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Tail probability via the tube formula when the critical radius is zero

AKIMICHI TAKEMURA[1] and SATOSHI KURIKI[2]

[1]*Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Bunkyo, Tokyo 113-0033, Japan. E-mail: takemura@stat.t.u-tokyo.ac.jp*
[2]*Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan. E-mail: kuriki@ism.ac.jp*

It has recently been established that the tube formula and the Euler characteristic method give an identical and valid asymptotic expansion of the tail probability of the maximum of a Gaussian random field when the random field has finite Karhunen–Loève expansion and the index set has positive critical radius. We show that the positiveness of the critical radius is an essential condition. When the critical radius is zero, we prove that only the main term is valid and that other higher-order terms are generally not valid in the formal asymptotic expansion based on the tube formula. This is done by first establishing an exact tube formula and comparing the formal tube formula with the exact formula. Furthermore, we show that the equivalence of the formal tube formula and the Euler characteristic method no longer holds when the critical radius is zero. We conclude by applying our results to some specific examples.

Keywords: chi-square field; Euler characteristic method; Karhunen–Loève expansion; Morse's theorem; second fundamental form; support cone

1. Introduction

Let M be a closed subset of the unit sphere S^{n-1} in $\mathbb{R}^n$. We consider the upper tail probability of the maximum of a random field $Z(u)$, $u = (u_1, \ldots, u_n)^{\mathrm{T}} \in M$, defined by

$$Z(u) = u^{\mathrm{T}} z = \sum_{i=1}^{n} u_i z_i, \tag{1}$$

where $z = (z_1, \ldots, z_n)^{\mathrm{T}}$ follows the standard multivariate normal distribution $N_n(0, I_n)$. This is the canonical form of a Gaussian random field with finite Karhunen–Loève expansion and constant variance, as discussed in Takemura and Kuriki (2002). Let $y = (y_1, \ldots, y_n)^{\mathrm{T}} = z/\|z\|$ follow the uniform distribution $\mathrm{Unif}(S^{n-1})$ on the unit sphere S^{n-1}. We also study the upper tail probability of the maximum of

$$Y(u) = u^{\mathrm{T}} y. \tag{2}$$

In Takemura and Kuriki (1997) we treated the situation of convex M in order to study the properties of the $\bar{\chi}^2$ distribution in the framework of testing against multivariate ordered alternatives. In Kuriki and Takemura (2001) we dealt with smooth M without boundary to study multilinear forms in normal variates. Unifying these cases, in Takemura and Kuriki (2002) we considered an index set M which is locally approximated by a convex cone. We established that in this case M has positive critical radius, and that the tube method due to Sun (1993) and the Euler characteristic method of Adler (1981) and Worsley (1995a; 1995b) lead to an identical valid asymptotic expansion of the upper tail probabilities. In a different setting Adler (2000), using results due to Piterbarg (1996), showed that the Euler characteristic method for isotropic Gaussian random fields on a piecewise smooth domain gives a valid asymptotic expansion.

These results might give the impression that the formal asymptotic expansion based on the tube formula is valid and identical to the Euler characteristic method for practically all regular cases. However, this is not the case if the critical radius of M is zero. The main purpose of this paper is to show that if the critical radius of M is zero, the asymptotic expansion based on the tube formula is generally incorrect except for the main term. Furthermore, the equivalence of the formal tube formula and the Euler characteristic method no longer holds. We also give some simple examples of index sets with zero critical radius, for which the formal tube formula and the Euler characteristic method give different asymptotic expansions and both are incorrect. A more substantial application of the results of the present paper is given in Takemura and Kuriki (2001), where a natural multivariate test statistic has an associated index set with zero critical radius.

One advantage of the Euler characteristic method over the tube formula is that it can be applied to non-Gaussian fields, whereas the tube formula is essentially restricted to Gaussian fields. See, for example Worsley (1994) and Cao and Worsley (1998; 1999) for applications of the Euler characteristic method to various non-Gaussian fields. However, the validity of the Euler characteristic method for non-Gaussian fields has not been established in general. Indeed, our example in Section 3.3 suggests that the validity of the Euler characteristic method for non-Gaussian fields is hard to prove in general. In Section 3.3 we first apply the formal tube formula to a Gaussian field whose index set has zero critical radius. However, this Gaussian field can be transformed to an equivalent χ^2 field with very regular index set, and we apply the Euler characteristic method to the resulting χ^2 field. It will be shown that the Euler characteristic method for this χ^2 field leads to an invalid asymptotic expansion, which is identical to the asymptotic expansion obtained by the formal tube formula for the original Gaussian field.

This paper is organized as follows. In Section 2, after a preliminary discussion of the properties of index sets with zero critical radius, we give some theoretical results on the asymptotic expansion based on the formal tube formula. In Section 3 we study some relevant examples in detail. Proofs and mathematical details are postponed to the appendices.

2. General results

In this section we first define a class of index sets M for which the tube formula can be defined. Then in Section 2.2 the difference between the formal tube formula and the exact tube formula for these index sets is clarified. The invalidity of the asymptotic expansion based on the formal tube formula is shown in Section 2.3.

2.1. A class of index sets

We consider a class of index sets M with the following property. At each point $x \in M$, M can be locally approximated by a cone, but the cone is not necessarily convex. We call sets M with this property *locally conic*. This class contains the boundary of a polyhedron and the union of submanifolds of S^{n-1} which are self-intersecting on S^{n-1}. Unfortunately the class of locally conic sets cannot be defined using standard manifold terminology because we allow self-intersection of the index set. Precise definitions of this class and other notions of this subsection are given in Appendix A.

The approximating cone of M at $x \in M$ is called the *support cone* of M at x and is denoted by $S_x(M)$. Let $C(S_x(M))$ denote the convex hull of $S_x(M)$. The dual cone of $S_x(M)$ (or equivalently, the dual cone of $C(S_x(M))$) in $\mathbb{R}^n$ is called the *normal cone* of M at x and is denoted by $N_x(M)$. As we shall show, the critical radius of M is zero if $S_x(M)$ is non-convex at some $x \in M$ because of the singularity of the projection onto M around x.

We discuss several simple examples to illustrate the above notions. Note that in our definition in Appendix A the support cone $S_x(M)$ and the normal cone $N_x(M)$ are defined with their vertices located at the origin.

Example 2.1. On the sphere $S^2 \subset \mathbb{R}^3$ consider the union of two great circles:

$$M = \{(x_1, x_2, x_3) \in S^2 | x_3 = 0\} \cup \{(x_1, x_2, x_3) \in S^2 | x_2 = 0\}. \tag{3}$$

Except for two points $(\pm 1, 0, 0)$, M is a regular one-dimensional manifold. However, at these two points M cannot be considered as a manifold using the standard terminology because of the self-intersection. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, 0) | x_2 \in \mathbb{R}\} \cup \{(0, 0, x_3) | x_3 \in \mathbb{R}\}$, $C(S_x(M)) = \{(0, x_2, x_3) | (x_2, x_3) \in \mathbb{R}^2\}$ and $N_x(M) = \mathrm{span}\{x\}$.

Example 2.2. On S^2 consider

$$M = \{(x_1, x_2, x_3) \in S^2 | x_2 x_3 \geqslant 0\},$$

whose boundary is M of Example 2.1. At $x = (x_1, x_2, x_3)$, with $x_2 x_3 > 0$, $S_x(M) = C(S_x(M))$ is the tangent plane $T_x(S^2)$ of S^2 at x and $N_x(M) = \mathrm{span}\{x\}$. At $x = (x_1, x_2, 0)$, with $|x_1| < 1$ and $x_2 > 0$, $S_x(M) = C(S_x(M)) = T_x(S^2) \cap \{(y_1, y_2, y_3) | y_3 \geqslant 0\}$ and $N_x(M) = \mathrm{span}\{x\} \oplus \{(0, 0, y_3) | y_3 \leqslant 0\}$, where '$\oplus$' is the orthogonal direct sum. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, x_3) | x_2 x_3 \geqslant 0\}$, $C(S_x(M)) = \{(0, x_2, x_3) | (x_2, x_3) \in \mathbb{R}^2\}$ and $N_x(M) = \mathrm{span}\{x\}$.

Example 2.3. Again on S^2, let M be the union of two semicircles $M = \{(x_1, x_2, 0) \in S^2 | x_2 \geqslant 0\} \cup \{(x_1, 0, x_3) \in S^2 | x_3 \geqslant 0\}$. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, 0) | x_2 \geqslant 0\} \cup \{(0, 0, x_3) | x_3 \geqslant 0\}$, $C(S_x(M)) = \{(0, x_2, x_3) | x_2 \geqslant 0, x_3 \geqslant 0\}$ and $N_x(M) = \{(0, x_2, x_3) | x_2 \leqslant 0, x_3 \leqslant 0\} \oplus \mathrm{span}\{x\}$.

