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Learning the distribution of latent variables in paired comparison models with round-robin scheduling

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Paired comparison data considered in this paper originate from the comparison of a large number N of individuals in couples. The dataset is a collection of results of contests between two individuals when each of them has faced n opponents, where $n \ll N$. Individuals are represented by independent and identically distributed random parameters characterizing their abilities. The paper studies the maximum likelihood estimator of the parameters distribution. The analysis relies on the construction of a graphical model encoding conditional dependencies of the observations which are the outcomes of the first n contests each individual is involved in. This graphical model allows to prove geometric loss of memory properties and deduce the asymptotic behavior of the likelihood function. This paper sets the focus on graphical models obtained from round-robin scheduling of these contests. Following a classical construction in learning theory, the asymptotic likelihood is used to measure performance of the maximum likelihood estimator. Risk bounds for this estimator are finally obtained by sub-Gaussian deviation results for Markov chains applied to the graphical model.

Keywords: latent variables; nonasymptotic risk bounds; nonparametric estimation; paired comparisons data

1. Introduction

Consider a paired comparison problem involving a large number N of individuals. For all $1 \le i \le N$, the ith individual is characterized by a *strength* (or *ability*) represented by an unknown parameter V_i. These parameters are indirectly observed through discrete valued scores $X_{i,j}$ describing the results of contests between individuals i and j. Given the values $V = (V_1, \dots, V_N)$, the random variables $X_{i,j}$ are assumed to be independent and for each i and j, the conditional distribution of $X_{i,j}$ given V depends only on V_i and V_j: there is a known function k such that, for all $1 \le i < j \le N$,

$$\mathbb{P}(X_{i,j} = x|V) = k(x, V_i, V_j).$$

The most classical example is the Bradley–Terry model [2,32] where $x \in \{0, 1\}$ and $k(1, V_i, V_j) = V_i/(V_i + V_j)$. In the seminal works [2,32], the problem was to recover the strengths

$(V_1, \dots, V_N)$ of a small number of players when the number of observed scores for each pair grows to infinity, see [7] for a review of these results in the original Bradley–Terry model and some of its extensions. More recently, [23] considered the problem of estimating each strength based on one score per pair in a tournament where the number N of players grows to infinity. This framework led to several developments in computational statistics for the Bradley–Terry model, see [14] and [4] for various extensions of this original model. The related Chen–Lu model was considered in [5] where the observations take values in $\{0, 1\}$ and where the function k is given by $k(1, V_i, V_j) = V_i V_j / (1 + V_i V_j)$. Using one observation per pair of nodes, it is proved in [5] that, with probability asymptotically larger than $1 - 1/N^2$, there exists a unique maximum likelihood estimator of the nodes strengths which is such that the supremum norm of the estimation error is upper bounded by $\sqrt{\log N/N}$.

Consider the random oriented graph $G = (\{1, \dots, N\}, E)$, where an edge is drawn from i to j in E if $X_{i,j} = 1$ when $i < j$ and if $X_{j,i} = 0$ when $i > j$. It is known since [32] that a necessary and sufficient condition for the existence of the maximum likelihood estimator (MLE) of $(V_1, \dots, V_N)$ in the Bradley–Terry model is that G is connected, that is, there is a path between every pair of nodes. This assumption implies some restrictions on the ratio between the strongest and the weakest strength [23]. This prevents the use of maximum likelihood estimation in a sparse setting where the objective is to predict the outcome of future comparisons based on few observations. This problem was for instance considered in [31] which analyzes the MLE of $(V_1, \dots, V_N)$ under the condition of existence of [32], but in a graph where some edges may be unobserved.

This paper sets the focus on the case where each individual is compared to n others, with possibly $n \ll N$ in such a way that the assumption of [32] may not hold. In other words, *the MLE of* $V_1, \dots, V_N$ *may not exist* in this setting. To the best of our knowledge, this kind of dataset has not been analyzed previously and it is not clear what quantities can be recovered from these observations. Our strategy is motivated by the *Bradley–Terry model in random environment* [6,24]. In this model, strengths are supposed to be realizations of independent and identically distributed random variables with common distribution $\pi_\star$. The paper [24] illustrated for example that an elementary parametric model for the strength can be used to make predictions regarding the teams scores at the end of baseball tournaments. The paper [6] recently proved that the player with maximal strength ends the tournament with the highest degree in the graph G if the tail of the nodes weights distribution is sufficiently convex.

The take-home message is that *the strengths distribution* $\pi_\star$ *is relevant to predict future outcomes* which motivates the estimation of $\pi_\star$. As every player is supposed to meet exactly n opponents, the observed graph is naturally n regular (every node has the same degree n). It is also assumed that players meet according to the round-robin scheduling (see Section 2 for a description of this algorithm), a famous method to build n-regular graphs recursively. The round-robin algorithm is routinely used for example, to manage scheduling in chess, bridge, sport and online gaming tournaments. The MLE of $\pi_\star$ is analyzed based on the observation of the scores of every contest of the first n rounds of the algorithm.

First, a graphical model encoding conditional dependencies between strengths and scores is built. This representation allows to approximate the likelihood function using a stationary hidden Markov model [3]. The asymptotic behavior of the normalized loglikelihood is analyzed using loss of memory properties of the hidden Markov process, following essentially the approach of [11]. Then, following [27], the limit of the normalized loglikelihood is used to define a risk function, see Section 4.1 for details on this construction. This risk is then bounded from above for finite values of the number N of nodes using concentration inequalities for Markov Chains [10]. The excess risk scales as Dudley's entropy of the underlying statistical model normalized by a term of order $\sqrt{N}$ when n is fixed and $N \to \infty$. From a learning perspective, Dudley's entropy bound is known to be suboptimal in general, it can be replaced by a majorizing measure bound [25] since it derives from a sub-Gaussian concentration inequality for the increments of the underlying process, see (28).

More generally, the methodology introduced in this paper leads the way to various research perspectives in several fields. For example, identifiability of nonparametric hidden Markov models with finite state spaces was established recently along with the first convergence properties of estimators of the unknown distributions, see [8] for a penalized least-squares estimator of the emission densities, [9,29,30] for consistent estimation of the posterior distributions of the states and posterior concentration rates for the parameters or [17] for order estimation. However, very few theoretical results are available for the nonparametric estimation of general state spaces hidden Markov models. In computational statistics, Bayesian estimators of the strengths have been studied in Bradley–Terry models [14] and other extensions, see for example [4]. In [16], the unknown distribution of hidden variables is analyzed in a Bayesian framework and contraction rates of the posterior distribution are obtained using the concentration inequality established in this paper. Designing new algorithms to compute the MLE of the prior would then be of great interest to derive empirical Bayes estimators [13,22].

The paper is organized as follows. Section 2 details the model, the maximum likelihood estimator of the strengths distribution and the round-robin algorithm. Section 3 presents preliminary results. The graphical model encoding conditional dependencies in round-robin graphs with latent variables is displayed, and the Markov chain associated with this representation is shown to be well approximated by a geometrically ergodic Markov chain. The main results are gathered in Section 4: convergence of the likelihood is established when the number N of nodes grows to $+\infty$ and risk bounds for the MLE are provided. Finally, Appendices A to C are devoted to the proofs of these results.

2. Setting

Graphs with latent variables

Let N be a positive integer, E a set of couples (i, j) with $1 \le i < j \le N$ and $G = (\{1, \dots, N\}, E)$ the corresponding oriented graph. Let $V_1, \dots, V_N$ denote independent and identically distributed (i.i.d.) random variables taking values in a measurable set $\mathcal{V}$ with common *unknown* distribution $\pi_\star$. For all $(i, j) \in E$, let $X_{i,j}$ denote a random variable taking values in a finite set $\mathcal{X}$ such that, conditionally on $V = (V_1, \dots, V_N)$, the random variables $(X_{i,j})_{(i,j)\in E}$ are independent with conditional distributions given by

$$\mathbb{P}(X_{i,j} = x | V) = k(x, V_i, V_j),$$

where $k : \mathcal{X} \times \mathcal{V} \times \mathcal{V} \to [0, 1]$ is a known function. In the following, the sets $\mathcal{X}$, $\mathcal{V}$ and the scores $(X_{i,j})_{(i,j)\in E}$ are available while the vector V is unknown and the objective is to estimate the distribution $\pi_\star$. The following examples of triplets $(\mathcal{X}, \mathcal{V}, k)$ have been considered in the literature.

Example 1 (Bradley–Terry model [2]). In this example, $\mathcal{V} = (0, \infty)$, $\mathcal{X} = \{0, 1\}$ and for all $x \in \mathcal{X}$,

$$k(x, V_i, V_j) = \left(\frac{V_i}{V_i + V_j}\right)^x \left(\frac{V_j}{V_i + V_j}\right)^{1-x}.$$

Example 2 (Extensions of Bradley–Terry model [4]). In the following examples, $\mathcal{V} = (0, \infty)$.

– Let $\theta > 0$ and $\mathcal{X} = \{0, 1\}$. In the Bradley–Terry model with home advantage, if i is home, for all $x \in \mathcal{X}$,

$$k(x, V_i, V_j) = \left(\frac{\theta V_i}{\theta V_i + V_j}\right)^x \left(\frac{V_j}{\theta V_i + V_j}\right)^{1-x}.$$

– In the Bradley–Terry model with ties [21], $\mathcal{X} = \{-1, 0, 1\}$ and

$$k(1, V_i, V_j) = \frac{V_i}{V_i + \theta V_j} \quad \text{and} \quad k(0, V_i, V_j) = \frac{(\theta^2 - 1) V_i V_j}{(\theta V_i + V_j)(V_i + \theta V_j)}.$$

Example 3 (Graphon model). The probability that two nodes i and j are connected in the graphon model (i.e., $(i, j) \in E$) is the random variable $\mathsf{W}(V_i, V_j)$ with $\mathsf{W} : \mathcal{V} \times \mathcal{V} \to [0, 1]$ and $\mathcal{V} \subset \mathbb{R}^+$. In the context of this paper, this boils down to choosing $\mathcal{X} = \{0, 1\}$ and setting by convention $X_{i,j} = 0$ if and only if $(i, j) \notin E$ with

$$k(x, V_i, V_j) = \mathsf{W}(V_i, V_j)^x \big(1 - \mathsf{W}(V_i, V_j)\big)^{1-x}.$$

The problem in the graphon model is to estimate the matrix of connection probabilities $(\mathsf{W}(V_i, V_j))_{1\le i,j\le N}$ using the observations of the adjacency matrix, and assuming that the distribution of V_i is given.

In our setting, the aim is different, we try to estimate $\pi_\star$, the law of the latent variables, from a partial observation E of the adjacency matrix and with a known function W.

