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Stratonovich type integration with respect to fractional Brownian motion with Hurst parameter less than $1/2$

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Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$ and $p : \mathbb{R} \to \mathbb{R}$ a polynomial function. The main purpose of this paper is to introduce a Stratonovich type stochastic integral with respect to B^H, whose domain includes the process $p(B^H)$. That is, an integral that allows us to integrate $p(B^H)$ with respect to B^H, which does not happen with the symmetric integral given by Russo and Vallois (*Probab. Theory Related Fields* **97** (1993) 403–421) in general. Towards this end, we combine the approaches utilized by León and Nualart (*Stochastic Process. Appl.* **115** (2005) 481–492), and Russo and Vallois (*Probab. Theory Related Fields* **97** (1993) 403–421), whose aims are to extend the domain of the divergence operator for Gaussian processes and to define some stochastic integrals, respectively. Then, we study the relation between this Stratonovich integral and the extension of the divergence operator (see León and Nualart (*Stochastic Process. Appl.* **115** (2005) 481–492)), an Itô formula and the existence of a unique solution of some Stratonovich stochastic differential equations. These last results have been analyzed by Alòs, León and Nualart (*Taiwanese J. Math.* **5** (2001) 609–632), where the Hurst paramert H belongs to the interval $(1/4, 1/2)$.

Keywords: derivative and divergence operators in the Malliavin calculus sense; Doss transformation; fractional integrals and derivatives; Itô formula; Malliavin calculus for fBm; Stratonovich stochastic differential equation; symmetric stochastic integration

1. Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B_t^H : t \in [0, T]\}$ with covariance function (see Mandelbrot and Van Ness [25])

$$R(t, s) := E\big(B_t^H B_s^H\big) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t - s|^{2H}\big), \quad s, t \in [0, T]. \tag{1.1}$$

It is well known that B^H is not a semimartingale for $H \neq 1/2$ (see, for instance, Liptser and Shiryaev [24], Nualart [28] or Rogers [33]). So, we cannot use the classical Itô's calculus to define stochastic integrals with respect to fBm as it is done for Brownian motion $B^{1/2}$. Therefore, it is necessary to apply or develop different approaches in order to consider some interpretations of stochastic integral with respect to B^H. Hence, in the literature, there are different points of view to deal with this problem. Thus, the main purpose of this paper is to define a stochastic

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integral of Stratonovich type and to analyze an Itô's formula for it, for any $H < 1/2$. Also, as an application of this Itô's formula, we show the existence and uniqueness for the solution to some Stratonovich stochastic differential equations driven by fBm.

For the Brownian motion (i.e., $H = 1/2$), important applications of the classical Itô's stochastic calculus to different areas of human knowledge are based on integrals of Itô and Stratonovich sense, and their change of variables formulae. But, sometimes the nature of the phenomenon that is being studied requires to work with integrals whose domains include processes not necessarily adapted to the underlying filtration, as it is in the analysis of finantial markets with an insider (see, for example, León, Navarro and Nualart [21]). In order to resolve this problem, several authors have employed the Skorohod integral (or divergence operator δ in the Malliavin calculus) and the forward integral introduced by Russo and Vallois [35]. Both are extensions of Itô's integral in the sense that they agree with it if the integrand is a square-integrable and adapted process to the filtration generated by $B^{1/2}$. Moreover, it is possible to get estimations of the moments of δ via the Malliavin calculus as it is done in Nualart [29], while the forward integral given by Russo and Vallois [35] is difficult to handle, in general, because it is a limit in probability. Fortunately, we can estimate the moments of the forward integral using the Malliavin calculus and its relation with the divergence operator δ, if we restrict its domain to a set of processes that satisfy suitable condition in the Malliavin calculus sense. Furthermore, Russo and Vallois [35] have introduced an integral in the Stratonovich sense, the so called symmetric integral, that is related to the operator δ via the Malliavin calculus.

For $H \in (0, 1)$, Decreusefond and Üstünel [12] have utilized the calculus of variations to consider the divergence operator and an integral in the Stratonovich sense with respect to B^H, and their Itô's formulae. It is worth mentioning that these formulae do not include the case analized in Theorem 4.1 below. For paths with q-variation along a sequence of partitions and q-times continuous differentiable functions, with $q \in 2\mathbb{N}$, Cont and Perkowski [8] construct a pathwise integration theory to get a change of variables formula, where the involved integral is defined as a pointwise limit of compensated Riemann sums. Carmona, Coutin and Montseny [5] have defined a stochastic integral with respect to B^H as the limit of integrals with respect to semimartingales. The construction of these semimartingales is based on the integral representation $B^H_\cdot = \int_0^\cdot K_H(\cdot, s)\, dB_s^{1/2}$ given in Decreusefond and Üstünel [12] (see also Nualart [28], or Mandelbrot and Van Ness [25]). That is, the semimartingales are obtained by smoothing the kernel K_H, and the Malliavin calculus techniques are used to handle with this limit and to analyze an Itô's formula for $H > 1/6$. Moreover, Alòs, Mazet and Nualart [3] have utilized the calculus of variations to study the divergence operator and a Stratonovich integral with respect to Gaussian processes of the form $\int_0^t K(t, s)\, dB_s^{1/2}$, $t \in [0, T]$. As an application, they obtain Itô's formulae for $H > 1/4$ (see also Decreusefond [11] for an associated analysis with Alòs, Mazet and Nualart [3]). In general, the forward and symmetric integrals in Russo and Vallois [35] are integrals with respect to processes that are not necessarily semimartingales. So, it is natural to consider these integrals with respect to either fBm, or another processes. For instance, the symmetric integral defined in Russo and Vallois [35] with respect to B^H, $H \geq 1/4$, (resp. $H > 1/6$) has been used to analyze an Itô's formula in Gradinaru, Russo and Vallois [17] (resp. Russo and Tudor [34]). For cubic variation continuous processes, Errami and Russo [15] work with the symmetric integral in Russo and Vallois [35] to get a change of variables formula and the existence of a unique solution to SDEs through Doss method (see Doss [13]).

In the case that $H > 1/2$, it is natural to interpret the integral with respect to B^H as a pathwise Riemann–Stieltjes integral (i.e., ω by ω), for any α-Hölder continuous stochastic process with $\alpha > 1 - H$ (see Young [38]), due to fBm having β-Hölder continuous paths, for every $\beta < H$. It turns out that the integral of Young type agrees with the forward and symmetric integrals (see Russo and Vallois [36]). Lin [23], and Dai and Heyde [10] have dealt with the $L^2(\Omega)$ convergence of the Riemann sums. Note that this approach is useful for this case (i.e., $H > 1/2$) when we are working with stochastic differential equations driven by fBm because the Riemann–Stieltjes integral, in general, has β-Hölder continuous paths, for every $\beta < H$. In order to improve this pathwise approach, Zähle [39] (resp. Zähle [40]) has employed the fractional calculus to give an extension of the Riemann–Stieltjes integral (resp. of the forward integral given by Russo and Vallois [35]). In Alòs and Nualart [4], it is developed a stochastic calculus for fBm via the Malliavin calculus. In particular, they have established that the forward and symmetric integrals in Russo and Vallois [35] are the same if the integrand satisfies some conditions involving the derivative operator in the Malliavin calculus sense. These integrals are equal to divergence operator plus a trace term (see also Duncan, Hu and Pasik-Duncan [14], where the integrals are defined as the limit of Riemann–Wick sums).

For $H < 1/2$, Alòs, León and Nualart [1] have pointed out that the forward integral $\int_0^T B_s^H \, dB_s^{H-}$ does not exist. But the Stratonovich integral $\int_0^T B_s^H \circ dB_s^H$, in the Russo and Vallois [35] sense, is always well-defined, which has been observed by Cheridito and Nualart [7]. The existence of a unique solution to SDEs driven by fBm and an Itô's formula similar to that in Theorem 4.1 below hold in the following situations:

- (i) $H > 1/4$ and the Stratonovich stochastic integral is the symmetric integral in Russo and Vallois [35], which is a limit in probabiliy. This is done by Alòs, León and Nualart [1] using the Malliavin calculus.
- (ii) $H > 1/4$ and the integral is defined by means of the rough path theory (see Coutin and Qian [9]).
- (iii) $H > 1/6$ and the symmetric integral defined by Russo and Vallois [35] is given as a uniformly limit in probability. The results can be found in Gradinaru et al. [16] and Nourdin [26].
- (iv) $H \leq 1/6$ and the integral is a renormalized Stratonovich integral. This is also stated in Gradinaru et al. [16] and Nourdin [26].

We remark that in Statements (iii) and (iv), the Itô's formula is satisfied for $f \in C^6(\mathbb{R})$ and $f \in C^{4r+2}(\mathbb{R})$, respectively. Here $r \geq 2$ is an integer such that $(2r + 1)H > 1/2$. Moreover, concerning SDEs, the diffusion coefficient belongs to $C^{n_H}(\mathbb{R})$, where $n_H \in \mathbb{N}$ depends on H. In the present paper, the Itô's formula is established for $f \in C^2(\mathbb{R})$ satisfying suitable growth conditions and the integral coincides with those considered in Statements (i) and (iii) if the integrand satisfies suitable conditions (see Theorem 4.1 below). On the other hand, still in the case $H < 1/2$, the approach utilized by Carmona, Coutin and Montseny [5] was followed by Alòs, Mazet and Nualart [2] to obtain sufficient conditions for the existence of an integral with respect to Gaussian processes of the form $\int_0^t (t-s)^\alpha \, dB_s^{1/2}$, where $\alpha \in (0, 1/2)$. Concerning the divergence operator with respect to B^H (resp. Gaussian processes), its domain has been extended by Cheridito and Nualart [7] (resp. by León and Nualart [22]). In Cheridito and Nualart [7], the authors use the extended divergence operator in order to establish that, for $H \in (1/6, 1/2)$ and

$g \in C^4(\mathbb{R})$, the Russo and Vallois symmetric integral $\int_0^T g'(B_s^H) \circ dB_s^H$ exists and is equal to $g(B_T^H) - g(0)$ (i.e., the Itô's formula is satisfied), while the integral $\int_0^T (B_s^H)^2 \circ dB_s^H$ does not exist for every $H \in (0, 1/6]$. Also, Hu, Jolis and Tindel [18] have considered an extension of the divergence operator with respect to a Gaussian process X to deal with change of variables formulae of Stratonovich and Skorohod type. To do so, they use the Malliavin calculus to see that the process $r \mapsto \nabla f(X_r)$ belongs to the domain of the extended divergence operator, for any $f \in C^2(\mathbb{R}^d, \mathbb{R})$ with suitable growth conditions. Then, they study the relation between the extended Skorohod integral and the Young integral given by the rough paths theory of $\nabla f(X)$ with respect to X (see Cass and Lim [6] for a similar relation). In this way, they get a Stratonovich change of variables formula for $f \in C^{2n}(\mathbb{R}^d, \mathbb{R})$, where n depends on the path regularity of the Gaussian process X. In Kruk and Russo [20], the authors examine problems similar to those in Hu, Jolis and Tindel [18]. Concerning B^H with $H \in (0, 1/2)$, Privault [31] has introduced Skorohod type integrals to obtain an Itô's formula via Malliavin calculus, for $f \in C_b^2(\mathbb{R})$. Furthermore, Nualart and Taqqu [30] have also treated with the Skorohod integral with respect to Gaussian processes to obtain an Itô's formula, which includes the fBm case with $H \in (1/4, 1/2)$.

