On the probability distribution of the local times of diagonally operator-self-similar Gaussian fields with stationary increments

KAMRAN KALBASI[*] and THOMAS MOUNTFORD[†]

Institute of Mathematics, EPFL (Ecole Polytechnique Fédérale de Lausanne), Lausanne, Switzerland.
E-mail: [*]kamran.kalbasi@alumni.epfl.ch; [†]thomas.mountford@epfl.ch

In this paper, we study the local times of vector-valued Gaussian fields that are 'diagonally operator-self-similar' and whose increments are stationary. Denoting the local time of such a Gaussian field around the spatial origin and over the temporal unit hypercube by Z, we show that there exists $\lambda \in (0, 1)$ such that under some quite weak conditions, $\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda}$ and $\lim_{x \to +\infty} \frac{-\log \mathbb{P}(Z > x)}{x^{\frac{1}{\lambda}}}$ both exist and are strictly positive (possibly $+\infty$). Moreover, we show that if the underlying Gaussian field is 'strongly locally nondeterministic', the above limits will be finite as well. These results are then applied to establish similar statements for the intersection local times of diagonally operator-self-similar Gaussian fields with stationary increments.

Keywords: fractional Brownian fields; Gaussian fields; local times; operator-self-similar random fields; probability tail decay

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $X_t := (X_t^1, \ldots, X_t^d)$, $t \in \mathbb{R}^N$ be an N-parameter d-dimensional centered Gaussian field on $(\Omega, \mathcal{F}, \mathbb{P})$, that is, each component X_t^i is a real-valued zero-mean Gaussian field indexed by $\mathbb{R}^N$. We call such a random field a centered Gaussian (N, d)-field.

We denote by $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Q}$ respectively the sets of strictly positive real numbers (> 0), strictly positive integers (≥ 1), and finally the rational numbers. Evidently $\mathbb{R}_{\geq 0}$ denotes the real numbers that are positive or zero. We denote the space of matrices of size $m \times n$ with real entries by $\mathbb{R}^{n \times m}$. For any two same-sized vectors $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, $\boldsymbol{u} \circ \boldsymbol{v}$ denotes their Schur product, that is, the vector $\boldsymbol{u} \circ \boldsymbol{v} := (u_1 v_1, \ldots, u_n v_n)$. For any square matrix $\mathbf{Y}$, we denote its trace (i.e., the sum of all its diagonal entries) by $\text{tr}(\mathbf{Y})$. For any matrix $\mathbf{Y}$, we denote its transpose by $\mathbf{Y}^\dagger$. For any matrices $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$, we define $\text{diag}(\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n})$ as the block diagonal matrix that has matrices $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$ on its diagonal (respecting the order) and is zero elsewhere.

For any $\boldsymbol{p} \in \mathbb{R}^N$ and $\boldsymbol{T} \in \mathbb{R}_+^N$, let $\mathcal{C}(\boldsymbol{p}, \boldsymbol{T})$ denote $\boldsymbol{p} + \prod_{i=1}^N [0, T_i]$, i.e., the N-dimensional cube of side lengths equal to $\{T_i\}_{i=1}^N$ and based at point $\boldsymbol{p}$. We also denote $[0, 1]$ by $\mathcal{I}$.

For any measurable subset $\mathcal{B} \subset \mathbb{R}^d$, we denote its Lebesgue measure by $\text{Vol}(\mathcal{B})$. For any subset $\mathcal{A}$ of an arbitrary set $\mathcal{X}$, we denote its indicator function by $\mathbf{1}_{\{\mathcal{A}\}}$, that is,

$$\mathbf{1}_{\{\mathcal{A}\}}(x) := \begin{cases} 1 & \text{for } x \in \mathcal{A}, \\ 0 & \text{for } x \notin \mathcal{A}. \end{cases}$$