Example 4 (Chen–Lu model). Consider a random graph where E is such that an edge is drawn between node i and node j (i.e., $(i, j) \in E$) with probability $V_i V_j/(1 + V_i V_j)$, with for all $1 \le k \le N$, $V_k \in \mathcal{V} = (0, \infty)$. In the context of this paper, this boils down to choosing $\mathcal{X} = \{0, 1\}$ and setting by convention $X_{i,j} = 0$ if and only if $(i, j) \notin E$ with

$$k(x, V_i, V_j) = \left(\frac{V_i V_j}{1 + V_i V_j}\right)^x \left(\frac{1}{1 + V_i V_j}\right)^{1-x}.$$

Maximum likelihood estimator

The aim of this paper is to estimate the distribution $\pi_\star$ of the hidden variables $V = (V_1, \dots, V_N)$ from the observations $X^E = (X_{i,j})_{(i,j)\in E}$. Let $\mathcal{A}$ be a σ-field on $\mathcal{V}$ and Π be a set of probability measures on $(\mathcal{V}, \mathcal{A})$. The statistical model is not assumed to be well specified that is, Π may not contain $\pi_\star$. For all $\pi \in \Pi$, the joint distribution of (X^E, V) is given, for any $x^E \in \mathcal{X}^{|E|}$ and all $A \in \mathcal{A}^{\otimes N}$ by

$$\mathbb{P}_\pi^E\big(X^E = x^E, V \in A\big) = \int \mathbb{1}_A(v) \prod_{(i,j)\in E} k\big(x_{i,j}^E, v_i, v_j\big)\pi^{\otimes N}(\mathrm{d}v), \tag{1}$$

where $\mathbb{1}_A$ is the indicator function of the set A. Using the convention $\log 0 = -\infty$, the log-likelihood is given, for all $\pi \in \Pi$, by

$$\ell^E(\pi) = \log \mathbb{P}_\pi^E\big(X^E\big) \quad \text{where } \mathbb{P}_\pi^E\big(X^E\big) = \mathbb{P}_\pi^E\big(X^E, V \in \mathcal{V}^N\big).$$

In this paper, $\pi_\star$ is estimated by the maximum likelihood estimator $\widehat{\pi}^E$ defined as any maximizer of the log-likelihood:

$$\widehat{\pi}^E \in \operatorname*{argmax}_{\pi\in\Pi}\big\{\ell^E(\pi)\big\}.$$

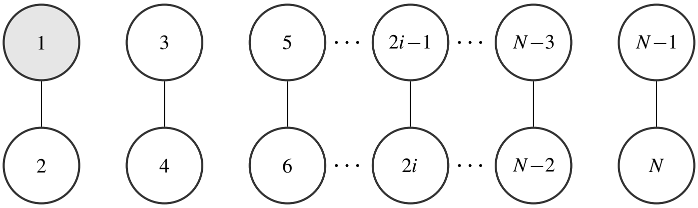
Round-robin (RR) scheduling

Assume that N is an even integer. In the case of a round-robin scheduling, at $t = 1$, $2i - 1$ is paired with $2i$, for all $i \in [N/2]$, as in Figure 1(a). At $t = 2$, the RR permutation $\mathcal{P}_{\mathrm{RR}}$ is performed: node 1 is fixed $\mathcal{P}_{\mathrm{RR}}(1) = 1$, $\mathcal{P}_{\mathrm{RR}}(2) = 3$, each odd integer $2i - 1 < N - 1$ satisfies $\mathcal{P}_{\mathrm{RR}}(2i - 1) = 2i + 1$, $\mathcal{P}_{\mathrm{RR}}(N - 1) = N$ and each even integer $2i > 2$ satisfies $\mathcal{P}_{\mathrm{RR}}(2i) = 2(i - 1)$. This permutation is illustrated by the graphical representation given in Figure 1(b). Then, the RR pairing is performed as in Figure 1(c). At each time $t > 2$, a RR permutation is performed as in Figure 1(b) and followed by a RR pairing. Let $n \geq 1$ denote an integer. The RR graph denoted by $E_{\mathrm{RR}}^{n,N}$ studied in detail in this paper contains all pairs collected in the first n pairings of the RR algorithm. Note that $E_{\mathrm{RR}}^{N-1,N}$ is the complete graph and that we focus on situations where $n \ll N$.

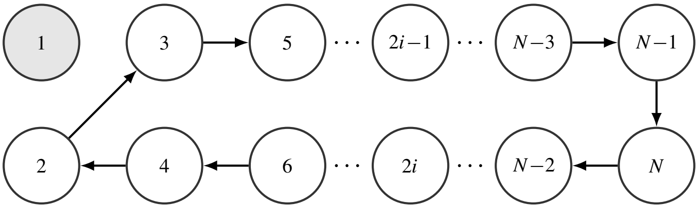
3. Conditional dependencies of round-robin graphs

Let d_0^E denote the graph distance in $(\{1, \dots, N\}, E)$, that is $d_0^E(i, j)$ is the minimal length of a path between nodes i and j. Write $\{V_1, \dots, V_N\} = \bigcup_{q=0}^N V_q^E$, where $V_0^E = \{V_1\}$ and, for any $q \geq 1$, V_q^E is the set of V_i such that $d_0^E(1, i) = q$. Let $\mathsf{q}_E + 1$ denote the maximal distance between 1 and $i \in \{1, \dots, N\}$:

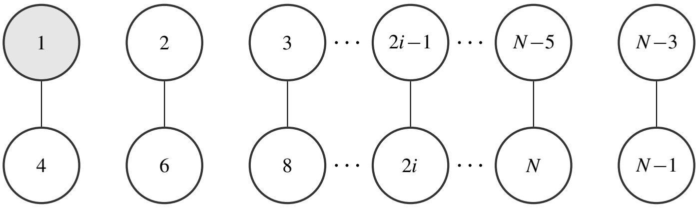
$$\mathsf{q}_E + 1 = \max_{1\leq i\leq N} d_0^E(1, i).$$



(a) Round-robin pairing, step 1.



(b) Round-robin permutation.



(c) Round-robin pairing, step 2.

Figure 1. Round-robin algorithm.

– For all $1 \le q \le \mathsf{q}_E + 1$, let

$$X^E_{q\leftrightarrow q} = \left\{X_{i,j} : (i,j) \text{ or } (j,i) \in E, i \in V^E_q, j \in V^E_q\right\}.$$

The set $X^E_{q\leftrightarrow q}$ gathers all $X_{i,j}$ such that i and j satisfy $d^E_0(1,i) = d^E_0(1,j) = q$.

– For all $0 \le q \le \mathsf{q}_E$, let

$$X^E_{q\leftrightarrow q+1} = \left\{X_{i,j} : (i,j) \text{ or } (j,i) \in E, i \in V^E_q, j \in V^E_{q+1}\right\}.$$

The set $X^E_{q\leftrightarrow q+1}$ gathers all $X_{i,j}$ such that $d^E_0(1,i) = q$ and $d^E_0(1,j) = q+1$.

Finally, for any $0 \le q \le \mathsf{q}_E$, let

$$X^E_q = X^E_{q\leftrightarrow q+1} \cup X^E_{q+1\leftrightarrow q+1}.$$

Following [15], the distribution $\mathbb{P}^E_\pi$, given in (1), can be factorized with respect to an oriented acyclic graph where graph separations represent conditional independence. The factorization

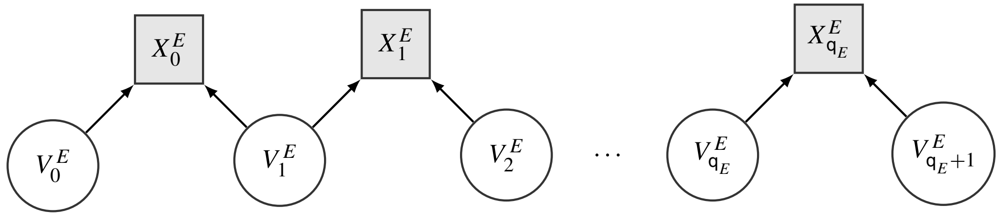


Figure 2. Graphical model of paired comparisons contests.

illustrates a global Markov property such that two sets of random variables U_1 and U_2 are independent given a third set Z if U_1 and U_2 are d-separated by Z in the oriented acyclic graph. The sets U_1 and U_2 are d-separated by Z if every path from U_1 to U_2 is blocked by Z:

- the path contains a node in Z, and the edges of the path do not meet head-to-head at this node.
- the path contains a node *not* in Z, none of its descendants are in Z, and the edges of the path *do* meet head-to-head at this node.

Conditional dependencies described by $\mathbb{P}_\pi^E$ can be represented in the graphical model of Figure 2.

For instance, V_1^E is independent of V_2^E ($Z = \varnothing$) as every path between them goes through X_1^E, which is not in Z, with two edges meeting head-to-head at X_1^E. For all $0 \le q \le \mathsf{q}_E$ any path between X_q^E and other vertices except V_q^E and V_{q+1}^E goes through V_q^E or V_{q+1}^E which means that X_q^E is independent of all other nodes given V_q^E and V_{q+1}^E ($Z = \{V_q^E, V_{q+1}^E\}$ and no head-to-head edges). In particular, for all $0 \le q \le \mathsf{q}_E$, and all $\pi \in \Pi$,

$$\mathbb{P}_\pi^E\big(X_q^E | V, X_{0:q-1}^E\big) = \mathbb{P}_\pi^E\big(X_q^E | V_q^E, V_{q+1}^E\big) = \prod_{(i,j):X_{i,j} \in X_q^E} k(X_{i,j}, V_i, V_j).$$

Lemma 1. *Let $N \ge n \ge 1$ and let $(\{1, \dots, N\}, E_{\mathrm{RR}}^{n,N})$ denote the corresponding round-robin graph defined in Section 2. Assume that $2 \le n < N/4$. Then, $\mathsf{q}_{E_{\mathrm{RR}}^{n,N}}$ is the quotient of the Euclidean division of $N/2 - 1$ by $n - 1$, that is*

$$N/2 - 1 = \mathsf{q}_{E_{\mathrm{RR}}^{n,N}}(n-1) + \mathsf{r}_N^n \quad \textit{with } 0 \le \mathsf{r}_N^n < n - 1.$$

Moreover, $(V_{q+1}^{E_{\mathrm{RR}}^{n,N}}, X_q^{E_{\mathrm{RR}}^{n,N}})_{2 \le q \le \mathsf{q}_{E_{\mathrm{RR}}^{n,N}} - 1}$ is a stationary Markov chain such that for all $2 \le q \le \mathsf{q}_{E_{\mathrm{RR}}^{n,N}} - 1$,

$$\big|V_q^{E_{\mathrm{RR}}^{n,N}}\big| = 2(n-1), \qquad \big|X_q^{E_{\mathrm{RR}}^{n,N}}\big| = n(n-1).$$

Lemma 1 is proved in Section A. It shows that RR graphs can be approximated by stationary hidden Markov models. When $E = E_{\mathrm{RR}}^{n,N}$, by Lemma 1, the joint sequence $(V_{q+1}^E, X_q^E)_{2 \le q \le \mathsf{q}_E - 1}$

is a stationary Markov chain which points toward the following decomposition of the likelihood.

$$\log \mathbb{P}_\pi^E(X^E) = \log \mathbb{P}_\pi^E(X^E_{2:\mathsf{q}_E-1}) + \log \mathbb{P}_\pi^E(X_0^E, X_1^E, X_{\mathsf{q}_E}^E | X^E_{2:\mathsf{q}_E-1}). \tag{2}$$

It is shown in Section 4 that under a minoration condition on the kernel k, the last term in (2) is $o(\mathsf{q}_E)$ when N grows to infinity. This implies that the first term is the leading term in the analysis of the likelihood's asymptotic behavior. The uniform minoration condition of k also ensures that the joint Markov chain $(V_{q+1}^E, X_q^E)_{q\geq 2}$ is uniformly ergodic and admits the whole space $\mathbb{V} \times \mathbb{X}$ as small set with stationary distribution on $\mathbb{V} \times \mathbb{X}$ given by $(A, x_0) \mapsto \int \mathbb{1}_A(v_1)\pi_V(\mathrm{d}v_1)\pi_V(\mathrm{d}v_0)k(x_0, v_0, v_1)$. The joint stationary Markov chain $(V_{q+1}^E, X_q^E)_{q\geq 2}$ may then be extended to a stationary process $(\mathbf{X^n}, \mathbf{V^n})$ indexed by $\mathbb{Z}$ with the same transition kernel. Hereafter, the distribution of this extended chain is denoted by $\mathbf{P}_\pi^n$.

4. Risk bounds for the MLE

Section 4.1 computes the limit likelihood function and shows why this limit defines a natural risk function to evaluate the MLE. Risk bounds for the MLE are obtained in Section 4.2 using concentration inequalities for Markov chains.

4.1. Asymptotic analysis of the likelihood

The problem being reduced to the analysis of the graphical model represented in Figure 2, convergence results follow from geometrically decaying mixing rates of the conditional laws of the strengths V_k^E given the observations. These rates are established under the following assumption. For any probability distribution π, denote by $\operatorname{supp}(\pi)$ the support of π.

H1 There exists $\varepsilon > 0$ such that for all $x \in \mathcal{X}$, $\pi \in \Pi \cup \{\pi_\star\}$ and $v_1, v_2 \in \operatorname{supp}(\pi)$, $k(x, v_1, v_2) \geq \varepsilon$.

Define also the shift operator ϑ on $(\mathcal{X}^{n(n-1)})^{\mathbb{Z}}$ by $(\vartheta x)_k = x_{k+1}$ for all $k \in \mathbb{Z}$ and all $x \in (\mathcal{X}^{n(n-1)})^{\mathbb{Z}}$. The following result establishes loss of memory properties of the extended hidden Markov chain $(\mathbf{X^n}, \mathbf{V^n})$ as well as the asymptotic behavior of the likelihood. This is the first main result of the paper.