The first purpose of this paper is to introduce a Stratonovich type stochastic integral with respect to the process B^H (i.e., an integral related to the symmetric integral in Russo and Vallois [35]), via the stochastic calculus of variations, in the case that $H < 1/2$. To do so, we manipulate the ideas developed by Cheridito and Nualart [7], and León and Nualart [22]. In this way, we define an integral such that $\int_0^T p(B_s^H) \circ dB_s^H$ exists, for every real polynomial function p. Also, for $H < 1/2$, we state a relation between this Stratonovich integral and the extended divergence operator studied in León and Nualart [22] following the ideas in the proof of Theorem 2 in Alòs, León and Nualart [1], which requires that H belongs to the interval $(1/4, 1/2)$. The second aim of this article is to study an Itô's formula for this integral, which gives an existence and uniqueness result for the solution of some SDEs driven by fBm.

In order to clarify the purposes of this paper, now we give an idea of the definition of our integral (see Definition 3.1 below). The reader can see Section 2 for details.

Let $\mathcal{H}$ be the reproducing kernel Hilbert space associated with B^H. Then, by Nualart [28], there is a linear operator $\mathcal{T} : \mathcal{H} \subset L^2([0, T]) \to L^2([0, T])$ such that $|\mathcal{T}h|_{L^2([0,T])} = |h|_{\mathcal{H}}$, for all $h \in \mathcal{H}$. Thus, we say that a suitable process u is weak Stratonovich integrable if and only if there exists a square-integrable random variable $\int_0^T u_t \circ dB_t^H$ such that

$$\left\langle F, \int_0^T u_t \circ dB_t^H \right\rangle_{L^2(\Omega)} = \lim_{\varepsilon \downarrow 0} \left\langle F, \frac{1}{2\varepsilon} \int_0^T u_s \left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H \right) ds \right\rangle_{L^2(\Omega)},$$

provided this limit exists for every smooth functional F such that DF belong to the domain of the operator $\mathcal{T}^*\mathcal{T}$. Here, D is the derivative operator with respect to B^H, in the Malliavin calculus sense, and $\mathcal{T}^*$ is the adjoint operator of $\mathcal{T}$. It turns out that such a family of smooth functionals F is large enough to characterize the random variable $\int_0^T u_t \circ dB_t^H$. Note that this definition follows the ideas developed in León and Nualart [22], and Russo and Vallois [35].

In order to get our Itô's formula, for $\varepsilon > 0$, we introduce the process

$$B_t^{H,\varepsilon} = \frac{1}{2\varepsilon} \int_0^t \left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H \right) ds, \quad t \in [0, T],$$

and use the fundamental theorem of calculus to have

$$\begin{aligned}
&f\left(t, B_t^{H,\varepsilon}\right) \\
&\quad = f(0,0) + \int_0^t \partial_t f\left(s, B_s^{H,\varepsilon}\right) ds + \frac{1}{2\varepsilon}\int_0^t \partial_x f\left(s, B_s^H\right)\left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H\right) ds \\
&\qquad + \frac{1}{2\varepsilon}\int_0^t \left(\partial_x f\left(s, B_s^{H,\varepsilon}\right) - \partial_x f\left(s, B_s^H\right)\right)\left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H\right) ds.
\end{aligned}$$

Hence, we only need to prove that

$$\left\langle F, \frac{1}{2\varepsilon}\int_0^t \left(\partial_x f\left(s, B_s^{H,\varepsilon}\right) - \partial_x f\left(s, B_s^H\right)\right)\left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H\right) ds\right\rangle_{L^2(\Omega)}$$

converges to 0, as $\varepsilon \downarrow 0$, where F is as before. Towards this end, we use the duality relation between the operator D and the extended divergence operator in Cheridito and Nualart [7], and León and Nualart [22]. But, unlike Alòs, León and Nualart [1], we do not need to apply a norm of a Sobolev space given by the Malliavin calculus to see this convergence since now it is enough to analyze it using basically the norm of the space $L^2(\Omega \times [0,T])$.

The paper is organized as follows. Section 2 contains the framework and the basic tool that we need to state our results. Section 3 is devoted to define the integral of Stratonovich type and to associate it with the extended divergence operator given in Cheridito and Nualart [7], and León and Nualart [22]. In Section 4, we establish an Itô's formula for the indefinite Stratonovich integral and we consider one-dimensional SDEs in the Stratonovich sense driven by fBm. Finally, in Section 4.2, we deal with some auxiliary results, which are part of the proof of the Itô's formula.

2. Preliminaries

In this section, we establish the framework that is considered in this paper. Although some results in this section are known, we prefer to provide a self-contained exposition for the convenience of the reader.

Throughout the article, C stands for a generic constant whose value may change from line to line.

2.1. Fractional integrals and derivatives

Consider $a < b$ and an $L^1([a,b])$-function f. For $t \in [a,b]$ and $\beta \in (0,1)$, the fractional integrals of f are defined as

$$I_{a+}^{\beta} f_t = \frac{1}{\Gamma(\beta)}\int_a^t (t-r)^{\beta-1} f_r\, dr, \quad \text{and} \quad I_{b-}^{\beta} f_t = \frac{1}{\Gamma(\beta)}\int_t^b (r-t)^{\beta-1} f_r\, dr.$$

For any $p \geq 1$, we denote by $I^{\beta}_{a+}(L^p)$ the image of $L^p([a,b])$ by I^{β}_{a+}, and similarly for $I^{\beta}_{b-}(L^p)$. The inverses of the operators I^{β}_{a+} and I^{β}_{b-} are called fractional derivatives, and are defined as follows. For $f \in I^{\beta}_{a+}(L^p)$ and $t \in [a,b]$ we set

$$D^{\beta}_{a+} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\beta)} \left(\frac{f_t}{(t-a)^{\beta}} + \beta \int_a^{t-\varepsilon} \frac{f_t - f_r}{(t-r)^{1+\beta}} \, dr \right), \tag{2.1}$$

where we apply the convention $f_r = 0$ on $[a,b]^c$. In the same way, for $f \in I^{\beta}_{b-}(L^p)$ and $t \in [a,b]$, we set

$$D^{\beta}_{b-} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\beta)} \left(\frac{f_t}{(b-t)^{\beta}} + \beta \int_{t+\varepsilon}^{b} \frac{f_t - f_r}{(r-t)^{1+\beta}} \, dr \right). \tag{2.2}$$

By Samko, Kilbas and Marichev [37] (Remark 13.2), we have that, for $p > 1$, $f \in I^{\beta}_{a+}(L^p)$ (resp. $f \in I^{\beta}_{b-}(L^p)$) if and only if $f \in L^p([a,b])$ and the limit in the right-hand side of (2.1) (resp. (2.2)) exists. In this case, $f = I^{\beta}_{a+}(D^{\beta}_{a+} f)$ (resp. $f = I^{\beta}_{b-}(D^{\beta}_{b-} f)$).

2.2. Fractional Brownian motion

The purpose of this section is to give the notation and results on fractional Brownian motion (fBm) that we use in this article. We refer to Nualart [28] or Nualart [29] for a detailed exposition of this subjet.

Henceforth, $T \in (0,\infty)$ and $B^H = \{B^H_t : t \in [0,T]\}$ is a fBm with Hurst parameter $H \in (0,1/2)$.

The reproducing kernel Hilbert space $\mathcal{H}$, associated with B^H, is the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s),$$

where R is introduced in (1.1). It is well known that $1_{[0,t]} \mapsto B^H_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space generated by B^H. This isometry is denoted by $\varphi \mapsto B^H(\varphi)$ and allows us to consider B^H as an isonormal Gaussian process on $\mathcal{H}$. Moreover, the space $\mathcal{H}$ is densely and continuously embedded in $L^2([0,T])$ and, with $\alpha = \frac{1}{2} - H$,

$$\mathcal{H} = \left\{ f : [0,T] \to \mathbb{R} : \exists \phi_f \in L^2\big([0,T]\big) \text{ such that } f(u) = u^{\alpha} \big(I^{\alpha}_{T-}\big(s^{-\alpha}\phi_f(s)\big)\big)(u) \right\},$$

is a Hilbert space equipped with the inner product

$$\langle f, g \rangle_{\mathcal{H}} = C_H \langle \phi_f, \phi_g \rangle_{L^2([0,T])}.$$

Here $C_H = \pi\alpha(2\alpha - 1)(\Gamma(1+2\alpha)\sin(-\pi\alpha))^{-1}$. Hence, we can use the linear operator $\mathcal{T} : \mathcal{H} \subset L^2([0,T]) \to L^2([0,T])$ given by $\mathcal{T} f = C_H^{1/2} \phi_f$. This operator have the following properties (see León and Nualart [22]):

(P1) $|\mathcal{T}h|_{L^2([0,T])} = |h|_{\mathcal{H}}$, for all $h \in \mathcal{H}$.
(P2) $\mathcal{J}_{\mathcal{H}} := \{h \in \mathcal{H} : \mathcal{T}h \in \mathcal{D}(\mathcal{T}^*)\}$ is a dense subset of $\mathcal{H}$, where $\mathcal{D}(\mathcal{T}^*)$ stands for the domain of the adjoint of the operator $\mathcal{T}$.
(P3) $\mathcal{J}_{L^2([0,T])} = \{\mathcal{T}^*\mathcal{T}h : h \in \mathcal{J}_{\mathcal{H}}\}$ is dense in $L^2([0,T])$.