In the above three examples the points $x = (\pm 1, 0, 0)$ exhibit a certain singularity. However, from the viewpoint of the spherical tube around M in S^2, the points $x = (\pm 1, 0, 0)$ in Example 2.3 contribute to the volume of the tube just as other points in the sense that the points in the direction of $N_x(M)$ from x are sent to x when projected onto M. On the other hand, in Examples 2.1 and 2.2, the points $x = (\pm 1, 0, 0)$ do not contribute to the volume of the spherical tube around M, because no point (other than x itself) is sent to x when projected onto M. In general, consider a spherical tube around M in S^{n-1}. A point $x \in M$ does not contribute to the volume of the tube around M if the dimension of $N_x(M)$ and the dimension of the support cone $S_x(M)$ do not add up to n. This consideration leads us to call $x \in M$ a *proper* d-dimensional boundary point of M if $S_x(M)$ contains a linear subspace L of dimension $d = n - \dim N_x(M)$. We define the dimension of M by the maximum value of d such that there exists a proper d-dimensional boundary point of M. Note that we use the term 'boundary' even if x belongs to the relative interior of M.

Let ∂M_d, $d = 0, \ldots, m$ $(= \dim M)$, denote the set of proper d-dimensional boundary points of M. We now make the following technical assumption on locally conic sets M.

Assumption 2.1. *For $d = 0, \ldots, m$, ∂M_d is a relatively open d-dimensional C^2-submanifold of $\mathbb{R}^n$. Let $I(M)$ denote the set of improper boundary points of M. The Lebesgue measure of $\bigcup_{u \in I(M)} N_u(M)$ is zero.*

Here we are assuming that ∂M_d is an open manifold embedded in $\mathbb{R}^n$. This assumption is satisfied if the set of improper boundary points is at most countable. If M satisfies this assumption we call it a 'set with piecewise smooth proper boundary'. In summary, we assume that the index set $M \subset S^{n-1}$ is a locally conic closed set with piecewise smooth proper boundary.

We now consider spherical projection onto M. For $x, y \in S^{n-1}$, let

$$\mathrm{dist}(x, y) = \arccos(x^{\mathrm{T}} y) \in [0, \pi]$$

be the geodesic distance and define

$$\mathrm{dist}(x, M) = \mathrm{dist}(x, x_M) = \min_{y \in M} \mathrm{dist}(x, y),$$

where x_M is the spherical projection of x onto M. Although x_M may not be unique, $\mathrm{dist}(x, M)$ is uniquely determined because M is closed. We are interested in the geometry of the set of points with a unique projection onto M:

$$R(M) = \{x | x_M \text{ is unique}\}. \tag{4}$$

For $u, v \in S^{n-1}$, $u^{\mathrm{T}} v = 0$ and $0 \leqslant \theta \leqslant \pi$, let

$$[u, u\cos\theta + v\sin\theta) = \begin{cases} \{u\cos t + v\sin t | 0 \leqslant t < \theta\}, & \text{if } \theta > 0, \\ \{u\}, & \text{if } \theta = 0, \end{cases} \tag{5}$$

denote the great circle segment joining u and $u\cos\theta + v\sin\theta$, which includes u and excludes $u\cos\theta + v\sin\theta$. Let

$$K(M) = \bigcup_{c \geqslant 0} cM$$

denote the smallest cone containing M in $\mathbb{R}^n$. For $u \in \partial M_d$, $d \leqslant n-2$, and $v \in N_u(K(M)) \cap S^{n-1}$, consider the semicircle

$$[u,\ u\cos\pi + v\sin\pi) \tag{6}$$

starting from $u \in M$ in the direction of v. In Appendix B it is shown that this semicircle is divided into two segments. The points on the first segment have u as the unique projection and the points on the second do not. More precisely, define

$$\bar{\theta}(u, v) = \sup\{0 \leqslant \theta < \pi | u\cos\theta + v\sin\theta \in R(M),\ (u\cos\theta + v\sin\theta)_M = u\}. \tag{7}$$

Then u is the unique projection of points on the segment $[u,\ u\cos\bar{\theta}(u, v))$ but u is not the unique projection of $u\cos\theta + v\sin\theta$ for $\bar{\theta}(u, v) < \theta \leqslant \pi$. At the boundary $u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(u, v)$, u may or may not be the unique projection.

For the case $d = n-1$, that is, if M contains non-empty interior in S^{n-1} and u is an interior point, $N_u(K(M)) = \{0\}$ and $N_u(K(M)) \cap S^{n-1} = \emptyset$. To simplify the notation in this case, we define $\bar{\theta}(u, v) \equiv 0$ and

$$\bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\ u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(u, v)) = \{u\}. \tag{8}$$

Henceforth we use this notational convention for the case $d = n-1$ throughout the paper. Now we state the following basic proposition concerning $R(M)$ in (4).

Proposition 2.1. *For a locally conic closed set M with piecewise smooth proper boundary,*

$$R(M) \supset \bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\ u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(u, v)) \tag{9}$$

and the complement in S^{n-1} of the right-hand side of (9) *has zero spherical volume.*

The proof of Proposition 2.1 is given in Appendix B.

2.2. Exact tube formula and a formal tube formula

The open spherical tube of radius θ around a closed set $M \subset S^{n-1}$ is defined by

$$M_\theta = \{x | \mathrm{dist}(x, M) < \theta\}.$$

Classifying the points of the tube in terms of their projection onto M and the direction of the projection, M_θ can be written as

$$M_\theta = \bigcup_{u \in M} \bigcup_{v \in N_u(M), \|v\|=1} [u,\ u\cos\theta + v\sin\theta).$$

Note that here the cross-section

$$C_u(\theta) = \bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\, u\cos\theta + v\sin\theta)$$

may overlap for different values of u. If we only count points with unique projection onto M, we obtain

$$\tilde{M}_\theta = \bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\, u\cos\theta' + v\sin\theta') \subset M_\theta,$$

where $\theta' = \min(\theta, \bar{\theta}(u, v))$. Note that, by Proposition 2.1, $M_\theta - \tilde{M}_\theta$ is a null set.

Writing the tube $\tilde{M}_\theta$ as above, we see that $R(M)$ in (9) is a generalization of the tube where the radius of the tube depends on $u \in M$ and on $v \in N_u(K(M))$. Define

$$\bar{\theta}(u) = \inf_{v \in N_u(K(M)), \|v\|=1} \bar{\theta}(u, v).$$

In Example 2.3 consider the points $u_0 = (\pm 1, 0, 0)$. Note that $\bar{\theta}(u_0) = \pi/2$, whereas $\bar{\theta}(u) \to 0$ as $u \to u_0$. This example shows that $\bar{\theta}(u)$ may not be a continuous function of $u \in M$. The critical radius (or angle) of $M \subset \mathbb{R}^n$ is

$$\bar{\theta} = \inf\left\{ \bar{\theta}(u) | u \in \bigcup_{d=0}^{n-2} \partial M_d \right\}.$$

In this definition we omit the interior ∂M_{n-1} of $M \subset S^{n-1}$ when M contains a non-empty interior in S^{n-1}. In the case of positive critical radius $\bar{\theta} > 0$, the constant-radius tube $\bigcup_{u \in M} C_u(\bar{\theta})$ was essential for obtaining an asymptotic expansion of the tail probability of the maximum of $Z(u)$ of (1) and $Y(u)$ of (2). See Kuriki and Takemura (2001) and Takemura and Kuriki (2002).

As already mentioned, we have the following simple lemma concerning the critical radius $\bar{\theta}$ of M.

Lemma 2.1. *The critical radius of M is zero if, for some $x \in M$, the support cone $S_x(M)$ is not convex.*

The proof is outlined in Appendix B.

Now we study the volume of the tube M_θ, when M is a locally conic closed set with piecewise smooth proper boundary. Let $H(x, v)$ denote the second fundamental form of M at x in the direction v. Then from Lemma 2.3 of Takemura an Kuriki (2002) the volume element $\mathrm{d}y$ of S^{n-1} (induced from the Lebesgue measure of $\mathbb{R}^n$) at $y = x\cos\theta + v\sin\theta$, $x \in \partial M_d$, $v \in N_x(K(M))$, is written as

$$\mathrm{d}y = \det(I_d \cos\theta + H(x, v)\sin\theta)\sin^{n-d-2}\theta \;\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}v,$$

where $\mathrm{d}x$ denotes the volume element of ∂M_d at x and $\mathrm{d}v$ denotes the volume element of $S^{n-d-2} = N_x(K(M)) \cap S^{n-1}$ at v. Note that, for $\theta < \bar{\theta}(x, v)$, the matrix $(I_d \cos\theta + H(x, v)\sin\theta)$ is positive definite. This can be seen from the discussion on focal points in

Appendix B. Therefore, by the standard derivation of the tube formula, the spherical volume of the tube M_θ is written as

$$V(M_\theta) = \int_{M-I(M)} \mathrm{d}x \int_{v \in N_x(K(M)) \cap S^{n-1}} \mathrm{d}v \int_0^{\min(\theta, \bar{\theta}(x,v))} \mathrm{d}\tau \det(I_d \cos\tau + H(x, v) \sin\tau) \sin^{n-d-2}\tau$$

$$= \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{v \in N_x(K(M)) \cap S^{n-1}} \mathrm{d}v \sum_{j=0}^{d} \mathrm{tr}_j H(x, v) \int_0^{\min(\theta, \bar{\theta}(x,v))} \cos^{d-j}\tau \sin^{n-d+j-2}\tau \, \mathrm{d}\tau,$$

where $\mathrm{tr}_j H$ denotes the jth elementary symmetric function of the characteristic roots of H. Using the fact that, for $0 \leqslant \theta \leqslant \pi/2$,

$$\int_0^\theta \cos^a\tau \sin^b\tau \, \mathrm{d}\tau = \frac{\Omega_{a+b+2}}{\Omega_{a+1}\Omega_{b+1}} \bar{B}_{(a+1)/2,(b+1)/2}(\cos^2\theta),$$

where $\bar{B}_{k,l}$ denotes the upper probability function of the beta distribution with parameters (k, l) and

$$\Omega_c = V(S^{c-1}) = \frac{2\pi^{c/2}}{\Gamma(c/2)}$$

is the volume of S^{c-1}, we have established the following theorem.

