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Explicit formulae for time-space Brownian chaos

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Let F be a square-integrable and infinitely weakly differentiable functional of a standard Brownian motion X: we show that the nth integrand in the time-space chaotic decomposition of F has the form $\mathbb{E}(\alpha_{(n)} D^n F | X_{t_1}, \ldots, X_{t_n})$, where $\alpha_{(n)}$ is a transform of Hardy type and D^n denotes the nth derivative operator. In this way, we complete the results of previous papers, and provide a time-space counterpart to the classic Stroock formulae for Wiener chaos. Our main tool is an extension of the Clark–Ocone formula in the context of initially enlarged filtrations. We discuss an application to the static hedging of path-dependent options in a continuous-time financial model driven by X. A formal connection between our results and the orthogonal decomposition of the space of square-integrable functionals of a standard Brownian bridge – as proved by Gosselin and Wurzbacher – is also established.

Keywords: Brownian bridge; Brownian motion; Clark–Ocone formula; enlargement of filtrations; Hardy operators; static hedging; Stroock's formula; time-space chaos

1. Introduction

Consider the canonical space $(C_{[0,1]}, \mathcal{C}, \mathbb{P})$, where $\mathbb{P}$ is the law of a real-valued standard Brownian motion initialized at zero, write

$$X := \{X_t : t \in [0, 1]\}$$

for the coordinate process, and denote by $L^2(X)$ the space of square-integrable functionals of X. The aim of this paper is to obtain explicit formulae for the *time-space chaotic decomposition* of $L^2(X)$ first described in Peccati (2001a). Indeed, denote by Δ^n the simplex contained in $[0, 1]^n$, and by $\{X_t^{(u)}, t \in [0, u]\}$, for fixed $u \in (0, 1]$, the martingale part of X, regarded as a semimartingale with respect to the enlarged filtration

$$\sigma(X_v, v \leqslant t) \vee \sigma(X_v, v \geqslant u), \qquad t \in [0, u]$$

(observe that $X^{(u)}$ is again a standard Brownian motion; see, for example, Jeulin and Yor 1979). In Peccati (2001a) we showed that, for every n and for every deterministic function $\psi(u_1, x_1; \ldots; u_n, x_n)$ on $\Delta^n \times \mathbb{R}^n$ such that

$$\int_{\Delta^n} \mathbb{E}(\psi(u_1, X_{u_1}; \ldots; u_n, X_{u_n})^2) \mathrm{d}u_n \ldots \mathrm{d}u_1 < +\infty, \tag{1}$$

multiple stochastic integrals of the type

$$\int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} \psi(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) \mathrm{d}X_{u_n}^{(u_{n-1})} \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \ldots \mathrm{d}X_{u_1} \tag{2}$$

are well defined as L^2 limits of standard, iterated stochastic integrals of progressive processes with respect to X (they are called *time-space multiple integrals* of nth order). Moreover, they provide an orthogonal decomposition of $L^2(X)$. As a matter of fact, if one defines, for every $n \geqslant 0$, $\overline{\Pi}_n$ to be the Hilbert subspace generated by the set

$$\Pi_n := \{f(X_{t_1}, \ldots, X_{t_n}); f \text{ Borel measurable and bounded on } \mathbb{R}^n; 0 \leqslant t_1 < \ldots < t_n \leqslant 1\},$$

and K_n to be the collection of random variables of the form (2), with $K_0 = \overline{\Pi}_0 = \mathbb{R}$, then $L^2(X)$ coincides with the Hilbert space generated by the union of the $\overline{\Pi}_n$, and also (see Peccati 2001a, Theorem 1)

$$\overline{\Pi}_n = \bigoplus_{i=0}^{n} K_i, \tag{3}$$

where $\bigoplus$ indicates an orthogonal summation, so that $K_n = \overline{\Pi}_n \ominus \overline{\Pi}_{n-1}$, $n \geqslant 1$, and therefore $L^2(X) = \bigoplus_n K_n$.

It follows that for every $F \in L^2(X)$ there exists a unique sequence of functions $\{\psi_n^F, n \geqslant 1\}$, defined on $\mathbb{R}^n \times \Delta^n$ for every n and satisfying condition (1), such that

$$F = \mathbb{E}(F) + \sum_{n \geqslant 1} \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} \psi_n^F(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) \mathrm{d}X_{u_n}^{(u_{n-1})} \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \ldots \mathrm{d}X_{u_1} \tag{4}$$

where the summation converges in $L^2(X)$. By analogy with *Wiener chaos* – see Wiener (1930) for the original result, as well as McKean (1973) and Stroock (1987) for more recent presentations – such a decomposition is named *time-space chaotic* (TSCD). This result yields, in particular, a unitary isomorphism between $L^2(X)$ and an appropriate restriction of the free Fock space over $L^2(\mu_1) := L^2([0, 1] \times \mathbb{R}, \mu_1)$, where

$$\mu_1(\mathrm{d}s, \mathrm{d}x) := \mathrm{d}s \mathbb{P}(X_s \in \mathrm{d}x)$$

(to prove such a claim, just use the isometric properties of stochastic integrals to calculate the covariance of a pair of random variables such as (2), and then carry out the standard change of variables $v_m = u_m$, $v_i = u_i - u_{i+1}$, $i = 1, \ldots, m-1$; it is also straightforward to check that the right restriction of the Fock space is given by

$$\mathbb{R} \oplus \bigoplus_{n \geqslant 1} L^2(\Gamma^n \times \mathbb{R}^n, \mu_1^{\otimes n})$$

where Γ^n is the translation of the set Δ^n by means of the above change of variables). To further justify our terminology, we recall our proof in Corollary 5 of Peccati (2001a) that $L^2(\mu_1)$ is generated by time-space harmonic functions.

The main achievement of the subsequent sections is the explicit construction, for every n, of a bounded Hardy operator, denoted by $\alpha_{(n)}$, from $L^2([0, 1]^n, \mathrm{d}u_1 \ldots \mathrm{d}u_n)$ to itself and such that the following result holds:

Theorem 1. *Let $F \in L^2(X)$ be infinitely differentiable in the sense of Shigekawa–Malliavin,* *with $D^n F$ denoting its nth derivative process for every n. Then the integrands $\{\psi_n^F,\ n \geqslant 1\}$ in the TSCD* (4) *of F are such that*

$$\psi_n^F(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) = \mathbb{E}(\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | X_{u_1}, \ldots, X_{u_n}) \tag{5}$$

for every $n \geqslant 1$.

Of course, since $D^n F$ is a member of $L^2([0, 1]^n \times C_{[0,1]}, \mathrm{d}u_1 \ldots \mathrm{d}u_n \otimes \mathrm{d}\mathbb{P})$, the symbol $\alpha_{(n)} D^n F$ has a precise meaning only on the set Ω' of probability 1 defined as

$$\Omega' := \{\omega \in C_{[0,1]} : D^n F(\omega) \in L^2([0, 1]^n, \mathrm{d}u_1 \ldots \mathrm{d}u_n)\}$$

and, for ω outside Ω', $\alpha_{(n)} D^n F(\omega)$ is set equal to zero by definition. Analogous conventions are used on many occasions throughout the paper, and we will henceforth no longer mention them explicitly.

A little inspection shows that Theorem 1 gives an exhaustive description of the integrands ψ_n^F for any $F \in L^2(X)$. As a matter of fact, define as below (see Section 4) $\mathbb{D}^n(X)$ ($\mathbb{D}^\infty(X)$) to be the space of n times (infinitely) differentiable, square-integrable functionals of X, and set

$$L^2(\mu_n) := L^2(\Delta^n \times \mathbb{R}^n, \mu_n),$$

where

$$\mu_n(\mathrm{d}u_1, \ldots, \mathrm{d}u_n; \mathrm{d}x_1, \ldots, \mathrm{d}x_n) := \mathrm{d}u_1 \ldots \mathrm{d}u_n \mathbb{P}(X_{u_1} \in \mathrm{d}x_1, \ldots, X_{u_n} \in \mathrm{d}x_n),$$

so that one can immediately verify that the operator, from $L^2(X)$ to $L^2(\mu_n)$, given by

$$F \mapsto \psi_n^F \tag{6}$$

is onto and continuous. Now observe that $\mathbb{D}^\infty(X)$ (and therefore $\mathbb{D}^n(X)$ for every n) is dense in $L^2(X)$, thus implying that the application in (6) is, for every n, the closure of the operator, from $\mathbb{D}^n(X)$ (endowed with the norm $\|.\|_{L^2(X)}$) to $L^2(\mu_n)$, defined as

$$F \mapsto \mathbb{E}(\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | X_{u_1} = x_1, \ldots, X_{u_n} = x_n).$$

We will show in Corollaries 9 and 10 below that, for F smooth, formula (5) translates into a very simple expression. It is also worth noting that the form of the integrands ψ_n^F in (5) parallels the well-known *Stroock formula* for Wiener chaos, proved in Stroock (1987) (and indeed valid in a more general context). For future reference, and for the sake of completeness, we present a version of such a result that is appropriate to our setting.

Theorem 2 (Stroock's formula). *Every $F \in \mathbb{D}^\infty(X)$ admits the (Wiener) chaotic decomposition*

$$F = \mathbb{E}(F) + \sum_{n \geqslant 1} \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} \phi_n^F(u_1, \ldots, u_n) \mathrm{d}X_{u_n} \mathrm{d}X_{u_{n-1}} \ldots \mathrm{d}X_{u_1},$$

where

$$\phi_n^F(u_1, \ldots, u_n) = \mathbb{E}(D^n_{u_1,\ldots,u_n} F) \tag{7}$$

for every $n \geqslant 1$.

As discussed in the next section, the initial impetus for the present investigation was provided by financial mathematics. More specifically, in Peccati (2001d) we studied (in a spirit close to that of Carr *et al.* 1998) the problem of static hedging of path-dependent options by means of simpler contingent claims (for instance, calls and puts). We pointed out that a proper use of the concept of time-space chaos, and of formula (5), may give an explicit representation of the *intrinsic risk* related to such static strategies. More details are given below; the reader is also referred to Peccati (2001b; 2002) and Peccati and Yor (2001) for further relations between Hardy operators, time-space chaos and principal values of Brownian local times.

