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Stationary covariances associated with exponentially convex functions

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We establish a bijection between exponentially convex functions and entire positive definite functions, and extend Loève's construction of stochastic processes associated with them. As an application, we derive parametric covariance models for locally stationary random fields.

Keywords: characteristic function; exponential family; exponentially convex; Laplace transform; locally stationary; positive definite; random field

1. Introduction

Let $Z(x)$, $x \in \mathbb{R}^d$, be a complex-valued stochastic process, and suppose that $Z(x)$ has finite variance for all x. Then the function $C(x_i, x_j) = \operatorname{cov}(Z(x_i), \overline{Z(x_j)})$ is defined for $x_i, x_j \in \mathbb{R}^d$ and called the *covariance function* of the process. It is well known that a complex-valued function C defined on the product space $\mathbb{R}^d \times \mathbb{R}^d$ is a covariance function if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \bar{a}_j \, \mathrm{C}(x_i, x_j) \geqslant 0 \tag{1}$$

for all finite sets of complex coefficients $a_1, \ldots, a_n$ and points $x_1, \ldots, x_n \in \mathbb{R}^d$.

A covariance function is called *stationary* if

$$C(x_i, x_j) = \varphi(x_i - x_j), \qquad x_i, x_j \in \mathbb{R}^d, \tag{2}$$

for a function φ defined on $\mathbb{R}^d$. Stationary processes and stationary covariance functions play major roles in the statistical analysis of time series and spatial data. We then say that φ is a *positive definite* function on $\mathbb{R}^d$, meaning that (1) holds for the function defined in (2). By a classical theorem of Bochner (1933), a continuous function is positive definite if and only if it is of the form

$$\varphi(t) = \int e^{\mathrm{i} r \cdot t} \, \mathrm{d}F(r), \qquad t \in \mathbb{R}^d, \tag{3}$$

where F is a non-negative measure on $\mathbb{R}^d$. Thus, a continuous function is positive definite if and only if it is the characteristic function of a non-negative finite measure. Similarly, a covariance function is called *exponentially convex* if

$$C(x_i, x_j) = \psi(x_i + x_j), \qquad x_i, x_j \in \mathbb{R}^d, \tag{4}$$

for a function ψ defined on $\mathbb{R}^d$. We then say that ψ is an *exponentially convex* function, meaning that (1) holds for the function defined in (4). Exponentially convex stochastic processes and exponentially convex covariances have been studied by Loève (1946; 1965), among others. Bernstein (1929), Widder (1934; 1946, pp. 273 and 275) and Devinatz (1955) showed that a continuous function is exponentially convex if and only if it is of the form

$$\psi(s) = \int e^{r \cdot s} \, dF(r), \qquad s \in \mathbb{R}^d, \tag{5}$$

where F is a non-negative finite measure on $\mathbb{R}^d$ and where the integral converges for all s. In other words, a continuous function is exponentially convex if and only if it is the Laplace transform of a non-negative finite measure.

Suppose that $\varphi(t)$, $t \in \mathbb{R}^d$, is a continuous positive definite function. In view of the representations (3) and (5), it is tempting to assume that we can naively put $\psi(s) = \varphi(-is)$, $s \in \mathbb{R}^d$, and obtain an exponentially convex function. However, this is not true in general; for example, $\varphi(t) = 1/(1 + t^2)$ is a positive definite function on $\mathbb{R}$ but $\psi(s) = \varphi(-is) = 1/(1 - s^2)$ is evidently not exponentially convex. Our key result, the bijection theorem in Section 2, shows that the plug-in procedure can be rigorously justified if φ is an entire function, and thereby establishes a bijection between entire positive definite functions and exponentially convex functions. Its main use lies in the construction of exponentially convex functions from positive definite functions, of which we give examples. In Section 3, we extend a closely related stochastic process construction of Loève (1946; 1965) to the case of random fields defined on $\mathbb{R}^d$, $d \geqslant 1$.