For any k-dimensional Gaussian random vector $\mathbf{Y} = (Y_1, \ldots, Y_k)$, we denote the determinant of its covariance matrix by $\text{detCov}[\mathbf{Y}]$; in other words

$$\text{detCov}[\mathbf{Y}] := \det\big[\mathbb{E}\big(\mathbf{Y}\mathbf{Y}^\dagger\big) - \mathbb{E}(\mathbf{Y})\mathbb{E}\big(\mathbf{Y}^\dagger\big)\big],$$

where $\mathbf{Y}$ is regarded as a $k \times 1$ matrix. For any finite family of vectors $\mathbf{y}_i = (y_1^i, \ldots, y_k^i)$, $i = 1, \ldots, n$, we call the following vector as their adjoined vector:

$$\big[y_1^1, \ldots, y_k^1, y_1^2, \ldots, y_k^2, \ldots, y_1^n, \ldots, y_k^n\big],$$

and we denote it by $[\mathbf{y}_1, \ldots, \mathbf{y}_n]$.

Once a centered Gaussian (N, d)-field $\mathbf{X}$ is fixed, for any positive integer k and any $\mathbf{t}_1, \ldots, \mathbf{t}_k \in \mathbb{R}^N$ we define

$$\text{K}_n^X(\mathbf{t}_1, \ldots, \mathbf{t}_n) := (2\pi)^{-\frac{nd}{2}} \big(\det\text{Cov}[\mathbf{X}_{\mathbf{t}_1}, \ldots, \mathbf{X}_{\mathbf{t}_n}]\big)^{-\frac{1}{2}}. \tag{1}$$

We use the following definition of local times which provides a pointwise characterisation. The more common definition of local times as the Radon–Nikodym derivative of the occupation measure of a random process (if it is absolutely continuous), only provides an almost-sure characterisation of the occupation density. In Section 3, we will see more on this and the link between the two definitions.

Definition 1.1. Let $\{X_t\}_t$ be a random field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. We define the local time of X at $x \in \mathbb{R}^d$ and over the cube $\mathcal{C}(p, T)$ as the following limit (if it exists)

$$L_x\big(X; \mathcal{C}(p, T)\big) := \lim_{\varepsilon \to 0} \int_{\mathcal{C}(p, T)} \frac{1}{\text{Vol}(\mathcal{B}_\varepsilon(x))} \mathbf{1}_{\{\|X_t - x\| < \varepsilon\}}(t)\,dt,$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$, and $\mathcal{B}_\varepsilon(x) := \{y \in \mathbb{R}^d; \|x - y\| < \varepsilon\}$.

We are interested in the tail-decay behavior of the probability distribution of $L_0(X; \mathcal{I}^N)$. The first work in this direction goes back to [6]. They consider a one-parameter one-dimensional ($N = d = 1$) Gaussian process $X(t)$ with stationary increments satisfying the local nondeterminism condition [11]. Moreover, defining $\sigma^2(t) = \mathbb{E}[(X_t - X_s)^2]$, they assume that $\sigma(t)$ is continuous and strictly increasing on the interval $[0, 1]$, that $\frac{1}{\sigma(t)}$ is integrable over $\mathcal{I} = [0, 1]$, and finally $\sigma(t)$ varies regularly at 0 with some exponent $0 < H < 1$, i.e., $\lim_{t \to 0} \frac{\sigma(\omega t)}{\sigma(t)} = \omega^H$ for every $\omega > 0$. In fact this latter condition is a gauge for asymptotic self-similarity near the origin.

Under these conditions they show that the local times of $X(\cdot)$ exist, and moreover

$$0 < \liminf_{x \to +\infty} \frac{-\log \mathbb{P}[L_0(X, \mathcal{I}) > x]}{\sigma^{-1}(\frac{1}{x})} \leq \limsup_{x \to +\infty} \frac{-\log \mathbb{P}[L_0(X, \mathcal{I}) > x]}{\sigma^{-1}(\frac{1}{x})} < +\infty.$$

When $\sigma(t) = t^H$, which corresponds to the fractional Brownian motion of Hurst parameter H, the exponential decay rate $\sigma^{-1}(\frac{1}{x})$ equals $x^{\frac{1}{H}}$.