The paper is organized as follows. In Section 2 we describe the links between the content of this paper, static hedging in a continuous-time financial model and the theory of weak Brownian motions (a notion introduced in Föllmer *et al.* 2000). In Section 3 we discuss some simple relations between Hardy operators on $L^2([0, 1], \mathrm{d}t)$ and Brownian motion, and construct the operators $\alpha_{(n)}$ that appear in the statement of Theorem 1. Section 4 is devoted to the proof of Theorem 1, as well as to some useful extension of the *Clark–Ocone formula* (see Clark 1970; Ocone 1984) in the context of initially enlarged filtrations.

We conclude the paper by discussing the connections between Theorem 1 and the chaotic representation result for real-valued Brownian bridges first proved in Gosselin and Wurzbacher (1997), and then extended in Peccati (2001c) to more general Gaussian processes. Indeed, Gosselin and Wurzbacher show that $L^2(X)$ is also spanned by the orthogonal summations of iterated stochastic integrals with respect to the 'adapted' Brownian bridge (of length 1 and from 0 to 0)

$$X_1^{0,0}(t) := \begin{cases} (1-t)\displaystyle\int_0^t \mathrm{d}X_s/(1-s), & 0 \leqslant t < 1, \\ 0 & t = 1, \end{cases} \tag{8}$$

and we shall point out here that the corresponding 'Stroock-type formulae' – i.e. the representation of the integrands in the chaos development as expected values of suitable transformations of the derivative processes – involve the same kind of Hardy operators that appear in Theorem 1; see Peccati (2001c; 2002) for a discussion in a more general framework, but we stress that the present paper is self-contained.

2. Motivations from financial mathematics and links with weak Brownian motions

Suppose that the price of some financial asset A evolves over time according to the equation

$$A_t = A(t, X_t), \qquad t \in [0, 1]$$

where $A(t, x)$ is a measurable and deterministic function from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}_+$. To simplify, we can assume that $A(t, x)$ is invertible in x for every t, and that the interest rate in this economy is constantly equal to zero. For instance, if $A(t, x) = \exp[\sigma x + (\mu - \sigma^2/2)t]$, then A_t represents the price process of the risky asset in a Black–Scholes model with constant volatility σ and drift μ. We call an *option* (or *contingent claim*) any square-integrable functional of the process A_t, so that the class of contingent claims coincides in this case with $L^2(X)$. In Peccati (2001d) we introduced a (partial) classification of options according to their *path-dependence degree* (p.d.d.): more precisely, we say that $H \in L^2(X)$ has p.d.d. of order n $(n \geqslant 1)$, if

(i) it can be *statically approximated* in the L^2 sense by portfolios (i.e. linear combinations) of contingent claims whose payoffs depend on the realizations of the price process A_t in *at most* n instants;
(ii) there is no way to approximate H by means of linear combinations of payoffs that depend on the realizations of A_t in at most $n - 1$ instants.

A standard density argument yields the following Hilbert space equivalent of (i) and (ii): $H \in L^2(X)$ has p.d.d. of order n if and only if (i′) $H \in \overline{\Pi}_n$ and (ii′) $\pi[H, K_n] \neq 0$, where the notation is the same as in the previous section, and $\pi[\cdot, \cdot]$ denotes the usual projection operator. Observe that in a constant volatility Black–Scholes model examples of options with p.d.d. of order one are vanilla European options such as calls and puts, whereas examples of order $n > 1$ are *barrier* and *Asian options* of the type

$$H_{\text{barrier}} = 1_{(\min(A_{t_1}, \ldots, A_{t_{n-1}}) > c)} \phi(A_1) \quad \text{and} \quad H_{\text{asian}} = \left(\frac{1}{n} \sum_{i=1}^{n} A_{t_i} - E \right)_+ \tag{9}$$

where $0 < t_1 < \ldots < t_n \leqslant 1$, $c < A_0$, ϕ is not identically zero and such that $\mathbb{E}(\phi(A_1)^2) < +\infty$ and $E > 0$. Now consider a generic option H^*: we say that an investor implements a *strategy of (purely) static* Π_n*-hedging* of H^* if at time 0 he writes H^* and forms a portfolio P_n of elements of Π_n, and then waits until time 1 without changing position. Observe that investors in real financial markets, due mainly to frictions and transaction costs (and most plausibly over a short period of time), are often forced to realize strategies of purely static hedging, although they are in general not optimal. For instance, a typical strategy of static Π_1-hedging consists of forming a portfolio of European calls and puts to counter the risk of a barrier option such as the first object in (9). Plainly, the terminal wealth of the above investor will be $P_n - H^* + c$, where the real constant c gives the difference between the price of H^* and that of P_n. It is also clear, since Π_n *is not total in* $L^2(X)$ for every n, that for any n there exists an option $H^* \neq 0$ – which can be chosen to be bounded: see Proposition 2 in Peccati (2001d) – such that, for any $P_n \in \overline{\Pi}_n$,

$$\mathbb{P}\{P_n - H^* + c \neq 0\} > 0.$$

It follows that every strategy of static Π_n-hedging of such a contract H^* yields an *intrinsic risk* that cannot be eliminated, whenever the hedger is forced to stay inside the space $\overline{\Pi}_n$. Such a risk can be made explicit by means of the results contained in this work.

More precisely, thanks to time-space chaos and Theorem 1, we are able to represent and quantify the *intrinsic quadratic risk* related to the static Π_n-hedging of H^*, defined as

$$Q_s(\Pi_n, H^*) = \inf_{H\in\overline{\Pi}_n} \mathbb{E}[(H - H^*)^2].$$

As a matter of fact, it is straightforward to show that, for any $H^* \in L^2(X)$,

$$\pi[H^*, \overline{\Pi}_n] = \mathbb{E}(H^*) + \sum_{k=1}^{n} \int_0^1 \cdots \int_0^{u_{k-1}} \psi_k^{H^*}(u_1, X_{u_1}; \ldots; u_k, X_{u_k}) \mathrm{d}X_{u_k}^{(u_{k-1})} \ldots \mathrm{d}X_{u_1}$$

and therefore

$$Q_s(\Pi_n, H^*) = \sum_{k\geqslant n+1} \|\psi_k^{H^*}\|^2_{L^2(\mu_k)},$$

where the $\psi_k^{H^*}$ are the integrands in the TSCD (4) of H^*, and moreover these terms are explicitly given by formula (5) when H^* is a member of $\mathbb{D}^\infty(X)$ (see also Proposition 8 below for an equivalent of Theorem 1 with less stringent assumptions). Note eventually that, for any n and for fixed $(u_1, x_1; \ldots; u_n, x_n) \in \Delta^n \times \mathbb{R}^n$, the quantity

$$\mathbb{E}(\alpha_{(n)} D_{u_1,\ldots,u_n} H^* | X_{u_1} = x_1, \ldots, X_{u_n} = x_n)$$

appearing in the statement of Theorem 1 is in several cases explicitly known. To see this, take a vector $(u_1, x_1; \ldots; u_n, x_n)$ as above and consider the law $\mathbb{P}^{(u_1,x_1;\cdots;u_n,x_n)}$ on $(C_{[0,1]}, \mathcal{C})$ induced by the process Y_t, $t \in [0, 1]$, defined as

$$Y_t = \sum_{i=1}^{n} X^{(x_{i+1},x_i),(u_i-u_{i+1})}(u_{i+1} - t)1_{(u_{i+1},u_i]}(t) + B(u_1 - t)1_{(u_1,1]}(t),$$

where $x_{n+1} = u_{n+1} = 0$, $\{X^{(x_{i+1},x_i),(u_i-u_{i+1})}, i = 1, \ldots, n\}$ is a collection of n independent Brownian bridges each of length $u_i - u_{i+1}$, from x_{i+1} to x_i, and B is a standard Brownian motion initialized at x_n and independent of the n bridges. Now, for every bounded functional Φ of X it is known that the function

$$(u_1, x_1; \ldots; u_n, x_n) \mapsto \mathbb{E}^{(u_1,x_1;\ldots;u_n,x_n)}(\Phi)$$

is measurable as an application from $\Delta^n \times \mathbb{R}^n$ to $\mathbb{R}$, and moreover,

$$\mathbb{E}[\Phi | X_{u_1}, \ldots, X_{u_n}] = \mathbb{E}^{(u_1,X_{u_1};\ldots;u_n,X_{u_n})}(\Phi)$$

$\mathrm{d}\mathbb{P} \otimes \mathrm{d}u_1 \ldots \mathrm{d}u_n$-almost every on $C_{[0,1]} \times \Delta^n$. Such a remark still holds for polynomial or exponential functionals. The reader is referred to Peccati (2001d) for a complete discussion of these topics, and to Lacoste (1996) and Barucci and Mancino (1998) for other applications of (Wiener) chaotic decompositions to financial modelling.

To conclude, it is interesting to note that the non-totality, for every n, of the class Π_n was first proved in Föllmer *et al.* (2000), in the context of the theory of *weak Brownian motions*. More precisely, we say – in the notation of this paper – that a process Y is a weak Brownian motion of order n if its n-dimensional marginal distributions coincide with those of X, although Y is not a Brownian motion. In Föllmer *et al.* (2000) it is then proved that, for every n, there exists an element Φ of $\overline{\Pi}_n^\perp$ that is different from zero and bounded, say, by $1/2$, and therefore that the probability measure

$$\mathbb{Q} := (1 + \Phi) \cdot \mathbb{P}$$

gives precisely the law of a weak Brownian motion of order n. As a matter of fact, with obvious notation and for every bounded function $F(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ and every $(t_1, \ldots, t_n) \in [0, 1]^n$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(F(X_{t_1}, \ldots, X_{t_n})) &= \mathbb{E}((1 + \Phi)F(X_{t_1}, \ldots, X_{t_n})) \\ &= \mathbb{E}(F(X_{t_1}, \ldots, X_{t_n}))\end{aligned}$$

since $\Phi \in \overline{\Pi}_n^\perp$.