Note that (P1) yields that $\mathcal{T}$ is a closed operator on $L^2([0,T])$. Therefore, $\mathcal{D}(\mathcal{T}^*)$ is a dense subset of $L^2([0,T])$ (see Reed and Simon [32], Theorem VIII.1). Furthermore, it is proven in León and Nualart [22] (Proposition 4.2) that if $g : [0,T] \to \mathbb{R}$ is such that $u \mapsto u^\alpha g(u)$ belongs to $I^\alpha_{0+}(L^q([0,T]))$ for some $q > \alpha^{-1} \vee H^{-1}$. Then, $g \in \mathcal{D}(\mathcal{T}^*)$ and for $u \in [0,T]$,

$$(\mathcal{T}^* g)(u) = C_H^{1/2} u^{-\alpha} D^\alpha_{0+}\big(s^\alpha g(s)\big)(u).$$

2.2.1. *The derivative operator*

Let $\mathcal{S}$ (resp. $\mathcal{S}(L^2([0,T]))$) be the class of all smooth random variables of the form

$$F = f\big(B^H(\phi_1), \dots, B^H(\phi_n)\big) \qquad \big(\text{resp. } F = f\big(B^H(\phi_1), \dots, B^H(\phi_n)\big)g\big), \tag{2.3}$$

where ϕ_i is in $\mathcal{H}$, $i = 1, \dots, n$, (resp. $g \in L^2([0,T])$) and $f \in C_p^\infty(\mathbb{R}^n)$. That is, $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞-function such that f and all its partial derivatives have polynomial growth.

The derivative of the smooth random variable F given by (2.3) is the $\mathcal{H}$ (resp. $\mathcal{H} \otimes L^2([0,T])$)-valued random variable DF defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(B^H(\phi_1), \dots, B^H(\phi_n)\big)\phi_i$$
$$\left(\text{resp. } DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(B^H(\phi_1), \dots, B^H(\phi_n)\big)\phi_i \otimes g\right).$$

It is well known that D is a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ (resp. from $L^2(\Omega; L^2([0,T]))$ into $L^2(\Omega; \mathcal{H} \otimes L^2([0,T]))$). The domain $\mathbb{D}^{1,2}$ (resp. $\mathbb{D}^{1,2}(L^2([0,T]))$) of the closure of D (also denoted by D) is the completion of $\mathcal{S}$ (resp. $\mathcal{S}(L^2([0,T]))$) with respect to the norm

$$\|F\|^2_{1,2} = E\big(|F|^2 + |DF|^2_{\mathcal{H}}\big) \qquad \big(\text{resp. } \|F\|^2_{1,2,L^2([0,T])} = E\big(|F|^2_{L^2([0,T])} + |DF|^2_{\mathcal{H}\otimes L^2([0,T])}\big)\big).$$

In this paper, we also consider the operator

$$D_{\mathcal{T}} : \mathcal{S}_{\mathcal{T}} \subset L^2(\Omega) \to L^2\big(\Omega \times [0,T]\big)$$

define by

$$D_{\mathcal{T}}(F) = \mathcal{T}^*\mathcal{T}DF, \quad F \in \mathcal{S}_{\mathcal{T}}, \tag{2.4}$$

where $\mathcal{S}_{\mathcal{T}}$ is the class of smooth random variables in $\mathcal{S}$ of the form (2.3), but ϕ_i is in $\mathcal{J}_{\mathcal{H}}$, $i = 1, \dots, n$. In the appendix of León and Nualart [22], it is stated that this operator is closable from $L^2(\Omega)$ into $L^2(\Omega \times [0,T])$. The domain of its closure (also denoted by $D_{\mathcal{T}}$) in $L^2(\Omega)$ is the set $\mathbb{D}^{1,2}_{\mathcal{T}}$. It means, $\mathbb{D}^{1,2}_{\mathcal{T}}$ is the completion of the smooth random variables $\mathcal{S}_{\mathcal{T}}$ with respect to the norm

$$\|F\|^2_{1,2,\mathcal{T}} = E\big(|F|^2 + \big|\mathcal{T}^*\mathcal{T}DF\big|^2_{L^2([0,T])}\big).$$

Moreover, it is proven in the appendix of León and Nualart [22] that if $F \in \mathbb{D}^{1,2}_{\mathcal{T}}$, then we have that $F \in \mathbb{D}^{1,2}$, $\mathcal{T}DF$ belongs to $\mathcal{D}(\mathcal{T}^*)$ w.p.1 and

$$D_{\mathcal{T}}F = \mathcal{T}^*\mathcal{T}DF.$$

2.2.2. *An extension of the divergence operator*

The divergence operator δ (with respect to B^H) is the adjoint of the derivative operator D given in Section 2.2.1. It means, a random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator, denoted by $\mathcal{D}(\delta)$, if and only if there is a square-integrable random variable $\delta(u)$ satisfying the duality relation

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_{\mathcal{H}}\big), \quad \text{for any } F \in \mathbb{D}^{1,2}. \tag{2.5}$$

The divergence operator satisfies the following result.

Lemma 2.1. *Let $F \in \mathbb{D}^{1,2}$ and $u \in \mathcal{D}(\delta)$ such that $Fu \in L^2(\Omega;\mathcal{H})$ and $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. Then, Fu belongs to $\mathcal{D}(\delta)$ and*

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}.$$

An extension of δ is obtained in León and Nualart [22] using the operator $D_{\mathcal{T}}$ introduced in (2.4). This extended divergence is defined in the following.

Definition 2.2. Let $u \in L^2(\Omega \times [0,T])$. We say that u belongs to $\mathcal{D}(\delta_{\mathcal{T}})$ if and only if there exists $\delta_{\mathcal{T}}(u) \in L^2(\Omega)$ such that

$$E\big(\langle D_{\mathcal{T}}F, u\rangle_{L^2([0,T])}\big) = E\big(F\delta_{\mathcal{T}}(u)\big), \quad \text{for every } F \in \mathcal{S}_{\mathcal{T}}. \tag{2.6}$$

In this case, the random variable $\delta_{\mathcal{T}}(u)$ is called the extended divergence of u.

Remarks 2.3.

(i) Property (P2) in Section 2.2 implies that the operator $\delta_{\mathcal{T}}$ is well-defined and we can figure out $\mathcal{D}(\delta_{\mathcal{T}})$ by means of Property (P3) and the chaos decomposition of a square-integrable process (see León and Nualart [22]).

(ii) León and Nualart [22] have pointed out that $B^H \in (\mathcal{D}(\delta_{\mathcal{T}}) \setminus \mathcal{D}(\delta))$, for $H \in (0, 1/4)$.

(iii) By León and Nualart [22] (Theorem 3.2), $\mathcal{D}(\delta) \subset \mathcal{D}(\delta_{\mathcal{T}})$ and $\delta_{\mathcal{T}}$ agrees with δ on $\mathcal{D}(\delta)$. We observe that this also follows from (2.5). Indeed, let $F \in \mathcal{S}_{\mathcal{T}}$, then

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_{\mathcal{H}}\big) = E\big(\langle \mathcal{T}DF, \mathcal{T}u\rangle_{L^2([0,T])}\big) = E\big(\langle D_{\mathcal{T}}F, u\rangle_{L^2([0,T])}\big).$$

3. The Stratonovich integral

The purpose of this section is to define our stochastic integral of Stratonovich type and to state a relation between this integral and $\delta_{\mathcal{T}}$.

Remember that the operator $\mathcal{T}:\mathcal{H}\subset L^2([0,T])\to L^2([0,T])$ and the set $\mathcal{S}_{\mathcal{T}}$ are given in Sections 2.2 and 2.2.1, respectively.

The following definition is inspired by that of Russo and Vallois [35], and by Definition 2.2.

Definition 3.1. Let $u=\{u_t : t\in[0,T]\}$ be a measurable process with integrable paths such that $E((\int_0^T |u_t|\,dt)^p)<\infty$ for some $p>2$. We say that u belong to $\mathcal{D}(\delta_S^{B^H})$ if there exists a square-integrable random variable $\int_0^T u_t\circ dB_t^H$ such that

$$\left\langle F,\int_0^T u_t\circ dB_t^H\right\rangle_{L^2(\Omega)}=\lim_{\varepsilon\downarrow 0}\left\langle F,\frac{1}{2\varepsilon}\int_0^T u_s\big(B^H_{(s+\varepsilon)\wedge T}-B^H_{(s-\varepsilon)\vee 0}\big)\,ds\right\rangle_{L^2(\Omega)},\tag{3.1}$$

provided this limit exists for every $F\in\mathcal{S}_{\mathcal{T}}$. In this case, $\int_0^T u_t\circ dB_t^H$ is called the weak Stratonovich integral of u with respect to the fBm B^H.

Remarks 3.2.

(i) Note that $\int_0^T u_s(B^H_{(s+\varepsilon)\wedge T}-B^H_{(s-\varepsilon)\vee 0})\,ds$ is a square-integrable randon variable due to $\sup_{0\le s\le T}|B_s^H|$ is in $L^p(\Omega)$, for any $p\ge 1$.

(ii) Property (P2) in Section 2.2 implies that there is at most one square-integrable random variable $\int_0^T u_t\circ dB_t^H$ such that (3.1) holds for every $F\in\mathcal{S}_{\mathcal{T}}$.

(iii) Let $G\in\mathcal{S}_{\mathcal{T}}$ be a bounded random variable and $u\in\mathcal{D}(\delta_S^{B^H})$. Then, Gu also belongs to $\mathcal{D}(\delta_S^{B^H})$ and

$$\int_0^T Gu_t\circ dB_t^H=G\int_0^T u_t\circ dB_t^H.$$

(iv) Consider a process $u=\{u_t : t\in[0,T]\}$ with β-Hölder continuous paths such that $\beta+H>1$ and $\|u\|_\beta+\|u\|_\infty=(\sup_{s,t\in[0,T]}\frac{|u_t-u_s|}{|t-s|^\beta})+\sup_{t\in[0,T]}|u_t|$ is in $L^p(\Omega)$, for some $p>2$. Then, by Russo and Vallois [36] (Proposition 3 and Lemma 1), we have that $u\in\mathcal{D}(\delta_S^{B^H})$ and $\int_0^T u_t\circ dB_t^H$ agrees with the integral given by Young [38] of u with respect to B^H.