Theorem 2. *Assume* **H1** *holds. Then, for all* $\mathsf{n}' > \mathsf{n} \geq q$ *and all* $p' < p < q$ *in* $\mathbb{Z}$,

$$\sup_{\pi\in\Pi} \left|\log \mathbf{P}_\pi^n(\mathbf{X}_q^n | \mathbf{X}^n_{q+1:\mathsf{n}}) - \log \mathbf{P}_\pi^n(\mathbf{X}_q^n | \mathbf{X}^n_{q+1:\mathsf{n}'})\right| \leq \varepsilon^{-n^2}(1-\varepsilon^{n^2})^{\mathsf{n}-q-1},$$

$$\sup_{\pi\in\Pi} \left|\log \mathbf{P}_\pi^n(\mathbf{X}_q^n | \mathbf{X}^n_{p:q-1}) - \log \mathbf{P}_\pi^n(\mathbf{X}_q^n | \mathbf{X}^n_{p':q-1})\right| \leq \varepsilon^{-n^2}(1-\varepsilon^{n^2})^{q-p}.$$

As a consequence, there exists a function ℓ_π^n *such that for all* q *in* $\mathbb{Z}$,

$$\sup_{\pi\in\Pi} \left|\log \mathbf{P}_\pi^n(\mathbf{X}_q^n | \mathbf{X}^n_{q+1:\mathsf{n}}) - \ell_\pi^n(\vartheta^q \mathbf{X^n})\right| \underset{\mathsf{n}\to\infty}{\longrightarrow} 0, \quad \mathbf{P}^n_{\pi_\star}\text{-}a.s. \tag{3}$$

Finally, when $E = E_{\mathrm{RR}}^{n,N}$, *for all* $\pi \in \Pi$, $\mathbf{P}_{\pi_\star}^n$*-a.s. and in* $\mathrm{L}^1(\mathbf{P}_{\pi_\star}^n)$,

$$\frac{1}{\mathsf{q}_E} \log \mathbb{P}_\pi^E\left(X^E\right) \underset{N\to\infty}{\longrightarrow} \mathsf{L}_{\pi_\star}^n(\pi) = \mathbb{E}_{\pi_\star}^n\left[\ell_\pi^n\left(\mathbf{X}^\mathbf{n}\right)\right]. \tag{4}$$

Theorem 2 is proved in Section C.1. It establishes convergence of the likelihood to the limit $\mathsf{L}_{\pi_\star}^n(\pi)$ when the number of nodes $N \to \infty$ while n remains fixed. The rate of almost sure convergence q_E is proportional to N in this case by Lemma 1. Equation (4) is the key to understand the definition of the risk function used in Section 4.2.

Let $Y, Y_1, \dots, Y_N$ denote i.i.d. observations in $\mathcal{Y}$, let F denote a set of parameters, and let $\ell : F \times \mathcal{Y} \to \mathbb{R}$ denote a loss function. The empirical risk minimizer is defined in this context by

$$\hat{f}_N^{\mathrm{ERM}} = \operatorname*{argmin}_{f\in F} \sum_{i=1}^N \ell(f, Y_i).$$

If $\mathbb{E}[\ell(f, Y_1)] < \infty$ for all $f \in F$, the performance of any $f \in F$ is measured by the *excess risk* [20]

$$R(f) = \mathbb{E}\left[\ell(f, Y)\right] - \mathbb{E}\left[\ell\left(f^*, Y\right)\right],$$

where Y is a copy of Y_1, independent of $Y_1, \dots, Y_N$ and f^* is the minimizer of $\mathbb{E}[\ell(f, Y)]$ over F. Note that, when $\mathbb{E}[\ell(f, Y_1)] < \infty$ for all $f \in F$, the normalized empirical criterion satisfies almost surely,

$$\frac{1}{N}\sum_{i=1}^N \ell(f, Y_i) \to \mathbb{E}\left[\ell(f, Y_1)\right].$$

Therefore, following for instance [27,28], the excess risk $R(f)$ in learning theory is the difference between the asymptotic normalized empirical loss evaluated at f and the minimizer of this quantity.

In this paper, the MLE minimizes over $\pi \in \Pi$ the loglikelihood $-\log \mathbb{P}_\pi^E(X^E)$. Using the identifications $\pi \sim f$, $\Pi \sim F$ and $-\log \mathbb{P}_\pi^E(X^E) \sim \sum_{i=1}^N \ell(f, Y_i)$, Theorem 2 suggests to use $-\mathsf{L}_{\pi_\star}^n(\pi)$ as a surrogate for $\mathbb{E}[\ell(f, Y)]$. Therefore, define, for all $\pi \in \Pi$,

$$R_{\pi_\star}^n(\pi) = \mathsf{L}_{\pi_\star}^n(\pi_\star) - \mathsf{L}_{\pi_\star}^n(\pi). \tag{5}$$

By Proposition 13, $\pi_\star$ is actually a minimizer of $-\mathsf{L}_{\pi_\star}^n(\pi)$ over $\Pi \cup \{\pi_\star\}$. Therefore, $R_{\pi_\star}^n$ is a natural extension of the excess risk associated with the likelihood function. Notice here that the model is non identifiable. Clearly, the observed distribution is not changed if the distribution π of V is replaced by the distribution of $\varphi(V)$, for any mapping $\varphi : \mathcal{V} \to \mathcal{V}$ such that $k(x, \varphi(v_1), \varphi(v_2)) = k(x, v_1, v_2)$ for any $x \in \mathcal{X}$, and v_1, v_2 in $\mathcal{V}$. For example, in the Bradley–Terry model, for any $\lambda > 0$, $k(x, \lambda v_1, \lambda v_2) = k(x, v_1, v_2)$ for any $x \in \mathcal{X}$, and v_1, v_2 in $\mathcal{V}$. It is not easy however to describe precisely the class of transformations that would leave the observed distribution invariant in general, specially for a fixed n. This is why, in the following, we focus on bounding the risk $R_{\pi_\star}^n(\hat{\pi})$ of the estimator $\hat{\pi}$ rather than trying to bound a distance between π^* and $\hat{\pi}$.

4.2. Non asymptotic deviation bounds for the MLE

The following theorem provides nonasymptotic deviation bounds for the excess risk of the MLE. This is the main result of this paper. Let $\|\cdot\|_{\mathsf{tv}}$ denote the total variation norm: for any signed measure π on $\mathcal{V}$,

$$\|\pi\|_{\mathsf{tv}} = \sup\left\{\int \pi(\mathrm{d}v) f(v) : f \text{ bounded and measurable on } \mathcal{V}, \|f\|_\infty = 1\right\}.$$

Theorem 3. *Assume* **H1** *holds and* $(\{1,\dots,N\}, E)$ *is the round-robin graph* (*that is* $E = E_{\mathrm{RR}}^{n,N}$). *For any probability measures* π *and* π', *define*

$$d\big(\pi,\pi'\big) = \begin{cases} \big\|\pi-\pi'\big\|_{\mathsf{tv}} \log\left(\dfrac{1}{\|\pi-\pi'\|_{\mathsf{tv}}}\right) & \text{if } \big\|\pi-\pi'\big\|_{\mathsf{tv}} < \mathrm{e}^{-1}, \\ \big\|\pi-\pi'\big\|_{\mathsf{tv}} & \text{if } \big\|\pi-\pi'\big\|_{\mathsf{tv}} \geq \mathrm{e}^{-1}. \end{cases} \tag{6}$$

Let $\mathsf{N}(\Pi\cup\{\pi_\star\}, d, \epsilon)$ *be the minimal number of balls of* d*-radius* ϵ *necessary to cover* $\Pi\cup\{\pi_\star\}$. *Then, there exists* $c>0$ *such that, for any* $t>0$ *and any* n, $N\geq 1$,

$$\mathbb{P}_{\pi_\star}^E\left(R_{\pi_\star}^n\big(\widehat{\pi}^E\big) > \frac{cn\varepsilon^{-6n^2}}{\sqrt{N}}\left[\int_0^{+\infty}\sqrt{\log \mathsf{N}\big(\Pi\cup\{\pi_\star\}, d, \epsilon\big)}\,\mathrm{d}\epsilon + t\right]\right) \leq \mathrm{e}^{-t^2}.$$

Theorem 3 is proved in Section C.3. It provides the first non asymptotic risk bounds for any estimator of $\pi_\star$. Besides, to the best of our knowledge, the "sparse" observation setting where each player only faces a few opponent has never been considered previously, neither in the Bradley–Terry model nor in any extensions. Theorem 3 demonstrates that the estimation of the distribution $\pi_\star$ of the parameters V is fundamentally different from the problem of estimating V that is usually considered, at least in Bradley–Terry models. While estimating nodes weights is possible under Zermelo's strong connectivity condition [23,31,32], the estimation of their distribution can be performed without such condition.

The quasi-metric d defined in (6) used to measure the entropy of Π is not intuitive. However, it is easy to check that $d(\pi,\pi') \lesssim_\alpha \|\pi-\pi'\|_{\mathrm{tv}}^{1-\alpha}$ for any $\alpha>0$. It follows that, for any class Π with polynomial entropy for the total variation distance, that is such that $\mathsf{N}(\Pi\cup\{\pi_\star\}, \|\cdot\|_{\mathrm{tv}}, \epsilon) \lesssim \epsilon^D$ for small ϵ, Dudley's entropy integral for d satisfies

$$\int_0^{+\infty}\sqrt{\log \mathsf{N}\big(\Pi\cup\{\pi_\star\}, d, \epsilon\big)}\,\mathrm{d}\epsilon \lesssim_\alpha \sqrt{D}.$$

Therefore, "slow rates" of convergence are obtained for the MLE. The polynomial growth $\mathsf{N}(\Pi\cup\{\pi_\star\}, \|\cdot\|_{\mathrm{tv}}, \epsilon) \lesssim \epsilon^D$ is extremely standard, see [26], pages 271–274, for various examples where this assumption is satisfied and our result applies. On the other hand, "fast" rates of convergence remain an open question. In particular, the margin condition [19] required to prove such rates would hold if the total variation distance between strengths distributions was bounded from above by the excess risk derived from the asymptotic of the likelihood.

Appendices

The remaining of the paper is devoted to the proof of the main results. Section A proves Lemma 1, describing precisely the structure of the graphical model given in Figure 2 in the case of a round-robin scheduling. Then, Section B establishes central tools for the analysis of the likelihood of stationary processes whose conditional dependences are described by the graphical model in Figure 2. These results are stated as independent lemmas as they might be of independent interest. Proofs of the main theorems are finally gathered in Section C.

Appendix A: Proof of Lemma 1

This section details the sets V_q^E and X_q^E for $0 \le q \le \mathsf{q}_E + 1$ when $E = E_{\mathrm{RR}}^{n,N}$ (cf. Figures 1(a)–1(c)). In the following, notations i are identified with V_i for all $1 \le i \le N$, we also use $E = E_{\mathrm{RR}}^{n,N}$ to shorten notations. Lemma 1 follows directly from Lemmas 4 and 5 below. To prove these lemmas, consider the following notations.

$$\mathcal{E} = \{4x - 1, 4x : x \in [\lfloor N/4 \rfloor]\} \quad \text{and} \quad \mathcal{O} = [N] \setminus \mathcal{E}.$$

The notation $\mathcal{E}$ (resp $\mathcal{O}$) comes from the fact that $\mathcal{E}$ (resp $\mathcal{O}$) contains all indices of the form $4x$ (resp. of the form $(2(2x + 1))$) which are paired with 1 after an *even* (resp *odd*) number $n \le N/4$ of permutations of the round-robin algorithm. For all $1 \le q \le \mathsf{q}_E$, let

$$V_{q,e}^E = V_q^E \cap \mathcal{E} \quad \text{and} \quad V_{q,o}^E = V_q^E \cap \mathcal{O}.$$

Lemma 4. *Let $n, N \ge 1$ and $(\{1, \dots, N\}, E)$ be the round-robin graph ($E = E_{\mathrm{RR}}^{n,N}$). Assume that $2 \le n < N/4$ and let $N/2 - 1 = \mathsf{q}_E(n - 1) + \mathsf{r}_E$ where $0 \le \mathsf{r}_E < n - 1$. Then,*

$$V_1^E = \{V_{2x} : x = 1, \dots, n\}, \tag{7}$$

and, for any $2 \le q \le \mathsf{q}_E$,

$$\begin{aligned} V_q^E = &\{V_{2x+1} : x \in [(q - 2)(n - 1) + 1, (q - 1)(n - 1)]\} \\ &\cup \{V_{2x} : x \in [2 + (q - 1)(n - 1), 1 + q(n - 1)]\}. \end{aligned} \tag{8}$$

Furthermore,

$$\begin{aligned} V_{\mathsf{q}_E+1}^E = &\{V_{2x+1} : x \in [(\mathsf{q}_E - 1)(n - 1) + 1, \mathsf{q}_E(n - 1) + \mathsf{r}_E]\} \\ &\cup \{V_{2x} : x \in [2 + \mathsf{q}_E(n - 1), 1 + \mathsf{r}_E + \mathsf{q}_E(n - 1)]\}. \end{aligned} \tag{9}$$

Therefore, $|V_0^E| = 1$, $|V_1^E| = n$ and for all $2 \le q \le \mathsf{q}_E$, $|V_q^E| = 2(n - 1)$.