In the next section we introduce the objects that are involved in the proof of Theorem 1.

3. Brownian motion and Hardy isomorphisms between Hilbert spaces

For a fixed vector $\mathbf{u} := (u_1, \ldots, u_m)$ such that

$$0 < u_m < \ldots < u_1 \leqslant 1$$

(i.e. such that $\mathbf{u} \in \Delta^m$), we introduce the following notation:

$$T_1 := L^2([0, 1], \mathrm{d}t)$$

$$T^{\mathbf{u}} := \left\{ f \in T_1 : \int_0^{u_j} f(t)\mathrm{d}t = 0, \forall j = 1, \ldots, m \right\}.$$

The scalar product on T_1, hence on $T^{\mathbf{u}}$, is denoted by $\langle \cdot, \cdot \rangle$ ($\|\cdot\|$ denotes the norm): observe that, if $m = 1$ and $\mathbf{u} \equiv u \in (0, 1]$, $T^{\mathbf{u}}$ coincides with the space of square-integrable functions on $[0, 1]$, such that $\int_0^u f(x)\mathrm{d}x = 0$.

To obtain the objects that are involved in the proof of Theorem 1, we start by introducing, for every $f \in T_1$ and for every $u \in (0, 1]$, the classic Hardy operator

$$H^{(u)}f(x) := \frac{1}{u - x}\int_x^u f(y)\mathrm{d}y.1_{(x<u)}, \qquad x \in [0, 1],$$

along with its adjoint,

$$\tilde{H}^{(u)}f(x) := \int_0^x \frac{f(y)}{u - y}\mathrm{d}y.1_{(x<u)}, \qquad x \in [0, 1].$$

It is well known (see, for example, Hardy *et al.* 1934) that, for every $u \in (0, 1]$, $H^{(u)}$ and $\tilde{H}^{(u)}$ are bounded operators from T_1 to itself, and moreover one has, for every $f \in T_1$, the celebrated *Hardy inequalities*

$$\|H^{(u)}f\| \leqslant 2\|f1_{[0,u]}\| \leqslant 2\|f\|,$$

$$\|\tilde{H}^{(u)}f\| \leqslant 2\|f1_{[0,u]}\| \leqslant 2\|f\|.$$

Now introduce the linear operators, defined for $f \in T_1$ and for a fixed $u \in (0, 1]$,

$$\alpha^{(u)}f(x) := f(x) - H^{(u)}f(x), \qquad x \in [0, 1], \tag{10}$$

$$\beta^{(u)}f(x) := f(x) - \tilde{H}^{(u)}f(x), \qquad x \in [0, 1],$$

as well as the following ones, defined for $f \in T_1$ and for a fixed $\mathbf{u} = (u_1, \ldots, u_m) \in \Delta^m$,

$$\alpha^{(\mathbf{u})}f(x) := \sum_{j=1}^{m} 1_{(u_{j+1},u_j]}(x)\alpha^{(u_j)}[f1_{(u_{j+1},u_j]}](x) + f(x)1_{(u_1,1]}(x), \qquad x \in [0, 1],$$

$$\beta^{(\mathbf{u})}f(x) := \sum_{j=1}^{m} 1_{(u_{j+1},u_j]}(x)\beta^{(u_j)}[f1_{(u_{j+1},u_j]}](x) + f(x)1_{(u_1,1]}(x), \qquad x \in [0, 1], \tag{11}$$

$$\eta^{(\mathbf{u})}f(x) := f(x) - \sum_{j=1}^{m}(u_j - u_{j+1})^{-1}\int_{u_{j+1}}^{u_j} f(y)\mathrm{d}y.1_{(u_{j+1},u_j]}(x), \qquad x \in [0, 1],$$

where we adopt (here, and for the rest of the paper) the convention $u_{m+1} := 0$.

Remark. The three objects defined in (11) are not to be confused with the operators $\alpha^{(\mathbf{u},n)}$, $\beta^{(\mathbf{u},n)}$, $\eta^{(\mathbf{u},n)}$ introduced in formula (8) of Peccati (2001c). As a matter of fact, the latter are bounded operators from $L^2([0, 1]^n, \mathrm{d}t_1 \ldots \mathrm{d}t_n)$ to itself, and are used in that paper to study the chaotic representation properties of general conditioned Gaussian measures.

The following result describes some reciprocal relations between $\alpha^{(\mathbf{u})}$, $\beta^{(\mathbf{u})}$ and $\eta^{(\mathbf{u})}$.

Proposition 3. *Let* $\mathbf{u}$ *be a fixed element of* Δ^m*, and let* $\alpha^{(\mathbf{u})}$*,* $\beta^{(\mathbf{u})}$ *and* $\eta^{(\mathbf{u})}$ *be defined as in* (11)*. Then*

(i) *the operator* $\eta^{(\mathbf{u})}$ *is a function from* T_1 *to* $T^{\mathbf{u}}$*, and the restriction of* $\eta^{(\mathbf{u})}$ *to* $T^{\mathbf{u}}$ *coincides with the identity operator;*
(ii) *the operators* $\alpha^{(\mathbf{u})}$ *and* $\beta^{(\mathbf{u})}$ *are mutually inverse unitary isomorphisms from* $T^{\mathbf{u}}$ *to* T_1 *and from* T_1 *to* $T^{\mathbf{u}}$*, respectively;*
(iii) *for every* $f \in T_1$*,*

$$\alpha^{(\mathbf{u})}\eta^{(\mathbf{u})}f = \alpha^{(\mathbf{u})}f$$

and therefore, for $f \in T_1$*,*

$$\beta^{(\mathbf{u})}\alpha^{(\mathbf{u})}f = \eta^{(\mathbf{u})}f.$$

Proof. Part (i) is an easy consequence of the definition of the $\eta^{(\mathbf{u})}$. To prove (ii), start with

$m = 1$. In this case, the fact that $\alpha^{(u)}$ and $\beta^{(u)}$ are mutually inverse unitary isomorphisms can be easily verified using integration by parts. For the general case, just use the two properties

$$1_{(u_{j+1},u_j]}\beta^{(u_j)}[f1_{(u_{j+1},u_j]}] = \beta^{(u_j)}[f1_{(u_{j+1},u_j]}], \qquad \forall f \in T_1, \forall j = 1, \ldots, m,$$

$$1_{(u_{j+1},u_j]}\alpha^{(u_j)}[f1_{(u_{j+1},u_j]}] = \alpha^{(u_j)}[f1_{(u_{j+1},u_j]}], \qquad \forall f \in T^{\mathbf{u}}, \forall j = 1, \ldots, m.$$

Finally, (iii) is a consequence of the equation, valid for every $j = 1, \ldots, m$,

$$1_{(u_{j+1},u_j]}\alpha^{(u_j)}[1_{(u_{j+1},u_j]}] = 0.$$

□

In what follows, we will write $X(f)$, $f \in T_1$, to indicate the stochastic integral $\int_0^1 f(s)\mathrm{d}X_s$: in other words, the class

$$\{X(f), f \in T_1\}$$

is the centred Gaussian family generated by X (also called the *first Wiener chaos*). Proposition 3 allows us then to introduce the following Gaussian family:

$$\begin{aligned} X^{(\mathbf{u})} &= \{X^{(\mathbf{u})}(f), f \in T_1\} \\ &:= \{X(\beta^{(\mathbf{u})}f), f \in T_1\}. \end{aligned}$$

It is easy to prove – as a consequence of Proposition 3 – that the process

$$X_t^{(\mathbf{u})} := X^{(\mathbf{u})}(1_{[0,t]}) = X_t - \sum_{j=1}^m \int_{u_{j+1}}^{t\wedge u_j} \frac{X_{u_j} - X_s}{u_j - s}\,\mathrm{d}s.1_{(t>u_{j+1})}, \tag{12}$$

where the second equality is a consequence of a stochastic Fubini theorem such as the one discussed in Stricker and Yor (1978), is a standard Brownian motion on [0, 1], with respect to the enlarged filtration

$$\mathcal{G}_t^{(u_1,\ldots,u_m)} = \mathcal{G}_t^{(\mathbf{u})} := \mathcal{F}_t(X) \vee \sigma(X_{u_1}, \ldots, X_{u_m}) = \mathcal{F}_t(X^{(\mathbf{u})}) \vee \sigma(X_{u_1}, \ldots, X_{u_m}), \tag{13}$$

where $\mathcal{F}_t(X^{(\mathbf{u})})$ and $\mathcal{F}_t(X)$ are the natural filtrations of $X_t^{(\mathbf{u})}$ and X_t respectively, completed with the $\mathbb{P}$-negligible sets of $\mathcal{C}$, and, in particular, $X_t^{(\mathbf{u})}$ is independent of $\sigma(X_{u_1}, \ldots, X_{u_m})$ – to show that $X_t^{(\mathbf{u})}$ is a $\mathcal{G}_t^{(\mathbf{u})}$-Brownian motion, one can also start with (12) and then use arguments similar to those of Jeulin and Yor (1979). Note also that, for every $j = 1, \ldots, m$, $X_t^{(\mathbf{u})}$ is a Brownian motion on the interval $[0, u_j]$, with respect to the filtration

$$\mathcal{F}_t(X) \vee \sigma(X_{u_{j+1}}, \ldots, X_{u_m}) \vee \sigma(X_v, v \geqslant u_j).$$

Moreover, for a given $k \geqslant 1$, fix two vectors $\mathbf{u} = (u_1, \ldots, u_m) \in \Delta^m$ and $\mathbf{v} = (v_1, \ldots, v_k) \in \Delta^k$ such that $v_1 < u_m$ and set $\mathbf{u} \vee \mathbf{v} := (u_1, \ldots, u_m, v_1, \ldots, v_k) \in \Delta^{m+k}$. Then (12) implies

$$X_t^{(\mathbf{u}\vee\mathbf{v})} = X_t^{(\mathbf{v})}, \qquad t \in [0, v_1]. \tag{14}$$

Of particular interest for our discussion is the case $m = 1$, $\mathbf{u} = u \in (0, 1]$. The process

$$X_t^{(u)} = X_t - \int_0^{t\wedge u} \frac{X_u - X_s}{u - s}\,\mathrm{d}s, \tag{15}$$

is indeed, for every u, a $\mathcal{G}_t^{(u)} = \mathcal{F}_t(X) \vee \sigma(X_u)$ Brownian motion on $[0, 1]$ and an $\mathcal{F}_t(X) \vee \sigma(X_v, v \geqslant u)$ Brownian motion on $[0, u]$, and appears in the expression of multiple time-space integrals such as (2). Its relations with the Hardy operators in (10) will play a vital role in the proof of Theorem 1 (see Section 4).