This research was prompted by our quest for parametric models of exponentially convex covariances. These are required when locally stationary covariance functions in the sense of Silverman (1957) are fitted, a situation discussed by Genton and Perrin (2002). Briefly, a covariance function is *locally stationary* in Silverman's sense if it is of the form

$$C(x_i, x_j) = \varphi(x_i - x_j)\, \psi(x_i + x_j), \qquad x_i, x_j \in \mathbb{R}^d, \tag{6}$$

where φ is positive definite. The property $C(x_i, x_i) \geqslant 0$ forces the function ψ to be positive, but not necessarily exponentially convex, even though (1) must hold for C. For instance, if $a > 0$ the covariance $C(x_i, x_j) = \exp(-2a(x_i^2 + x_j^2))$ can be written as the product of $\exp(-a(x_i - x_j)^2)$ and $\exp(-a(x_i + x_j)^2)$, where the first term is positive definite, but the second is positive without being exponentially convex. Note that the product of a stationary covariance function φ by a positive function ψ as in (6) does not necessarily yield a positive definite covariance C. However, φ and ψ can be simultaneously modified in order to make C positive definite; see the matching theorem in Silverman (1959). It is nevertheless natural and convenient to build admissible locally stationary models from positive definite and exponentially convex functions, respectively.

2. A bijection between exponentially convex and entire positive definite functions

2.1. Exponential families

Natural exponential families (Lehmann 1986, Section 2.7) provide an appropriate framework for the questions at hand, and we begin with a review of their properties. Let F be a non-negative finite measure on $\mathbb{R}^d$, let

$$\varphi(t) = \int e^{i\, t\cdot x}\, dF(x), \qquad t \in \mathbb{R}^d, \tag{7}$$

denote its Fourier transform or characteristic function, and let

$$\psi(s) = \int e^{s\cdot x}\, dF(x), \qquad s \in \mathbb{R}^d, \tag{8}$$

denote its Laplace transform. Then

$$\Theta = \Theta_F = \left\{ \theta \in \mathbb{R}^d : \psi(\theta) = \int e^{\theta\cdot x}\, dF(x) < \infty \right\}$$

is a non-empty convex subset of $\mathbb{R}^d$ that contains 0. For simplicity, we will assume that Θ is open. We can then associate F with a *natural exponential family* $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ of finite measures, by setting $dF_\theta(x) = e^{\theta\cdot x}\, dF(x)$ for $\theta \in \Theta$. The characteristic function of F_θ is given by

$$\varphi_\theta(t) = \int e^{i\, t\cdot x}\, dF_\theta(x) = \int e^{(\theta + i t)\cdot x}\, dF(x), \qquad t \in \mathbb{R}^d,$$

where the integral is convergent and analytic as a function of $\zeta = \theta + it$ on $\Theta_1 = \Theta + i\mathbb{R}^d \subseteq \mathbb{C}^d$ (Lehmann 1986, Theorem 9, p. 52). Thus, it represents the unique analytic continuation of the Laplace transform (8) to this region,

$$\psi(\zeta) = \int e^{\zeta\cdot x}\, dF(x) = \varphi_\theta(t), \qquad \zeta \in \Theta_1.$$

Similarly, since $\theta + it = i(t - i\theta)$ we may extend the characteristic function (7) into an analytic function of $\omega = t - i\theta$ on $\Theta_2 = \mathbb{R}^d - i\Theta \subseteq \mathbb{C}^d$,

$$\varphi(\omega) = \int e^{i\, \omega\cdot x}\, dF(x) = \varphi_\theta(t), \qquad \omega \in \Theta_2.$$

It is then immediate that

$$\psi(\theta + it) = \varphi(t - i\theta), \qquad \theta \in \Theta,\ t \in \mathbb{R}^d. \tag{9}$$

If $\theta \in \Theta$ then $\varphi(t - i\theta) = \varphi_\theta(t)$ is evidently a positive definite function. Furthermore, if $\theta \in \mathbb{R}^d$, $s \in \mathbb{R}^d$, and $\theta + s \in \Theta$ then

$$\psi(\theta+s) = \int \mathrm{e}^{(\theta+s)\cdot x}\,\mathrm{d}F(x) = \int \mathrm{e}^{s\cdot x}\,\mathrm{d}F_\theta(x).$$

In other words, the shifted function $s \mapsto \psi(\theta+s)$ is the Laplace transform of F_θ, and it follows from the results of Devinatz (1955) that it is exponentially convex when restricted to the appropriate domain.