Proof. To ease the reading of this proof, one can check its arguments on Figures 3 and 4 illustrating the case $n = 3$.

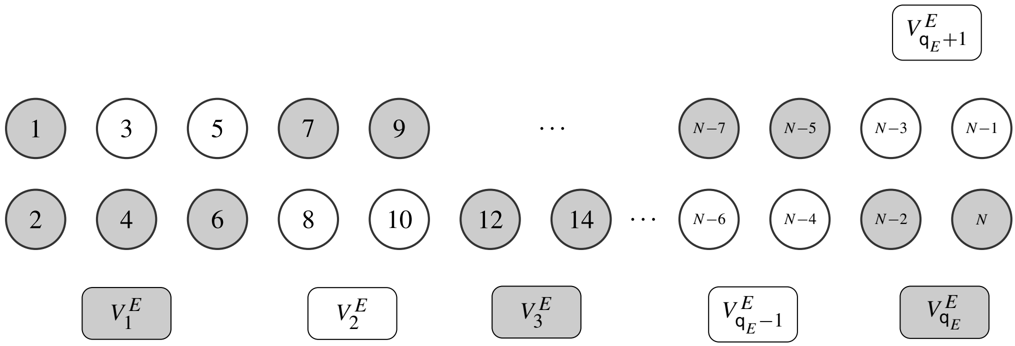


Figure 3. Elements of $\mathcal{V}^E$, case $n = 3$, $r_E = 0$.

We proceed by induction on q. The definition of V_1^E given by (7) is straightforward. Then, V_2^E contains:

– all V_i paired with some $V_j \in V_1^E$ before the first RR permutation besides V_1 that does not belong to V_2^E. These are all $\{V_{2x+1} : x = 1, \dots, n-1\}$;
– all V_i paired with V_2 and V_4 that are not in $V_0^E \cup V_1^E$. After n RR permutations, all V_i paired with V_2 are $\{V_1, V_{4x+2} : x = 1, \dots, n-1\}$ and those with V_4 are $\{V_1, V_3, V_{4x} : x = 2, \dots, n-2\}$.

Therefore,

$$V_2^E \supset \{V_{2x+1} : x = 1, \dots, n-1\} \cup \{V_{2x} : x = n+1, \dots, 2n-1\}.$$

On the other hand, by induction, for all $i \notin \{N - 2x + 1, x = 1, \dots, 2(n-1)\} \cup \{2x : x = 1, \dots, 2n-1\}$,

$$\begin{aligned} &\text{if } i \text{ is odd, it is paired with } \{V_{i+4x+1} : x = 0, \dots, n-1\}, \\ &\text{if } i \text{ is even, it is paired with } \{V_{i-4x-1} : x = 0, \dots, n-1\}. \end{aligned} \tag{10}$$

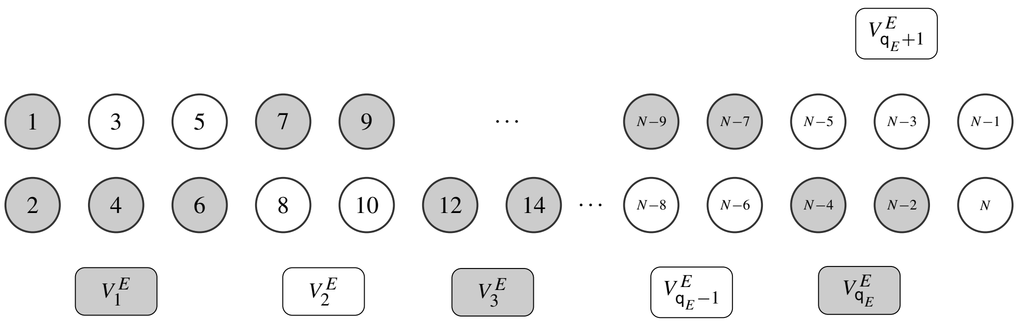


Figure 4. Elements of $\mathcal{V}^E$, case $n = 3$, $r_E = 1$.

This implies that there is no even number $i \geq 4n$ nor odd number $i > 2n-1$ such that $V_i \in V_2^{n,N}$, which yields:

$$V_2^E = \{V_{2x+1} : x = 1, \dots, n-1\} \cup \{V_{2x} : x = n+1, \dots, 2n-1\}.$$

Equation (8) is obtained by induction using the same arguments and (9) is a direct consequence of the round-robin algorithm. The last claim follows by noting that for all $q \in [2, \mathsf{q}_E]$,

$$\left|V_{q,e}^E\right| = \left|V_{q,o}^E\right| = n-1.$$

Indeed, one of the following cases holds.

– $n-1 = 2p$ for some $p \in \mathbb{N}$. In this case,

$$\left|\left\{j : V_j \in V_{q,e}^E, j \in 2\mathbb{Z}\right\}\right| = \left|\left\{i : V_i \in V_{q,e}^E, i \in 2\mathbb{Z}+1\right\}\right| = p.$$

– $n-1 = 2p+1$ for some $p \in \mathbb{N}$. In this case, either

$$\left|\left\{j : V_j \in V_{q,e}^E, j \in 2\mathbb{Z}\right\}\right| = p \quad \text{and} \quad \left|\left\{i : V_i \in V_{q,e}^E, i \in 2\mathbb{Z}+1\right\}\right| = p+1,$$

or

$$\left|\left\{j : V_j \in V_{q,e}^E, j \in 2\mathbb{Z}\right\}\right| = p+1 \quad \text{and} \quad \left|\left\{i : V_i \in V_{q,e}^E, i \in 2\mathbb{Z}+1\right\}\right| = p.$$

□

Lemma 5. *Let $n, N \geq 1$ and $(\{1, \dots, N\}, E)$ be the round-robin graph ($E = E_{\mathrm{RR}}^{n,N}$). Then, for all $2 \leq q \leq \mathsf{q}_E - 1$,*

$$\left|X_q^E\right| = n(n-1).$$

Proof. The proof essentially consists in building the graphical model of Figure 5 from the one displayed in Figure 2.

Edges involving the first node are decomposed as:

$$X_{0\leftrightarrow 1,e}^E = \left\{X_{1,4x} : x = 1, \dots, \lfloor n/2 \rfloor\right\} = \left\{X_{1,i} : V_i \in V_{1,e}^E\right\} \quad \text{and} \quad X_{0\leftrightarrow 1,o}^E = \left\{X_{1,i} : V_i \in V_{1,o}^E\right\}.$$

Edges involving nodes in V_1^E that are both different from 1 are described as follows.

– Edges between two nodes in V_1^E denoted by:

$$X_{1\leftrightarrow 1,e}^E = \left\{X_{4x,4y} : (x,y) \in \left[\lfloor n/2 \rfloor\right], x < y\right\} = \left\{X_{i,j} : V_i, V_j \in V_{1,e}^E, i < j\right\},$$
$$X_{1\leftrightarrow 1,o}^E = \left\{X_{i,j} : V_i, V_j \in V_{1,o}^E, i < j\right\}.$$

Note that there is no edge between any $V_i \in V_{1,e}^E$ and a node $V_j \in V_{q,o}^E$ for any $q \geq 1$. In particular, there is no edge between any $V_i \in V_{1,e}^E$ and $V_j \in V_{1,o}^E$. Therefore, $X_{1\leftrightarrow 1,e}^E \cup X_{1\leftrightarrow 1,o}^E$ describes all edges between nodes in V_1^E.

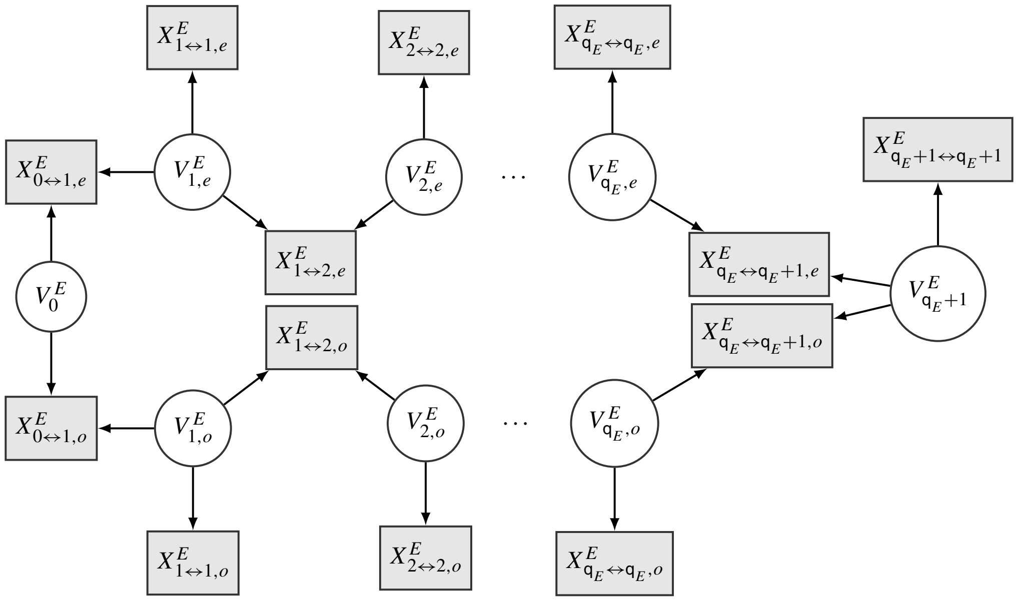


Figure 5. Graphical model of the round-robin algorithm.

– Edges between $V_i \in V_1^E$ and $V_j \in V_2^E$ are described as follows:

$$
\begin{aligned}
X^E_{1\leftrightarrow 2,e} &= \{X_{4y-1-4k,4y} : y \in [\lfloor n/2 \rfloor], k < y\} \\
&\quad \cup \{X_{4x,4y} : x \in [\lfloor n/4 \rfloor], y \in [\lfloor n/2 \rfloor + 1, n - x]\} \\
&= \{X_{i,j} : V_i \in V^E_{1,e}, V_j \in V^E_{2,e}, j \in 2\mathbb{Z}+1, j > i\} \\
&\quad \cup \{X_{i,j} : V_i \in V^E_{1,e}, V_j \in V^E_{2,e}, j \in 2\mathbb{Z} \cap [4n - i]\}, \\
X^E_{1\leftrightarrow 2,o} &= \{X_{i,j} : V_i \in V^E_{1,o}, V_j \in V^E_{2,o}, j \in 2\mathbb{Z}+1, j > i\} \\
&\quad \cup \{X_{i,j} : V_i \in V^E_{1,o}, V_j \in V^E_{2,o}, j \in 2\mathbb{Z} \cap [4n - i]\}.
\end{aligned}
$$

By (10), for any $q \in [2, \mathsf{q}_E]$, edges between V_i and V_j both in V_q^E are:

$$
\begin{aligned}
X^E_{q\leftrightarrow q,e} &= \{X_{i,j} : V_i \in V^E_{q,e}, i \in 2\mathbb{Z}+1, V_j \in V^E_{q,e}, j \in 2\mathbb{Z}\}, \\
X^E_{q\leftrightarrow q,o} &= \{X_{i,j} : V_i \in V^E_{q,o}, i \in 2\mathbb{Z}+1, V_j \in V^E_{q,o}, j \in 2\mathbb{Z}\}.
\end{aligned}
$$

Note that (10) shows also that there is no edge between $V_i \in V^E_{q,e}$ and $V_j \in V^E_{q,o}$. For all $2 \le q \le \mathsf{q}_E$ and all $V_i \in V_q^E$ and $V_j \in V^E_{q+1}$,

$$
\begin{aligned}
X^E_{q\leftrightarrow q+1,e} = &\{X_{i,j} : V_i \in V^E_{q,e}, i \in (2\mathbb{Z}+1), V_j \in V^E_{q+1,e}, j \in 2\mathbb{Z} \cap [i + 4n - 3]\} \\
&\cup \{X_{i,j} : V_i \in V^E_{q,e}, i \in 2\mathbb{Z}, V_j \in V^E_{q+1,e}, j \in 2\mathbb{Z}+1 \cap [i]\},
\end{aligned}
$$

$$X^E_{q\leftrightarrow q+1,o} = \big\{X_{i,j} : V_i \in V^E_{q,o}, i \in (2\mathbb{Z}+1), V_j \in V^E_{q+1,o}, j \in 2\mathbb{Z} \cap [i+4n-3]\big\}$$
$$\cup \big\{X_{i,j} : V_i \in V^E_{q,o}, i \in 2\mathbb{Z}, V_j \in V^E_{q+1,o}, j \in (2\mathbb{Z}+1) \cap [i]\big\}.$$