We conclude this section by constructing the operator $\alpha_{(n)}$ involved in the statement of Theorem 1. To do this, introduce the notation

$$L^2([0, 1]^n, \mathrm{d}t_1 \ldots \mathrm{d}t_n) := T_n, \qquad n \geqslant 2,$$

consider again the operator $\alpha^{(u)}$ appearing in (10), and extend its definition to any n and to any $f \in T_n$ in the following way. We may always assume that the application

$$s \mapsto f_{(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n)}(s) := f(t_1, \ldots, t_{k-1}, s, t_{k+1} \ldots, t_n)$$

is an element of T_1 for every $k \in \{1, \ldots, n\}$ and for $\mathrm{d}\lambda_{n-1}$-a.e. $(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$ in $[0, 1]^{n-1}$, where λ_{n-1} indicates Lebesgue measure. As a consequence, we can fix a vector $(u_1, \ldots, u_n) \in (0, 1]^n$ and define, for every $k \in \{1, \ldots, n\}$, for every $(t_1, \ldots, t_n) \in [0, 1]^n$ and every $f \in T_n$,

$$\alpha_k^{(u_k)} f(t_1, \ldots, t_n) := [\alpha^{(u_k)} f_{(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n)}(t_k)] 1_{(f_{(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n)} \in T_1)}.$$

It is clear that, for every $f \in T_n$, $\alpha_k^{(u_k)} f$ is in T_n, so that it is meaningful to define, for $(u_1, \ldots, u_n) \in (0, 1]^n$ fixed as above, the operator, from T_n to itself,

$$\alpha^{[u_1,\ldots,u_n]} f := \alpha_1^{(u_1)} \left[\alpha_2^{(u_2)} \ldots \left(\alpha_n^{(u_n)} f\right)\right],$$

and eventually the bounded operator, from T_n to itself,

$$\alpha_{(n)} f(t_1, \ldots, t_n) := \alpha^{[0,t_1,\ldots,t_{n-1}]} f(t_1, \ldots, t_n).$$

Remark. In Peccati (2001c, 2001d), the operator $\alpha^{[u_1,\ldots,u_n]}$ is written $\alpha^{(u_1,\ldots,u_n)}$: this minor change has been made to avoid confusion with the operator $\alpha^{(\mathbf{u})}$ appearing in (11).

4. Proof of the main results

We define, for every n, the (probability) measure on $\left(C_{[0,1]} \times [0, 1]^n, \mathcal{C} \otimes \mathcal{B}([0, 1]^n)\right)$

$$\nu_n(\mathrm{d}\omega; \mathrm{d}u_1, \ldots, \mathrm{d}u_n) := \mathbb{P}(\mathrm{d}\omega) \otimes \mathrm{d}u_1, \ldots, \mathrm{d}u_n$$

and consider the application

$$\begin{aligned} X_{(n)} &: \Omega \times [0, 1]^n \mapsto [0, 1]^n \times \mathbb{R}^n \\ &: (\omega; u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n; X_{u_1}(\omega), \ldots, X_{u_n}(\omega)). \end{aligned}$$

In the following – as well as in the statement of Theorem 1 – given a measurable process $f(\omega; u_1, \ldots, u_n)$ we write, by a slight abuse of notation,

$$\mathbb{E}(f(u_1, \ldots, u_n)|X_{u_1}(\omega), \ldots, X_{u_n}(\omega)) := \nu_n(f|X_{(n)}). \tag{16}$$

Since we will deal with predictable projection operators, we shall first show that the filtration

$$\{\mathcal{G}_t^{(\mathbf{u})}, t \in [0, 1]\},$$

as defined in (13), satisfies the usual conditions.

Proposition 4. *Let the above notation prevail, and fix* $\mathbf{u} \in \Delta^m$. *Then the filtration* $\mathcal{G}_t^{(\mathbf{u})}$ *is* $\mathbb{P}$*-complete and right-continuous.*

Proof. Only the right-continuity is to be proved: to do this, consider for every t the class

$$\mathcal{H}_t^{(\mathbf{u})} := \sigma(X_{(t+s)\wedge 1}^{(\mathbf{u})} - X_t^{(\mathbf{u})}, s \geqslant 0);$$

it is clear that for every $\varepsilon > 0$, $\mathcal{H}_{t+\varepsilon}^{(\mathbf{u})}$ is independent of $\mathcal{G}_{t+\varepsilon}^{(\mathbf{u})}$ and therefore it is independent of

$$\mathcal{G}_{t+}^{(\mathbf{u})} := \bigcap_{\varepsilon} \mathcal{G}_{t+\varepsilon}^{(\mathbf{u})};$$

moreover, since the family $\mathcal{H}_t^{(\mathbf{u})}$ increases as t decreases and, $\mathbb{P}$-almost surely,

$$X_{(t+s)\wedge 1}^{(\mathbf{u})} - X_t^{(\mathbf{u})} = \lim_{\varepsilon \downarrow 0} \left(X_{(t+\varepsilon+s)\wedge 1}^{(\mathbf{u})} - X_{t+\varepsilon}^{(\mathbf{u})} \right)$$

we have

$$\mathcal{H}_t^{(\mathbf{u})} = \bigvee_{\varepsilon > 0} \mathcal{H}_{t+\varepsilon}^{(\mathbf{u})},$$

and therefore $\mathcal{H}_t^{(\mathbf{u})}$ is independent of $\mathcal{G}_{t+}^{(\mathbf{u})}$. Now take three bounded random variables: $F \in \mathcal{G}_t^{(\mathbf{u})}$, $G \in \mathcal{G}_{t+}^{(\mathbf{u})}$ and $H \in \mathcal{H}_t^{(\mathbf{u})}$. Then

$$\begin{aligned} \mathbb{E}(FGH) &= \mathbb{E}(FG)\mathbb{E}(H) \\ &= \mathbb{E}\big(F\mathbb{E}\big(G|\mathcal{G}_t^{(\mathbf{u})}\big)\big)\mathbb{E}(H) \\ &= \mathbb{E}\big(FH\mathbb{E}\big(G|\mathcal{G}_t^{(\mathbf{u})}\big)\big), \end{aligned}$$

and, since $\mathcal{G}_t^{(\mathbf{u})} \vee \mathcal{H}_t^{(\mathbf{u})} = \mathcal{C}$, this implies that every bounded $\mathcal{G}_{t+}^{(\mathbf{u})}$-measurable random variable must equal $\mathbb{P}$-a.s. a $\mathcal{G}_t^{(\mathbf{u})}$-measurable functional. As the filtrations we consider are complete by construction, this gives the desired result. □

Remark. One can easily verify that $\mathcal{G}_t^{(\mathbf{u})}$ is also left-continuous.

Given a standard Brownian motion Y, we denote by $L^2(Y)$ the space of its square-integrable functionals, and we define the collection of differentiable functionals of Y in the following (standard) way: as in Section 1.2 of Nualart (1995), we introduce the nth derivative operator, $n \geqslant 1$, on the class $\mathcal{S}(Y)$ of smooth functionals of Y, which is indeed an application from $\mathcal{S}(Y)$ to $L^2(C_{[0,1]} \times [0, 1]^n, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t_1 \ldots \mathrm{d}t_n)$; then, denoting by $D^n F(Y)$ the nth derivative process of a given $F \in \mathcal{S}(Y)$, we define, for every p, the class $\mathbb{D}^{k,p}(Y)$ as the completion of $\mathcal{S}(Y)$ with respect to the seminorm

$$\|F\|_{k,p} := \left[\mathbb{E}(|F|^p) + \sum_{j=1}^{k} \mathbb{E}\left(\|D^j F\|^p_{L^2([0,1]^j, \mathrm{d}t_1 \ldots \mathrm{d}t_j)} \right) \right]^{1/p}. \tag{17}$$

In what follows, we are exclusively interested in the case $p = 2$, and we will use the symbol $\mathbb{D}^k(Y)$ instead of $\mathbb{D}^{k,2}(Y)$ to simplify the notation. If $F \in \mathbb{D}^k(Y)$ we will say that F is *k times (weakly) differentiable with respect to Y* in the Shigekawa–Malliavin sense – see Nualart (1995) or Ocone (1984) for further details – whereas $D^j F(Y)$, $j = 1, \ldots, k$, will indicate *the jth derivative process of F as a functional of Y.*

Note that in what follows, given a certain operator γ, whether we write $\gamma D_t F(Y)$ or $(\gamma DF(Y))_t$ will depend purely on notational convenience.