2.2. Bijection theorem

We now establish a bijection between exponentially convex functions and entire positive definite functions that justifies the aforementioned plug-in procedure, when applied to the associated power series representation. Recall that a complex-valued function $\varphi(t)$, $t \in \mathbb{R}^d$, is *entire* if it can be extended to a necessarily unique analytic function $\varphi(z)$, $z \in \mathbb{C}^d$.

Theorem 1. *If $\psi(s)$, $s \in \mathbb{R}^d$, is an exponentially convex function, then it is entire, and $\varphi(t) = \psi(\mathrm{i}t)$, $t \in \mathbb{R}^d$, is a positive definite function. Conversely, if $\varphi(t)$, $t \in \mathbb{R}^d$, is an entire positive definite function, then $\psi(s) = \varphi(-\mathrm{i}s)$, $s \in \mathbb{R}^d$ is an exponentially convex function.*

Proof. Suppose that $\psi(s)$, $s \in \mathbb{R}^d$, is an exponentially convex function. Then it admits the representation (5) and the arguments in Section 2.1 apply. In particular, ψ is entire and it follows from (9) with $\theta = 0$ along with the comments thereafter that $\varphi(t) = \psi(\mathrm{i}t)$, $t \in \mathbb{R}^d$, is positive definite.

Conversely, suppose that $d = 1$ and let $\varphi(t)$, $t \in \mathbb{R}$, be an entire positive definite function. Then φ admits the representation (3) for a non-negative finite measure F, and Theorem 1.12.6 of Bisgaard and Sasvári (2000) shows that $\Theta = \Theta_F = \mathbb{R}$. Clearly, $\psi(s) = \int \mathrm{e}^{sx}\,\mathrm{d}F(x)$, $s \in \mathbb{R}$, is exponentially convex on $\mathbb{R}$, and (9) with $\theta = s$ and $t = 0$ implies that $\psi(s) = \varphi(-\mathrm{i}s)$ for $s \in \mathbb{R}$. Next suppose that $d > 1$ and let $\varphi(t)$, $t \in \mathbb{R}^d$, be an entire positive definite function. Again, φ admits the representation (3) for a non-negative finite measure F, and we need to show that $\Theta = \Theta_F = \mathbb{R}^d$. Let F_k denote the kth marginal measure of F with characteristic function ϕ_k and let e_k denote the kth unit vector in $\mathbb{R}^d$, for $k = 1, \ldots, d$. Then

$$\phi_k(s) = \int \mathrm{e}^{\mathrm{i}su}\,\mathrm{d}F_k(u) = \int \mathrm{e}^{\mathrm{i}s\,e_k\cdot x}\,\mathrm{d}F(x) = \varphi(se_k), \qquad s \in \mathbb{R},$$

is a positive definite function and the restriction to $\mathbb{R}$ of an entire function $z \mapsto \varphi(ze_k)$ on $\mathbb{C}$. From the case $d = 1$ we know that $\Theta_{F_k} = \mathbb{R}$ for $k = 1, \ldots, d$, and therefore

$$\int \mathrm{e}^{re_k\cdot x}\,\mathrm{d}F(x) = \int \mathrm{e}^{ru}\,\mathrm{d}F_k(u) < \infty \qquad \text{for all } r \in \mathbb{R},\ k = 1, \ldots, d.$$

Thus Θ_F contains all coordinate axes, and since Θ_F is convex we have $\Theta_F = \mathbb{R}^d$. In view of (9) with $\theta = s$ and $t = 0$ and the comments thereafter, the proof of the theorem is complete. $\square$