Therefore, for all $2 \le q \le \mathsf{q}_E$,

$$\big|X^E_{q\leftrightarrow q,e}\big| = \big|\big\{i : V_i \in V^E_{q,e}, i \in 2\mathbb{Z}+1\big\}\big|\big|\big\{j : V_j \in V^E_{q,e}, j \in 2\mathbb{Z}\big\}\big|$$
$$= \begin{cases} p^2 & \text{if } n-1 = 2p, \\ p(p+1) & \text{if } n-1 = 2p+1. \end{cases}$$

The same holds for $|X^E_{q\leftrightarrow q,o}|$ so that $|X^E_{q\leftrightarrow q}| = 2p^2$ if $n-1 = 2p$ and $|X^E_{q\leftrightarrow q}| = 2p(p+1)$ if $n-1 = 2p+1$. On the other hand,

$$\big|X^E_{q\leftrightarrow q+1,e}\big| = \sum_{i:V_i\in V^E_{q,e}, i\in(2\mathbb{Z}+1)} \big|\big\{j : V_j \in V^E_{q+1,e} j \in 2\mathbb{Z} \cap [i+4n-3]\big\}\big|$$
$$+ \sum_{i:V_i\in V^E_{q,e}, i\in 2\mathbb{Z}} \big|\big\{j : V_j \in V^E_{q+1,e}, j \in 2\mathbb{Z}+1 \cap [i]\big\}\big|$$
$$= \begin{cases} 2\sum_{i=1}^{p} i = p(p+1) & \text{if } n-1 = 2p, \\ \sum_{i=1}^{p} i + \sum_{i=1}^{p+1} i = (p+1)^2 & \text{if } n-1 = 2p+1. \end{cases}$$

As the same holds for $|X^E_{q\leftrightarrow q+1,o}|$, $|X^E_{q\leftrightarrow q+1}| = 2p(p+1)$ if $n-1 = 2p$ and $|X^E_{q\leftrightarrow q+1}| = 2(p+1)^2$ if $n-1 = 2p+1$. The proof is completed by writing $|X^E_q| = |X^E_{q\leftrightarrow q+1}| + |X^E_{q+1\leftrightarrow q+1}|$. □

Appendix B: Probabilistic study of the graphical model

This section analyses stochastic processes whose conditional dependences are encoded in the graphical model of Figure 2. To ease applications of these general results to our problem, we focus on a restricted class of such stochastic processes.

Let $\mathsf{n} \in \mathbb{N} \setminus \{0\}$, π_V be a distribution on a measurable space $\mathbb{V}$ and $\mathbb{X}$ be a discrete space. Let K_i denote non-negative functions defined on $\mathbb{X} \times \mathbb{V}^2$ such that all $K_i(., v, w)$ are probability distributions on $\mathbb{X}$. Let $\mathbb{P}_{\pi_V}$ be the distribution on $\mathbb{V}^{\mathsf{n}+1} \times \mathbb{X}^{\mathsf{n}}$ defined by:

$$\mathbb{P}_{\pi_V}(V_{1:\mathsf{n}+1} \in A_{1:\mathsf{n}+1}, X_{1:\mathsf{n}}) = \int \prod_{i=1}^{\mathsf{n}+1} \mathbb{1}_{A_i}(v_i) \prod_{i=1}^{\mathsf{n}+1} \pi_V(\mathrm{d}v_i) \prod_{i=1}^{\mathsf{n}} K_i(X_i, v_i, v_{i+1}). \tag{11}$$

The random variables $(V_i)_{i\in\{1,\ldots,\mathsf{n}+1\}}$ are i.i.d. taking values in $\mathbb{V}$ with common distribution π_V and $(X_i)_{i\in\{1,\ldots\mathsf{n}\}}$ is a stochastic process taking values in a discrete set $\mathbb{X}$ such that $(X_i)_{i\in\{1,\ldots,\mathsf{n}\}}$

are independent conditionally on V and

$$\mathbb{P}_{\pi_V}(X_i = x | V_{1:\mathsf{n}+1}) = \mathbb{P}_{\pi_V}(X_i = x | V_i, V_{i+1}) = K_i(x, V_i, V_{i+1}), \quad \forall i \in \{1, \mathsf{n}\}, \forall x \in \mathbb{X}.$$

Therefore, $\mathbb{P}_{\pi_V}$ is a generic probability distribution with conditional dependences encoded by the graphical model of Figure 2. Assume that there exist $\nu_i > 0$ such that

$$\nu_i \leq K_i(x, v, w) \leq 1, \quad \forall x \in \mathbb{X}, \forall i \in \mathbb{Z}, \forall v, w \in \mathbb{V}. \tag{12}$$

For some results, the following assumption is required.

$$\forall i \in \{1, \dots, \mathsf{n}\}, \quad K_i = K. \tag{13}$$

Whenever Assumption (13) holds, we shall denote by ν a real number such that

$$\nu \leq K(x, v, w) \leq 1, \quad \forall x \in \mathbb{X}, \forall v, w \in \mathbb{V}.$$

Note that by (11), the sequence $(V_{k+1}, X_k)_{k \geq 0}$ is a Markov chain with transition kernel on $\mathbb{V} \times \mathbb{X}$ such that:

$$\mathbb{P}_{\pi_V}(V_{k+1} \in A, X_k | V_k, X_{k-1}) = \int \mathbb{1}_A(v_{k+1}) \pi_V(\mathrm{d}v_{k+1}) K_k(X_k, V_k, v_{k+1}) \geq \nu_k \pi_V(A).$$

This uniform minoration condition ensures that the joint Markov chain $(V_{k+1}, X_k)_{k \geq 0}$ is geometrically ergodic and admits the whole space $\mathbb{V} \times \mathbb{X}$ as small set. Note also that, as defined by (11), $\mathbb{P}_{\pi_V}$ is the law of this Markov chain started from stationarity, the stationary distribution on $\mathbb{V} \times \mathbb{X}$ being $(A, x_0) \mapsto \int \mathbb{1}_A(v_1) \pi_V(\mathrm{d}v_1) \pi_V(\mathrm{d}v_0) k(x_0, v_0, v_1)$.

Lemma 6 first shows that, conditionally on the observations, $V_1, \dots, V_{\mathsf{n}}$ is a backward Markov chain admitting the all state space as small set.

Lemma 6. *For any $q \geq 1$, conditionally on $X_{q:\mathsf{n}}$, $(V_{\mathsf{n}}, \dots, V_1)$ is a Markov chain. Its transition kernels $(K^{V|X}_{\pi_V,k,q})_{q \leq k < \mathsf{n}}$ are such that, for all $q \leq k < \mathsf{n}$, there exists a measure $\mu_{k,q}$ satisfying for all measurable set A:*

$$K^{V|X}_{\pi_V,k,q}(V_{k+1}, A) = \mathbb{P}_{\pi_V}(V_k \in A | V_{k+1:\mathsf{n}}, X_{q:\mathsf{n}}) = \mathbb{P}_{\pi_V}(V_k \in A | V_{k+1}, X_{q:\mathsf{n}}) \geq \nu_k \mu_{k,q}(A).$$

On the other hand, for all $1 \leq k < q$,

$$K^{V|X}_{\pi_V,k,q}(V_{k+1}, A) = \mathbb{P}_{\pi_V}(V_k \in A | V_{k+1:\mathsf{n}}, X_{q:\mathsf{n}}) = \pi_V(A).$$

Proof. The Markov property is immediate. The case $1 \leq k < q$ follows from the independence of V_k and $(V_{k+1:\mathsf{n}}, X_{q:\mathsf{n}})$. Then, for any $q \leq k < \mathsf{n}$ and all measurable set A,

$$\begin{aligned}\mathbb{P}_{\pi_V}(V_k \in A | V_{k+1:\mathsf{n}}, X_{q:\mathsf{n}}) &= \mathbb{P}_{\pi_V}(V_k \in A | V_{k+1}, X_{q:k}) \\ &= \frac{\int \mathbb{1}_A(v_k) \pi_V(\mathrm{d}v_k) K_k(X_k, v_k, V_{k+1}) \mathbb{P}_{\pi_V}(X_{q:k-1} | v_k)}{\int \pi_V(\mathrm{d}v_k) K_k(X_k, v_k, V_{k+1}) \mathbb{P}_{\pi_V}(X_{q:k-1} | v_k)},\end{aligned}$$

with the conventions $\mathbb{P}_{\pi_V}(X_{q:q-1}|V_q) = 1$. By Assumption **H1**,

$$\mathbb{P}_{\pi_V}(V_k \in A|V_{k+1}, X_{q:\mathsf{n}}) \geq \nu_k \frac{\int \mathbb{1}_A(v_k)\pi_V(\mathrm{d}v_k)\mathbb{P}_{\pi_V}(X_{q:k-1}|v_k)}{\int \pi_V(\mathrm{d}v_k)\mathbb{P}_{\pi_V}(X_{q:k-1}|v_k)}.$$

The proof is then completed by choosing:

$$\mu_{k,q}(A) = \frac{\int \mathbb{1}_A(v_k)\pi_V(\mathrm{d}v_k)\mathbb{P}_{\pi_V}(X_{q:k-1}|v_k)}{\int \pi_V(\mathrm{d}v_k)\mathbb{P}_{\pi_V}(X_{q:k-1}|v_k)}.$$ □

Lemma 7 shows the contraction properties of the Markov kernel of the chain V conditionally on the observations. It is a direct consequence of the minoration condition given in Lemma 6, see for instance [18], Sections III.9 to III.11 or [3], Corollary 4.3.9 and Lemma 4.3.13. Let $\|\cdot\|_{\mathsf{tv}}$ be the total variation norm defined, for any measurable set $(\mathsf{Z}, \mathcal{Z})$ and any finite signed measure ξ on $(\mathsf{Z}, \mathcal{Z})$, by

$$\|\xi\|_{\mathsf{tv}} = \sup\left\{\int f(z)\xi(\mathrm{d}z);\, f \text{ measurable real function on } \mathsf{Z} \text{ such that } \|f\|_\infty = 1\right\}.$$

Lemma 7. *For all measures* μ_1, μ_2 *and all* $1 \leq q \leq k < \mathsf{n}$,

$$\left\|\int \mu_1(\mathrm{d}x)K^{V|X}_{\pi_V,k,q}(x,\cdot) - \int \mu_2(\mathrm{d}x)K^{V|X}_{\pi_V,k,q}(x,\cdot)\right\|_{\mathsf{tv}} \leq (1-\nu_k)\|\mu_1 - \mu_2\|_{\mathsf{tv}} \leq (1-\nu_k).$$

In particular, by induction,

$$\left\|\int \{\mu_1(\mathrm{d}v_\mathsf{n}) - \mu_2(\mathrm{d}v_\mathsf{n})\}K^{V|X}_{\pi_V,\mathsf{n}-1,q}(v_\mathsf{n}, \mathrm{d}v_{\mathsf{n}-1})\cdots K^{V|X}_{\pi_V,k,q}(v_{k+1},\cdot)\right\|_{\mathsf{tv}} \leq \prod_{i=k}^{\mathsf{n}-1}(1-\nu_i). \tag{14}$$

Lemma 8 proves a key loss of memory property of the backward chain X_q, with geometric rate of convergence. Whenever it is necessary, we adopt the convention $\prod_{k=\ell}^m a_k = 1$ for any $(a_\ell, \ldots, a_m)$ and any $\ell > m$.