Theorem 2.1. *For a locally conic closed set $M \subset S^{n-1}$ with piecewise smooth proper boundary, the spherical volume of the tube M_θ, $\theta \leqslant \pi/2$, is given by*

$$V(M_\theta) = \Omega_n \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{v \in N_x(K(M)) \cap S^{n-1}} \mathrm{d}v$$

$$\cdot \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{(d-j+1)/2,(n-d+j-1)/2}(\cos^2 \min(\theta, \bar{\theta}(x, v))). \tag{10}$$

Theorem 2.1 can be generalized to the case $\min(\theta, \bar{\theta}(x, v)) > \pi/2$ as in Proposition 2.1 of Takemura and Kuriki (2002). Exact tube formula for a submanifold of a Riemannian manifold is given in Lemma 8.3 of Gray (1990). Now we define a formal tube formula for $\theta \leqslant \pi/2$ by setting $\bar{\theta}(x, v) = \pi/2$.

Definition 2.1. *A formal tube formula approximation to the exact volume of the tube M_θ in* (10) *is defined by*

$$\tilde{V}(M_\theta) = \Omega_n \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{v \in N_x(K(M)) \cap S^{n-1}} \mathrm{d}v$$

$$\cdot \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{(d-j+1)/2,(n-d+j-1)/2}(\cos^2\theta). \tag{11}$$

For sets with positive critical radius $\bar{\theta} > 0$, (11) is the usual tube formula and coincides with the exact volume (10) for $\theta < \bar{\theta}$. For the case of zero critical radius there is actually an alternative way to define the formal tube formula, by requiring equivalence to the Euler characteristic method. We will discuss these points at the end of this section.

Since $V(M_\theta)/\Omega_n$ gives the exact tail probability of $\max_{u\in M} Y(u)$, we have the following.

Corollary 2.1. *For* $t \geqslant 0$,

$$P\left(\max_{u\in M} Y(u) \geqslant t\right) = \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{v\in N_x(K(M))\cap S^{n-1}} \mathrm{d}v \cdot \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{(d-j+1)/2,(n-d+j-1)/2}(\max(t^2, \bar{t}(x, v)^2)), \tag{12}$$

where $\bar{t}(x, v) = \cos\bar{\theta}(x, v)$.

We can also derive the exact tail probability for the maximum of $Z(u)$ in (1). Let g_k and G_k denote the density and the cumulative distribution function of the χ^2 distribution with k degrees of freedom, and write

$$Q_{k,l}(a, b) = \int_a^\infty g_k(x) G_l(bx)\mathrm{d}x.$$

Theorem 2.2. *Let* $M \subset S^{n-1}$ *be a locally conic closed set with piecewise smooth proper boundary. For* $t \geqslant 0$,

$$P\left(\max_{u\in M} Z(u) \geqslant t\right) = \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{N_x(K(M))\cap S^{n-1}} \mathrm{d}v \cdot \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \mathrm{Q}_{d-j+1,n-d+j-1}(t^2, \tan^2\bar{\theta}(x, v)). \tag{13}$$

Proof. Since, for $z \sim N_n(0, I_n)$, $y = z/\|z\|$ and $\|z\|$ are independent, $P(\max_{u\in M} Z(u) \geqslant t) = P(\max_{u\in M} u^{\mathrm{T}} z \geqslant t)$ is calculated by substituting $t := t/\|z\|$ in (12) and taking expectation with respect to $\|z\|^2 \sim \chi^2(n)$. Let B be a random variable distributed as $B(k, l)$, the beta distribution with parameters (k, l). Then, for $k + l = n$,

$$\begin{aligned} \mathrm{E}[\bar{B}_{k,l}(\max(t^2/\|z\|^2, \bar{t}^2))] &= P(\|z\|^2 B \geqslant t^2,\ B \geqslant \bar{t}^2) \\ &= P(\|z\|^2 B \geqslant t^2,\ \|z\|^2 B(1-\bar{t}^2)/\bar{t}^2 \geqslant \|z\|^2(1-B)) \\ &= Q_{k,l}(t^2, (1-\bar{t}^2)/\bar{t}^2), \end{aligned}$$

since $\|z\|^2 B$ and $\|z\|^2(1-B)$ are independently distributed according to $\chi^2(k)$ and $\chi^2(l)$, respectively. $\square$

The formal asymptotic expansion by the tube formula is obtained by letting $\bar{\theta}(x, v) = \pi/2$. In this case

$$Q_{d-j+1,n-d+j-1}(t^2, \infty) = \bar{G}_{d-j+1}(t^2) = 1 - G_{d-j+1}(t^2),$$

and the formal expansion is given by

$$\tilde{P}\left(\max_{u \in M} Z(u) \geqslant t\right) = \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}x \int_{N_x(K(M)) \cap S^{n-1}} \mathrm{d}v \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{G}_{d-j+1}(t^2), \qquad (14)$$

where $\bar{G}_k$ denotes the upper probability function of the χ^2 distribution with k degrees of freedom.

As mentioned above, there is an alternative definition of the formal tube formula for the case of zero critical radius. Here we give a brief discussion of this point. For piecewise smooth M with convex support cone (and hence with positive critical radius $\bar{\theta} > 0$), Takemura and Kuriki (2002) established the equivalence of the tube formula and the Euler characteristic method in the sense that

$$V(M_\theta) = \int_{S^{n-1}} \chi(A(y, \theta))\mathrm{d}y, \qquad \theta < \bar{\theta},$$

where

$$A(y, \theta) = \{u \in M | u^{\mathrm{T}} y \geqslant \cos\theta\} \qquad (15)$$

is the excursion set, $\chi(\cdot)$ is the Euler characteristic, and $\mathrm{d}y$ is the volume element of S^{n-1}. For M with smooth boundary it is a consequence of the kinematic fundamental formula (Santaló 1976, IV.18.3). An alternative definition of the formal tube formula for the case $\bar{\theta} = 0$ may be given by requiring the equivalence to the Euler characteristic method – that is, we define

$$\hat{V}(M_\theta) = \int_{S^{n-1}} \chi(A(y, \theta))\mathrm{d}y. \qquad (16)$$

In convex analysis, the tube formula for a convex body K in $\mathbb{R}^n$ is called the Steiner formula and the coefficients of the tube formula are called curvature measures of K. The notion of the curvature measures of convex bodies can be generalized in various ways. When M belongs to the convex ring (the set of finite unions of convex bodies), $\tilde{V}(M_\theta)$ of Definition 2.1 corresponds to the absolute curvature measures of M, whereas the alternative definition $\hat{V}(M_\theta)$ in (16) corresponds to additive extension of curvature measures of M. These notions are discussed in Matheron (1975, Section 4.7), Schneider (1993, Section 4.4) and Stoyan *et al*. (1995, Section 7.3.4). See also Cheeger *et al*. (1986) for the alternative definition (16) when M is piecewise linear.

Our example in Section 3.1 shows that $\tilde{V}(M_\theta)$ and $\hat{V}(M_\theta)$ are different in general for M with zero critical radius, and furthermore both of them lead to incorrect expansion of the exact volume $V(M_\theta)$. The reason for adopting $\tilde{V}(M_\theta)$ as our definition is that its discrepancy from the exact volume $V(M_\theta)$ in (10) is easier to study. At present we know of no integral expression for $\hat{V}(M_\theta)$ similar to (10) for $\bar{\theta} = 0$ in the literature.

2.3. Invalidity of formal expansion

In this subsection we show that when the critical radius $\bar{\theta}$ is zero, the formal tube formula only gives a valid main term, with other higher-order expansion terms not being valid in general. Concerning the tail probability of $\max_{u\in M} Y(u)$, we let $\theta\downarrow 0$ and compare the Taylor expansion of (10) and (11). Similarly, we let $t\to\infty$ and compare (13) and (14).

