Bernoulli **27**(1), 2021, 218–220
https://doi.org/10.3150/20-BEJ1234

Corrigendum to "Simple simulation of diffusion bridges with application to likelihood inference for diffusions"

MOGENS BLADT[1,*], MARCIN MIDER[2] and MICHAEL SØRENSEN[1,†]

[1] *Dept. of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark. E-mail:* [*] *bladt@math.ku.dk;* [†] *michael@math.ku.dk*
[2] *Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig, Germany. E-mail: marcin.mider@mis.mpg.de*

We correct an error in Theorem 2.1 in Bladt and Sørensen (*Bernoulli* **20** (2014) 645–675), where the initial distribution of an auxiliary diffusion process that is used to describe the distribution of the proposed approximate diffusion bridge is wrong. As a consequence, we also correct the pseudo marginal Metropolis-Hastings algorithm that has an exact diffusion bridge as its target distribution. The same auxiliary diffusion plays a central role in the algorithm.

Keywords: diffusion bridge; discretely sampled diffusions; pseudo-marginal MCMC

We correct Theorem 2.1 in Bladt and Sørensen [1] that gives the distribution of the approximate diffusion bridge proposed in the paper. First, we briefly describe the model and the proposed approximate diffusion bridge.

Let $X = \{X_t\}_{t\geq 0}$ be a one-dimensional diffusion given by the stochastic differential equation

$$dX_t = \alpha(X_t)\,dt + \sigma(X_t)\,dW_t, \tag{1}$$

where W is a Wiener process, and where the coefficients α and $\sigma > 0$ are sufficiently regular to ensure that the equation has a unique weak solution that is a strong Markov process. If a and b are given points in the state space of X, a solution of (1) in the interval $[0, \Delta]$ such that $X_0 = a$ and $X_\Delta = b$ is called an (a, b) diffusion bridge.

Let X^1 and X^2 be independent solutions of (1) with initial conditions $X^1_0 = a$ and $X^2_0 = b$ and define

$$Z_t = \begin{cases} X^1_t & \text{if } 0 \leq t \leq \tau \\ X^2_{\Delta-t} & \text{if } \tau < t \leq \Delta, \end{cases}$$

where $\tau = \inf\{0 \leq t \leq \Delta \mid X^1_t = X^2_{\Delta-t}\}$ $(\inf \varnothing = +\infty)$. Conditional on the event $\{\tau \leq \Delta\}$, $\{Z_t\}_{0\leq t\leq\Delta}$ is an approximate (a, b) diffusion bridge. The distribution of the approximate bridge is given by the following theorem, which replaces the incorrect Theorem 2.1 in Bladt and Sørensen [1].

Theorem 1. *Suppose the speed measure of the diffusion given by* (1) *is finite. Then the distribution of* $\{Z_t\}_{0\leq t\leq\Delta}$*, conditional on the event* $\{\tau \leq \Delta\}$*, equals the distribution of an* (a, b) *diffusion bridge, conditional on the event that the bridge is hit by an independent diffusion with stochastic differential equation* (1) *and initial distribution equal to the invariant probability measure.*

The condition of the theorem can often be verified by using that the density function of the speed measure is given by

$$m(x) = \frac{1}{\sigma^2(x)} \exp\left(2 \int_z^x \frac{\alpha(y)}{\sigma^2(y)} \,\mathrm{d}y\right),$$

where z is an arbitrary point in the state space. The density function of the invariant probability measure is proportional to $m(x)$.

What has been changed relative to Bladt and Sørensen [1] is the initial distribution of the independent diffusion. The reason for the error was a misinterpretation of a conditional probability. In order to avoid such misinterpretations, we give the proof of the theorem in more detail.