Lemma 8. *For any* $1 \leq q \leq \mathsf{n}-1$,

$$\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}})\right| \leq \log(\nu_q^{-1}). \tag{15}$$

For all $\ell \geq 1$, $1 \leq q \leq \mathsf{n}-1$,

$$\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}+\ell})\right| \leq \nu_q^{-1}\prod_{k=q+1}^{\mathsf{n}-1}(1-\nu_k). \tag{16}$$

Proof. To prove (16), for $1 \le q < \mathsf{n}$, note that by Lemma 6,

$$\begin{aligned}
&\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}})\\
&\quad = \int \mathbb{P}_{\pi_V}(\mathrm{d}v_\mathsf{n}|X_{q+1:\mathsf{n}})\left(\prod_{k=q+1}^{\mathsf{n}-1} K^{V|X}_{\pi_V,k,q+1}(v_{k+1},\mathrm{d}v_k)\right)\pi_V(\mathrm{d}v_q)K_q(X_q,v_q,v_{q+1}). \qquad (17)
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}+\ell})\\
&\quad = \int \mathbb{P}_{\pi_V}(\mathrm{d}v_\mathsf{n}|X_{q+1:\mathsf{n}+\ell})\left(\prod_{k=q+1}^{\mathsf{n}-1} K^{V|X}_{\pi_V,k,q+1}(v_{k+1},\mathrm{d}v_k)\right)\pi_V(\mathrm{d}v_q)K_q(X_q,v_q,v_{q+1}). \qquad (18)
\end{aligned}$$

Then, by Lemma 6 and (14), combining (17) and (18) yields:

$$\begin{aligned}
&\left|\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}+\ell}) - \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}})\right|\\
&\quad \le \left(\prod_{k=q+1}^{\mathsf{n}-1}(1-\nu_k)\right)\sup_{v_{q+1}\in\mathbb{V}}\left|\int \pi_V(\mathrm{d}v_q)K_q(X_q,v_q,v_{q+1})\right| \le \prod_{k=q+1}^{\mathsf{n}-1}(1-\nu_k).
\end{aligned}$$

Inequality (16) is then a direct consequence of (17), (18) and the fact that for all $x, y > 0$, $|\log x - \log y| \le |x-y|/x \wedge y$. Inequality (15) follows from (17). □

Lemma 9 is the crucial result to bound the increments of the log-likelihood.

Lemma 9. *For all distributions $\pi_V, \pi'_V \in \Pi \cup \{\pi^\star\}$ and any $1 \le q \le \mathsf{n}$,*

$$\begin{aligned}
&\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi'_V}(X_q|X_{q+1:\mathsf{n}})\right|\\
&\quad \le 2\sum_{\ell=0}^{\mathsf{n}+1-q}(\nu_q\nu_{q+\ell-1}\nu_{q+\ell})^{-1}\left(\prod_{k=q+1}^{q+\ell-1}(1-\nu_k)\right)\|\pi_V - \pi'_V\|_{\mathsf{tv}}.
\end{aligned}$$

Proof. When $q = \mathsf{n}$,

$$\mathbb{P}_{\pi_V}(X_\mathsf{n}) - \mathbb{P}_{\pi'_V}(X_\mathsf{n}) = \int \left\{\pi_V'^{\otimes 2}(\mathrm{d}v_{\mathsf{n}:\mathsf{n}+1}) - \pi_V^{\otimes 2}(\mathrm{d}v_{\mathsf{n}:\mathsf{n}+1})\right\}K_\mathsf{n}(X_\mathsf{n},v_\mathsf{n},v_{\mathsf{n}+1}).$$

Thus $|\mathbb{P}_{\pi_V}(X_\mathsf{n}) - \mathbb{P}_{\pi'_V}(X_\mathsf{n})| \le 2\|\pi_V - \pi'_V\|_{\mathsf{tv}}$. When $1 \le q \le \mathsf{n}-1$,

$$\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\pi'_V}(X_q|X_{q+1:\mathsf{n}}) = \sum_{\ell=0}^{\mathsf{n}+1-q}\left\{\mathbb{P}_\ell(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(X_q|X_{q+1:\mathsf{n}})\right\},$$

where $\mathbb{P}_\ell$ is the joint distribution of $(X_{q:\mathsf{n}}, V_{q:\mathsf{n}+1})$ when $(V_q, \dots, V_{q+\ell-1})$ are i.i.d. π'_V and $(V_{q+\ell}, \dots, V_{\mathsf{n}+1})$ are i.i.d. π_V. The first term in the telescopic sum is given by:

$$\mathbb{P}_0(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_1(X_q|X_{q+1:\mathsf{n}}) = \int \mathbb{P}_0(\mathrm{d}v_{q+1}|X_{q+1:\mathsf{n}}) \int \pi'_V(\mathrm{d}v_q) K_q(X_q, v_q, v_{q+1}) \\ - \int \mathbb{P}_0(\mathrm{d}v_{q+1}|X_{q+1:\mathsf{n}}) \int \pi_V(\mathrm{d}v_q) K_q(X_q, v_q, v_{q+1}),$$

where $\mathbb{P}_0(V_{q+1}|X_{q+1:\mathsf{n}})$ is the distribution of V_{q+1} conditionally on $X_{q+1:\mathsf{n}}$ when $(V_q, \dots, V_{\mathsf{n}+1})$ are i.i.d. π_V. As V_q is independent of $(V_{q+1}, X_{q+1:\mathsf{n}})$, this distribution is the same as the distribution of V_{q+1} conditionally on $X_{q+1:\mathsf{n}}$ when $V_q \sim \pi'_V$ and $(V_{q+1}, \dots, V_{\mathsf{n}+1})$ are i.i.d. π_V.

$$\left|\mathbb{P}_0(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_1(X_q|X_{q+1:\mathsf{n}})\right| \le \left\|\pi_V - \pi'_V\right\|_{\mathsf{tv}}.$$

Then, for all $1 \le \ell \le \mathsf{n} + 2 - q$,

$$\mathbb{P}_\ell(X_q|X_{q+1:\mathsf{n}}) \\ = \int \mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) \left(\prod_{k=q+1}^{q+\ell-1} K^{V|X}_{\pi'_V,k,q+1}(v_{k+1}, \mathrm{d}v_k)\right) \int \pi'_V(\mathrm{d}v_q) K_q(X_q, v_q, v_{q+1}).$$

Therefore, by (14),

$$\left|\mathbb{P}_\ell(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(X_q|X_{q+1:\mathsf{n}})\right| \\ \le \left(\prod_{k=q+1}^{q+\ell-1} (1 - \nu_k)\right) \left\|\mathbb{P}_\ell(V_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(V_{q+\ell}|X_{q+1:\mathsf{n}})\right\|_{\mathsf{tv}},$$

where $\mathbb{P}_\ell(V_{q+\ell}|X_{q+1:\mathsf{n}})$ is the distribution of $V_{q+\ell}$ conditionally on $X_{q+1:\mathsf{n}}$ when $(V_q, \dots, V_{q+\ell-1})$ are i.i.d. π'_V and $(V_{q+\ell}, \dots, V_{\mathsf{n}+1})$ are i.i.d. π_V. It remains to show that

$$\left\|\mathbb{P}_\ell(V_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(V_{q+\ell}|X_{q+1:\mathsf{n}})\right\|_{\mathsf{tv}} \le 2(\nu_q \nu_{q+\ell-1} \nu_{q+\ell})^{-1} \left\|\pi_V - \pi'_V\right\|_{\mathsf{tv}}$$

which amounts to showing that for all f such that $\|f\|_\infty \le 1$,

$$\left|\int f(v_{q+\ell})\left\{\mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}})\right\}\right| \le 2(\nu_q \nu_{q+\ell-1} \nu_{q+\ell})^{-1} \left\|\pi_V - \pi'_V\right\|_{\mathsf{tv}}.$$

Write, for all $1 \le \ell \le \mathsf{n} + 2 - q$,

$$\mathrm{L}_\ell(\mathrm{d}v, X) = \prod_{m=q+1}^{q+\ell-1} \pi'_V(\mathrm{d}v_m) \prod_{m=q+\ell}^{\mathsf{n}+1} \pi_V(\mathrm{d}v_m) \prod_{m=q+1}^{\mathsf{n}} K_m(X_m, v_m, v_{m+1}). \tag{19}$$

We have

$$\int f(v_{q+\ell}) \mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) = \frac{\int f(v_{q+\ell}) L_\ell(\mathrm{d}v, X)}{\int L_\ell(\mathrm{d}v, X)}.$$

Therefore,

$$
\begin{aligned}
&\int f(v_{q+\ell})\big\{\mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}})\big\} \\
&\quad = \int f(v_{q+\ell})\bigg(\frac{L_\ell(\mathrm{d}v, X)}{\int L_\ell(\mathrm{d}v, X)} - \frac{L_{\ell+1}(\mathrm{d}v, X)}{\int L_{\ell+1}(\mathrm{d}v, X)}\bigg), \\
&\quad = \int f(v_{q+\ell})\frac{L_\ell(\mathrm{d}v, X) - L_{\ell+1}(\mathrm{d}v, X)}{\int L_\ell(\mathrm{d}v, X)} \\
&\qquad + \int f(v_{q+\ell})\frac{L_{\ell+1}(\mathrm{d}v, X)}{\int L_{\ell+1}(\mathrm{d}v, X)}\frac{\int[L_{\ell+1}(\mathrm{d}v, X) - L_\ell(\mathrm{d}v, X)]}{\int L_\ell(\mathrm{d}v, X)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\bigg|\int f(v_{q+\ell})\big\{\mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}})\big\}\bigg| \\
&\quad \le 2\frac{|\int\{L_\ell(\mathrm{d}v, X) - L_{\ell+1}(\mathrm{d}v, X)\}|}{\int L_\ell(\mathrm{d}v, X)}.
\end{aligned}
\tag{20}
$$

By (19), $1 \le \ell \le \mathsf{n} + 1 - q$,

$$
\begin{aligned}
&\bigg|\int\big\{L_\ell(\mathrm{d}v, X) - L_{\ell+1}(\mathrm{d}v, X)\big\}\bigg| \\
&\quad = \bigg|\int \prod_{m=q+1}^{q+\ell-1} \pi'_V(\mathrm{d}v_m)\big\{\pi_V(\mathrm{d}v_{q+\ell}) - \pi'_V(\mathrm{d}v_{q+\ell})\big\} \\
&\qquad \times \prod_{m=q+\ell+1}^{\mathsf{n}+1} \pi_V(\mathrm{d}v_m) \prod_{m=q+1}^{\mathsf{n}} K_m(X_m, v_m, v_{m+1})\bigg|.
\end{aligned}
$$

As $K_{q+\ell-1}$ and $K_{q+\ell}$ are upper bounded by 1,

$$
\begin{aligned}
&\bigg|\int\big\{L_\ell(\mathrm{d}v, X) - L_{\ell+1}(\mathrm{d}v, X)\big\}\bigg| \\
&\quad \le \bigg(\int \prod_{m=q+1}^{q+\ell-1} \pi'_V(\mathrm{d}v_m) \prod_{m=q+1}^{q+\ell-2} K_m(X_m, v_m, v_{m+1})\bigg) \\
&\qquad \times \|\pi_V - \pi'_V\|_{\mathrm{tv}}\bigg(\int \prod_{m=q+\ell+1}^{n+1} \pi_V(\mathrm{d}v_m) \prod_{m=q+\ell+1}^{n} K_m(X_m, v_m, v_{m+1})\bigg).
\end{aligned}
$$

Similarly, since $K_{q+\ell-1}$ and $K_{q+\ell}$ are respectively lower bounded by $\nu_{q+\ell-1}$ and $\nu_{q+\ell}$,

$$\int L_\ell(\mathrm{d}v, X) \geq \left(\int \prod_{m=q+1}^{q+\ell-1} \pi'_V(\mathrm{d}v_m) \prod_{m=q+1}^{q+\ell-2} K_m(X_m, v_m, v_{m+1})\right) \\ \times \nu_{q+\ell-1}\nu_{q+\ell}\left(\int \prod_{m=q+\ell+1}^{n+1} \pi_V(\mathrm{d}v_m) \prod_{m=q+\ell+1}^{n} K_m(X_m, v_m, v_{m+1})\right).$$

Plugging these bounds in (20) yields, for $1 \leq \ell \leq \mathsf{n} + 1 - q$,

$$\left|\int f(v_{q+\ell})\left\{\mathbb{P}_\ell(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\ell+1}(\mathrm{d}v_{q+\ell}|X_{q+1:\mathsf{n}})\right\}\right| \leq 2(\nu_{q+\ell-1}\nu_{q+\ell})^{-1}\left\|\pi_V - \pi'_V\right\|_{\mathrm{tv}}.$$

The proof is completed using the fact that for all $x, y > 0$, $|\log x - \log y| \leq |x - y|/x \wedge y$. □

Lemma 10 is a key ingredient to prove bounded difference properties for log-likelihood based processes.