First, we consider the main terms of the expansions. In (10) the main term is given by $d=m$, $j=0$. The case $m=n-1$ is trivial, because in this case (10) and (11) converge to $V(M)=V(\partial M_{n-1})>0$. Therefore, let $m<n-1$. Then

$$V(M_\theta)\sim\int_{\partial M_m}\mathrm{d}x\int_{v\in N_x(K(M))\cap S^{n-1}}\mathrm{d}v\,\frac{\Omega_n}{\Omega_{m+1}\Omega_{n-m-1}}\bar{B}_{(m+1)/2,(n-m+1)/2}(\cos^2\min(\theta,\,\bar{\theta}(x,\,v))).$$

Write $a=(m+1)/2,\ b=(n-m-1)/2$. Ignoring the constant, which is common to (10) and (11), consider

$$\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v\int_{\cos^2\min(\theta,\,\bar{\theta}(x,\,v))}^{1}\xi^{a-1}(1-\xi)^{b-1}\,\mathrm{d}\xi$$

$$\begin{aligned}&\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v\int_{\cos^2\min(\theta,\,\bar{\theta}(x,\,v))}^{1}(1-\xi)^{b-1}\,\mathrm{d}\xi\\&=\frac{1}{b}\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v\sin^{2b}\min(\theta,\,\bar{\theta}(x,\,v))\\&=\frac{\theta^{2b}}{b}\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v\,\frac{\sin^{2b}\min(\theta,\,\bar{\theta}(x,\,v))}{\theta^{2b}}.\end{aligned}$$

Now for each fixed $(x,\,v)$ such that $x\notin I(M)$, $\bar{\theta}(x,\,v)>0$, because M is locally conic and $N_x(K(M))\neq\{0\}$. Therefore,

$$\frac{\sin^{2b}\min(\theta,\,\bar{\theta}(x,\,v))}{\theta^{2b}}\to 1,\qquad \theta\to 0,$$

and by the dominated convergence theorem we have

$$\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v\,\frac{\sin^{2b}\min(\theta,\,\bar{\theta}(x,\,v))}{\theta^{2b}}\to\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v.$$

Taking the constant into account again, we obtain

$$V(M_\theta)\sim\frac{\theta^{n-m-1}}{n-m-1}\int_{\partial M_m}\mathrm{d}x\int_{N_x(K(M))\cap S^{n-1}}\mathrm{d}v,\qquad \theta\to 0.$$

However, this is the main term of $\tilde{V}(M_\theta)$ as well. Therefore, we have shown that (10) and (11) have the same main term.

Proving that (13) and (14) have the same main term

$$P\left(\max_{u\in M} Z(u) \geqslant t\right) \sim \frac{\Gamma((n-m-1)/2)}{2^{(m+3)/2}\,\pi^{n/2}}\, t^{m-1}\, \mathrm{e}^{-t^2/2} \int_{\partial M_m} \mathrm{d}x \int_{N_x(K(M))\cap S^{n-1}} \mathrm{d}v, \qquad t \to \infty,$$

is entirely similar, by noting that, for each (x, v), $x \notin I(M)$,

$$\frac{Q_{a,n-a}(t^2, \tan^2 \bar{\theta}(x, v))}{\bar{G}_a(t^2)} \to 1, \qquad t \to \infty.$$

We proceed to show that in general the higher-order terms of (10) and (11) or of (13) and (14) are not equal. The arguments for these two cases are entirely similar. Here we discuss only the difference between (13) and (14). In order to show the discrepancy we only consider expansion terms arising from the term $d = m$, $j = 0$, in the summation of (13) and (14). Ignoring $1/(\Omega_{m+1}\Omega_{n-m-1})$, the difference of these two terms is written as

$$\int_{\partial M_m} \mathrm{d}x \int_{N_x(K(M))\cap S^{n-1}} \mathrm{d}v \int_{t^2}^{\infty} g_{m+1}(\xi)\, \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x, v))\, \mathrm{d}\xi. \tag{17}$$

Define

$$A(t) = \{(x, v) | x \in \partial M_m,\ v \in N_x(K(M)) \cap S^{n-1},\ \tan \bar{\theta}(x, v) \leqslant 1/t\}.$$

Now assume that there exists some $k > 0$ such that

$$\int_{A(t)} \mathrm{d}x\, \mathrm{d}v = O(t^{-k}). \tag{18}$$

Fix $c > 1$. Then (17) is bounded below by

$$\begin{aligned}
&\int_{\partial M_m} \mathrm{d}x \int_{N_x(K(M))\cap S^{n-1}} \mathrm{d}v \int_{t^2}^{ct^2} g_{m+1}(\xi) \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x, v))\, \mathrm{d}\xi \\
&\qquad \geqslant \int_{A(t)} \mathrm{d}x\, \mathrm{d}v \int_{t^2}^{ct^2} g_{m+1}(\xi)\, \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x, v))\, \mathrm{d}\xi \\
&\qquad \geqslant \bar{G}_{n-m-1}(c) \int_{A(t)} \mathrm{d}x\, \mathrm{d}v \int_{t^2}^{ct^2} g_{m+1}(\xi)\, \mathrm{d}\xi \\
&\qquad = O(t^{-k} \bar{G}_{m+1}(t^2)) = O(\bar{G}_{m+1-k}(t^2)).
\end{aligned}$$

However, the term of order $O(\bar{G}_{m+1-k}(t^2))$ is not distinguishable from higher-order expansion terms of (13) or (14). Therefore, we have shown that the higher-order terms of (13) and (14) are not generally equal when (18) holds.

It may be the case that k in (18) is large and that many terms of the formal asymptotic expansion are correct. In this case we may want to approximate the tail probability using only the correct terms of the asymptotic expansion. Therefore it is important to determine the value of k in (18) for a given problem.

We now argue that in certain regular cases k in (18) is simply the difference between $m = \dim M$ and the dimension of the set of points with non-convex support cone.

We require the technical assumption that there exists $c \geqslant 1$ such that, on ∂M_m,

$$\liminf_{t\to 0} \inf_{x:\tan\bar\theta(x)\leqslant 1/(ct)} \int_{v\in N_x(K(M))\cap S^{n-1},\tan\bar\theta(x,v)\leqslant 1/t} \mathrm{d}v > 0. \tag{19}$$

This condition implies that, for sufficiently small t, the angle of $N_x(K(M)) \cap \{v|\bar\theta(x, v) \leqslant 1/t\}$ is bounded away from 0 for all $x \in \partial M_m$ with $\bar\theta(x) \leqslant 1/(ct)$. Now for $c \geqslant 1$,

$$\int_{x:\tan\bar\theta(x)\leqslant 1/(ct)} \mathrm{d}x \int_{v\in N_x(K(M))\cap S^{n-1},\tan\bar\theta(x,v)\leqslant 1/t} \mathrm{d}v \leqslant \int_{A(t)} \mathrm{d}x\,\mathrm{d}v \leqslant \Omega_{n-m} \int_{x:\tan\bar\theta(x)\leqslant 1/t} \mathrm{d}x.$$

Therefore, under assumption (19),

$$\int_{A(t)} \mathrm{d}x\,\mathrm{d}v = O(t^{-k}) \Leftrightarrow \int_{x:\tan\bar\theta(x)\leqslant 1/t} \mathrm{d}x = O(t^{-k})$$

and k can be evaluated from the volume of the set $\{x \in \partial M_m | \tan\bar\theta(x) \leqslant 1/t\}$.

Let $\overline{M}$ denote the set of points on the relative boundary of ∂M_m with non-convex support cone. We now make a second assumption that $\overline{M}$ forms a C^2-submanifold of $\mathbb{R}^n$ of dimension l. Finally, we assume that, for $x \in \partial M_m$, $\bar\theta(x) = O(1/t)$ if and only if $\mathrm{dist}(x, \overline{M}) = O(1/t)$. Under these assumptions the set $\{x \in \partial M_m|\bar\theta(x) \leqslant 1/t\}$ is basically a tube around $\overline{M}$ in ∂M of radius $O(1/t)$. Therefore the m-dimensional volume of this tube is proportional to $O(t^{-k})$ with

$$k = m - l = \dim M - \dim \overline{M}.$$

3. Examples

The formulae for exact tail probabilities in Section 2.2 are of theoretical importance. However, they may be difficult to evaluate explicitly for a given problem. Therefore in this section we investigate in detail some simple examples in which the exact tail probability as well as the formal expansion by the tube formula and the Euler characteristic method can be explicitly evaluated and the discrepancy between them can be clearly understood. A detailed treatment of a more complicated but statistically natural example is given in Takemura and Kuriki (2001).

3.1. Boundary of polyhedral cone

Here we consider simple examples of the tail probability of the maximum of $Y(u)$ in (2) corresponding to Examples 2.1 and 2.3. Consider the uniform distribution $\mathrm{Unif}(S^2)$ on the sphere S^2 in $\mathbb{R}^3$. For simplicity of notation we avoid subscripts and let (x, y, z) denote a vector on $\mathbb{R}^3$ or on S^2. Note that $\Omega_3 = 4\pi$, $\Omega_2 = 2\pi$, $\Omega_1 = 2$.