For a fixed $\mathbf{u}$, we will now study the classes of weakly differentiable functionals of X and $X^{(\mathbf{u})}$: the reciprocal relations between the two sets are explained in the following:

Proposition 5. *For a fixed* $\mathbf{u} \in \Delta^m$,

$$\mathbb{D}^1(X^{(\mathbf{u})}) \subset \mathbb{D}^1(X),$$

and moreover, for $F \in \mathbb{D}^1(X^{(\mathbf{u})})$,

$$DF(X^{(\mathbf{u})}) = \alpha^{(\mathbf{u})} DF(X).$$

Proof. A random variable F is in $\mathbb{D}^1(X^{(\mathbf{u})})$ if, and only if, there exists a sequence of smooth functionals of the type

$$F_n = f(X^{(\mathbf{u})}(h_1), \ldots, X^{(\mathbf{u})}(h_k))$$

(note that f, the h_i and k may depend, in general, on n) where $h_i \in T_1$, $i = 1, \ldots, k$, and $f \in C_b^\infty(\mathbb{R}^k)$ (the symbol C_b^∞ indicates the class of infinitely differentiable functions, whose partial derivatives of any order are bounded) such that F_n converges to F in $L^2(X^{(\mathbf{u})})$, and the sequence of processes

$$D_t F_n(X^{(\mathbf{u})}) = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} f(X^{(\mathbf{u})}(h_1), \ldots, X^{(\mathbf{u})}(h_k)) h_i(t), \qquad t \in [0, 1]$$

converges to a process $a(t, \omega) := D_t F(X^{(\mathbf{u})})(\omega)$ in the space $L^2(C_{[0,1]} \times [0, 1], \mathrm{d}\mathbb{P} \otimes \mathrm{d}t)$.

Now observe that, for every n, we may write by definition

$$F_n = f(X(\beta^{(\mathbf{u})} h_1), \ldots, X(\beta^{(\mathbf{u})} h_k))$$

so that $F_n \in \mathcal{S}(X) \subset \mathbb{D}^1(X)$ and, more to the point,

$$DF_n(X) = \beta^{(\mathbf{u})} DF_n(X^{(\mathbf{u})}).$$

Since $\beta^{(\mathbf{u})}$ is an isomorphism, the sequence $DF_n(X)$ converges in $L^2(C_{[0,1]} \times [0, 1], \mathrm{d}\mathbb{P} \otimes \mathrm{d}t)$ to $\beta^{(\mathbf{u})} a = \beta^{(\mathbf{u})} DF(X^{(\mathbf{u})})$, and therefore $F \in \mathbb{D}^1(X)$.

The last assertion in the proposition derives from

$$\begin{aligned} \alpha^{(\mathbf{u})} DF(X) &= \alpha^{(\mathbf{u})} \beta^{(\mathbf{u})} DF(X^{(\mathbf{u})}) \\ &= DF(X^{(\mathbf{u})}). \end{aligned}$$

□

The next result provides a version of the well-known *Clark–Ocone formula* that is appropriate to our setting. Note that in the following the symbol

$$^{(p;u_1,\ldots,u_m)}[\cdot] = {}^{(p;\mathbf{u})}[\cdot]$$

denotes the predictable projection operator (see, for example, Elliot 1982) with respect to the filtration $\mathcal{G}_t^{(\mathbf{u})}$, with $\mathbf{u} = (u_1, \ldots, u_m) \in \Delta^m$, whereas

$$\mathbb{D}^\infty(Y) := \bigcap_k \mathbb{D}^k(Y)$$

indicates the class of *infinitely weakly differentiable functionals* of a given Brownian motion Y. We will also need the following result (see Nualart 1995, Corollary 1.5.1): the class, denoted by $\textsc{Pol}(Y)$, of functionals of the form

$$F = q(Y(f_1), \ldots, Y(f_m)),$$

where $f_i \in T_1,\ i = 1, \ldots, m,$ and q is a polynomial of m variables, is dense in $\mathbb{D}^k(Y)$ for every $k \geqslant 1$.

We prove the following proposition:

Proposition 6 (Clark–Ocone formula for enlarged filtrations). *Let* $\mathbf{u}$ *be a fixed element of* Δ^m. *Then every* $F \in \mathbb{D}^1(X)$ *has the representation*

$$F = \mathbb{E}(F | X_{u_1}, \ldots, X_{u_m}) + \int_0^1 {}^{(p,\mathbf{u})}\left[\alpha^{(\mathbf{u})} DF(X)\right]_s \mathrm{d}X_s^{(\mathbf{u})}. \tag{18}$$

Proof. Consider first a functional $F \in \textsc{Pol}(X)$ of the form:

$$F = (X(z_1))^{k_1} \ldots (X(z_n))^{k_n}$$

where $z_i \in T_1,\ i = 1, \ldots, n,$ and the k_i are natural numbers. As

$$X(z_i) = X(\eta^{(\mathbf{u})} z_i) + X\left(\sum_{j=1}^m c(j, z_i) 1_{(u_{j+1}, u_j]}\right),$$

where $c(j, z) := (u_j - u_{j+1})^{-1} \int_{u_{j+1}}^{u_j} z(t)\mathrm{d}t$, we may write, thanks to the binomial formula,

$$(X(z_i))^{k_i} = \sum_{l=0}^{k_i} \binom{k_i}{l} \left[X\left(\eta^{(\mathbf{u})} z_i\right)\right]^l \left[X\left(\sum_{j=1}^{m} c(j, z_i) 1_{(u_{j+1}, u_j]}\right)\right]^{k_i - l},$$

and this implies that F can be represented as a linear combination of functionals of the type

$$\begin{aligned} H &= q(X_{u_m}, X_{u_{m-1}} - X_{u_m}, \ldots, X_{u_1} - X_{u_2}) \prod_{i=1}^{n} \left[X\left(\eta^{(\mathbf{u})} z_i\right)\right]^{\gamma_i} \\ &:= Q^{(\mathbf{u})} \times H^{(\mathbf{u})} \end{aligned} \tag{19}$$

where the γ_i are natural numbers, $q(\cdot)$ is a polynomial of m variables, and

$$\begin{aligned} H^{(\mathbf{u})} &:= \prod_{i=1}^{n} \left[X\left(\eta^{(\mathbf{u})} z_i\right)\right]^{\gamma_i}, \\ Q^{(\mathbf{u})} &:= q(X_{u_m}, X_{u_{m-1}} - X_{u_m}, \ldots, X_{u_1} - X_{u_2}). \end{aligned} \tag{20}$$

Since, according to Proposition 3,

$$\eta^{(\mathbf{u})} z_i = \beta^{(\mathbf{u})} \alpha^{(\mathbf{u})} z_i,$$

we also have that

$$H^{(\mathbf{u})} = \prod_{i=1}^{n} [X^{(\mathbf{u})}(\alpha^{(\mathbf{u})} z_i)]^{\gamma_i},$$

so that $H^{(\mathbf{u})} \in \mathbb{D}^1(X^{(\mathbf{u})})$, and therefore, according to the Clark–Ocone formula, in the version of Theorem 3.1 and Corollary 3.2 of Ocone (1984), and to Proposition 5 above, we have

$$H = Q^{(\mathbf{u})} \mathbb{E}(H^{(\mathbf{u})}) + Q^{(\mathbf{u})} \int_0^1 \mathbb{E}[(\alpha^{(\mathbf{u})} D H^{(\mathbf{u})}(X))_s | \mathcal{F}_s(X^{(\mathbf{u})})] \mathrm{d}X_s^{(\mathbf{u})}.$$

Now independence, as well as the fact that $X^{(\mathbf{u})}$ is a $\mathcal{G}^{(\mathbf{u})}$ martingale and $\alpha^{(\mathbf{u})}$ a linear operator, yields

$$Q^{(\mathbf{u})} \mathbb{E}(H^{(\mathbf{u})}) = \mathbb{E}(Q^{(\mathbf{u})} H^{(\mathbf{u})} | X_{u_1}, \ldots, X_{u_m})$$

and

$$\begin{aligned} Q^{(\mathbf{u})} \int_0^1 \mathbb{E}[(\alpha^{(\mathbf{u})} D H^{(\mathbf{u})}(X))_s | \mathcal{F}_s(X^{(u)})] \mathrm{d}X_s^{(\mathbf{u})} &= \int_0^1 Q^{(\mathbf{u})} \mathbb{E}[(\alpha^{(\mathbf{u})} D H^{(\mathbf{u})}(X))_s | \mathcal{F}_s(X^{(\mathbf{u})})] \mathrm{d}X_s^{(\mathbf{u})} \\ &= \int_0^1 \mathbb{E}[(\alpha^{(\mathbf{u})} (Q^{(\mathbf{u})} D H^{(\mathbf{u})}(X)))_s | \mathcal{G}_s^{(\mathbf{u})}] \mathrm{d}X_s^{(\mathbf{u})}. \end{aligned}$$

Observe also that $Q^{(\mathbf{u})}$, $H^{(\mathbf{u})}$ and $Q^{(\mathbf{u})} H^{(\mathbf{u})}$ are all elements of $\mathcal{S}(X)$ and that consequently

$$
\begin{aligned}
DH(X) &= D\big(Q^{(\mathbf{u})}H^{(\mathbf{u})}\big)(X) \\
&= Q^{(\mathbf{u})}DH^{(\mathbf{u})}(X) + H^{(\mathbf{u})}\sum_{j=1}^{m}\frac{\partial}{\partial x_{m-j+1}}q(X_{u_m}, X_{u_{m-1}} - X_{u_m}, \ldots, X_{u_1} - X_{u_2})1_{(u_{j+1},u_j]}.
\end{aligned}
$$

Moreover, since for $j = 1, \ldots, m$,

$$\alpha^{(u_j)}(1_{(u_{j+1},u_j]})1_{(u_{j+1},u_j]} = 0$$

we have

$$1_{(u_{j+1},u_j]}\alpha^{(u_j)}[Q^{(\mathbf{u})}DH^{(\mathbf{u})}(X)1_{(u_{j+1},u_j]}] = 1_{(u_{j+1},u_j]}\alpha^{(u_j)}[DH(X)1_{(u_{j+1},u_j]}]$$

so that, by linearity,

$$\alpha^{(\mathbf{u})}(Q^{(\mathbf{u})}DH^{(\mathbf{u})}(X)) = \alpha^{(\mathbf{u})}DH(X)$$

and (18) holds for F and for an arbitrary element of $\textsc{Pol}(X)$: as a matter of fact, one can easily show that, for $F \in \textsc{Pol}(X)$, a version of ${}^{(p;\mathbf{u})}[(\alpha^{(\mathbf{u})}DF(X))]$ is given by

$$\{\mathbb{E}[(\alpha^{(\mathbf{u})}DF(X))_s|\mathcal{G}_s^{(\mathbf{u})}],\ \ s \in [0, 1]\}$$