Lemma 10. *For all $1 \leq q \leq \mathsf{n}$ and all $q \leq \tilde{q} \leq \mathsf{n}$, let $\widetilde{X}^{\tilde{q}}_{q:\mathsf{n}}$ be such that $\widetilde{X}^{\tilde{q}}_{\tilde{q}} \in \mathbb{X}$ and $\widetilde{X}^{\tilde{q}}_k = X_k$ for all $q \leq k \leq \mathsf{n}$ such that $k \neq \tilde{q}$. For any $1 \leq q \leq \tilde{q} \leq \mathsf{n}$,*

$$\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}\left(\widetilde{X}^{\tilde{q}}_q|\widetilde{X}^{\tilde{q}}_{q+1:\mathsf{n}}\right)\right| \leq \nu_q^{-1}\prod_{k=q+1}^{\tilde{q}-1}(1 - \nu_k).$$

Proof. If $q = \tilde{q} = \mathsf{n}$, then

$$\begin{aligned}\left|\mathbb{P}_{\pi_V}(X_\mathsf{n}) - \mathbb{P}_{\pi_V}\left(\widetilde{X}^n_n\right)\right| &= \left|\int \pi_V(\mathrm{d}v_\mathsf{n})\pi_V(\mathrm{d}v_{\mathsf{n}+1})\left\{K_\mathsf{n}(X_\mathsf{n}, v_\mathsf{n}, v_{\mathsf{n}+1}) - K_\mathsf{n}\left(\widetilde{X}^\mathsf{n}_\mathsf{n}, v_\mathsf{n}, v_{\mathsf{n}+1}\right)\right\}\right| \\ &\leq 1 - \nu_\mathsf{n} \leq 1.\end{aligned}$$

Assume now that $1 \leq q < \mathsf{n}$. When $\tilde{q} = q$,

$$\begin{aligned}&\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\pi_V}\left(\widetilde{X}^q_q|\widetilde{X}^q_{q+1:\mathsf{n}}\right) \\ &\quad = \int \mathbb{P}_{\pi_V}\left(\mathrm{d}v_{q+1}|\widetilde{X}^q_{q+1:\mathsf{n}}\right)\pi_V(\mathrm{d}v_q)\left\{K_q(X_q, v_q, v_{q+1}) - K_q\left(\widetilde{X}^q_q, v_q, v_{q+1}\right)\right\},\end{aligned}$$

which ensures that $|\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\pi_V}(\widetilde{X}^q_q|\widetilde{X}^q_{q+1:\mathsf{n}})| \leq 1 - \nu_q \leq 1$. When $\tilde{q} \geq q + 1$, as for all $q + 1 \leq k \leq \tilde{q} - 1$ the Markov transition kernel $K^{V|X}_{\pi_V,k,q+1}$ depends only on π_V, K_k and

$X_{q+1:k}$,

$$\begin{aligned}
&\mathbb{P}_{\pi_V}\big(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\big) \\
&\quad = \int \mathbb{P}_{\pi_V}\big(\mathrm{d}v_{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\big)\left(\prod_{k=q+1}^{\tilde{q}-1} K_{\pi_V,k,q+1}^{V|X}(v_{k+1},\mathrm{d}v_k)\right)\pi_V(\mathrm{d}v_q)K_q(X_q,v_q,v_{q+1}).
\end{aligned}$$

By Lemma 7, it follows that

$$\begin{aligned}
&\big|\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \mathbb{P}_{\pi_V}\big(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\big)\big| \\
&\quad \le \left(\prod_{k=q+1}^{\tilde{q}-1}(1-\nu_k)\right)\sup_{v_{q+1}\in\mathbb{V}}\left|\int \pi_V(\mathrm{d}v_q)K_q(X_q,v_q,v_{q+1})\right|.
\end{aligned}$$

The proof is completed using the fact that for all $x, y > 0$, $|\log x - \log y| \le |x-y|/x\wedge y$. $\square$

Let π_V^* denote a probability distribution on $\mathbb{V}$ and let

$$Z_{\pi_V}(X_{1:\mathsf{n}}) = \frac{1}{\mathsf{n}}\sum_{q=1}^{\mathsf{n}}\big[\log\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \mathbb{E}_{\pi_V^*}\big[\log\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}})\big]\big].$$

Lemma 11 shows the concentration of $Z_{\pi_V}(X_{1:\mathsf{n}})$ around its expectation.

Lemma 11. *Assume that $K_i = K$ for all $i\in\mathbb{Z}$, let $\mathcal{P}$ denote a class of probability distributions on $\mathbb{V}$. There exists $c>0$ such that for all $t>0$,*

$$\mathbb{P}_{\pi_V^*}\left(\left|\sup_{\pi_V\in\mathcal{P}}\big\{Z_{\pi_V}(X_{1:\mathsf{n}})\big\} - \mathbb{E}_{\pi_V^*}\Big[\sup_{\pi_V\in\mathcal{P}}\big\{Z_{\pi_V}(X_{1:\mathsf{n}})\big\}\Big]\right| \ge c\nu^{-2}\frac{t}{\sqrt{\mathsf{n}}}\right) \le 2e^{-t^2}.$$

Proof. The proof relies on the bounded difference inequality for Markov chains [10], Theorem 0.2. To apply this result, $\sup_{\pi_V\in\mathcal{P}}\{Z_{\pi_V}(X_{1:\mathsf{n}})\}$ has to be separately bounded. For all $1\le q\le\mathsf{n}$ and all $q\le\tilde{q}\le\mathsf{n}$, let $\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}$ such that $\widetilde{X}_{\tilde{q}}^{\tilde{q}}\in\mathbb{X}$ and $\widetilde{X}_k^{\tilde{q}} = X_k$ for all $1\le k\le\mathsf{n}$ such that $k\ne\tilde{q}$. Then,

$$\begin{aligned}
&\left|\sup_{\pi_V\in\mathcal{P}}\big\{Z_{\pi_V}(X_{1:\mathsf{n}})\big\} - \sup_{\pi_V\in\mathcal{P}}\big\{Z_{\pi_V}\big(\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}\big)\big\}\right| \\
&\quad \le \sup_{\pi_V\in\mathcal{P}}\left|\frac{1}{\mathsf{n}}\sum_{q=1}^{\mathsf{n}}\big[\log\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log\mathbb{P}_{\pi_V}\big(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\big)\big]\right| \\
&\quad \le \sup_{\pi_V\in\mathcal{P}}\left|\frac{1}{\mathsf{n}}\sum_{q=1}^{\tilde{q}}\big[\log\mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log\mathbb{P}_{\pi_V}\big(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\big)\big]\right|.
\end{aligned}$$

By Lemma 10, for any distribution $\pi_V \in \mathcal{P}$ and any $1 \le q \le \mathsf{n}$,

$$\left|\frac{1}{\mathsf{n}}\sum_{q=1}^{\mathsf{n}}\left[\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}\left(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\right)\right]\right| \le \frac{1}{\mathsf{n}}\sum_{q=1}^{\tilde{q}} \nu^{-1}(1-\nu)^{\tilde{q}-q-1}.$$

Hence, there exists $c > 0$ such that,

$$\left|\sup_{\pi_V \in \mathcal{P}}\left\{Z_{\pi_V}(X_{1:\mathsf{n}})\right\} - \sup_{\pi_V \in \mathcal{P}}\left\{Z_{\pi_V}\left(\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}\right)\right\}\right| \le \frac{c}{\nu^2 \mathsf{n}}.$$

The proof is concluded by [10], Theorem 0.2. □

Lemma 12 shows the subgaussian concentration inequality of the increments of $Z_{\pi_V}(X_{1:\mathsf{n}})$.

Lemma 12. *Assume that $K_i = K$ for all $i \in \mathbb{Z}$, let π_V, π_V' denote two probability distributions on $\mathbb{V}$. Then, there exists $c > 0$ such that for all $n \ge 1$, $t > 0$,*

$$\mathbb{P}_{\pi_V^*}\left(\left|\sqrt{\mathsf{n}}\left\{Z_{\pi_V}(X_{1:\mathsf{n}}) - Z_{\pi_V'}(X_{1:\mathsf{n}})\right\}\right| > t\right) \le \exp\left[-\frac{t^2}{(c\nu^{-5}d(\pi,\pi'))^2}\right]. \tag{21}$$

Proof. To prove that the increments $Z_{\pi_V} - Z_{\pi_V'}$ are separately bounded, consider, for all $1 \le \tilde{q} \le \mathsf{n}$, $\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}$ such that $\widetilde{X}_{\tilde{q}}^{\tilde{q}} \in \mathbb{X}$ and $\widetilde{X}_k^{\tilde{q}} = X_k$ for all $1 \le k \le \mathsf{n}$ such that $k \ne \tilde{q}$. Then, by Lemma 10,

$$\begin{aligned}\left|Z_{\pi_V}(X_{1:\mathsf{n}}) - Z_{\pi_V}\left(\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}\right)\right| &= \left|\frac{1}{\mathsf{n}}\sum_{q=1}^{\mathsf{n}}\left[\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}\left(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\right)\right]\right| \\ &\le \frac{1}{\mathsf{n}}\sum_{q=1}^{\tilde{q}}\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}\left(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\right)\right|.\end{aligned}$$

On one hand, by Lemma 9,

$$\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V'}(X_q|X_{q+1:\mathsf{n}})\right| \le 2\nu^{-4}\left\|\pi_V - \pi_V'\right\|_{\mathsf{tv}}.$$

On the other hand, by Lemma 10, for any $1 \le q \le \tilde{q} \le \mathsf{n}$,

$$\left|\log \mathbb{P}_{\pi_V}(X_q|X_{q+1:\mathsf{n}}) - \log \mathbb{P}_{\pi_V}\left(\widetilde{X}_q^{\tilde{q}}|\widetilde{X}_{q+1:\mathsf{n}}^{\tilde{q}}\right)\right| \le \nu^{-1}(1-\nu)^{\tilde{q}-q-1}.$$

Thus,

$$\begin{aligned}&\left|\left(Z_{\pi_V}(X_{1:\mathsf{n}}) - Z_{\pi_V'}(X_{1:\mathsf{n}})\right) - \left(Z_{\pi_V}\left(\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}\right) - Z_{\pi_V'}\left(\widetilde{X}_{1:\mathsf{n}}^{\tilde{q}}\right)\right)\right| \\ &\quad \le \frac{2\nu^{-4}}{\mathsf{n}}\sum_{q=1}^{\tilde{q}}\left[\left\|\pi_V - \pi_V'\right\|_{\mathsf{tv}} \wedge (1-\nu)^{\tilde{q}-q}\right] \le \frac{2\nu^{-5}}{\mathsf{n}}d\left(\pi,\pi'\right).\end{aligned}$$

Equation (21) follows by plugging these bounded differences properties in [10], Theorem 0.2. □

Appendix C: Proofs of the main results

When **H1** holds and $E = E_{\mathrm{RR}}^{n,N}$, $(V_{2:\mathsf{q}_E}^E, X_{2:\mathsf{q}_E-1}^E)$ satisfies the assumptions of Section B with

$$\pi_V = \pi^{\otimes n-1}, \qquad K_i\big(X_i^E, V_i^E, V_{i+1}^E\big) = \prod_{X_{i,j}\in X_i^E} k(X_{i,j}, V_i, V_j), \qquad \nu_i = \varepsilon^{|X_i^E|}.$$

Moreover, it is proved in Section A that $|X_q^E| = n(n-1)$ for $2 \le q \le \mathsf{q}_E - 1$, which implies that

$$\nu_i \ge \varepsilon^{n^2}. \tag{22}$$

Throughout the proofs, the following conventions are used. For all $0 \le k \le \mathsf{q}_E$,

$$v_k^E \in \mathcal{V}^{|V_k^E|}, \quad \pi\big(\mathrm{d}v_k^E\big) = \prod_{i: V_i \in V_k^E} \pi(\mathrm{d}v_i).$$

C.1. Proof of Theorem 2

The first inequality is a direct conclusion of Lemma 8. The proof of the second inequality follows the same lines. Then, the log-likelihood is decomposed as follows

$$\begin{aligned}\log \mathbb{P}_\pi^E\big(X^E\big) &= \log \mathbb{P}_\pi^E\big(X_{2:\mathsf{q}_E-1}^E\big) + \log \mathbb{P}_\pi^E\big(X_0^E, X_1^E, X_{\mathsf{q}_E}^E | X_{2:\mathsf{q}_E-1}^E\big)\\ &= \sum_{q=2}^{\mathsf{q}_E-1} \log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E-1}^E\big) + \log \mathbb{P}_\pi^E\big(Z^E | X_{2:\mathsf{q}_E-1}^E\big).\end{aligned} \tag{23}$$