First, we discuss Example 2.1. If M is as in (3), then

$$\max_{u\in M} Y(u) = \max\left(\sqrt{x^2+y^2},\ \sqrt{x^2+z^2}\right)$$
$$= \max\left(\sqrt{1-z^2},\ \sqrt{1-y^2}\right)$$

and

$$P\left(\max_{u\in M} Y(u) \geqslant \cos\theta\right) = \frac{V(M_\theta)}{4\pi}.$$

$Y(u)$ corresponds to the maximum of two correlated beta variables. This type of statistic is commonly used in change-point analysis or multiple comparisons. M consists of four arcs of length π and two crossing points at $(\pm 1, 0, 0)$. These two points are improper and do not contribute to the volume of the tube. On the other hand, the four arcs form the one-dimensional proper boundary of M. Without loss of generality, consider points on the arc $u = (\cos\tau, \sin\tau, 0)$, $0 \leqslant \tau \leqslant \pi$. $N_u(K(M)) \cap S^2 = (0, 0, \pm 1)$ and the cross-section at u is the arc

$$C_u(\theta) = \cos\xi(\cos\tau, \sin\tau, 0) + \sin\xi(0, 0, 1), \qquad |\xi| < \theta.$$

u is the unique projection of points in $C_u(\theta)$ if and only if

$$|\sin\xi| < \cos\xi \sin\tau.$$

Therefore, for $v = (0, 0, \pm 1)$,

$$\bar\theta(u, v) = \arctan(\sin\tau).$$

Now in (10) and (11) $m = d = 1$, $\int_{N_u(K(M))\cap S^2} \mathrm{d}v = 2$, $H = 0$, $\mathrm{tr}_0 H = 1$, $\mathrm{tr}_1 H = 0$, and

$$\bar B_{1,1/2}(t^2) = \frac{1}{2}\int_{t^2}^{1} (1-\xi)^{-1/2}\,\mathrm{d}\xi = (1-t^2)^{1/2}.$$

The point which makes the largest angle from M is $(0, 1, 1)/\sqrt{2}$, and this angle is $\pi/4$.

We first consider the formal tube formula, because it is simpler. We have

$$\tilde V(M_\theta) = 2\,\Omega_3 \int_{4\text{ arcs}} \mathrm{d}u\, \frac{1}{\Omega_1\Omega_2}\, \bar B_{1,1/2}(\cos^2\theta)$$
$$= 8\sin\theta \int_0^{\pi} \mathrm{d}\tau = 8\pi\sin\theta.$$

This is the sum of the spherical areas of two bands around the two great circles of M.

In this example the alternative definition $\hat V(M_\theta)$ can also be explicitly computed. The difference between $\tilde V(M_\theta)$ and $\hat V(M_\theta)$ comes from excursion sets $A(y, \theta)$ of (15) for y near two crossing points. Figure 1 depicts the excursion set $A(y, \theta)$ for y close to a crossing point, which is placed at the origin. In $\tilde V(M_\theta)$, points in the 'square' $[-\theta, \theta] \times [-\theta, \theta]$ are counted twice. On the other hand, in $\hat V(M_\theta)$ the points in the spherical circle of radius θ are counted once, because the Euler characteristic of the latter is 1 and hence $\chi(A(y, \theta)) = 1$ for y in the spherical circle. Note that points y' in the square outside the circle (shaded area in Figure 1) are counted twice, because $A(y', \theta)$ consists of two line

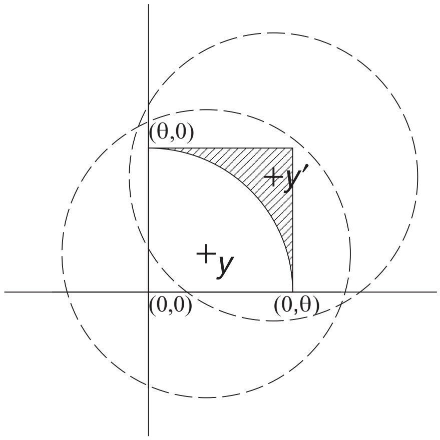

Figure 1. Euler characteristic around a crossing point.

segments and $\chi(A(y', \theta)) = 2$. The area of a spherical circle of radius θ is given by $2\pi(1 - \cos\theta)$. Hence

$$\hat{V}(M_\theta) = \tilde{V}(M_\theta) - 4\pi(1 - \cos\theta) = 8\pi\sin\theta - 4\pi(1 - \cos\theta).$$

We now consider the true volume $V(M_\theta)$. We only consider $\theta \leqslant \pi/4$. Write $\theta = \arctan(\sin\tau_0)$ or $\tau_0 = \arcsin(\tan\theta)$. Then

$$\min(\theta, \bar{\theta}(u, v)) = \begin{cases} \arctan(\sin\tau), & \text{if } 0 \leqslant \tau \leqslant \tau_0, \\ \theta, & \text{if } \tau_0 < \tau < \pi - \tau_0, \\ \arctan(\sin\tau), & \text{if } \pi - \tau_0 \leqslant \tau \leqslant \pi. \end{cases}$$

The contribution of the middle case to the volume is

$$8\sin\theta\,(\pi - 2\tau_0) = 8\pi\sin\theta - 16\sin\theta\arcsin(\tan\theta).$$

The contribution from the region where $\bar{\theta}(u, v) < \theta$ is

$$16\int_0^{\tau_0}\sin(\arctan(\sin\tau))\,\mathrm{d}\tau = 16\int_0^{\tau_0}\frac{\sin\tau}{\sqrt{1 + \sin^2\tau}}\,\mathrm{d}\tau.$$

Let $w = \sin^2\tau$, with $\mathrm{d}w = 2\sin\tau\cos\tau\,\mathrm{d}\tau$. Then

$$\begin{aligned} 16\int_0^{\tau_0}\frac{\sin\tau}{\sqrt{1 + \sin^2\tau}}\,\mathrm{d}\tau &= 8\int_0^{\sin^2\tau_0}\frac{1}{\sqrt{1 + w}\sqrt{1 - w}}\,\mathrm{d}w \\ &= 8\int_0^{\tan^2\theta}\frac{1}{\sqrt{1 - w^2}}\,\mathrm{d}w \\ &= 8\arcsin(\tan^2\theta). \end{aligned}$$

Therefore, we obtain

$$V(M_\theta) = 8\pi \sin\theta - 16 \sin\theta \arcsin(\tan\theta) + 8 \arcsin(\tan^2\theta).$$

Note that both $V(M_\theta)$ and $\tilde{V}(M_\theta)$ are $O(\theta)$ and they differ in the term of order $O(\theta^2)$:

$$\tilde{V}(M_\theta) = V(M_\theta) + 8\theta^2 + o(\theta^2).$$

$\hat{V}(M_\theta)$ also differs from $V(M_\theta)$ in the term of order $O(\theta^2)$:

$$\hat{V}(M_\theta) = V(M_\theta) + (8 - 2\pi)\theta^2 + o(\theta^2).$$

$\bar{\theta}(u, v)$ tends to zero around two crossing points of M. Note that in this example the conditions of Section 2.3 are satisfied with $c = 1$ in (19). The volume (actually the length in this example) of points $u \in \partial M_1$ with $\bar{\theta}(u) \leqslant 1/t$ is $O(1/t)$. Therefore, $k = 1$ in (18). This corresponds to the difference between $1 = \dim M$ and 0, which is the dimension of these two points.

We now turn to Example 2.3. The necessary calculation has already been done. The formal tube formula $\tilde{V}(M_\theta)$ consists of the sum of two areas of half bands and the area of two quarter spherical circles corresponding to the normal cone at $(\pm 1, 0, 0)$. Therefore

$$\tilde{V}(M_\theta) = 4\pi \sin\theta + \pi(1 - \cos\theta).$$

We obtain $\hat{V}(M_\theta)$ by subtracting the area of two spherical quarter circles from $\tilde{V}(M_\theta)$:

$$\hat{V}(M_\theta) = 4\pi \sin\theta + \pi(1 - \cos\theta) - \pi(1 - \cos\theta) = 4\pi \sin\theta.$$

On the other hand, the true volume $V(M_\theta)$ is obtained by further subtracting twice the shaded area of Figure 1:

$$V(M_\theta) = 4\pi \sin\theta - \pi(1 - \cos\theta) + 4 \sin\theta \arcsin(\tan\theta) - 2 \arcsin(\tan^2\theta).$$

We again see that these three are all different and the difference appears at the order $O(\theta^2)$.

3.2. Direct product of two cones

Here we consider a generalization of Examples 2.1 and 2.3. For $i = 1, 2$, let K_i be smooth cones in $\mathbb{R}^{n_i}$ such that $M_i = K_i \cap S^{n_i - 1}$ is a d_i-dimensional closed manifold. Define the cone of direct product

$$K = K_1 \times K_2 = \{(x_1, x_2) | x_1 \in K_1,\ x_2 \in K_2\} \subset \mathbb{R}^n$$

and its intersection with the unit sphere $M = K \cap S^{n-1}$, where $n = n_1 + n_2$. Note that M is of dimension $d = d_1 + d_2 + 1$ and is expressed as

$$M = \{(u_1 \cos\varphi,\ u_2 \sin\varphi) | u_1 \in M_1,\ u_2 \in M_2,\ \varphi \in [0, \pi/2]\}. \tag{20}$$

Examples 2.1 and 2.3 correspond the case where $K_1 = \mathbb{R}^1$ and

$$K_2 = \begin{cases} \{(0, y) \in \mathbb{R}^2 | y \in \mathbb{R}^1\} \cup \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}^1\} & \text{(Example 2.1)}, \\ \{(0, y) \in \mathbb{R}^2 | y \geqslant 0\} \cup \{(x, 0) \in \mathbb{R}^2 | x \geqslant 0\} & \text{(Example 2.3)}. \end{cases}$$

As seen in Examples 2.1 and 2.3, M in (20) generally has zero critical radius, although

M_1 and M_2 have positive critical radii. This is because the support cone of M at $u = (u_1, 0) \in M,\ u_1 \in M_1,$ is

$$S_u(M) = T_{u_1}(M_1) \times K_2,$$

which is not convex unless K_2 is convex. Conversely, the set of regular points of M is

$$M - I(M) = \{(u_1 \cos\varphi,\ u_2 \sin\varphi) | u_1 \in M_1,\ u_2 \in M_2,\ \varphi \in (0, \pi/2)\}$$

when both of K_1 and K_2 are non-convex.