The results at the end of Section 2.1 show that if $\psi(s)$, $s \in \mathbb{R}^d$, is an exponentially convex function, then $\psi_\theta(s) = \psi(\theta + s)$, $s \in \mathbb{R}^d$, is exponentially convex for all $\theta \in \mathbb{R}^d$. Similarly, if an entire function $\varphi(t)$, $t \in \mathbb{R}^d$, is positive definite then $\varphi_\theta(t) = \varphi(t - \mathrm{i}\theta)$, $t \in \mathbb{R}^d$, is positive definite for all $\theta \in \mathbb{R}^d$. Thus, the theorem in fact establishes a bijection between a family of entire positive definite functions $\varphi_\theta(t) = \varphi(t - \mathrm{i}\theta)$, $t \in \mathbb{R}^d$, which is indexed by $\theta \in \mathbb{R}^d$, and a family of exponentially convex functions $\psi_\theta(s) = \varphi(-\mathrm{i}(\theta + s))$, $s \in \mathbb{R}^d$, which is also indexed by $\theta \in \mathbb{R}^d$. For ease of exposition, we have restricted attention to entire characteristic functions for which the finiteness set Θ_F equals $\mathbb{R}^d$. Using the results in Section 1.12 of Bisgaard and Sasvári (2000), it is not difficult to extend the bijection theorem to the case where Θ_F is a convex subset of $\mathbb{R}^d$ containing the origin as an interior point. This considerably extends the class of covariance functions that can be modelled by means of the present approach.

2.3. Applications

The key application of Theorem 1 lies in the explicit construction of exponentially convex functions from positive definite functions, of which we give examples below. This is of practical importance, since there is a vast body of literature on the construction of positive definite functions, both in terms of characteristic functions and in terms of covariance functions, and it is often easy to prove that a given function is entire. In contrast, the literature on exponentially convex functions is minimal.

However, the bijection theorem can also be applied in the other direction, and we can sometimes derive results about positive definite functions from the properties of exponentially convex functions. For instance, if p is an even polynomial of degree $4k + 2$, $k = 0, 1, 2, \ldots$, and $\varphi(t) = p(t)\mathrm{e}^{-t^2}$, $t \in \mathbb{R}$, is a characteristic function, then the leading coefficient of p is necessarily negative. Indeed, φ is entire and Theorem 1 implies that $\theta(s) = p(\mathrm{i}s)\mathrm{e}^{s^2}$ is exponentially convex, hence non-negative.

In light of the example, we turn to a discussion of basic properties of exponentially convex functions. It is immediate from the representation (5) that an exponentially convex function ψ on $\mathbb{R}$ is real-valued, non-negative, and analytic with derivatives

$$\psi^{(n)}(s) = \int r^n \mathrm{e}^{rs}\,\mathrm{d}F(r), \qquad s \in \mathbb{R}.$$

The even-order derivatives are non-negative and ψ is strictly convex unless it is constant. Analogous statements hold for exponentially convex functions in $\mathbb{R}^d$. Radial functions are of particular interest in applications, and Nussbaum (1972) showed that a continuous radial function $\psi(s) = \psi_0(|s|)$, $s \in \mathbb{R}^d$, is exponentially convex if and only if it is of the form

$$\psi(s) = \int_{[0,\infty)} \Gamma(d/2)\left(\frac{2}{\mathrm{i}r|s|}\right)^{(d-2)/2} J_{(d-2)/2}(\mathrm{i}r|s|)\,\mathrm{d}F(r), \qquad s \in \mathbb{R}^d, \tag{10}$$

where J is a Bessel function. This corresponds to Schoenberg's (1938) classical representation of continuous radial positive definite functions,

$$\varphi(t) = \int_{[0,\infty)} \Gamma(d/2)\left(\frac{2}{r|t|}\right)^{(d-2)/2} J_{(d-2)/2}(r|t|)\, \mathrm{d}F(r), \qquad t \in \mathbb{R}^d, \tag{11}$$

and (10) and (11) are special cases of (5) and (3), respectively. If d is odd, the function under the integral sign in (10) can be expressed in terms of hyperbolic functions; if $d = 1$ it reduces to the hyperbolic cosine, thereby showing that an even exponentially convex function $\psi(s)$, $s \in \mathbb{R}$, is non-negative, convex, and non-decreasing in $s > 0$.