Russo and Vallois [35] have introduced the symmetric integral of u with respect to B^H as

$$\lim_{\varepsilon\downarrow 0}\left(\frac{1}{2\varepsilon}\int_0^T u_s\big(B^H_{(s+\varepsilon)\wedge T}-B^H_{(s-\varepsilon)\vee 0}\big)\,ds\right),$$

where the limit is in probability. In Cheridito and Nualart [7], it has been pointed out that B^H is in the domain of this integral, but $(B^H)^2$ is not in this domain if $H\in(0,1/6]$. Moreover, Alòs, León and Nualart [1] (Theorem 2) have stated a relation between the symmetric integral in Russo and Vallois [35], and the divergence operator with respect to B^H. In particular, they have proven that the equality

$$\int_0^T B_s^H \circ dB_s^H = \delta\big(B^H\big) + \frac{1}{2}T^{2H} \tag{3.2}$$

is satisfied for $H \in (1/4, 1/2)$.

In our case, using the ideas of the proof of Theorem 2 in Alòs, León and Nualart [1], we have the following result.

Proposition 3.3. *Let $H < \frac{1}{2}$. Then, B^H belongs to $\mathcal{D}(\delta_S^{B^H})$ and*

$$\int_0^T B_s^H \circ dB_s^H = \delta_{\mathcal{T}}\big(B^H\big) + \frac{1}{2}T^{2H}.$$

Remark 3.4. As we have already pointed out, León and Nualart [22] have showed that $B^H \in (\mathcal{D}(\delta_{\mathcal{T}}) \setminus \mathcal{D}(\delta))$, for $H < 1/4$. So, we have that (3.2) holds even for $H < 1/4$ if we write $\delta_{\mathcal{T}}$ instead of δ. It means, we now utilize that $B^H \in \mathcal{D}(\delta_{\mathcal{T}})$. Note that León and Nualart [22] (Theorem 3.2 and Proposition 4.4) and this proposition imply

$$\int_0^T B_s^H \circ dB_s^H = \frac{1}{2}\big(I_2(1\otimes 1) + T^{2H}\big) = \frac{1}{2}\big(B_T^H\big)^2$$

is true even for $H < 1/4$. Furthermore, In Section 4 (Theorem 4.1), in particular, we see that $p(B^H) \in \mathcal{D}(\delta_S^{B^H})$, for any real polynomial function p. The proof of this fact does not require that the integrand is in $\mathcal{D}(\delta_{\mathcal{T}})$. We think that the proof of Proposition 3.3, together with the one of Theorem 4.1, explains how we can handle the existence of a Stratonovich integral introduced in Definition 3.1.

Proof. Let $\varepsilon > 0$. Then, Lemma 2.1 yields

$$\begin{aligned}
&\int_0^T B_s^H\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds \\
&\quad= \int_0^T \delta\big(B_s^H 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\big)\,ds + \int_0^T \langle 1_{[0,s]}, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\rangle_{\mathcal{H}}\,ds \\
&\quad= \int_0^T \delta\big(B_s^H 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\big)\,ds + \int_0^T \big(R\big(s,(s+\varepsilon)\wedge T\big) - R\big(s,(s-\varepsilon)\vee 0\big)\big)\,ds.
\end{aligned}$$

Hence, (1.1), Remark 2.3.(*iii*) and Fubini theorem imply that, for any $F \in \mathcal{S}_{\mathcal{T}}$,

$$\begin{aligned}
&\frac{1}{2\varepsilon}E\left(F\int_0^T B_s^H\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds\right) \\
&\quad= \frac{1}{2\varepsilon}E\left(\int_0^T \big\langle \mathcal{D}_{\mathcal{T}}F(\cdot), B_s^H 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\big\rangle_{L^2([0,T])}\,ds\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\varepsilon}E\bigg(F\int_0^T\big[\big((s+\varepsilon)\wedge T\big)^{2H}-\big((s-\varepsilon)\vee 0\big)^{2H}\\
&-\big((s+\varepsilon)\wedge T-s\big)^{2H}+\big(s-(s-\varepsilon)\vee 0\big)^{2H}\big]\,ds\bigg)\\
={}&E\bigg(\int_0^T D_{\mathcal{T}}F(r)\bigg(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T}B_s^H\,ds\bigg)\,dr\bigg)\\
&+\frac{1}{4\varepsilon}E\bigg(F\int_0^T\big[\big((s+\varepsilon)\wedge T\big)^{2H}-\big((s-\varepsilon)\vee 0\big)^{2H}\\
&-\big((s+\varepsilon)\wedge T-s\big)^{2H}+\big(s-(s-\varepsilon)\vee 0\big)^{2H}\big]\,ds\bigg),
\end{aligned}
$$

which converges to $E(\int_0^T D_{\mathcal{T}}F(s)B_s^H\,ds+HF\int_0^T s^{2H-1}\,ds)$, as $\varepsilon\downarrow 0$. Therefore, León and Nualart [22] (Proposition 4.4) and (2.6) give that B belongs to $\mathcal{D}(\delta_S^{B^H})$ and

$$
\int_0^T B_s^H\circ dB_s^H=\delta_{\mathcal{T}}(B)+\frac{1}{2}T^{2H}.
$$

Thus, the proof is complete. □

Remark 3.5. In the introduction we have already pointed out that the forward integral $\int_0^T B_s^H\,dB_s^{H-}$ does not exist. That is, $I^-(\varepsilon):=\frac{1}{\varepsilon}\int_0^T B_s^H(B_{(s+\varepsilon)\wedge T}^H-B_s^H)\,ds$ does not converge in probability, as $\varepsilon\downarrow 0$. Indeed, for ε small enough, proceeding as in the last proof we can get that $E(FI^-(\varepsilon))$ is equal to a suitable term plus the quantity

$$
-\frac{T}{2}E(F)\varepsilon^{2H-1}, \tag{3.3}
$$

which diverges to $-\infty$, as $\varepsilon\downarrow 0$. Similarly, for the backward integral $\int_0^T B_s^H\,dB_s^{H+}$, we consider the integral $I^+(\varepsilon):=\frac{1}{\varepsilon}\int_0^T B_s^H(B_s^H-B_{(s-\varepsilon)\vee 0}^H)\,ds$. So, proceeding as in the last proof, we have that $E(FI^+(\varepsilon))$ is equal to a suitable term plus

$$
\frac{T}{2}E(F)\varepsilon^{2H-1}, \tag{3.4}
$$

which diverges to ∞.

The Stratonovich integral is well-defined because

$$
\frac{1}{2\varepsilon}\int_0^T B_s^H\big(B_{(s+\varepsilon)\wedge T}^H-B_{(s-\varepsilon)\vee 0}^H\big)\,ds=\frac{1}{2}\big(I^-(\varepsilon)+I^+(\varepsilon)\big),
$$

and, consequently, the terms (3.3) and (3.4) cancel each other.

Notice that Remark 3.5 shows the importance of the fact that the integral studied in this paper is of Stratonovich type. Other consequence of the last proof is the following result, which is quite similar to Theorem 2 in Alòs, León and Nualart [1].

Theorem 3.6. *Let $p > 2$ and $u \in \mathbb{D}^{1,2}(L^2([0,T])) \cap L^p(\Omega \times [0,T]) \cap \mathcal{D}(\delta_{\mathcal{T}})$ a process such that*

(i) *For each $\varepsilon > 0$ small enough, we have that*

$$\left(u_s\left(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\right) - \langle Du_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\rangle_{\mathcal{H}}\right) \in L^2(\Omega),$$

for almost all $s \in [0,T]$.

(ii) *There exist a square-integrable random variable $TrDu$ such that*

$$E(FTrDu) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}E\left(F\int_0^T \langle Du_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\rangle_{\mathcal{H}}\right),$$

for any $F \in \mathcal{S}_{\mathcal{T}}$.

Then, u belongs to $\mathcal{D}(\delta_S^{B^H})$ and

$$\int_0^T u_s \circ dB^H_s = \delta_{\mathcal{T}}(u) + TrDu.$$

Proof. In order to see that the result is satisfied, in the proof of Proposition 3.3, we only need to change B^H_s and $1_{[0,s]}$ by u_s and Du_s, respectively. Indeed, remember that $DB^H_s = 1_{[0,s]}$. □

4. An Itô's formula

In this section, we analyze an Itô formula for the Stratonovich type stochastic integral given in Definition 3.1.

Henceforth, $C_e^{1,2}([0,T]\times\mathbb{R})$ stands for all functions f such that $f \in C^{1,2}([0,T]\times\mathbb{R})$ and

$$\max\left\{\left|f(t,x)\right|, \left|\partial_t f(t,x)\right|, \left|\partial_x f(t,x)\right|, \left|\partial_x^2 f(t,x)\right|\right\} \le c\exp\left(C|x|\right),$$

for $(t,x) \in [0,T]\times\mathbb{R}$. Here, c and C are two positive constants. Also, we use the conventions

$$B_t^{H,\varepsilon} := \frac{1}{2\varepsilon}\int_0^t \left(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\right)ds, \quad \text{for } t \in [0,T] \text{ and } \varepsilon > 0, \tag{4.1}$$

and

$$\int_0^t u_s \circ dB^H_s := \int_0^T \left(u_s 1_{[0,t]}(s)\right)\circ dB^H_s, \quad \text{for } t \in [0,T].$$

Now, we are ready to establish the Itô formula. Some details of its proof are provided in Section 4.2 as auxiliary lemmas so that the main ideas used in this proof can be appreciated.

Theorem 4.1. *Let $f \in C_e^{1,2}([0,T]\times\mathbb{R})$. Then, for $t\in[0,T]$, $\partial_x f(\cdot, B^H_\cdot)1_{[0,t]}(\cdot)\in\mathcal{D}(\delta_S^{B^H})$ and*

$$f\big(t, B_t^H\big) = f(0,0) + \int_0^t \partial_t f\big(s, B_s^H\big)\,ds + \int_0^t \partial_x f\big(s, B_s^H\big)\circ dB_s^H. \tag{4.2}$$

Remark 4.2. The symmetric integral in Russo and Vallois [35] of $(B^H)^2$ with respect to B^H does not exist for any $H \le 1/6$ (see Cheridito and Nualart [7]). But, as a consequence of (4.2), we have that the integral $\int_0^T p(B_s^H)\circ dB_s^H$ is well-defined, for any polynomial function $p:\mathbb{R}\to\mathbb{R}$.