The following proposition, proved in Appendix C, gives the formal tube formula for M.

Proposition 3.1. *Assume that the formal tube formula for M_i, $i = 1, 2$, is given by*

$$V((M_i)_\theta) = \Omega_{n_i} \sum_{j=0}^{d_i} w_{ij} \bar{B}_{(d_i - j + 1)/2, (n_i - d_i + j - 1)/2}(\cos^2\theta),$$

which is exact for $0 \leqslant \theta \leqslant \theta_{ic}$, where $\theta_{ic} > 0$ is the critical radius of M_i. Then the formal tube formula (11) *for M is*

$$\tilde{V}(M_\theta) = \Omega_n \sum_{j_1=0}^{d_1} \sum_{j_2=0}^{d_2} w_{1j_1} w_{2j_2} \bar{B}_{(d - j_1 - j_2 + 1)/2,\, (n - d + j_1 + j_2 - 1)/2}(\cos^2\theta).$$

As an example, consider the cones in $\mathbb{R}^6$ given by

$$K_1 = K_2 = \{x \otimes y \in \mathbb{R}^6 | x \in \mathbb{R}^2,\ y \in \mathbb{R}^3\},$$

where $\otimes$ denotes the Kronecker product. Cones of this type are defined by bilinear forms and investigated fully in Kuriki and Takemura (2001). The formal tube formula for $M_i = K_i \cap S^{6-1}$, $i = 1, 2$, is given by

$$V((M_i)_\theta) = \Omega_6 \Big\{ 2\bar{B}_{4/2,2/2}(\cos^2\theta) - 2\bar{B}_{2/2,4/2}(\cos^2\theta) \Big\}, \tag{21}$$

which is exact for $0 \leqslant \theta \leqslant \pi/4$. Write $K = K_1 \times K_2$ and $M = K \cap S^{12-1}$ as before.

Let $z = (z_1, z_2) \in \mathbb{R}^{12}$, $z_1, z_2 \in \mathbb{R}^6$, be random vectors consisting of independent standard normal random variables. Then $\max_{u \in M_i} u^{\mathrm{T}} z_i = \sqrt{\ell_1(W_i)}$, $i = 1, 2$, and

$$\max_{u \in M} u^{\mathrm{T}} z = \max_{u_i \in M_i, \varphi \in [0, \pi/2]} (u_1^{\mathrm{T}} z_1 \cos\varphi + u_2^{\mathrm{T}} z_2 \sin\varphi) = \sqrt{\ell_1(W_1) + \ell_1(W_2)},$$

where W_1, W_2 are 2×2 matrices independently following the Wishart distribution with three degrees of freedom and scale matrix I_2, $\mathrm{Wis}_2(3, I_2)$, and are a function of z_1 and z_2. Here $\ell_1(\cdot)$ denotes the largest eigenvalue of the matrix. The tube formula (21) gives an asymptotic expansion for the upper tail probability

$$P(\ell_1(W_i) \geqslant t) \sim 2\bar{G}_4(t) - 2\bar{G}_2(t), \qquad t \to \infty.$$

By Proposition 3.1 and (14), the formal tube formula is given by

$$\tilde{P}(\ell_1(W_1) + \ell_1(W_2) \geqslant t) = 4\bar{G}_8(t) - 8\bar{G}_6(t) + 4\bar{G}_4(t). \tag{22}$$

In the case of this example the exact distribution of $\ell_1(W_i)$ can be obtained in a simple form. The joint density of the eigenvalues (ℓ_1, ℓ_2), $\ell_1 \geqslant \ell_2 \geqslant 0$, of a $\mathrm{Wis}_2(3, I_2)$ matrix is known to be

$$\frac{1}{4}\,\mathrm{e}^{-(\ell_1+\ell_2)/2}(\ell_1-\ell_2), \qquad \ell_1 \geqslant \ell_2 \geqslant 0. \tag{23}$$

By integrating this over the region $\ell_1 \geqslant t$, $\ell_1 \geqslant \ell_2 \geqslant 0$, we have

$$P(\ell_1 \geqslant t) = 2\overline{G}_4(t) - 2\overline{G}_2(t) + \overline{G}_2(2t),$$

with moment generating function $\mathrm{E}(\mathrm{e}^{\theta l_1}) = 2(1-2\theta)^{-2} - 2(1-2\theta)^{-1} + (1-\theta)^{-1}$. Square the moment generating function and for the term $(1-2\theta)^{-\nu/2}(1-\theta)^{-1}$ use the asymptotic relation

$$P\left(\chi_\nu^2 + \frac{1}{2}\chi_2^2 \geqslant t\right) = 2\overline{G}_\nu(t) + O(\mathrm{e}^{-t/2}\,t^{\nu/2-2}), \qquad t \to \infty,$$

where χ_ν^2 and χ_2^2 are independent χ^2 random variables. Then the tail probability of the sum of the two largest eigenvalues is evaluated as

$$P(\ell_1(W_1) + \ell_1(W_2) \geqslant t) = 4\overline{G}_8(t) - 8\overline{G}_6(t) + 8\overline{G}_4(t) + O(\overline{G}_2(t)).$$

Therefore, we see that the formal tube formula (22) is invalid in the term of order $O(\overline{G}_4(t))$. This was expected since $k = \dim M - \dim I(M) = 7 - 3 = 4$.

3.3. Euler characteristic method applied to χ^2 field

We continue to examine the example of sum of the largest eigenvalues of two independent Wishart matrices $\ell_1(W_1) + \ell_1(W_2)$, $W_1, W_2 \sim \mathrm{Wis}_2(3, I_2)$.

Define a χ^2 field with index set $S^1 \times S^1$:

$$\begin{aligned} X(u) &= \sum_{i=1}^{2} (\cos(\phi_i/2), \sin(\phi_i/2))\, W_i \begin{pmatrix} \cos(\phi_i/2) \\ \sin(\phi_i/2) \end{pmatrix} \\ &= \sum_{i=1}^{2} \left(\frac{w_{i11} - w_{i22}}{2} \cos\phi_i + w_{i12}\sin\phi_i + \frac{w_{i11} + w_{i22}}{2} \right), \end{aligned}$$

where $u = (\cos\phi_1, \sin\phi_1, \cos\phi_2, \sin\phi_2)$, $0 \leqslant \phi_1, \phi_2 < 2\pi$. The (j_1, j_2)th element of W_i is denoted by $w_{ij_1j_2}$. This is a χ^2 field in a sense that, for each u fixed, $X(u)$ is distributed as χ^2 with two degrees of freedom. In the following we apply the Euler characteristic method to approximate the upper tail probability of

$$\max_{u \in S^1 \times S^1} X(u) = \ell_1(W_1) + \ell_1(W_2).$$

The Euler characteristic method approximates the tail probability by

$$P(\ell_1(W_1) + \ell_1(W_2) \geqslant t) \approx \mathrm{E}[\chi(A(t))],$$

where

$$A(t) = \{u \in S^1 \times S^1 | X(u) \geqslant t\}$$

is the excursion set. The expectation $\mathrm{E}[\chi(A(t))]$ can be evaluated with the help of Morse theory.

Define a (random) function on $S^1 \times S^1$ by $f(u) = -X(u)$. With probability one there exist four critical points of f:

$$u^* = (\epsilon_1 \cos\phi_1^*, \epsilon_1 \sin\phi_1^*, \epsilon_2 \cos\phi_2^*, \epsilon_2 \sin\phi_2^*), \qquad \epsilon_1 = \pm 1, \epsilon_2 = \pm 1,$$

where

$$\phi_i^* = \tan^{-1} \frac{2w_{i12}}{w_{i11} - w_{i22}}, \qquad i = 1, 2.$$

The Hessian at each critical point is shown to be

$$\det\left(\frac{\partial^2 f(u)}{\partial\phi_i \partial\phi_j}\right)\bigg|_{u=u^*} = \prod_{i=1}^{2} \epsilon_i \sqrt{\left(\frac{w_{i11} - w_{i22}}{2}\right)^2 + w_{i12}^2},$$

which is non-zero with probability one. Therefore $f(u)$ is a Morse function with probability one. Then, by Morse's theorem (Worsley 1995b, Theorem 1; Takemura and Kuriki 2002, Proposition 3.1), we have