2.4. Examples of the bijection

We first give examples on $\mathbb{R}$:

(i) $\varphi(t) = \mathrm{e}^{\mathrm{i}at - bt^2}$, $t \in \mathbb{R}$,	$\psi(s) = \mathrm{e}^{as+bs^2}$, $s \in \mathbb{R}$;
(ii) $\varphi(t) = (\mathrm{e}^{\mathrm{i}t} - 1)/(\mathrm{i}t)$, $t \in \mathbb{R}$,	$\psi(s) = (\mathrm{e}^{s} - 1)/s$, $s \in \mathbb{R}$;
(iii) $\varphi(t) = (1 - t^2)\mathrm{e}^{-t^2/2}$, $t \in \mathbb{R}$,	$\psi(s) = (1 + s^2)\mathrm{e}^{s^2/2}$, $s \in \mathbb{R}$;
(iv) $\varphi(t) = \sin(t)/t$, $t \in \mathbb{R}$,	$\psi(s) = \sinh(s)/s$, $s \in \mathbb{R}$;
(v) $\varphi(t) = 2(1 - \cos(t))/t^2$, $t \in \mathbb{R}$	$\psi(s) = 2(\cosh(s) - 1)/s^2$, $s \in \mathbb{R}$.

Examples (i), (ii), (iv), and (v) are based on Feller's (1966, p. 503) table of characteristic functions, and example (iii) corresponds to $\mathrm{d}F(r) = (2\pi)^{-1/2} r^2 \mathrm{e}^{-r^2/2}\,\mathrm{d}r$. In examples (i) and (iii), F is supported on the real line, and in examples (ii), (iv), and (v), F has compact support. In example (i) we require that $a \in \mathbb{R}$ and $b \geqslant 0$.

Radially symmetric and related functions in $\mathbb{R}^d$, especially $d = 2$ and $d = 3$, are of particular interest in the aforementioned statistical framework, which calls for parametric families of exponentially convex functions.

Example (i) with $a \in \mathbb{R}^d$ and $b \geqslant 0$ in fact carries over to $\mathbb{R}^d$, $d \geqslant 1$, and Example (iv) applies in $\mathbb{R}^d$, $d \leqslant 3$, as follows.

(i) $\varphi(t) = \mathrm{e}^{\mathrm{i}a\cdot t - b|t|^2}$, $t \in \mathbb{R}^d$	$\psi(s) = \mathrm{e}^{a\cdot s + b|s|^2}$, $s \in \mathbb{R}^d$
(iv) $\varphi(t) = \sin(|t|)/|t|$, $t \in \mathbb{R}^3$	$\psi(s) = \sinh(|s|)/|s|$, $s \in \mathbb{R}^3$.

Example (iii) also allows for an interesting generalization. Consider the entire function

$$\varphi(t) = \sigma^2(1 - a|t|^2)\mathrm{e}^{-b|t|^2}, \qquad t \in \mathbb{R}^d,$$

which is positive definite if and only if $\sigma^2 \geqslant 0$, $b \geqslant 0$, and $0 \leqslant a \leqslant 2b/d$ (Cambanis *et al.* 1981). The associated parametric family of radial exponentially convex functions in $\mathbb{R}^d$ is given by

$$\psi(s) = \sigma^2(1 + a|s|^2)\mathrm{e}^{b|s|^2}, \qquad s \in \mathbb{R}^d,$$

with the same restrictions on the parameters. This leads to interesting examples of parametric models for locally stationary covariances of the form (6), by multiplying a stationary covariance by any of the above exponentially convex covariances ψ in (i)–(v). For instance, the family

$$C(x_i, x_j) = \sigma^2\big(1 - a_1|x_i - x_j|^2\big)\mathrm{e}^{-b_1|x_i - x_j|^2}\,\big(1 + a_2|x_i + x_j|^2\big)\mathrm{e}^{b_2|x_i + x_j|^2}, \qquad x_i, x_j \in \mathbb{R}^d,$$