For the general case, consider a sequence of functionals $F_n \in \textsc{Pol}(X)$ converging to F in the norm $\|.\|_{1,2}$ defined in (17). Then, for a given, positive constant K (since $\alpha^{(\mathbf{u})}$ is a bounded operator)

$$
\begin{aligned}
0 = \lim_{n\uparrow\infty}\|F_n - F\|_{1,2}^2 &\geqslant \lim_{n\uparrow\infty}\mathbb{E}[\|DF_n - DF\|^2] \\
&\geqslant K\lim_{n\uparrow\infty}\mathbb{E}[\|(\alpha^{(\mathbf{u})}D(F_n - F)(X))\|^2] \\
&\geqslant K\lim_{n\uparrow\infty}\int_0^1\mathbb{E}({}^{(p;\mathbf{u})}[(\alpha^{(\mathbf{u})}D(F_n - F)(X))]_s^2)\mathrm{d}s
\end{aligned}
$$

(recall that $\|f\|^2 := \int_0^1 f^2(x)\mathrm{d}x$, $f \in T_1$), as, thanks to Jensen's inequality, for fixed s and n,

$$
\begin{aligned}
\mathbb{E}[({}^{(p;\mathbf{u})}[(\alpha^{(\mathbf{u})}D(F_n - F)(X))]_s)^2] &= \mathbb{E}[(\mathbb{E}[(\alpha^{(\mathbf{u})}D(F_n - F)(X))_s|\mathcal{G}_s^{(\mathbf{u})}])^2] \\
&\leqslant \mathbb{E}[(\alpha^{(\mathbf{u})}D(F_n - F)(X))_s^2].
\end{aligned}
$$

□

The following modification of Proposition 1.2.4 in Nualart (1995) will also be needed:

Lemma 7. *Let $F \in \textsc{Pol}(X)$, fix $s \in [0, 1]$ and define*

$$\Gamma^{(\mathbf{u})}(F, s) := \mathbb{E}(F|\mathcal{G}_s^{(\mathbf{u})}).$$

Then

$$\Gamma^{(\mathbf{u})}(F, s) \in \mathbb{D}^1(X)$$

and

$$\alpha^{(\mathbf{u})} D_t \Gamma^{(\mathbf{u})}(F, s)(X) = \mathbb{E}(\alpha^{(\mathbf{u})} D_t F(X) | \mathcal{G}_s^{(\mathbf{u})}) 1_{[0,s]}(t)$$

$\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-*a.e. on* $C_{[0,1]} \times [0, 1]$.

Proof. Consider a smooth random variable of the type

$$F = \prod_{i=1}^{n} (X(z_i))^{k_i},$$

where the k_i are natural numbers and $z_i \in T_1$. Thanks again to the binomial formula, we know that F may be represented as a linear combination of random variables of the form

$$H = q(X_{u_m}, X_{u_{m-1}} - X_{u_m}, \ldots, X_{u_1} - X_{u_2}) H^{(\mathbf{u})} := Q^{(\mathbf{u})} H^{(\mathbf{u})},$$

where the notation is that of (20). In particular, $H^{(\mathbf{u})}$ is a polynomial smooth functional of $X^{(\mathbf{u})}$. As

$$\Gamma^{(\mathbf{u})}(H, s) = Q^{(\mathbf{u})} \mathbb{E}(H^{(\mathbf{u})} | \mathcal{F}_s(X^{(\mathbf{u})})),$$

and

$$\Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s) := \mathbb{E}(H^{(\mathbf{u})} | \mathcal{F}_s(X^{(\mathbf{u})}))$$

can be shown to be an element of $\mathrm{P{\small OL}}(X)$, it is clear that $\Gamma^{(\mathbf{u})}(H, s)$, and therefore $\Gamma^{(\mathbf{u})}(F, s)$, belongs to $\mathbb{D}^1(X)$.

Moreover, $\Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s)$ is an element of $\mathbb{D}^1(X^{(\mathbf{u})})$, and we also have (see Nualart 1995, Proposition 1.2.4)

$$D_t \Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s)(X^{(\mathbf{u})}) = \mathbb{E}(D_t H^{(\mathbf{u})}(X^{(\mathbf{u})}) | \mathcal{F}_s(X^{(\mathbf{u})})) 1_{[0,s]}(t)$$

$\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-a.e. on $C_{[0,1]} \times [0, 1]$. Proposition 5, along with an independence argument, shows that the last relation implies

$$\alpha^{(\mathbf{u})}[Q^{(\mathbf{u})} D_t \Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s)(X)] = \mathbb{E}(\alpha^{(\mathbf{u})}[Q^{(\mathbf{u})} D_t H^{(\mathbf{u})}(X)] | \mathcal{G}_s^{(\mathbf{u})}) 1_{[0,s]}(t)$$

$\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-a.e. on $C_{[0,1]} \times [0, 1]$, hence the desired result, since it is easily shown (by arguments analogous to those rehearsed in the proof of Proposition 6) that

$$\begin{aligned} \alpha^{(\mathbf{u})}[Q^{(\mathbf{u})} D \Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s)(X)] &= \alpha^{(\mathbf{u})} D[Q^{(\mathbf{u})} \Lambda^{(\mathbf{u})}(H^{(\mathbf{u})}, s)](X) \\ &= \alpha^{(\mathbf{u})} D[\Gamma^{(\mathbf{u})}(H, s)](X) \end{aligned}$$

and

$$\begin{aligned} \alpha^{(\mathbf{u})}[Q^{(\mathbf{u})} D H^{(\mathbf{u})}](X) &= \alpha^{(\mathbf{u})} D[Q^{(\mathbf{u})} H^{(\mathbf{u})}](X) \\ &= \alpha^{(\mathbf{u})} D H(X). \end{aligned}$$

□

We can now complete the proof of Theorem 1.

Conclusion of the proof of Theorem 1. Suppose for convenience that $\mathbb{E}(F) = 0$ and that $F \in \mathrm{P{\scriptsize OL}}(X)$. Then the Clark–Ocone formula yields

$$F = \int_0^1 \mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X)) \mathrm{d}X_{u_1}.$$

It is, moreover, clear that for every u_1 the functional $\mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X))$ is an element of $\mathbb{D}^1(X)$ (it is actually a polynomial smooth functional of X), and also that there exist two $\mathcal{F}_{u_1}(X)$-progressively measurable applications

$$(u_1, \omega) \mapsto \phi_1(u_1, \omega),$$

$$(u_1, \omega) \mapsto \phi_2(u_1, \omega),$$

such that, for a fixed u_1, the two terms are versions respectively of

$$\mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X)) \quad \text{and} \quad \mathbb{E}(D_{u_1} F | X_{u_1}).$$

As a consequence, we have, thanks to Proposition 6 in the special case $m = 1$ and $\mathbf{u} \equiv u_1$, the representation

$$\begin{aligned}
\mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X)) &= \mathbb{E}(D_{u_1} F | X_{u_1}) \\
&\quad + \int_0^{u_1} \mathbb{E}[\alpha^{(u_1)} D_{u_2} \mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X)) | \mathcal{G}_{u_2}^{(u_1)}] \mathrm{d}X_{u_2}^{(u_1)} \\
&= \mathbb{E}(D_{u_1} F | X_{u_1}) \\
&\quad + \int_0^{u_1} \mathbb{E}[\mathbb{E}(\alpha^{[0,u_1]} D_{u_1,u_2}^2 F | \mathcal{F}_{u_1}(X)) | \mathcal{G}_{u_2}^{(u_1)}] \mathrm{d}X_{u_2}^{(u_1)} \\
&= \mathbb{E}(D_{u_1} F | X_{u_1}) + \int_0^{u_1} \mathbb{E}[\alpha_{(2)} D_{u_1,u_2}^2 F | \mathcal{G}_{u_2}^{(u_1)}] \mathrm{d}X_{u_2}^{(u_1)},
\end{aligned} \tag{21}$$

using the equality

$$\alpha^{(u_1)} D_{u_2} \mathbb{E}(D_{u_1} F | \mathcal{F}_{u_1}(X)) = \mathbb{E}(\alpha^{[0,u_1]} D_{u_1,u_2} F | \mathcal{F}_{u_1}(X)) 1_{[0,u_1]}$$

which is a consequence of Lemma 7 and of the fact that $\mathcal{F}_{u_1}(X) = \mathcal{G}_{u_1}^{(u_1)}$, as well as the equation

$$F = \int_0^1 \mathbb{E}(D_{u_1} F | X_{u_1}) \mathrm{d}X_{u_1} + \int_0^1 \int_0^{u_1} \mathbb{E}[\alpha_{(2)} D_{u_1,u_2}^2 F | \mathcal{G}_{u_2}^{(u_1)}] \mathrm{d}X_{u_2}^{(u_1)} \mathrm{d}X_{u_1}. \tag{22}$$

Note that, in (22), the double stochastic integral is well defined because, since $F \in \mathrm{P{\scriptsize OL}}(X)$, for every n there exists a measurable application

$$(\omega; u_1, \ldots, u_n) \mapsto \phi_{u_1,\ldots,u_n}(X_{u_1}(\omega), \ldots, X_{u_{n-1}}(\omega); X_s(\omega), s \leqslant u_n)$$

such that, for fixed $u_1 > u_2 > \ldots > u_n$, $\phi_{u_1,\ldots,u_n}$ is a version of

$$\mathbb{E}[\alpha_{(n)} D_{u_1,\ldots,u_n}^n F | \mathcal{G}_{u_n}^{(u_1,\ldots,u_{n-1})}] = \mathbb{E}[\alpha_{(n)} D_{u_1,\ldots,u_n}^n F | X_{u_1}, \ldots, X_{u_{n-1}}, \mathcal{F}_{u_n}(X^{(u_{n-1})})] \tag{23}$$

and in Section 2 of Peccati (2001a) it is shown that iterated integrals of the type

$$\int_0^1 \cdots \int_0^{u_{n-1}} \phi_{u_1,\ldots,u_n}(X_{u_1}, \ldots, X_{u_{n-1}}; X_s, s \leqslant u_n) \mathrm{d}X_{u_n}^{(u_{n-1})} \ldots \mathrm{d}X_{u_1}$$

are well defined as L^2 limits of standard stochastic integrals (of progressively measurable processes) with respect to X. Note that, in our case, (21) ensures that the application

$$(\omega, u_1) \mapsto \int_0^{u_1} \mathbb{E}[\alpha_{(2)} D^2_{u_1,u_2} F | \mathcal{G}_{u_2}^{(u_1)}] \mathrm{d}X_{u_2}^{(u_1)}$$

is actually a progressively measurable process.