Let us first bound from above the last term in (23).

$$\begin{aligned}\mathbb{P}_\pi^E\big(Z^E | X_{2:\mathsf{q}_E-1}^E\big) &= \int \mathbb{P}_\pi^E\big(Z^E, \mathrm{d}v_{0:2}^E, \mathrm{d}v_{\mathsf{q}_E:\mathsf{q}_E+1}^E | X_{2:\mathsf{q}_E-1}^E\big)\\ &= \int \mathbb{P}_\pi^E\big(\mathrm{d}v_{0:2}^E, \mathrm{d}v_{\mathsf{q}_E:\mathsf{q}_E+1}^E | X_{2:\mathsf{q}_E-1}^E\big)\Bigg\{\prod_{X_{i,j}\in Z^E} k(X_{i,j}, v_i, v_j)\Bigg\}.\end{aligned}$$

By Assumption **H1**

$$\varepsilon^{3n^2} \le \mathbb{P}_\pi^E\big(Z^E | X_{2:\mathsf{q}_E-1}^E\big) \le 1. \tag{24}$$

In particular, the last term in (23) is $O(1)$ when N grows to infinity. On the other hand,taking the limit as $\ell \to \infty$ in Lemma 8 and recalling that $\nu_i \ge \varepsilon^{n^2}$, see (22), for any $\pi \in \Pi$,

$$\frac{1}{\mathsf{q}_E}\sum_{q=2}^{\mathsf{q}_E-1}\big|\log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E-1}^E\big) - \ell_\pi^n\big(\vartheta^q \mathbf{X}^{\mathbf{n}}\big)\big| \le \frac{1}{\mathsf{q}_E}\sum_{q=2}^{\mathsf{q}_E-1}\frac{(1-\varepsilon^{n^2})^{q_E-q-2}}{\varepsilon^{n^2}} \le \frac{\varepsilon^{-3n^2}}{\mathsf{q}_E}. \tag{25}$$

By (15), $|\ell_\pi^n(\mathbf{X^n})| \leq n^2 \log(\varepsilon^{-1})$, thus ℓ_π^n is integrable. Therefore, the ergodic theorem [1], Theorem 24.1, can be applied to $\sum_{q=2}^{\mathsf{q}_E-1} \ell_\pi^n(\vartheta^q \mathbf{X^n})/\mathsf{q}_E$ and (4) follows.

C.2. $R_{\pi_\star}$ is the excess risk function

The following result shows that $R_{\pi_\star}^n$ is a non-negative function.

Proposition 13. *For all* $\pi \in \Pi$ *and all* $n \geq 1$, $R_{\pi_\star}^n(\pi) \geq 0$.

Proof. Let $\pi \in \Pi$ and $n \geq 1$. By (3),

$$\mathsf{L}_{\pi_\star}^n(\pi) = \mathbb{E}_{\pi_\star}\Big[\lim_{N\to\infty} \log \mathbb{P}_\pi^E\big(X_2^E | X_{3:\mathsf{q}_E-1}^E\big)\Big].$$

By Lebesgue's bounded convergence theorem

$$\begin{aligned}\mathsf{L}_{\pi_\star}^n(\pi) &= \lim_{N\to\infty} \mathbb{E}_{\pi_\star}\big[\log \mathbb{P}_\pi^E\big(X_2^E | X_{3:\mathsf{q}_E-1}^E\big)\big] \\ &= \lim_{N\to\infty} \mathbb{E}_{\pi_\star}\big[\mathbb{E}_{\pi_\star}\big[\log \mathbb{P}_\pi^E\big(X_2^E | X_{3:\mathsf{q}_E-1}^E\big) | X_{3:\mathsf{q}_E-1}^E\big]\big].\end{aligned}$$

Therefore,

$$R_{\pi_\star}^n(\pi) = \lim_{N\to\infty} \big\{\mathbb{E}_{\pi_\star}\big[\mathbb{E}_{\pi_\star}\big[\log \mathbb{P}_{\pi_\star}^E\big(X_2^E | X_{3:\mathsf{q}_E-1}^E\big) - \log \mathbb{P}_\pi^E\big(X_2^E | X_{3:\mathsf{q}_E-1}^E\big) | X_{3:\mathsf{q}_E-1}^E\big]\big]\big\},$$

and the latter is non negative since the term in the expectation is a Kullback–Leibler divergence. □

C.3. Proof of Theorem 3

As that for any $\pi \in \Pi \cup \{\pi_\star\}$, $\ell^E(\pi) = \log \mathbb{P}_\pi^E(X^E)$, the excess loss satisfies:

$$\begin{aligned}R_{\pi_\star}^n\big(\widehat{\pi}^E\big) = {} & \mathsf{L}_{\pi_\star}^n(\pi_\star) - \mathbb{E}_{\pi_\star}\left[\frac{1}{\mathsf{q}_E}\ell^E(\pi_\star)\right] + \mathbb{E}_{\pi_\star}\left[\frac{1}{\mathsf{q}_E}\ell^E(\pi_\star)\right] - \frac{1}{\mathsf{q}_E}\ell^E(\pi_\star) \\ & + \frac{1}{\mathsf{q}_E}\ell^E(\pi_\star) - \frac{1}{\mathsf{q}_E}\ell^E\big(\widehat{\pi}^E\big) + \frac{1}{\mathsf{q}_E}\ell^E\big(\widehat{\pi}^E\big) - \mathbb{E}_{\pi_\star}\left[\frac{1}{\mathsf{q}_E}\ell^E\big(\widehat{\pi}^E\big)\right] \\ & + \mathbb{E}_{\pi_\star}\left[\frac{1}{\mathsf{q}_E}\ell^E\big(\widehat{\pi}^E\big)\right] - \mathsf{L}_{\pi_\star}^n\big(\widehat{\pi}^E\big).\end{aligned}$$

By definition $\ell^E(\pi_\star) - \ell^E(\widehat{\pi}^E) \leq 0$. Thus,

$$R_{\pi_\star}^n\big(\widehat{\pi}^E\big) \leq 2 \sup_{\pi \in \Pi\cup\{\pi^*\}} \left\{\left|\mathsf{L}^{\pi_\star}(\pi) - \frac{\mathbb{E}_{\pi_\star}[\ell^E(\pi)]}{\mathsf{q}_E}\right| + \left|\frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\big[\ell^E(\pi)\big] - \frac{\ell^E(\pi)}{\mathsf{q}_E}\right|\right\}.$$

For all $\pi \in \Pi$, as, for any $q \in \mathbb{Z}$, $\mathbb{E}_{\pi_\star}[\ell_\pi^n(\mathbf{X^n})] = \mathbb{E}_{\pi_\star}[\ell_\pi^n(\vartheta^q \mathbf{X^n})]$,

$$\mathsf{L}^{\pi_\star}(\pi) = \frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\left[\sum_{q=2}^{\mathsf{q}_E-1} \ell_\pi^n\big(\vartheta^q \mathbf{X^n}\big)\right] + \frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\big[2\ell_\pi^n\big(\mathbf{X^n}\big)\big].$$

Moreover, if $Z^E = X_0^E \cup X_1^E \cup X_{\mathsf{q}_E}^E$,

$$\ell^E(\pi) = \log \mathbb{P}_\pi^E\big(X^E\big) = \sum_{q=2}^{\mathsf{q}_E-1} \log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E-1}^E\big) + \log \mathbb{P}_\pi^E\big(Z^E | X_{2:\mathsf{q}_E-1}^E\big).$$

Therefore,

$$\left|\mathsf{L}^{\pi_\star}(\pi) - \frac{\mathbb{E}_{\pi_\star}[\ell^E(\pi)]}{\mathsf{q}_E}\right| \le \frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\left[\sum_{q=2}^{\mathsf{q}_E-1} \left|\ell_\pi^n\big(\vartheta^q \mathbf{X^n}\big) - \log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E-1}^E\big)\right|\right]$$
$$+ \frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\big[\big|2\ell_\pi^n\big(\mathbf{X^n}\big)\big| + \big|\log \mathbb{P}_\pi^E\big(Z^E | X_{2:\mathsf{q}_E-1}^E\big)\big|\big].$$

Then, by (25), (15) and (24) and the inequality $x \le e^x$, there exists c such that:

$$\sup_{\pi \in \Pi \cup \{\pi^*\}} \left|\mathsf{L}^{\pi_\star}(\pi) - \frac{\mathbb{E}_{\pi_\star}[\ell^E(\pi)]}{\mathsf{q}_E}\right| \le \frac{c\varepsilon^{-3n^2}}{\mathsf{q}_E}.$$

This yields:

$$R_{\pi_\star}^n\big(\widehat{\pi}^E\big) \le \frac{c\varepsilon^{-3n^2}}{\mathsf{q}_E} + 2 \sup_{\pi \in \Pi \cup \{\pi^*\}} \left|\frac{1}{\mathsf{q}_E}\mathbb{E}_{\pi_\star}\big[\ell^E(\pi)\big] - \frac{1}{\mathsf{q}_E}\ell^E(\pi)\right|,$$

and therefore, by (24),

$$R_{\pi_\star}^n\big(\widehat{\pi}^E\big) \le \frac{c\varepsilon^{-3n^2}}{\mathsf{q}_E} + 2 \sup_{\pi \in \Pi \cup \{\pi^*\}} |Z_{\pi_V}|, \tag{26}$$

where

$$Z_\pi = \frac{1}{\mathsf{q}_E}\sum_{q=2}^{\mathsf{q}_E-1} \big[\log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E}^E\big) - \mathbb{E}_{\pi_\star}\big[\log \mathbb{P}_\pi^E\big(X_q^E | X_{q+1:\mathsf{q}_E}^E\big)\big]\big].$$

Lemma 11 applies by assumption **H1** since $E = E_{\mathrm{RR}}^{n,N}$, therefore, there exists $c > 0$ such that,

$$\mathbb{P}_{\pi_\star}\left(\left|\sup_{\pi \in \Pi \cup \{\pi^*\}} Z_\pi - \mathbb{E}_{\pi_\star}\left[\sup_{\pi \in \Pi \cup \{\pi^*\}} Z_\pi\right]\right| > c\varepsilon^{-2n^2}\frac{t}{\sqrt{\mathsf{q}_E}}\right) \le e^{-t^2}, \quad \forall t > 0. \tag{27}$$

Furthermore, by Lemma 12, the increments of Z_π have subgaussian tails.

$$\mathbb{P}_{\pi_\star}\left(\sqrt{\mathsf{q}_E}|Z_\pi - Z_{\pi'}| > t\right) \le \exp\left(-\frac{t^2}{(c\varepsilon^{-5n^2} d(\pi^{\otimes|V_2^E|}, (\pi')^{\otimes|V_2^E|}))^2}\right), \quad \forall t > 0.$$

Now it is easy to check that

$$\left\|\pi^{\otimes|V_2^E|} - (\pi')^{\otimes|V_2^E|}\right\|_{\mathsf{tv}} \le \left|V_2^E\right| \|\pi - \pi'\|_{\mathsf{tv}}.$$

Therefore, $d(\pi^{\otimes|V_2^E|}, (\pi')^{\otimes|V_2^E|}) \le cn^2 d(\pi, \pi') \le c\varepsilon^{-n^2} d(\pi, \pi')$, thus

$$\mathbb{P}_{\pi_\star}\left(\sqrt{\mathsf{q}_E}|Z_\pi - Z_{\pi'}| > t\right) \le \exp\left(-\frac{t^2}{(c\varepsilon^{-6n^2} d(\pi, \pi'))^2}\right), \quad \forall t > 0. \tag{28}$$

Then, by Dudley's entropy bound, see [12] or [25], Proposition 2.1,

$$\mathbb{E}_{\pi_\star}\left[\sup_{\pi \in \Pi \cup \{\pi_\star\}} Z_\pi\left(X^E\right)\right] \le \frac{c e^{-6n^2}}{\sqrt{\mathsf{q}_E}} \int_0^{+\infty} \sqrt{\log \mathsf{N}\left(\Pi \cup \{\pi_\star\}, d, \epsilon\right)}\, \mathrm{d}\epsilon. \tag{29}$$

Plugging (27) and (29) into (26) concludes the proof.

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