Proof. Let $t\in[0,T]$ and $\varepsilon>0$. Then, the fundamental theorem of calculus leads to write

$$\begin{aligned}
&f\big(t, B_t^{H,\varepsilon}\big)\\
&\quad= f(0,0) + \int_0^t \partial_t f\big(s, B_s^{H,\varepsilon}\big)\,ds + \frac{1}{2\varepsilon}\int_0^t \partial_x f\big(s, B_s^{H,\varepsilon}\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds\\
&\quad= f(0,0) + \int_0^t \partial_t f\big(s, B_s^{H,\varepsilon}\big)\,ds + \frac{1}{2\varepsilon}\int_0^t \partial_x f\big(s, B_s^{H}\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds\\
&\qquad+ \frac{1}{2\varepsilon}\int_0^t \big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds.
\end{aligned} \tag{4.3}$$

Note that Lemmas 2.1 and 4.14, León, Navarro and Nualart [21] (Lemma 2.1), Nourdin [27] (property (4.13)), Nualart [28] (Section 2.1) and (4.14)–(4.16) below lead us to write

$$\begin{aligned}
I_t^\varepsilon &:= \frac{1}{2\varepsilon}\int_0^t \big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds\\
&= \frac{1}{2\varepsilon}\int_0^t \delta\big(\big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\big)\,ds\\
&\quad+ \frac{1}{2\varepsilon}\int_0^t \big\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big)DB_s^{H,\varepsilon} - \partial_x^2 f\big(s, B_s^{H}\big)1_{[0,s]}, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\big\rangle_{\mathcal{H}}\,ds.
\end{aligned} \tag{4.4}$$

Now, we deal with the first term in the right-hand side of last equality. From Remark 2.3.(*iii*) and Fubini theorem, we can deduce that, for $F\in\mathcal{S}_{\mathcal{T}}$,

$$\begin{aligned}
&E\bigg(\frac{F}{2\varepsilon}\int_0^t \delta\big(\big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\big)\,ds\bigg)\\
&\quad= E\bigg(\frac{1}{2\varepsilon}\int_0^t \big\langle D_{\mathcal{T}}F, \big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\big\rangle_{L^2([0,T])}\,ds\bigg)\\
&\quad\le \bigg(E\int_0^t \big(\partial_x f\big(s, B_s^{H,\varepsilon}\big) - \partial_x f\big(s, B_s^{H}\big)\big)^2\,ds\bigg)^{1/2}\\
&\qquad\times\bigg(E\int_0^t\bigg(\frac{1}{2\varepsilon}\int_{(s-\varepsilon)\vee 0}^{(s+\varepsilon)\wedge T} D_{\mathcal{T}}F(r)\,dr\bigg)^2\,ds\bigg)^{1/2}
\end{aligned}$$

$$\le \left(E\int_0^t \left(\partial_x f\left(s, B_s^{H,\varepsilon}\right) - \partial_x f\left(s, B_s^H\right)\right)^2 ds\right)^{1/2} \left(E\int_0^t \frac{1}{2\varepsilon}\int_{(s-\varepsilon)\vee 0}^{(s+\varepsilon)\wedge T} \left(D_{\mathcal{T}}F(r)\right)^2 dr\, ds\right)^{1/2}$$

$$\le \left(E\int_0^t \left(\partial_x f\left(s, B_s^{H,\varepsilon}\right) - \partial_x f\left(s, B_s^H\right)\right)^2 ds\right)^{1/2}$$

$$\times \left(E\int_0^{(t+\varepsilon)\wedge T} \left(D_{\mathcal{T}}F(r)\right)^2 dr\right)^{1/2}.$$

Hence, using that f is in $C_e^{1,2}([0,T]\times\mathbb{R})$, Nourdin [27] (property (4.13)), together with (4.16), (4.4) and Lemmas 4.15 and 4.16 below, we get $\lim_{\varepsilon\downarrow 0} E(F I_t^\varepsilon)) = 0$, for every $F \in \mathcal{S}_{\mathcal{T}}$. Therefore, due to (4.3) and Lemma 4.13, $s \mapsto \partial_x f(s, B_s^H) 1_{[0,t]}(s)$ belongs to $\mathcal{D}(\delta_S^{B^H})$ and (4.2) holds. Thus, the proof is complete. □

4.1. Stochastic differential equations of Stratonovich type

The aim of this section is to study the existence and uniqueness for the solution of some Stratonovich type stochastic differential equations.

Consider the stochastic differential equation

$$X_t = x_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s) \circ dB_s^H, \quad t \in [0,T]. \tag{4.5}$$

Here, $x_0 \in \mathbb{R}$ and $b, \sigma : [0,T]\times\mathbb{R} \to \mathbb{R}$ are two measurable functions.

Definition 4.3. We say that a measurable process $X = \{X_t, t \in [0,T]\}$ is a solution to (4.5) if and only if, for each $t \in [0,T]$, $(\sigma(\cdot, X_\cdot)1_{[0,t]}(\cdot)) \in \mathcal{D}(\delta_S^{B^H})$ (see Definition 3.1) and equality (4.5) holds w.p.1.

4.1.1. *Some linear differential equations*

Here, we deal with the existence of a unique solution to the linear stochastic differential equation of the form

$$X_t = x_0 + \int_0^t b(s) X_s\,ds + \int_0^t \sigma X_s \circ dB_s^H, \quad t \in [0,T], \tag{4.6}$$

with $x_0, \sigma \in \mathbb{R}$ and $b \in L^1([0,T])$.

As an application of Theorem 4.1, we can state the following result.

Proposition 4.4. *Let $b : [0,T] \to \mathbb{R}$ be a continuous function. Then, the process*

$$X_t = x_0 \exp\left(\int_0^t b(s)\,ds + \sigma B_t^H\right), \quad t \in [0,T], \tag{4.7}$$

is a solution to equation (4.6).

Remark 4.5. Note that X is a continuous process that belongs to $L^p(\Omega \times [0,T])$, for any $p \geq 2$.

Proof. The result is an immediate consequence of Theorem 4.1. Indeed, we only need to observe that the function

$$f(t,x) = x_0 \exp\left(\int_0^t b(s)\,ds + \sigma x\right), \quad (t,x) \in [0,T] \times \mathbb{R},$$

belong to $C_e^{1,2}([0,T] \times \mathbb{R})$. □

In order to establish the uniqueness for the solution to equation (4.6), we now obtain some properties of the process X introduced in (4.7). Towards this end, we proceed as in Kohatsu-Higa, León and Nualart [19].

In this section, we use the notation

$$Y_t^\varepsilon = x_0 + \int_0^t b(s) Y_s\,ds + \frac{1}{2\varepsilon} \int_0^t \sigma Y_s \bigl(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\bigr)\,ds, \quad t \in [0,T],$$

where $\varepsilon > 0$ and Y is a process with integrable paths. Furthermore, we consider a function $\psi : \mathbb{R} \to [0,1]$ in $C_b^\infty(\mathbb{R})$ (i.e., ψ and all its derivatives are bounded), such that

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

In the following result, for $m \in \mathbb{N}$, $\psi_m : \mathbb{R} \to \mathbb{R}$ represents the function $\psi_m(x) = \psi(x/m)x$ and $\{F_n : n \in \mathbb{N}\} \subset \mathcal{S}(L^2([0,T]))$ is a sequence that converges to B^H in $L^2(\Omega; L^2([0,T]))$ and almost surely, where F_n has the form

$$F_n = \sum_{i=1}^{N_n} f_{i,n}\bigl(B^H(\phi_{1,n}), \ldots, B^H(\phi_{i_n,n})\bigr) g_{i,n}, \tag{4.8}$$

with $g_{i,n} \in C^1([0,T])$ and $f_{i,n}(B^H(\phi_{1,n}), \ldots, B^H(\phi_{i_n,n})) \in \mathcal{S}_\mathcal{T}$. Note that there is such a sequence due to $B^H \in L^2(\Omega; L^2([0,T]))$ and Property (P2). We point out that we can have that $F_n(0) = 0$. Indeed, we can change F_n by $F_n \tilde{\psi}_n$, where $\tilde{\psi}_n : \mathbb{R}_+ \to [0,1]$ is a function in $C_b^\infty(\mathbb{R}_+)$ such that

$$\tilde{\psi}_n(t) = \begin{cases} 1, & \text{if } t \geq \dfrac{2}{n}, \\ 0, & \text{if } 0 \leq t \leq \dfrac{1}{n}. \end{cases}$$

Lemma 4.6. *Let X be the process defined in* (4.7). *Then,*

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{\varepsilon \downarrow 0} E\bigl(F X_t^\varepsilon \exp\bigl(-\sigma \psi_m\bigl(F_n(t)\bigr)\bigr)\bigr) = E\bigl(F X_t \exp\bigl(-\sigma B_t^H\bigr)\bigr), \tag{4.9}$$

for almost all $t \in [0,T]$, for all $F \in \mathcal{S}_\mathcal{T}$.

Remark 4.7. The set $\{t \in [0,T] : (4.9) \text{ holds}\}$ is independent of the random variable F.

Proof. Since $F \exp(-\sigma \psi_m(F_n(t)))$ belongs to $\mathcal{S}_{\mathcal{T}}$, for $t \in [0,T]$ and $n, m \in \mathbb{N}$, then

$$\lim_{\varepsilon \downarrow 0} E\big(F X_t^{\varepsilon} \exp\big(-\sigma \psi_m\big(F_n(t)\big)\big)\big) = E\big(F X_t \exp\big(-\sigma \psi_m\big(F_n(t)\big)\big)\big).$$

Consequently, now it is easy to finish the proof using the definitions of ψ_m and F_n. □

Lemma 4.8. *Let X be the process defined in (4.7). Then,*

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{\varepsilon\downarrow 0} E\bigg[\frac{F}{2\varepsilon} \int_0^t X_s \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\, ds$$
$$- F \int_0^t X_s^{\varepsilon} \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big) \psi_m'\big(F_n(s)\big) F_n'(s)\, ds\bigg] = 0,$$

for almost all $t \in [0,T]$, for all $F \in \mathcal{S}_{\mathcal{T}}$.