is a valid locally stationary covariance function when $\sigma^2 \geqslant 0$, $b_1 \geqslant 0$, $b_2 \geqslant 0$, $0 \leqslant a_1 \leqslant 2b_1/d$, and $0 \leqslant a_2 \leqslant 2b_2/d$. It can be used to model departures from a stationary process through the exponentially convex covariance ψ. It is also worth noting that a differentiable covariance function $C(x_i, x_j)$ is exponentially convex, that is, satisfies (4), if and only if its first partial derivatives satisfy $C^{(1,0)}(x_i, x_j) = C^{(0,1)}(x_i, x_j)$ for all (x_i, x_j). Genton and Perrin (2002) characterize non-stationary covariances $C(x_i, x_j)$ on the real line that can be reduced to exponentially convex ones by a bijective deformation of the coordinates.

Note that exponentially convex and entire stationary covariance functions correspond to analytic processes that allow for everywhere convergent power series representations, such as (14) below. This limits their practical applicability, since analyticity is generally an unrealistic assumption in the modelling of real-world phenomena. A locally stationary covariance function of the form (6) also corresponds to an analytic process if both φ and ψ are analytic, as is the case in the aforementioned example. We intend to discuss other parametric models for locally stationary covariances elsewhere.

3. Loève's construction

Loève (1946; 1965) proposed a neat, little-known stochastic process construction that embeds stationary and exponentially convex processes into a unified structure. This final section extends his univariate construction to the case of random fields defined on $\mathbb{R}^d$, $d \geqslant 1$. Technicalities are omitted, but can be added easily along the lines of Loève's work.

Consider the mean-zero Gaussian process $Z(x)$, $x \in \mathbb{R}^d$, with exponentially convex covariance function (4), where ψ is entire and admits the representation (5) with a non-negative finite measure F. Let W be a Gaussian process defined on the class $\mathcal{B}$ of the Borel sets in $\mathbb{R}^d$ such that:

(i) $W(B) \sim \mathcal{N}(0, F(B))$ for $B \in \mathcal{B}$;
(ii) $W(B_1)$, $W(B_2)$ are independent if $B_1, B_2 \in \mathcal{B}$ are disjoint;
(iii) $W(B_1 \cup B_2) = W(B_1) + W(B_2)$ almost surely if $B_1, B_2 \in \mathcal{B}$ are disjoint.

Then the exponentially convex process $Z(x)$, $x \in \mathbb{R}^d$, admits the spectral representation

$$Z(x) = \int \mathrm{e}^{x \cdot y}\, \mathrm{d}W(y), \qquad x \in \mathbb{R}^d. \tag{12}$$

By Taylor expansion of the entire function $x \mapsto \mathrm{e}^{x \cdot y}$, $x \in \mathbb{R}^d$, we find that

$$\begin{aligned} Z(x) &= \int \left(\sum_{k=0}^{\infty} \frac{1}{k!} (x \cdot y)^k \right) \mathrm{d}W(y) \\ &= \sum_n \frac{x^n}{n!} \int y^n \, \mathrm{d}W(y) = \sum_n \frac{x^n}{n!} Z^{(n)}(0), \quad x \in \mathbb{R}^d, \end{aligned}$$

in the L^2 sense, where the summation is over all multi-indices $n = (n_1, \ldots, n_d)^{\mathrm{T}}$, $x^n = x_1^{n_1} \cdots x_d^{n_d}$, $n! = n_1! \cdots n_d!$, and where $Z^{(n)}$ stands for the associated partial derivative.