More to the point, the main result of Peccati (2001a) implies that the two summands on the right of (22) are the orthogonal projections of F on K_1 and $K_1^\perp$ respectively, as defined in the Introduction.

To conclude, introduce the following recursive assumption:

(A_{n-1}) Every $F \in \mathrm{Pol}(X)$ such that $\mathbb{E}(F) = 0$ admits the representation

$$\begin{aligned}
F = &\int_0^1 \mathbb{E}(D_{u_1} F | X_{u_1}) \mathrm{d}X_{u_1} \\
&+ \int_0^1 \int_0^{u_1} \mathbb{E}[\alpha_{(2)} D^2_{u_1,u_2} F | X_{u_1}, X_{u_2}] \mathrm{d}X_{u_2}^{(u_1)} \mathrm{d}X_{u_1} \\
&\cdots \\
&+ \int_0^1 \cdots \int_0^{u_{n-2}} \mathbb{E}[\alpha_{(n-1)} D^{n-1}_{u_1,\ldots,u_{n-1}} F | X_{u_1}, X_{u_2}, \ldots, X_{u_{n-1}}] \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \ldots \mathrm{d}X_{u_1} \\
&+ \int_0^1 \cdots \int_0^{u_{n-1}} \mathbb{E}[\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | \mathcal{G}_{u_n}^{(u_1,\ldots,u_{n-1})}] \mathrm{d}X_{u_n}^{(u_{n-1})} \ldots \mathrm{d}X_{u_1}.
\end{aligned}$$

We will show that (A_{n-1}) implies (A_n). To see this, fix $\mathbf{u} = (u_1, \ldots, u_n) \in \Delta^n$, and observe that, since $X^{(\mathbf{u})}$ equals $X^{(u_n)}$ on the interval $[0, u_n]$, and $\mathcal{G}_{u_n}^{(u_1,\ldots,u_{n-1})} = \mathcal{G}_{u_n}^{(u_1,\ldots,u_n)}$,

$$\begin{aligned}
\mathbb{E}[\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | \mathcal{G}_{u_n}^{(u_1,\ldots,u_{n-1})}] &= \mathbb{E}[\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | X_{u_1}, \ldots, X_{u_n}] \\
&\quad + \int_0^{u_n} \mathbb{E}[\alpha^{(u_n)} D_{u_{n+1}} \mathbb{E}[\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | \mathcal{G}_{u_n}^{(u_1,\ldots,u_{n-1})}] | \mathcal{G}_{u_{n+1}}^{(u_1,\ldots,u_n)}] \mathrm{d}X_{u_{n+1}}^{(u_n)} \\
&= \mathbb{E}[\alpha_{(n)} D^n_{u_1,\ldots,u_n} F | X_{u_1}, \ldots, X_{u_n}] \\
&\quad + \int_0^{u_n} \mathbb{E}[\alpha_{(n+1)} D^{n+1}_{u_1,\ldots,u_{n+1}} F | \mathcal{G}_{u_{n+1}}^{(u_1,\ldots,u_n)}] \mathrm{d}X_{u_{n+1}}^{(u_n)},
\end{aligned}$$

where the first equality derives from Proposition 6 in dimension n and with $\mathbf{u} \equiv (u_1, \ldots, u_n)$, whereas the second is a consequence of Lemma 7. This yields the desired implication $(\mathrm{A}_{n-1}) \Rightarrow (\mathrm{A}_n)$, and therefore shows that the decomposition of a polynomial smooth functional with respect to the spaces $K_1, \ldots, K_n$ and $\overline{\Pi}_n^{\perp}$ coincides with the one that is stated in Theorem 1, which is therefore completely proved in this case, since in the first part of the proof we have verified (A_1) (note that (A_0) is simply the Clark–Ocone formula).

Consider now a random variable $F \in \mathbb{D}^{\infty}(X)$, and fix $k \geqslant 1$: we write

$$(\omega; u_1, \ldots, u_k) \mapsto \psi_k^F(u_1, X_{u_1}(\omega), \ldots, u_k, X_{u_k}(\omega))$$

for the kth integrand in the TSCD of F, and we consider a sequence of functionals $F_N \in \mathrm{Pol}(X)$, converging to F in the norm $\|.\|_{k,2}$. It is clear that, denoting by $\psi_k^{F_N}$ the kth TSCD integrand of F_N, ψ_k^F must equal the limit of the sequence $\psi_k^{F_N}$ in the Hilbert space

$$L^2(C_{[0,1]} \times \Delta^k, \mathrm{d}\mathbb{P} \otimes \mathrm{d}u_1 \ldots \mathrm{d}u_k);$$

moreover, we know, according to the first part of the proof, that

$$\psi_k^{F_N}(u_1, X_{u_1}, \ldots, u_k, X_{u_k}) = \mathbb{E}(\alpha_{(k)} D^k_{u_1,\ldots,u_k} F_N | X_{u_1}, \ldots, X_{u_k}).$$

Now observe that, since the application

$$f \mapsto \alpha_{(k)} f$$

defines a bounded operator from $L^2(\Delta^k, \mathrm{d}t_1 \ldots \mathrm{d}t_k)$ to itself, there exists a positive constant K such that

$$\begin{aligned} 0 = \lim_{N\uparrow\infty} \|F_N - F\|^2_{(k,1)} &\geqslant \lim_{N\uparrow\infty} \mathbb{E}[\|D^k(F_N - F)\|^2_{L^2(\Delta^k, \mathrm{d}u_1 \ldots \mathrm{d}u_k)}] \\ &\geqslant K \lim_{N\uparrow\infty} \mathbb{E}[\|\alpha_{(k)}(D^k F_N - D^k F)\| L^2_{(\Delta^k, \mathrm{d}u_1 \ldots \mathrm{d}u_k)}] \\ &\geqslant K \lim_{N\uparrow\infty} \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{k-1}} \mathbb{E}[\mathbb{E}(\alpha_{(k)} D^k_{u_1,\ldots,u_k}(F_N - F) | X_{u_1}, \ldots, X_{u_k})^2] \mathrm{d}u_k \, \mathrm{d}u_{k-1} \ldots \mathrm{d}u_1, \end{aligned}$$

thanks again to Jensen's inequality. This concludes the proof of Theorem 1. □

We can also obtain similar results for functionals that are less regular, but with finite p.d.d.

Proposition 8. *Suppose that, for* $n \geqslant 1$,

$$F \in \mathbb{D}^n(X) \cap \overline{\Pi}_n.$$

Then the k*th integrand in the TSCD of* F *is given by*

$$\psi_k^F(u_1, X_{u_1}, \ldots, u_k, X_{u_k}) = \begin{cases} \mathbb{E}(\alpha_{(k)} D^k_{u_1,\ldots,u_k} F | X_{u_1}, \ldots, X_{u_k}), & k \leqslant n, \\ 0, & k > n. \end{cases}$$

As anticipated, the application of Theorem 1 to smooth functionals of X leads to even simpler formulae. We first introduce the following definition: given $f(x_1, \ldots, x_n) \in C^m(\mathbb{R}^n)$, that is, a continuous function on $\mathbb{R}^n$, having continuous partial derivatives up to the mth order, we define

$$f^{(k)}_{i_1 \ldots i_k}(x_1, \ldots, x_n) := \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(x_1, \ldots, x_n)$$

for every $k \leqslant m$, for every $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$. We also set $C^m_{\mathrm{b}}(\mathbb{R}^n)$ to be the subset of $C^m(\mathbb{R}^n)$ composed of functions such that their partial derivatives are bounded.

Corollary 9. *Consider a functional of the form*

$$F = f(X_{t_1}, \ldots, X_{t_n}),$$

where $0 \leqslant t_1 < \ldots < t_n \leqslant 1$ *are fixed and* $f \in C^m_{\mathrm{b}}(\mathbb{R}^n)$ *for some* $m \geqslant n$*. Then the kth integrand in the time-space decomposition of F is given, for* $k \leqslant n$*, by*

$$\psi^F_k(u_1, X_{u_1}; \ldots; u_k, X_{u_k}) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \mathbb{E}[f^{(k)}_{i_1 \ldots i_k}(X_{t_1}, \ldots, X_{t_n}) | X_{u_1}, \ldots, X_{u_k}]$$

$$\times \prod_{j=1}^{k} 1_{(t_{i_{k-j}}, t_{i_{k+1-j}})}(u_j) \prod_{j=2}^{k} \frac{u_{j-1} - t_{i_{k+1-j}}}{u_{j-1} - u_j},$$

where $t_0 := 0$.

Remark. Corollary 9 ensures that the integrands in the TSCD of $F = f(X_{t_1}, \ldots, X_{t_n})$ can be calculated by considering iterated derivatives of f where the differentiation is done *at most once* with respect to each variable: for n very large, this renders time-space decompositions rather appealing from a computational point of view. Note that this also provides an a posteriori proof of the fact that, for every n, functionals of such a form are contained in the space $\bigoplus_{i=0}^{n} K_i$.