Proof. Let $t \in [0,T]$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$. So, the fundamental theorem of calculus yields

$$\begin{aligned} X_t^{\varepsilon} \exp\big(-\sigma \psi_m\big(F_n(t)\big)\big) = x_0 &+ \int_0^t b(s) X_s \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big)\, ds \\ &+ \frac{\sigma}{2\varepsilon} \int_0^t X_s \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big)\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\, ds \\ &- \sigma \int_0^t X_s^{\varepsilon} \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big) \psi_m'\big(F_n(s)\big) F_n'(s)\, ds. \end{aligned} \tag{4.10}$$

Therefore, by Lemma 4.6, we only need to show that

$$\begin{aligned} &\lim_{m\to\infty} \lim_{n\to\infty} E\bigg[F \int_0^t b(s) X_s \exp\big(-\sigma \psi_m\big(F_n(s)\big)\big)\, ds\bigg] \\ &\quad = E\bigg[F \int_0^t b(s) X_s \exp\big(-\sigma B_s^H\big)\, ds\bigg]. \end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned} &E\bigg[|F| \int_0^t \big|b(s) X_s\big| \big|\exp\big(-\sigma \psi_m\big(F_n(s)\big)\big) - \exp\big(-\sigma B_s^H\big)\big|\, ds\bigg] \\ &\quad \le C\bigg(E \int_0^T |F X_s|^2\, ds\bigg)^{1/2} \bigg(E \int_0^T \big(\exp\big(-\sigma \psi_m\big(F_n(s)\big)\big) - \exp\big(-\sigma B_s^H\big)\big)^2\, ds\bigg)^{1/2} \\ &\quad \le C\bigg(E \int_0^T \big(\exp\big(-\sigma \psi_m\big(F_n(s)\big)\big) - \exp\big(-\sigma \psi_m\big(B_s^H\big)\big)\big)^2\, ds\bigg)^{1/2} \end{aligned}$$

$$+C\left(E\int_0^T \left(\exp\left(-\sigma\psi_m\left(B_s^H\right)\right)-\exp\left(-\sigma B_s^H\right)\right)^2 ds\right)^{1/2},$$

which implies that (4.11) holds. Thus, the proof is complete. □

Now, we imitate the ideas developed in Kohatsu-Higa, León and Nualart [19] to prove the uniqueness for the solution of (4.6). So, we introduce the family $\mathcal{A}$ of all the processes Y such that

(i) Y is a continuous process that is in $L^p(\Omega\times[0,T])$, for some $p>2$.
(ii) There exists a sequence $\{F_n : n\in\mathbb{N}\}\subset\mathcal{S}(L^2([0,T]))$ that converges to B^H in $L^2(\Omega; L^2([0,T]))$ and almost surely. Moreover, we assume that F_n is as in (4.8), with $g_{i,n}\in C^1([0,T])$, $g_{i,n}(0)=0$ and $f_{i,n}(B^H(\phi_{1,n}),\dots,B^H(\phi_{i_n,n}))\in\mathcal{S}_{\mathcal{T}}$.
(iii) For almost all $t\in[0,T]$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}E\left(FY_t^\varepsilon\exp\left(-\sigma\psi_m\left(F_n(t)\right)\right)\right)=E\left(FY_t\exp\left(-\sigma B_t^H\right)\right),$$

for all $F\in\mathcal{S}_{\mathcal{T}}$.
(iv) For almost all $t\in[0,T]$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}E\left[\frac{F}{2\varepsilon}\int_0^t Y_s\exp\left(-\sigma\psi_m\left(F_n(s)\right)\right)\left(B^H_{(s+\varepsilon)\wedge T}-B^H_{(s-\varepsilon)\vee 0}\right)ds\right.$$
$$\left.-F\int_0^t Y_s^\varepsilon\exp\left(-\sigma\psi_m\left(F_n(s)\right)\right)\psi_m'\left(F_n(s)\right)F_n'(s)\,ds\right]=0,$$

for all $F\in\mathcal{S}_{\mathcal{T}}$.

We are ready to show the uniqueness for the solution to equation (4.6).

Proposition 4.9. *Let $Y\in\mathcal{A}$ be a solution of equation* (4.6). *Then, $Y=X$ in $L^p(\Omega\times[0,T])$.*

Proof. We have that (4.10) is also true when we write Y and Y^ε instead of X and X^ε, respectively. Hence, using the definition of the family $\mathcal{A}$ and proceeding as in the proof of Lemma 4.8, we can establish the equality

$$E\left(FY_t\exp\left(-\sigma B_t^H\right)\right)=E\left[F\left(x_0+\int_0^t b(s)Y_s\exp\left(-\sigma B_s^H\right)ds\right)\right],$$

for almost all $t\in[0,T]$, for all $F\in\mathcal{S}_{\mathcal{T}}$. Finally, since $\mathcal{S}_{\mathcal{T}}$ is a dense set of $L^2(\Omega)$, then, we have

$$Y_t\exp\left(-\sigma B_t^H\right)=x_0+\int_0^t b(s)Y_s\exp\left(-\sigma B_s^H\right)ds,\quad \text{almost surely.}$$

Therefore, the continuity of the process Y gives that $Y_t\exp(-\sigma B_t^H)=x_0\exp(\int_0^t b(s)\,ds)$, which yields the result. □

4.1.2. *Reduced stochastic differential equations*

Now we consider the equation

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s^H, \quad t \in [0,T]. \tag{4.12}$$

Here and in the remaining of this section, $\sigma \in \mathcal{C}_b^2(\mathbb{R})$.

An auxiliary tool to deal with equation (4.12) is the solution to the ordinary differential equation

$$\begin{aligned} \partial_x \alpha(x,y) &= \sigma\big(\alpha(x,y)\big), \quad x \in \mathbb{R} \setminus \{0\}, \\ \alpha(0,y) &= y. \end{aligned} \tag{4.13}$$

Note that this equation has a unique solution because σ is a Lipschitz function. By Doss [13], we have

$$\partial_y \alpha(x,y) = \exp\biggl(\int_0^x \sigma'\big(\alpha(s,y)\big)\,ds\biggr).$$

Then, following the pathwise representation for one-dimensional Stratonovich stochastic differential equations due to Doss [13], we state the following result.

Proposition 4.10. *Assume that $\sigma \in \mathcal{C}_b^2(\mathbb{R})$. Then, the process*

$$X_t = \alpha\big(B_t^H, x_0\big), \quad t \in [0,T],$$

is a continuous solution to equation (4.12).

Proof. We claim that α belongs to $C_e^{1,2}([0,T]\times\mathbb{R})$. Indeed, (4.13) imply

$$\big|\alpha(x,x_0)\big| = \left|x_0 + \int_0^x \sigma\big(\alpha(u,x_0)\big)\,du\right| \le |x_0| + |x|\|\sigma\|_\infty, \quad x \in \mathbb{R}.$$

Using (4.13) again, we also have $|\partial_x \alpha(x,x_0)| = |\sigma(\alpha(x,x_0))| \le \|\sigma\|_\infty$ and $|\partial_x^2 \alpha(x,x_0)| = |\sigma'(\alpha(x,x_0))\sigma(\alpha(x,x_0))| \le \|\sigma\|_\infty \|\sigma'\|_\infty$. Thus, our claim is satisfied. Hence, the result is an immediate consequence of Theorem 4.1. $\square$

As in Section 4.1.1, now we analyze some properties of the process X in Proposition 4.10.

Let $\varepsilon > 0$. The solution of the equation

$$X_{t,\varepsilon} = x_0 + \frac{1}{2\varepsilon}\int_0^t \sigma(X_{s,\varepsilon})\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds, \quad t \in [0,T],$$

is the process $X_{t,\varepsilon} = \alpha(B_t^{H,\varepsilon}, x_0)$, which, for $p > 2$, is in $L^p(\Omega \times [0,T])$ and, for $t \in [0,T]$, $X_{t,\varepsilon}$ goes to X_t in $L^p(\Omega)$ due to $\alpha(\cdot, x_0) \in C_e^{1,2}([0,T]\times\mathbb{R})$ and (4.16) below. Moreover, Doss [13]

(Lemma 2) and the estimations for α obtained in the proof of Proposition 4.10 allow us to get

$$\lim_{\tilde{\varepsilon}\downarrow 0}\lim_{\varepsilon\downarrow 0} E\big(F\alpha\big(-B_t^{H,\tilde{\varepsilon}}, X_{t,\varepsilon}\big)\big) = E\big(F\alpha\big(-B_t^H, X_t\big)\big) = E\big(F\alpha\big(-B_t^H, \alpha\big(B_t^H, x_0\big)\big)\big) = E(Fx_0),$$

for every $F \in \mathcal{S}_{\mathcal{T}}$.

As a consequence, we have the following lemma.

Lemma 4.11. *Let X be given in Proposition 4.10, $t \in [0,T]$ and $F \in \mathcal{S}_{\mathcal{T}}$. Then,*

$$\lim_{\tilde{\varepsilon}\downarrow 0}\lim_{\varepsilon\downarrow 0} E\bigg(F\bigg[-\frac{1}{2\tilde{\varepsilon}}\int_0^t \sigma\big(\alpha\big(-B_s^{H,\tilde{\varepsilon}}, X_{s,\varepsilon}\big)\big)\big(B^H_{(s+\tilde{\varepsilon})\wedge T} - B^H_{(s-\tilde{\varepsilon})\vee 0}\big)\,ds \\ + \frac{1}{2\varepsilon}\int_0^t \partial_y\alpha\big(-B_s^{H,\tilde{\varepsilon}}, X_{s,\varepsilon}\big)\sigma(X_{s,\varepsilon})\big(B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}\big)\,ds\bigg]\bigg) = 0.$$

Proof. As in the proof of Lemma 4.8, we only need to apply the fundamental theorem of calculus to the process $s \mapsto \alpha(-B_s^{H,\tilde{\varepsilon}}, \alpha(B_s^{H,\varepsilon}, x_0))$. □

Now we take advantage of above properties of the process X (given in Proposition 4.10) in order to introduce the set $\tilde{\mathcal{A}}$. We say that a process Y belongs to the family $\tilde{\mathcal{A}}$ if and only if

(i) Y is a continuous process in $L^p(\Omega\times[0,T])$, for some $p>2$.

(ii) There exist two sequences $\{F_n : n \in \mathbb{N}\}$ and $\{\tilde{F}_n : n \in \mathbb{N}\}$ of processes such that
  - (a) $F_n, \tilde{F}_n \in C^1([0,T])$ and $F_n(0) = \tilde{F}_n(0) = 0$, for all $w \in \Omega$ and $n \in \mathbb{N}$.
  - (b) F_n and $\tilde{F}_n$ go to B^H in $L^2(\Omega; L^2([0,T]))$.