Essentially the same arguments show that (12) can be extended to the complex-valued Gaussian process

$$Z(\zeta) = \int e^{\zeta \cdot y} dW(y), \qquad \zeta \in \mathbb{C}^d, \tag{13}$$

that admits the representation

$$Z(\zeta) = \sum_n \frac{\zeta^n}{n!} Z^{(n)}(0), \qquad \zeta \in \mathbb{C}^d, \tag{14}$$

where the summation and the terms therein are defined as before. This process has covariance function

$$\begin{aligned} C(\zeta_1, \zeta_2) &= \mathrm{E}\big(Z(\zeta_1)\overline{Z(\zeta_2)}\big) \\ &= \iint e^{\zeta_1 \cdot x + \overline{\zeta_2} \cdot y} \, \mathrm{E} dW(x)\, dW(y) = \int e^{(\zeta_1 + \overline{\zeta_2}) \cdot x} \, dF(x) = \psi\big(\zeta_1 + \overline{\zeta_2}\big) \end{aligned}$$

where

$$\psi(\zeta) = \int e^{\zeta \cdot x} \, dF(x), \qquad \zeta \in \mathbb{C}^d,$$

denotes the analytic continuation of the exponentially convex function $\psi(s)$, $s \in \mathbb{R}^d$, with representation (5).

We can then fix $\mathrm{Im}(\zeta) = b \in \mathbb{R}^d$ in (13) and (14), respectively and consider the process $Z(x + \mathrm{i}b)$, $x \in \mathbb{R}^d$. For all $b \in \mathbb{R}^d$, this process is exponentially convex with covariance function $C(x_i, x_j) = \psi(x_i + x_j)$. Alternatively, we might fix $\mathrm{Re}(\zeta) = a/2 \in \mathbb{R}^d$ and consider the process $Z(a/2 + \mathrm{i}x)$, $x \in \mathbb{R}^d$. This process is stationary with covariance function $C(x_i, x_j) = \psi(a + \mathrm{i}(x_i - x_j))$, and we conclude that $\varphi(t) = \psi(a - \mathrm{i}t)$, $t \in \mathbb{R}^d$, is positive definite. Thus, putting $a = 0$ provides another perspective on the bijection theorem in Section 2.2.

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References

Bernstein, S. (1929) Sur les fonctions absolument monotones. *Acta Math.*, **52**, 1–66.

Bisgaard, T.M. and Sasvári, Z. (2000) *Characteristic Functions and Moment Sequences, Positive Definiteness in Probability*. Huntington, NY: Nova Science Publishers.

Bochner, S. (1933) Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. *Math. Ann.*, **108**, 378–410.

Cambanis, S., Huang, S. and Simons, G. (1981) On the theory of elliptically contoured distributions. *J. Multivariate Anal.*, **11**, 368–385.

Devinatz, A. (1955) The representation of functions as Laplace–Stieltjes integrals. *Duke Math. J.*, **22**, 185–191.

Feller, W. (1966) *An Introduction to Probability Theory and Its Applications, Vol. II*. New York: Wiley.

Genton, M.G. and Perrin, O. (2002) On a time deformation reducing nonstationary stochastic processes to local stationarity. Technical report, North Carolina State University.

Lehmann, E.L. (1986) *Testing Statistical Hypotheses*, 2nd edn. London: Chapman & Hall.

Loève, M. (1946) Fonctions aléatoires a décomposition orthogonale exponentielle. *Rev. Sci.*, **84**, 159–162.

Loève, M. (1965) Fonctions aléatoires a caractère exponentiel. In P. Lévy, *Processus stochastiques et mouvement Brownien*, 2nd edn., pp. 409–420. Paris: Gauthiers-Villars.

Nussbaum, A.E. (1972) Radial exponentially convex functions. *J. Anal. Math.*, **25**, 277–288.

Schoenberg, I.J. (1938) Metric spaces and completely monotone functions. *Ann. Math.*, **39**, 811–841.

Silverman, R.A. (1957) Locally stationary random processes. *IRE Trans. Inform. Theory*, **3**, 182–187.

Silverman, R.A. (1959) A matching theorem for locally stationary random processes. *Comm. Pure Appl. Math.*, **12**, 373–383.

Widder, D.V. (1934) Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral. *Bull. Amer. Math. Soc.*, **40**, 321–326.

Widder, D.V. (1946) *The Laplace Transform*. Princeton, NJ.: Princeton University Press.

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