Proof of Corollary 9. For F as in the statement, the kth derivative process is given by a linear combination of random processes of the type

$$C(\omega) \prod_{i=1}^{k} f_i(u_i),$$

where the f_i are indicator functions of time intervals, and also

$$\alpha_{(k)}\left(C(\omega) \prod_{i=1}^{k} f_i \right)(u_1, \ldots, u_k) = C(\omega) \prod_{i=1}^{k} \alpha^{(u_{i-1})} f_i(u_i),$$

where $u_0 := 0$ and the $\alpha^{(u)}$ are as before. It is now clear that – thanks to a recurrence argument – the result is proved in general, once it is shown for the case $n = 2$. Thus, we consider a functional of the form

$$F = f(X_{t_1}, X_{t_2})$$

for f as in the statement and fixed $t_1 < t_2$. Since $\alpha_{(1)}$ coincides with the identity operator, we may immediately deal with the second derivative process of f:

$$D^2_{u_1,u_2}F = \sum_{(i,j)\in\{1,2\}^2} \frac{\partial^2}{\partial x_i \partial x_j} f(X_{t_1}, X_{t_2}) 1_{(0,t_i)}(u_1) 1_{(0,t_j)}(u_2).$$

Now, if $j \geqslant i$,

$$\begin{aligned} 1_{(u_1>u_2)}\alpha_{(2)}[1_{(0,t_i)}1_{(0,t_j)}](u_1, u_2) &= 1_{(u_1>u_2)}1_{(0,t_i)}(u_1)\left[1_{(0,t_j)}(u_2) - \frac{u_1 \wedge t_j - u_2}{u_1 - u_2} 1_{(u_2<u_1\wedge t_j)}\right] \\ &= 0 \end{aligned}$$

and finally, for $j = 1,\ i = 2$

$$\begin{aligned} 1_{(u_1>u_2)}\alpha_{(2)}[1_{(0,t_2)}1_{(0,t_1)}](u_1, u_2) &= 1_{(u_1>u_2)}1_{(0,t_2)}(u_1)\left[1_{(0,t_1)}(u_2) - \frac{u_1 \wedge t_1 - u_2}{u_1 - u_2} 1_{(u_2<u_1\wedge t_1)}\right] \\ &= 1_{(0,t_1)\times(t_1,t_2)}(u_2, u_1)\frac{u_1 - t_1}{u_1 - u_2}, \end{aligned}$$

which proves the result. $\square$

Another consequence of Theorem 1 is the following:

Corollary 10. *Let* $h \in T_1$ *and*

$$F = \exp(X(h)).$$

Then the nth integrand in the chaotic time-space decomposition of F is given by

$$\psi_n^F(u_1, X_{u_1}; \ldots; u_n, X_n) = \mathbb{E}(F|X_{u_1}, \ldots, X_{u_n}) \prod_{i=1}^n \alpha^{(u_{i-1})} h(u_i)$$

with $\alpha^{(0)} := id$. *In particular, if*

$$F = \exp\left(\int_0^1 X_s \mathrm{d}s\right),$$

then

$$\psi_n^F(u_1, X_{u_1}; \ldots; u_n, X_n) = \frac{1}{2^{n-1}} \mathbb{E}(F|X_{u_1}, \ldots, X_{u_n}) \prod_{i=1}^n (u_{i-1} - u_i),$$

where $u_0 = 1$.

Proof. The first part derives straightforwardly from Theorem 1 and the definition of F, whereas the second is a consequence of

$$\int_0^1 X_s \,\mathrm{d}s = \int_0^1 \mathrm{d}X_s(1 - s),$$

and, for $u > s$,

$$\alpha^{(u)}(1-\cdot)(s)=\frac{1}{2}(u-s).$$

□

5. Bridge chaoses and time-space chaoses

For the sake of completeness we present a slight generalization of the main result of Gosselin and Wurzbacher (1997), discussed in the Introduction, concerning the chaotic representation property of Brownian bridges. More to the point, we state a 'Stroock-type formula' for this kind of decomposition, thus showing that Hardy operators, such as those presented in Section 3, also appear quite naturally in this case.

To start, we observe that Proposition 3 implies that, for every $\mathbf{u}\in\Delta^m$, the process

$$X_{\mathbf{u}}^{0,0}(t):=X\big(\alpha^{(\mathbf{u})}1_{[0,t]}\big),\qquad t\in[0,1],$$

satisfies the relation

$$\{X_{\mathbf{u}}^{0,0}(t),\ t\in[0,1]\}\overset{\text{law}}{=}\left\{\sum_{j=1}^{m}\hat{X}_j(t)1_{(u_{j+1},u_j]}(t)+X(t-u_1)1_{(u_1,1]}(t),\ \ t\in[0,1]\right\},\tag{24}$$

where the $\hat{X}_j$ are mutually independent Brownian bridges of length u_j-u_{j+1}, from 0 to 0, such that the family $(\hat{X}_j)_{j=1,\dots,m}$ is independent of X. Note that, for $\mathbf{u}=u=1$, $X_u^{0,0}$ is a standard Brownian bridge of length one, from 0 to 0, whose expression is given in formula (8) – this is the case studied in Gosselin and Wurzbacher (1997).

Now let $\mathbf{u}$ be a fixed element of Δ^m. The process $X_{\mathbf{u}}^{0,0}$ is of course a semimartingale; moreover, by defining

$$X_{\mathbf{u}}^{0,0}(f):=\int_0^1 f(s)\mathrm{d}X_{\mathbf{u}}^{0,0}(s)=\int_0^1\alpha^{(\mathbf{u})}f(s)\mathrm{d}X_s$$

for $f\in T^{\mathbf{u}}$, the Gaussian family

$$X_{\mathbf{u}}^{0,0}:=\{X_{\mathbf{u}}^{0,0}(f),\ f\in T^{\mathbf{u}}\}$$

is an *isonormal Gaussian process* (or a *Gaussian measure*: see Nualart 1995) over $T^{\mathbf{u}}$, and since $\alpha^{(\mathbf{u})}$ is a unitary isomorphism from $T^{\mathbf{u}}$ onto T_1, this implies that the space $L^2(X)$ is equal to $L^2(X_{\mathbf{u}}^{0,0})$ and that it is spanned by the orthogonal summations of multiple stochastic integrals with respect to the Gaussian measure $X_{\mathbf{u}}^{0,0}$. Now denote by $I_n^Y(f)$ the multiple stochastic integral of the nth order of an appropriate f with respect to a given Gaussian measure Y. Then, the above discussion – as well as the classic results on Wiener chaos – imply that, for every $F\in L^2(X)$, there exist two sequences of functions $\{f_n,\ n\geqslant 1\}$ and $\{g_n,\ n\geqslant 1\}$, with $f_n\in(T_1)^{\circ n}$ and $g_n\in(T^{\mathbf{u}})^{\circ n}$, where for every n $(T_1)^{\circ n}$ and $(T^{\mathbf{u}})^{\circ n}$ denote the nth symmetric tensor product respectively of T_1 and of $T^{\mathbf{u}}$, such that

$$F=\mathbb{E}(F)+\sum_{n\geqslant 1}I_n^{X_{\mathbf{u}}^{0,0}}(g_n)=\mathbb{E}(F)+\sum_{n\geqslant 1}I_n^X(f_n),\tag{25}$$

and, moreover,

$$g_n = \beta^{[\mathbf{u}^n]} f_n, \qquad f_n = \alpha^{[\mathbf{u}^n]} g_n.$$

Here, the operators $\alpha^{[\cdot]}$ and $\beta^{[\cdot]}$ are constructed as follows: for $f \in T_n$, write $\gamma^{(\mathbf{u})}$ for $\alpha^{(\mathbf{u})}$ or $\beta^{(\mathbf{u})}$, set

$$\gamma_k^{(\mathbf{u})} f(t_1, \ldots, t_n) := \gamma^{(\mathbf{u})} f_{(t_1, \ldots t_{k-1}, t_{k+1}, \ldots, t_n)}(t_k) 1_{(f_{(t_1, \ldots t_{k-1}, t_{k+1}, \ldots, t_n)} \in T_1)},$$

where the notation is the same as in Section 3, and finally

$$\gamma^{[\mathbf{u}^n]} f := \gamma_1^{(\mathbf{u})}[\gamma_2^{(\mathbf{u})} \ldots (\gamma_n^{(\mathbf{u})} f)].$$

It is straightforward to verify that, for $\mathbf{u}$ fixed as above, $\alpha^{[\mathbf{u}^n]}$ and $\beta^{[\mathbf{u}^n]}$ are mutually inverse unitary isomorphisms from $(T^{\mathbf{u}})^{\circ n}$ to $(T_1)^{\circ n}$ and from $(T_1)^{\circ n}$ to $(T^{\mathbf{u}})^{\circ n}$ respectively.

Remark. Observe that $(T^{\mathbf{u}})^{\circ n} \subset (T_1)^{\circ n} \subset T_n$.

Note that – as pointed out by Gosselin and Wurzbacher and just as in the case of X – quantities as $I_n^{X_{\mathbf{u}}^{0,0}}(g_n)$ have a nice interpretation in terms of standard, iterated stochastic integrals with respect to the continuous semimartingale $X_{\mathbf{u}}^{0,0}$. Moreover, when $F \in \mathbb{D}^{\infty}(X)$, we can write f_n and g_n explicitly, thanks to the *Stroock-type formulae*

$$f_n = \frac{1}{n!}\mathbb{E}(D^n F(X)), \qquad g_n = \frac{1}{n!}\mathbb{E}(\beta^{[\mathbf{u}^n]} D^n F(X)),$$

that are a consequence of the following relations, valid for every $n \geqslant 1$:

$$\mathbb{D}^n(X) = \mathbb{D}^n(X_{\mathbf{u}}^{0,0}),$$

$$D^n F(X_{\mathbf{u}}^{0,0}) = \beta^{[\mathbf{u}^n]} D^n F(X),$$

$$D^n F(X) = \alpha^{[\mathbf{u}^n]} D^n F(X_{\mathbf{u}}^{0,0}).$$

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