(iii) The solution of the equation

$$Y_t^n = x_0 + \int_0^t \sigma\big(Y_s^n\big)F_n'(s)\,ds, \quad t \in [0,T],$$

is such that
  - (c) $\lim_{n\to\infty} E(FY_t^n) = E(FY_t)$, for all $t \in [0,T]$ and $F \in \mathcal{S}_{\mathcal{T}}$.
  - (d) For all $t \in [0,T]$ and $F \in \mathcal{S}_{\mathcal{T}}$,

$$\lim_{m\to\infty}\lim_{n\to\infty} E\big(F\alpha\big(-\tilde{F}_m(t), Y_t^n\big)\big) = E\big(F\alpha\big(-B_t^H, Y_t\big)\big).$$

  - (e) For all $t \in [0,T]$ and $F \in \mathcal{S}_{\mathcal{T}}$,

$$\lim_{m\to\infty}\lim_{n\to\infty} E\bigg(F\bigg[-\int_0^t \sigma\big(\alpha\big(-\tilde{F}_m(s), Y_s^n\big)\big)\tilde{F}_m'(s)\,ds \\ + \int_0^t \partial_y\alpha\big(-\tilde{F}_m(s), Y_s^n\big)\sigma\big(Y_s^n\big)F_n'(s)\,ds\bigg]\bigg) = 0.$$

We are ready to state the uniqueness for the solution to equation (4.12).

Proposition 4.12. *Let $Y \in \tilde{\mathcal{A}}$ be a solution to equation* (4.12). *Then, $Y = X$, where X is defined in Proposition* 4.10.

Proof. Applying the fundamental theorem of calculus to the process $s \mapsto (\alpha(-\tilde{F}_m(s), Y_s^n))$ and the definition of the family $\tilde{\mathcal{A}}$, we can deduce that

$$\alpha\left(-B_t^H, Y_t\right) = x_0, \quad t \in [0, T].$$

Finally, we obtain the assertion of the proposition by utilizing Doss [13] (Lemma 2) in order to see that the last equality is equivalent to $Y_t = \alpha(B_t^H, x_0) = X_t$. It means, the proof is complete. □

4.2. Auxiliary results

The purpose of this section is to study some auxiliary lemmas in order to avoid a long and tedious proof of Theorem 4.1.

Lemma 4.13. *Let $f \in C_e^{1,2}([0, T] \times \mathbb{R})$ and $t \in [0, T]$. Then, in $L^2(\Omega)$, we have*

$$f\left(t, B_t^H\right) = \lim_{\varepsilon \downarrow 0} f\left(t, B_t^{H,\varepsilon}\right) \quad \text{and} \quad \int_0^t \partial_t f\left(s, B_s^H\right) ds = \lim_{\varepsilon \downarrow 0} \int_0^t \partial_t f\left(s, B_s^{H,\varepsilon}\right) ds.$$

Proof. Let $t \in (0, T]$ and $0 < \varepsilon < t$. Then, (4.1) and the change of variables formula imply

$$B_t^{H,\varepsilon} := \frac{1}{2\varepsilon}\left(\int_{t-\varepsilon}^{t+\varepsilon} B_{s\wedge T}^H \, ds - \int_0^\varepsilon B_s^H \, ds\right). \tag{4.14}$$

Hence, the continuity of B^H implies that $B_t^{H,\varepsilon}$ converges to B_t^H, as $\varepsilon \downarrow 0$, w.p.1. Moreover, for $0 \le t \le \varepsilon$, we have

$$B_t^{H,\varepsilon} = \frac{1}{2\varepsilon} \int_\varepsilon^{t+\varepsilon} B_{s\wedge T}^H \, ds, \tag{4.15}$$

which, together with (4.14), implies

$$\sup_{s\in[0,T]} \left|B_s^{H,\varepsilon}\right| \le 2 \sup_{s\in[0,T]} \left|B_s^H\right|. \tag{4.16}$$

Thus, the result is a consequence of the facts that, for any $c > 0$, $\exp(c \sup_{s\in[0,T]} |B_s^H|) \in L^2(\Omega)$ (see Theorem 4.2 in Nourdin [27]) and $f \in C_e^{1,2}([0, T] \times \mathbb{R})$, and the dominated convergence theorem. □

Lemma 4.14. *Let $a, b \in [0, T]$ and $\varepsilon \ge 0$ be such that $a \le b + \varepsilon$. Then, $\int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds$ belongs to $\mathbb{D}^{1,2}$ and*

$$\left\langle D \int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds, \phi\right\rangle_{\mathcal{H}} = \int_a^{b+\varepsilon} \langle 1_{[0,s\wedge T]}, \phi\rangle_{\mathcal{H}} \, ds, \quad \text{for } \phi \in \mathcal{H}.$$

Proof. From the continuity of B^H and $(\sup_{s\in[0,T]}|B_s^H|) \in L^2(\Omega)$, we obtain

$$\int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} B_{t_i\wedge T}^H (t_{i+1} - t_i),$$

where the limit is in $L^2(\Omega)$ and $\pi = \{a = t_0 < t_1 < \cdots < t_n = b + \varepsilon\}$ is a partition of the interval $[a, b+\varepsilon]$. Consequently, $\int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds$ is a square-integrable random variable in the chaos of order 1 and, therefore, it is in $\mathbb{D}^{1,2}$.

Finally, the Fubini theorem and (2.5) imply that, for $\phi \in \mathcal{H}$,

$$\begin{aligned}
&\left\langle D \int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds, \phi \right\rangle_{\mathcal{H}} \\
&\quad = E\left(\left\langle D \int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds, \phi \right\rangle_{\mathcal{H}} \right) = E\left(\delta(\phi) \int_a^{b+\varepsilon} B_{s\wedge T}^H \, ds \right) \\
&\quad = \int_a^{b+\varepsilon} E\big(\delta(\phi) B_{s\wedge T}^H\big) \, ds = \int_a^{b+\varepsilon} \langle 1_{[0,s\wedge T]}, \phi \rangle_{\mathcal{H}} \, ds.
\end{aligned}$$

Thus, the proof is complete. □

Lemma 4.15. *Let $f \in C_e^{1,2}([0,T] \times \mathbb{R})$. Then*

$$\lim_{\varepsilon \downarrow 0} E\left(\left(\frac{1}{2\varepsilon} \int_0^\varepsilon \left| \left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D B_s^{H,\varepsilon} - \partial_x^2 f\big(s, B_s^H\big) 1_{[0,s]}, 1_{[(s-\varepsilon)\vee 0, (s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} \right| ds \right)^2 \right) = 0.$$

Proof. Let $\varepsilon < T/2$. Then, (4.1) and the change of variables formula give

$$B_t^{H,\varepsilon} = \frac{1}{2\varepsilon} \int_\varepsilon^{t+\varepsilon} B_s^H \, ds, \quad t \in [0, \varepsilon].$$

Consequently, from the fact that $f \in C_e^{1,2}([0,T] \times \mathbb{R})$, (1.1), (4.16) and Lemma 4.14, we have

$$\begin{aligned}
&\frac{1}{2\varepsilon} \int_0^\varepsilon \left| \left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D B_s^{H,\varepsilon} - \partial_x^2 f\big(s, B_s^H\big) 1_{[0,s]}, 1_{[(s-\varepsilon)\vee 0, (s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} \right| ds \\
&\quad = \frac{1}{2\varepsilon} \int_0^\varepsilon \left| \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) \frac{1}{2\varepsilon} \int_\varepsilon^{s+\varepsilon} \langle 1_{[0,u]}, 1_{[0,s+\varepsilon]} \rangle_{\mathcal{H}} \, du - \partial_x^2 f\big(s, B_s^H\big) \langle 1_{[0,s]}, 1_{[0,s+\varepsilon]} \rangle_{\mathcal{H}} \right| ds \\
&\quad = \frac{1}{2\varepsilon} \int_0^\varepsilon \left| \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) \frac{1}{2\varepsilon} \int_\varepsilon^{s+\varepsilon} R(u, s+\varepsilon) \, du - \partial_x^2 f\big(s, B_s^H\big) R(s, s+\varepsilon) \right| ds \\
&\quad \le \frac{C}{\varepsilon} \exp\Big(C \sup_{s\in[0,T]} \big|B_s^H\big| \Big) \int_0^\varepsilon \left(\frac{1}{2\varepsilon} \int_\varepsilon^{s+\varepsilon} \big|R(u, s+\varepsilon)\big| \, du + \big|R(s, s+\varepsilon)\big| \right) ds \\
&\quad \le C \exp\Big(C \sup_{s\in[0,T]} \big|B_s^H\big| \Big) \varepsilon^{2H},
\end{aligned}$$

which goes to 0 in $L^2(\Omega)$, as $\varepsilon \downarrow 0$, because $\exp(C \sup_{s\in[0,T]} |B_s^H|) \in L^2(\Omega)$ (see Theorem 4.2 in Nourdin [27]). Thus, the result is true. $\square$

Lemma 4.16. *Let $f \in C_e^{1,2}([0,T]\times\mathbb{R})$ and $t \in [0,T]$. Then,*

$$\lim_{\varepsilon\downarrow 0} E\left(\left(\frac{1}{2\varepsilon}\int_{t\wedge\varepsilon}^{t} \left|\left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D B_s^{H,\varepsilon} - \partial_x^2 f\big(s, B_s^H\big) 1_{[0,s]}, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}}\right| ds\right)^2\right) = 0.$$

Proof. Let $0 < \varepsilon < t/2 \le T/2$. Then, (4.14) yields

$$\begin{aligned}
&\frac{1}{2\varepsilon}\int_{\varepsilon}^{t} \left|\left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D B_s^{H,\varepsilon} - \partial_x^2 f\big(s, B_s^H\big) 1_{[0,s]}, 1_{[s-\varepsilon,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}}\right| ds \\
&\quad\le \frac{1}{2\varepsilon}\int_{\varepsilon}^{t} \left|\left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D\left\{\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} B_{u\wedge T}^H\, du\right\} - \partial_x^2 f\big(s, B_s^H\big) 1_{[0,s]}, 1_{[s-\varepsilon,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}}\right| ds \\
&\qquad + \frac{1}{4\varepsilon^2}\int_{\varepsilon}^{t} \left|\left\langle \partial_x^2 f\big(s, B_s^{H,\varepsilon}\big) D\int_0^{\varepsilon} B_u^H\, du, 1_{[s-\varepsilon,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}}\right| ds \\
&\quad = I_1^\varepsilon + I_2^\varepsilon.
\end{aligned} \tag{4.17}$$