$$\begin{aligned}
\chi(A(t)) &= \sum_{u:\text{critical point}} I(f(u) \leqslant -t) \operatorname{sgn}\det\left(\frac{\partial^2 f(u)}{\partial\phi_i \partial\phi_j}\right) \\
&= \sum_{u:\text{critical point}} \epsilon_1\epsilon_2\, I\left(\sum_{i=1}^{2}\left(\epsilon_i \sqrt{\left(\frac{w_{i11} - w_{i22}}{2}\right)^2 + w_{i12}^2} + \frac{w_{i11} + w_{i22}}{2}\right) \geqslant t\right) \\
&= I(\ell_1(W_1) + \ell_1(W_2) \geqslant t) - I(\ell_1(W_1) + \ell_2(W_2) \geqslant t) \\
&\quad - I(\ell_2(W_1) + \ell_1(W_2) \geqslant t) + I(\ell_2(W_1) + \ell_2(W_2) \geqslant t),
\end{aligned}$$

with probability one, where $\ell_1(W_i) \geqslant \ell_2(W_i)$ are ordered eigenvalues of W_i. Note that the joint distribution of the eigenvalues is given in (23). Simple calculations yield $P(\ell_2 \geqslant t) = \overline{G}_2(2t)$,

$$P(\ell_1 \geqslant t) - P(\ell_2 \geqslant t) = 2\overline{G}_4(t) - 2\overline{G}_2(t),$$

and hence

$$\begin{aligned}
\mathrm{E}[\chi(A(t))] &= P(\ell_1(W_1) + \ell_1(W_2) \geqslant t) - P(\ell_1(W_1) + \ell_2(W_2) \geqslant t) \\
&\quad - P(\ell_2(W_1) + \ell_1(W_2) \geqslant t) + P(\ell_2(W_1) + \ell_2(W_2) \geqslant t) \\
&= 4\overline{G}_8(t) - 8\overline{G}_6(t) + 4\overline{G}_4(t).
\end{aligned}$$

The last equality can be easily verified by the moment generating function. This result coincides with the formal tube approximation (22), and the coefficient of the lowest term of order $O(\overline{G}_4(t))$ is incorrect.

Appendix A. Definition of locally conic set and related notions

Here we give precise definitions of various notions in Section 2.1. Throughout Section 2 we consider spherical tubes around $M \subset S^{n-1}$. Here we begin by considering $M \subset \mathbb{R}^n$ and the volume of tubes in $\mathbb{R}^n$ for simplicity. Once we have a proof for tubes in $\mathbb{R}^n$, it is straightforward to adapt it to the spherical tube.

Let M be a closed subset of $\mathbb{R}^n$. For each $x \in M$, we assume that M is locally approximated by a cone in the following definition.

Definition A.1. *A closed subset M of $\mathbb{R}^n$ is locally conic of class C^2 if, for each $x \in M$, there exist an open neighbourhood $U(x) \subset \mathbb{R}^n$ of x, $\epsilon = \epsilon_x > 0$, a C^2-diffeomorphism $\phi_x : (-\epsilon, \epsilon)^n \to U(x)$ with $\phi_x(0) = x$ and a closed cone $K = K_{\phi_x}$ of $\mathbb{R}^n$ such that $M \cap U(x)$ is the image of $K \cap (-\epsilon, \epsilon)^n$ by ϕ_x:*

$$M \cap U(x) = \phi_x(K \cap (-\epsilon, \epsilon)^n).$$

Furthermore, if $V = U(x) \cap U(x') \neq \emptyset$ for $x, x' \in M$, then $\phi_{x'}^{-1} \circ \phi_x$: $\phi_x^{-1}(V) \to \phi_{x'}^{-1}(V)$ is a C^2-diffeomorphism.

In Definition A.1 we are following the standard definition of a differentiable manifold. However, M may not be a standard manifold because we allow self-intersections in M. The definition of the locally conic set is the same when M is a subset of S^{n-1}.

For locally conic M we define the supporting cone and the normal cone at each $x \in M$ as follows. The support cone (or the tangent cone) of M at $x \in M$ is the image of K by the differential $\mathrm{d}\phi$ at the origin:

$$S_x(M) + x = \mathrm{d}\phi_{|(0,\ldots,0)} K. \tag{24}$$

Note that '+' on the left-hand side of (24) is the vector sum and hence $S_x(M)$ is defined with its vertex located at the origin. Let $C(S_x(M))$ be the convex hull of $S_x(M)$. The normal cone $N_x(M)$ of M at x is the dual cone of $C(S_x(M))$ in $\mathbb{R}^n$:

$$N_x(M) = \{y | y^{\mathrm{T}} v \leqslant 0, \forall v \in S_x(M)\} = \{y | y^{\mathrm{T}} v \leqslant 0, \forall v \in C(S_x(M))\}.$$

Here note that by definition the dual cone of $S_x(M)$ coincides with the dual cone of $C(S_x(M))$. For the case of geodesically convex M, the notions of support cone and normal cone given here coincide with the standard notions in convex analysis (Schneider 1993, Section 2.2; Takemura and Kuriki 1997, Section 2.3). Takemura and Kuriki (2002) assumed that $S_x(M)$ is convex for all $x \in M$ and proved the validity and the equivalence of the tube method and the Euler characteristic method. In the present paper $S_x(M)$ may not be convex and the distinction between $S_x(M)$ and its convex hull is important.

For $x \in M$, let

$$d = n - \dim N_x(M)$$

be the codimension of $N_x(M)$. Note that d is the dimension of the largest linear subspace contained in $C(S_x(M))$:

$$L = C(S_x(M)) \cap (-C(S_x(M))).$$

If L is contained in $S_x(M)$, then clearly L is the unique largest linear subspace contained in $S_x(M)$, and in this sense L is the tangent space $T_x(M)$ of M at x. On the other hand, if L is not contained in $S_x(M)$, then there are two non-nested linear subspaces contained in $S_x(M)$ and M does not possess a tangent space at x. In the tube formula the n-dimensional volume of the tube is obtained by integrating the product of the volume element of $N_x(M)$, the volume element of the tangent space $T_x(M)$ and the Jacobian containing the second fundamental form at x. This implies that if L is not contained in $S_x(M)$, then there should be no contribution to the volume of the tube from x. On the other hand, if L is contained in $S_x(M)$, the contribution of points in $N_x(M)$ to the volume of the tube is the same for convex or non-convex $S_x(M)$. This is the motivation for the definition of the proper boundary in Section 2.1; for convenience we now give a formal definition.

Definition A.2. *Let M be locally conic and, for $x \in M$, let $d = n - \dim N_x(M)$. x is a proper d-dimensional boundary point if $L = C(S_x(M)) \cap (-C(S_x(M)))$ is contained in $S_x(M)$.*

Appendix B. Proofs of Proposition 2.1 and Lemma 2.1

We begin with a proof of Proposition 2.1. We then state a version of Proposition 2.1 for tubes in $\mathbb{R}^n$. Finally, we outline the proof of Lemma 2.1.

Proof of Proposition 2.1. We first show that the complement of $R(M)$ in S^{n-1} has zero spherical volume. Let $x \notin R(M)$. Then there are at least two equidistant projections y_1, y_2 of x onto M. By Assumption 2.1 it suffices to consider the case where both y_1 and y_2 are proper boundary points of M. We need to distinguish two cases of non-uniqueness of projection. One case is that x is a 'focal point' of y_1 or y_2 in the sense of Milnor (1963, p. 33). Arguments similar to those in Milnor (1963, Corollary 6.2) show that the set of focal points is of zero spherical volume. In the second case there exist neighbourhoods $V(y_1)$, $V(y_2) \subset M$ of y_1 and y_2, respectively, such that y_i is the locally unique projection of x on $V(y_i)$. It can be easily seen that there exists a neighbourhood $U(x) \subset S^{n-1}$ of x such that projections π_i: $U(x) \to V(y_i)$, $i = 1, 2$, are of class C^2. Let

$$E(x) = \{z \in U(x) | z^{\mathrm{T}}\pi_1(z) = z^{\mathrm{T}}\pi_2(z)\}.$$

Then $E(x)$ is the set of points $z \in U(x)$ which are equidistant from $V(y_1)$ and $V(y_2)$. Let $g(z) = z^{\mathrm{T}}\pi_1(z) - z^{\mathrm{T}}\pi_2(z)$. Note that $\pi_i(\mathrm{d}z)$ belongs to the tangent space $T_{\pi_i(z)}(M)$ of M at $\pi_i(z)$ and hence $z^{\mathrm{T}}\pi_i(\mathrm{d}z) = 0$. It follows that

$$\operatorname{grad} g = \pi_2(z) - \pi_1(z) \neq 0.$$

Therefore, by the implicit function theorem, $E(x)$ is an $(n-2)$-dimensional submanifold of class C^2 in $U(x)$ and hence has zero spherical volume. Combining the above two cases, we have shown that the complement of $R(M)$ has zero spherical volume.

