambda}_{12}\hat{\beta}_j, \qquad \delta^{[j]} = \Lambda_{j,j}^{1/2}\delta, \qquad \mu^{[j]} = \mu_j, \qquad \Lambda^{[j]} = 1$$

together with

$$\hat{\beta}_j = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_d)^{\mathrm{T}},$$

$$\hat{\Lambda}_{12} = (\Lambda_{j,1}, \ldots, \Lambda_{j,j-1}, \Lambda_{j,j+1}, \ldots, \Lambda_{j,d}), \qquad \hat{\Lambda}_{21} = \hat{\Lambda}_{12}^{\mathrm{T}},$$

and $\hat{\Lambda}_{22}$ *denoting the* $(d-1)\times(d-1)$ *matrix equal to* Λ *with jth row and jth column removed.*

Proof. It suffices to apply Blæsild and Jensen (1981, Theorem 1(a)) after arranging the components of X as $(X_j, X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_d)^{\mathrm{T}}$ (and similarly for β and μ), and Λ as

$$\hat{\Lambda} = \begin{pmatrix} \Lambda_{j,j} & \hat{\Lambda}_{12} \\ \hat{\Lambda}_{21} & \hat{\Lambda}_{22} \end{pmatrix}.$$

□

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References

Abramowitz, M. and Stegun, I.A. (eds) (1968) *Handbook of Mathematical Functions.* New York: Dover.

Barndorff-Nielsen, O.E. (1977) Exponentially decreasing distributions for the logarithm of particle size. *Proc. Roy. Soc. Lond. Ser. A*, **353**, 401–419.

Barndorff-Nielsen, O.E. (1998) Processes of normal inverse Gaussian type. *Finance Stochastics*, **2**, 41–68.

Barndorff-Nielsen, O.E. (2001) Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab. Appl.*, **45**, 175–194.

Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1998) Some stationary processes in discrete and continuous time. *Adv. Appl. Probab.*, **30**, 989–1007.

Barndorff-Nielsen, O.E., Pedersen, J. and Sato, K. (2001) Multivariate subordination, selfdecomposability and stability. *Adv. Appl. Probab.*, **33**, 160–187.

Barndorff-Nielsen, O.E. and Shephard, N. (2001a) Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc. Ser. B*, **63**, 167–241.

Barndorff-Nielsen, O.E. and Shephard, N. (2001b) Modelling by Lévy processes for financial econometrics. In O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (eds), *Levy Processes: Theory and Applications*, pp. 283–318. Boston: Birkhäuser.

Bhattacharya, R.N. (1982) On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **60**, 185–201.

Blæsild, P. and Jensen, J.L. (1981) Multivariate distributions of hyperbolic type. In C. Taillie, G.P. Patil and B. Baldessari (eds), *Statistical Distributions in Scientific Work*. Dordrecht: Reidel.

Doukhan, P. (1994) *Mixing. Properties and Examples*, Lecture Notes in Statist. 85. New York: Springer-Verlag.

Fort, G. and Moulines, E. (2000) V-subgeometric ergodicity for a Hasting–Metropolis algorithm. *Statist. Probab. Lett.*, **49**, 401–410.

Girsanov, I.V. (1960) Strongly-Feller processes. I. General properties. *Theory Probab. Appl.*, **5**, 5–24.

Halgreen, C. (1979) Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **47**, 13–17.

Jacod, J. (1985) Grossissement de filtration et processus d'Ornstein–Uhlenbeck généralisé. In T. Jenlin and M. Yor (eds), *Grossissements de Filtrations: Exemples et Applications*, Lecture Notes in Math. 1118, pp. 36–44. Berlin: Springer-Verlag.

Jørgensen, B. (1982) *Statistical Properties of the Generalized Inverse Gaussian Distribution*, Lecture Notes in Statist. 9. New York: Springer-Verlag.

Jurek, Z.J. and Mason, J.D. (1993) *Operator-Limit Distributions in Probability Theory.* New York: Wiley.

Kwon, Y. and Lee, C. (1999) Strong Feller property and irreducibility of diffusions with jumps. *Stochastics Stochastics Rep.*, **67**, 147–157.

Lin, T.F. (1989) Some properties on strong Feller semigroups on L^2-spaces. *Semigroup Forum*, **39**, 95–103.

Meyn, S.P. and Tweedie, R.L. (1992) Stability of Markovian processes. I. Criteria for discrete-time chains. *Adv. Appl. Probab.*, **24**, 542–574.

Nummelin, E. and Tuominen, P. (1982) Geometric ergodicity of Harris recurrent Markov chains with application to renewal theory. *Stochastic Process. Appl.*, **12**, 187–202.

Picard, J. (1996) On the existence of smooth densities for jump processes. *Probab. Theory Related Fields*, **105**, 481–511.

Prause, K. (1998) The generalized hyperbolic models: estimation, financial derivatives and risk measurement. Doctoral thesis, Faculty of Mathematics, University of Freiburg.

Sato, K. (1987) Strictly operator-stable distributions. *J. Multivariate Anal.*, **22**, 278–295.

Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Sato, K. (2001) Subordination and selfdecomposability. *Statist. Probab. Lett.*, **54**, 317–324.

Sato, K. and Yamazato, M. (1983) Stationary processes of Ornstein–Uhlenbeck type. In K. Ito and J.V. Prokhorov (eds), *Probability Theory and Mathematical Statistics: Proceedings of the Fourth USSR–Japan Symposium*, Lecture Notes in Math., 1021, pp. 541–551. Berlin: Springer-Verlag.

Sato, K. and Yamazato, M. (1984) Operator-self-decomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. *Stochastic Process. Appl.*, **17**, 73–100.

Shanbhag, D.N. and Sreehari, M. (1979) An extension of Goldie's result and further results in infinite divisibility. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **47**, 19–25.

Sharpe, M. (1969) Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.*, **136**, 51–65.

Tuominen, P. and Tweedie, R.L. (1979) Exponential decay and ergodicity of general Markov processes and their discrete skeletons. *Adv. Appl. Probab.*, **11**, 784–803.

Tuominen, P. and Tweedie, R.L. (1994) Subgeometric rates of convergence of f-ergodic Markov chains. *Adv. Appl. Probab.*, **26**, 775–798.

Tweedie, R.L. (1983) Criteria for rates of convergence of Markov chains with application to queueing and storage theory. In J.F.C. Kingman and G.E.H. Reuter (eds), *Probability, Statistics and Analysis*, London Math. Soc. Lecture Note Ser. 79. Cambridge: Cambridge University Press.

Veretennikov, A.Yu. (1997) On polynomial mixing bounds for stochastic differential equations. *Stochastic Process. Appl.*, **70**, 115–127.

Veretennikov, A.Yu. (1999) On polynomial mixing and convergence rate for stochastic difference and differential equations. *Theory Probab. Appl.*, **44**, 361–374.

Wolfe, S.J. (1982) On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. *Stochastic Process. Appl.*, **12**, 301–312.

Yamazato, M. (1983) Absolute continuity of operator-selfdecomposable distributions on $\mathbb{R}^d$. *J. Multivariate Anal.*, **13**, 550–560.

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