Now, we decompose the proof into three parts.
Step 1: Here, we deal with the convergence to zero of I_2^ε in $L^2(\Omega)$, as $\varepsilon \downarrow 0$.
By Lemma 4.14 and (4.16), we obtain

$$\begin{aligned}
I_2^\varepsilon &= \frac{1}{4\varepsilon^2}\int_{\varepsilon}^{t} \left|\partial_x^2 f\big(s, B_s^{H,\varepsilon}\big)\int_0^{\varepsilon}\big(R\big(u,(s+\varepsilon)\wedge T\big) - R(u, s-\varepsilon)\big)\, du\right| ds \\
&\le \frac{C\exp(C\sup_{s\in[0,T]}|B_s^H|)}{\varepsilon^2}\bigg(\int_{\varepsilon}^{2\varepsilon}\int_0^{\varepsilon}\big|R\big(u,(s+\varepsilon)\wedge T\big) - R(u,s-\varepsilon)\big|\, du\, ds \\
&\qquad + \int_{2\varepsilon}^{t}\int_0^{\varepsilon}\big|R\big(u,(s+\varepsilon)\wedge T\big) - R(u,s-\varepsilon)\big|\, du\, ds\bigg) \\
&= \frac{C\exp(C\sup_{s\in[0,T]}|B_s^H|)}{\varepsilon^2}\big(I_{2,1}^\varepsilon + I_{2,2}^\varepsilon\big).
\end{aligned} \tag{4.18}$$

Note that (1.1) implies

$$I_{2,1}^\varepsilon \le C\varepsilon^{2H}\int_{\varepsilon}^{2\varepsilon}\int_0^{\varepsilon} du\, ds = C\varepsilon^{2+2H} \tag{4.19}$$

and, for $u \in (0,\varepsilon)$ and $v \in (\varepsilon, T]$,

$$\big|\partial_v R(v,u)\big| = H\big((v-u)^{2H-1} - v^{2H-1}\big)$$

which, together with the mean value theorem, allows us to obtain,

$$I_{2,2}^{\varepsilon} \le C\varepsilon \int_{2\varepsilon}^{t}\int_{0}^{\varepsilon} \{(s-2\varepsilon)^{2H-1} - ((s+\varepsilon)\wedge T)^{2H-1}\}\,du\,ds$$

$$= C\varepsilon^2 \int_{2\varepsilon}^{t} \{(s-2\varepsilon)^{2H-1} - ((s+\varepsilon)\wedge T)^{2H-1}\}\,ds. \tag{4.20}$$

Therefore, for $0 < t < T$ and ε small enough, we have

$$I_{2,2}^{\varepsilon} \le C\varepsilon^2 \int_{2\varepsilon}^{t} \{(s-2\varepsilon)^{2H-1} - (s+\varepsilon)^{2H-1}\}\,ds$$

$$= C\varepsilon^2\big[(t-2\varepsilon)^{2H} - (t+\varepsilon)^{2H} + (3\varepsilon)^{2H}\big] \le C\varepsilon^{2+2H}. \tag{4.21}$$

Also we could have that $t = T$. In this case, (4.20) gives that, for ε small enough,

$$I_{2,2}^{\varepsilon} \le C\varepsilon^2 \int_{2\varepsilon}^{T-\varepsilon} \{(s-2\varepsilon)^{2H-1} - (s+\varepsilon)^{2H-1}\}\,ds$$

$$+ C\varepsilon^2 \int_{T-\varepsilon}^{T} \{(s-2\varepsilon)^{2H-1} - (T)^{2H-1}\}\,ds$$

$$= C\varepsilon^2 \left| \frac{1}{2H}\big((T-2\varepsilon)^{2H} - (T)^{2H} + (3\varepsilon)^{2H}\big) - T^{2H-1}\varepsilon \right| \le C\varepsilon^{2+2H}. \tag{4.22}$$

Resuming, we have showed that I_2^{ε} converges to zero in $L^2(\Omega)$, as $\varepsilon \to 0$, due to inequalities (4.18)–(4.22) and Nourdin [27] (Theorem 4.2).

Step 2: In this part of the proof, we prove that I_1^{ε} goes to zero in $L^2(\Omega)$, as $\varepsilon \to 0$.

From (4.17), we can write

$$I_1^{\varepsilon} \le \frac{1}{2\varepsilon}\int_{\varepsilon}^{t} \big|\big\langle \big(\partial_x^2 f(s, B_s^{H,\varepsilon}) - \partial_x^2 f(s, B_s^H)\big) 1_{[0,s]}, 1_{[s-\varepsilon,(s+\varepsilon)\wedge T]}\big\rangle_{\mathcal{H}}\big|\,ds$$

$$+ \frac{1}{2\varepsilon}\int_{\varepsilon}^{t} \left|\left\langle \partial_x^2 f(s, B_s^{H,\varepsilon})\left(D\left\{\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} B_{u\wedge T}^H\,du\right\} - 1_{[0,s]}\right), 1_{[s-\varepsilon,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}}\right|\,ds$$

$$= I_{1,1}^{\varepsilon} + I_{1,2}^{\varepsilon}. \tag{4.23}$$

We first consider $I_{1,1}^{\varepsilon}$ in the case that $t < T$. Then, for ε small enough, the mean value theorem and (1.1) imply

$$I_{1,1}^{\varepsilon} = \frac{1}{2\varepsilon}\int_{\varepsilon}^{t} \big|\big(\partial_x^2 f(s, B_s^{H,\varepsilon}) - \partial_x^2 f(s, B_s^H)\big)\big(R(s,s+\varepsilon) - R(s,s-\varepsilon)\big)\big|\,ds$$

$$= \frac{1}{4\varepsilon}\int_{\varepsilon}^{t} \big|\partial_x^2 f(s, B_s^{H,\varepsilon}) - \partial_x^2 f(s, B_s^H)\big|\big((s+\varepsilon)^{2H} - (s-\varepsilon)^{2H}\big)\,ds$$

$$\le H\int_{\varepsilon}^{t} \big|\partial_x^2 f(s, B_s^{H,\varepsilon}) - \partial_x^2 f(s, B_s^H)\big|(s-\varepsilon)^{2H-1}\,ds. \tag{4.24}$$

Similarly, for $t = T$ and ε small enough, the mean value theorem gives

$$\begin{aligned}
I_{1,1}^{\varepsilon} &= \frac{1}{4\varepsilon}\int_{\varepsilon}^{T-\varepsilon} \left|\partial_x^2 f\left(s, B_s^{H,\varepsilon}\right) - \partial_x^2 f\left(s, B_s^H\right)\right|\left((s+\varepsilon)^{2H} - (s-\varepsilon)^{2H}\right) ds \\
&\quad + \frac{1}{4\varepsilon}\int_{T-\varepsilon}^{T} \left|\partial_x^2 f\left(s, B_s^{H,\varepsilon}\right) - \partial_x^2 f\left(s, B_s^H\right)\right|\left(T^{2H} - (s-\varepsilon)^{2H} + \varepsilon^{2H} - (T-s)^{2H}\right) ds \\
&\le H\int_{\varepsilon}^{T} \left|\partial_x^2 f\left(s, B_s^{H,\varepsilon}\right) - \partial_x^2 f\left(s, B_s^H\right)\right|(s-\varepsilon)^{2H-1}\, ds \\
&\quad + H\int_{T-\varepsilon}^{T} \left|\partial_x^2 f\left(s, B_s^{H,\varepsilon}\right) - \partial_x^2 f\left(s, B_s^H\right)\right|\left((T-s)^{2H-1} + (s-\varepsilon)^{2H-1}\right) ds,
\end{aligned}$$

which, together with Nourdin [27] (Theorem 4.2), the fact that $f \in C_e^{1,2}([0,T]\times\mathbb{R})$ and (4.24), allows us to deduce that $I_{1,1}^{\varepsilon}$ goes to zero in $L^2(\Omega)$, as $\varepsilon \to 0$. So, in order to see that the claim of Step 2 holds, we only need to see that $I_{1,2}^{\varepsilon}$ also goes to zero in $L^2(\Omega)$, as $\varepsilon \downarrow 0$, because of (4.23). Toward this end, we first assume that $t < T$. In this case, we use Lemma 4.14 and the notation $G = C\exp(C\sup_{s\in[0,T]}|B_s^H|)$ to establish that, for ε small enough,

$$\begin{aligned}
I_{1,2}^{\varepsilon} &\le \frac{G}{4\varepsilon^2}\int_{\varepsilon}^{t}\left|\int_{s-\varepsilon}^{s+\varepsilon}\left[R(u,s+\varepsilon) - R(u,s-\varepsilon) - R(s,s+\varepsilon) + R(s,s-\varepsilon)\right]du\right| ds \\
&\le C\frac{G}{\varepsilon^2}\int_{\varepsilon}^{t}\left|\int_{s-\varepsilon}^{s+\varepsilon}\left[(s+\varepsilon-u)^{2H} - \left(u-(s-\varepsilon)\right)^{2H}\right]du\right| ds = 0.
\end{aligned} \tag{4.25}$$

Thus, for $t = T$ and ε small enough, we can conclude

$$\begin{aligned}
I_{1,2}^{\varepsilon} &\le C\frac{G}{\varepsilon^2}\int_{T-\varepsilon}^{T}\left|\int_{s-\varepsilon}^{s+\varepsilon}\left[R\left(u\wedge T, (s+\varepsilon)\wedge T\right)\right.\right. \\
&\quad \left.\left. - R(u\wedge T, s-\varepsilon) - R\left(s, (s+\varepsilon)\wedge T\right) + R(s,s-\varepsilon)\right]du\right| ds \\
&= C\frac{G}{\varepsilon^2}\int_{T-\varepsilon}^{T}\left|\int_{s-\varepsilon}^{s+\varepsilon}\left[(T-u\wedge T)^{2H}\right.\right. \\
&\quad \left.\left. - \left(u\wedge T - (s-\varepsilon)\right)^{2H} + \varepsilon^{2H} - (T-s)^{2H}\right]du\right| ds \\
&\le C\frac{G}{\varepsilon^2}\int_{T-\varepsilon}^{T}\int_{s-\varepsilon}^{s+\varepsilon}\varepsilon^{2H}\, du\, ds = CG\varepsilon^{2H}.
\end{aligned}$$

Hence, (4.25) yields that $I_{1,2}^{\varepsilon} \to 0$ in $L^2(\Omega)$, as $\varepsilon \downarrow 0$.

Step 3: Finally, (4.17), and Steps 1 and 2 imply that the result is true. Consequently, the proof is finished. □

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