We now investigate points in $R(M)$. For $x \in R(M)$, let

$$v = \frac{x - (x^{\mathrm{T}} x_M) x_M}{\|x - (x^{\mathrm{T}} x_M) x_M\|} \in S^{n-1}.$$

Consider the segment of the great circle joining x_M and x, and let $u = x_M \cos\theta + x \sin\theta$, $0 < \theta < \mathrm{dist}(x, x_M)$, be an interior point of this segment. We claim that $u \in R(M)$, and that the projection of u coincides with x_M. Assume the contrary. Then there exists $\tilde{y} \neq x_M$, $\tilde{y} \in M$, such that

$$\mathrm{dist}(u, \tilde{y}) \leqslant \mathrm{dist}(u, x_M).$$

By the triangular inequality,

$$\begin{aligned} \mathrm{dist}(x, \tilde{y}) &\leqslant \mathrm{dist}(x, u) + \mathrm{dist}(u, \tilde{y}) \\ &\leqslant \mathrm{dist}(x, u) + \mathrm{dist}(u, x_M) \\ &= \mathrm{dist}(x, x_M). \end{aligned}$$

However, this contradicts the assumption that x_M is the unique projection of x onto M. Therefore, u has the unique projection x_M onto M.

Consider the semicircle of (6). The above argument shows that this semicircle is divided into two intervals. The points on the first interval have the unique projection u and the points on the second do not. Define $\bar{\theta}(u, v)$ by (7). Note that $\bar{\theta}(u, v) = 0$ corresponds to the case where no point other than u itself has u as the unique projection. $\bar{\theta}(u, v) \geqslant \pi/2$ corresponds to the case where all the points on the quarter circle from u in the direction v have u as unique projection, which is equivalent to

$$v^{\mathrm{T}} x \leqslant 0, \qquad \forall x \in M. \tag{25}$$

That is to say, M is entirely contained in one side of the hyperplane in $\mathbb{R}^n$ defined by the normal v.

Since M is assumed to be locally conic, u is a projection of $u\cos\theta + v\sin\theta$ for sufficiently small $\theta > 0$ if and only if $v \in N_u(K(M))$. Therefore we have

$$\begin{aligned} &\bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\, u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(u, v)) \subset R(M) \\ &\quad \subset \bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u,\, u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(y, v)], \end{aligned} \tag{26}$$

where

$$[u,\, u\cos\theta + v\sin\theta] = [u,\, u\cos\theta + v\sin\theta) \cup \{u\cos\theta + v\sin\theta\}$$

denotes the right closed segment of the great circle. Suppose there exist points x in the difference of $R(M)$ and the left-hand side of (26), that is, $x \in R(M)$ such that

$$x = u\cos\bar{\theta}(u, v) + v\sin\bar{\theta}(u, v), \qquad \bar{\theta}(u, v) > 0,$$

where $u = x_M$ is the unique projection of x. We now prove that x is a focal point of u. Assume the contrary, that is, assume that $\bar{\theta}(u, v)$ is smaller than the radius of curvature of M at u with respect to the direction v. Then there exist a neighbourhood $V(u) \subset M$ of u and $\epsilon > 0$ such that the restricted projection of points on the extended segment

$$u\cos\theta + v\sin\theta, \qquad \bar{\theta}(u, v) < \theta < \bar{\theta}(u, v) + \epsilon,$$

onto $V(u)$ is u and is unique in $V(u)$. Consider a sequence of points $x_n = u\cos(\bar{\theta}(u, v) + 1/n) + v\sin(\bar{\theta}(u, v) + 1/n)$ converging to x. By definition of $\bar{\theta}(u, v)$ there exists $y_n \in M$ such that $\mathrm{dist}(x_n, y_n) \leqslant \mathrm{dist}(x_n, u)$. Furthermore, by the above argument, $y_n \notin V(u)$. Since the sequence $\{y_n\}$ is bounded, there exists an accumulation point y_0 of $\{y_n\}$. Taking a subsequence if necessary, we can without loss of generality assume that $y_n \to y_0 \notin V(u)$. Then

$$\mathrm{dist}(x, y_0) = \lim \mathrm{dist}(x_n, y_n) \leqslant \lim \mathrm{dist}(x_n, u) = \mathrm{dist}(x, u).$$

However, this contradicts the assumption that $u = x_M$ is the unique projection of x onto M. Therefore x is a focal point of x_M. We have now shown that the difference of $R(M)$ and the left-hand side of (26) is contained in the set of focal points and hence has zero spherical volume. □

We now state a version of Proposition 2.1 for tubes in $\mathbb{R}^n$. Let $M \subset \mathbb{R}^n$ be a locally conic set satisfying Assumption 2.1. Let $R(M)$ denote the set of points of $\mathbb{R}^n$ with unique projection onto M. For $y \in M$ and $v \in N_y(M)$, $\|v\| = 1$, let

$$[y, y + rv) = \begin{cases} \{y + tv | 0 \leqslant t < r\}, & \text{if } r > 0, \\ \{y\}, & \text{if } r = 0, \end{cases}$$

denote the right open line segment starting from y in the direction v. Define

$$\bar{r}(y, v) = \sup\{r \geqslant 0 | y + rv \in R(M), (y + rv)_M = y\}. \tag{27}$$

Then we have the following proposition.

Proposition B.1. *For a locally conic closed set M with piecewise smooth proper boundary,*

$$R(M) \supset \bigcup_{y \in M} \bigcup_{v \in N_y(M), \|v\|=1} [y, y + v\bar{r}(y, v)), \tag{28}$$

and the complement of the right-hand side of (28) *has zero Lebesgue measure. Here, as in* (8)*, we define $\bigcup_{v \in N_y(M), \|v\|=1} [y, y + v\bar{r}(y, v)) = \{y\}$ for $y \in \partial M_n$.*

As in (25), $\bar{r}(y, v) = \infty$ is equivalent to

$$v^{\mathrm{T}}(x - y) \leqslant 0, \qquad \forall x \in M, \tag{29}$$

that is, M is entirely contained in one side of the hyperplane in $\mathbb{R}^n$ defined by the normal v.

Finally, we give an outline of the proof of a version of Lemma 2.1. Suppose that $x \in M$

has a non-convex support cone $S_x(M)$. It suffices to show that $\inf_{y\in U(x)} \bar{\theta}(y) = 0$, where $U(x)$ is a neighbourhood of x. If we take $U(x)$ sufficiently small, then M is approximated by the support cone $S_x(M)$. Therefore the essential point of the proof is to consider projection onto M around the point x and to show Lemma 2.1 for $M = K = S_x(M)$, which is a non-convex cone in $\mathbb{R}^n$. Consider $y \in K \cap S^{n-1}$. Let $\bar{r}(y, v)$ be defined by (27). Using (29), it can be easily shown that

$$\inf_{y\in K\cap S^{n-1}, v\in N_y(K), \|v\|=1} \bar{r}(y, v) = \infty$$

if and only if K is a convex cone. Since K is assumed to be non-convex, there exist $y \in K \cap S^{n-1}$ and $v \in N_y(M)$ such that $\bar{r}(y, v) < \infty$. By the proof of Proposition B.1, $x = y + \bar{r}(y, v)v$ has at least two equidistant projections onto M. Because of the scale invariance of the geometry of the cone, $\epsilon x = \epsilon y + \epsilon\bar{r}(y, v)v = \epsilon y + \bar{r}(\epsilon y, v)v$ has the same property for every $\epsilon > 0$. Therefore $\lim_{\epsilon\downarrow 0} \bar{r}(\epsilon y, v) = 0$, and this proves that the critical radius of M is zero.

Appendix C. Proof of Proposition 3.1

In the following the index i is assumed to run over $\{1, 2\}$.

Let $u_i \in M_i$ and let the volume element of M_i at u_i be denoted by $\mathrm{d}u_i$. At

$$u = (u_1 \cos\varphi, u_2 \sin\varphi) \in M, \qquad \varphi \in [0, \pi/2], \tag{30}$$

the volume element of M is given by

$$\mathrm{d}u = \cos^{d_1}\varphi \, \sin^{d_2}\varphi \, \mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}\varphi. \tag{31}$$

Let $v_i \in T_{u_i}(K_i)^\perp \cap S^{n_i-1}$ and let the volume element of $T_{u_i}(K_i)^\perp \cap S^{n_i-1}$ be denoted by $\mathrm{d}v_i$, where '$\perp$' denotes orthogonal complement. Then at

$$v = (v_1 \cos\tilde{\varphi}, v_2 \sin\tilde{\varphi}) \in T_u(K)^\perp \cap S^{n-1}, \qquad \tilde{\varphi} \in [0, \pi/2], \tag{32}$$

the volume element of $T_u(K)^\perp \cap S^{n-1}$ is given by

$$\mathrm{d}v = \cos^{n_1-d_1-2}\tilde{\varphi} \, \sin^{n_2-d_2-2}\tilde{\varphi} \, \mathrm{d}v_1 \, \mathrm{d}v_2 \, \mathrm{d}\tilde{\varphi}. \tag{33}$$

Let $H_i(u_i, v_i)$ be the second fundamental form of M_i at u_i with respect to the normal direction $v_i \in T_{u_i}(K_i)^\perp \cap S^{n_i-1}$. Then the second fundamental form of M at u in (30) with respect to the normal direction v in (32) is given by

$$H(u, v) = \mathrm{diag}\left(\frac{\cos\tilde{\varphi}}{\cos\varphi} H_1(u_1, v_1), \frac{\sin\tilde{\varphi}}{\sin\varphi} H_2(u_2, v_2), 0\right). \tag{34}$$

Substituting (31), (33), and (34) into (11), and integrating it over $0 < \tau, \varphi, \tilde{\varphi} < \pi/2$, we prove the proposition.

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