Proof of Theorem 1. Assume that X^1, X^2 and X^3 are independent, where X^1 and X^2 are as above, while X^3 solves (1) with initial distribution equal to the invariant probability measure. Define $\rho = \inf\{0 \le t \le \Delta \mid X_t^1 = X_t^3\}$ and

$$Y_t = \begin{cases} X_t^1 & \text{if } 0 \le t \le \rho \\ X_t^3 & \text{if } \rho < t \le \Delta. \end{cases}$$

Since the speed measure is finite, we have by Lemma 2.2 in Bladt and Sørensen [1] that

$$\{X_{\Delta-t}^2\}_{0 \le t \le \Delta} \sim \{X_t^3\}_{0 \le t \le \Delta} \mid X_\Delta^3 = b,$$

where $\sim$ denotes that distributions are equal, so

$$P(Z \in \cdot \mid \tau \le \Delta) = P\big(Y \in \cdot \mid X_\Delta^3 = b, \rho \le \Delta\big) = P(Y \in \cdot \mid Y_\Delta = b, \rho \le \Delta).$$

By the strong Markov property $Y \sim X^1$, and moreover, since ρ depends only on the sample path $\{Y_t\}_{0 \le t \le \rho} = \{X_t^1\}_{0 \le t \le \rho}$ (and $\{X_t^3\}_{0 \le t \le \rho}$, of course), we have that

$$(Y, \rho) \sim \big(X^1, \rho\big).$$

In particular, the joint distribution of $(Y_{t_1}, \ldots, Y_{t_n}, Y_\Delta, \rho)$ equals the joint distribution of $(X_{t_1}^1, \ldots, X_{t_n}^1, X_\Delta^1, \rho)$, for $0 < t_1 < \cdots < t_n < \Delta$, which implies that

$$P(Y \in \cdot \mid Y_\Delta = b, \rho \le \Delta) = P\big(X^1 \in \cdot \mid X_\Delta^1 = b, \rho \le \Delta\big).$$

In conclusion,

$$P(Z \in \cdot \mid \tau \le \Delta) = P\big(X^1 \in \cdot \mid X_\Delta^1 = b, \rho \le \Delta\big).$$

The event $\{X_\Delta^1 = b, \rho \le \Delta\}$ is the event that X^1 is an (a, b) diffusion bridge and that the diffusion bridge is hit by X^3. The process X^3 is independent of X^1, so its initial distribution is not changed by the condition $\{X_\Delta^1 = b\}$. □

The brief discussion of the symmetric definition of an approximate diffusion bridge must be changed similarly: the initial distribution of X^3 must be the invariant distribution. In the rest of the paper the words "diffusion with initial distribution $p_\Delta(b, \cdot)$" must in all cases be changed to "diffusion with initial distribution equal to the invariant distribution". We refer to such a process as a stationary diffusion, so the word "$p_\Delta(b, \cdot)$-diffusion" must in all cases be changed to "stationary diffusion". In particular, this redefines the probabilities $\pi_\Delta(x)$ and π_Δ in the alternative formulation of the conclusion of Theorem 2.1, on page 652: $\pi_\Delta(x)$ is the probability that the trajectory x is hit by a stationary diffusion, and π_Δ is the probability that an (a, b) diffusion bridge is hit by an independent stationary diffusion.

With these changes, the results in Section 2.2 still hold. In the pseudo marginal Metropolis-Hastings algorithm in Section 2.3, the definition of the probability $\pi_\Delta(x)$ has been changed as explained above, so the sequence of diffusions $Y^{(1)}, Y^{(2)}, \ldots$ that define the geometric random variables must be independent stationary diffusions. With this change, the target distribution is the distribution of an exact diffusion bridge. Most of the simulation study in Section 3 is concerned with the approximate method, and is hence not affected by the error in Theorem 2.1. The few simulation results on the probability π and on the Metropolis-Hastings algorithm are obviously affected, but are still of interest, because when the length of the time interval Δ is sufficiently large, the distribution of a $p_\Delta(b, \cdot)$-diffusion is not far from that of a stationary diffusion.

References

[1] Bladt, M. and Sørensen, M. (2014). Simple simulation of diffusion bridges with application to likelihood inference for diffusions. *Bernoulli* **20** 645–675. MR3178513 https://doi.org/10.3150/12-BEJ501

Received May 2020