More recently, [1] considers the one-parameter d-dimensional fractional Brownian motion $\boldsymbol{B}^H(t) = (B_1^H(t), \ldots, B_d^H(t))$ and also d-dimensional fractional Riemann–Liouville process $\boldsymbol{W}^H(t) = (W_1^H(t), \ldots, W_d^H(t))$ where $\{B_i^H\}_{i=1}^d$ ($\{W_i^H\}_{i=1}^d$) are d independent copies of a fractional Brownian motion (fractional Riemann–Liouville process) with Hurst parameter H. They show that the following limits exist

$$\lim_{x \to +\infty} x^{-\frac{1}{dH}} \log \mathbb{P}\big[L_{\boldsymbol{0}}(\boldsymbol{B}^H, \mathcal{I}) > x\big] \quad \text{and} \quad \lim_{x \to +\infty} x^{-\frac{1}{dH}} \log \mathbb{P}\big[L_{\boldsymbol{0}}(\boldsymbol{W}^H, \mathcal{I}) > x\big].$$

We will prove the existence of this exponential tail-decay limit for the class of Gaussian fields that have stationary increments (Property $\mathfrak{A}_2$ below) and are 'diagonally self-similar' as defined in Property $\mathfrak{A}_3$ below.

Throughout the paper, we assume that the random field X has both of the following two properties ($\mathfrak{A}_0$ and $\mathfrak{A}_1$).

Property $\mathfrak{A}_0$. There exists a positive constant $c_0 > 0$ such that $\mathrm{var}(X_t^i) \leq c_0$ for every $t \in [0, 1]^N$ and $i = 1, \ldots, d$.

As we do not assume any kind of continuity of X or its covariance matrix, the boundedness of its variance (Property $\mathfrak{A}_0$), seems inevitable.

Property $\mathfrak{A}_1$. The (N, d)-Gaussian field X has the property that for any positive integer n, the expression $\mathrm{K}_n^X(t_1, \ldots, t_n)$ is integrable over $(\mathcal{I}^N)^n$.

Property $\mathfrak{A}_1$ guarantees the existence of the local times at every point, that is, $L_x(X; \mathcal{I}^N)$, and the finiteness of all their moments, see Proposition 4.2. In fact, $\mathfrak{A}_1$ is the weakest-known sufficient condition for the existence of local time at the origin and the finiteness of all its moments.

Next, we have the following two properties that form our main framework.

Property $\mathfrak{A}_2$ (Stationary increments). The random field X is zero at the origin and has stationary increments, that is, for any $\boldsymbol{p} \in \mathbb{R}^N$ we have the following equality for every $s, t \in \mathbb{R}^N$ and $i, j \in \{1, \ldots, d\}$

$$\mathbb{E}\big[X_s^{(i)} X_t^{(j)}\big] = \mathbb{E}\big[\big(X_{s+p}^{(i)} - X_p^{(i)}\big)\big(X_{t+p}^{(j)} - X_p^{(j)}\big)\big].$$

Property $\mathfrak{A}_3$ (Diagonal self-similarity). There exist a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$ and a matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$ with positive trace ($\mathrm{tr}(\mathbf{H}) > 0$) such that for every $\omega > 0$ we have

$$X_{t \circ \omega^\alpha} \overset{d}{=} \omega^{\mathbf{H}} X_t; \quad \forall \omega \in \mathbb{R}_+, \tag{2}$$

where $\omega^{\boldsymbol{\alpha}} := (\omega^{\alpha_1}, \ldots, \omega^{\alpha_1})$, the values of X_t are considered as $d \times 1$ matrices, $\overset{d}{=}$ means equality in finite dimensional distributions for the two random fields, and $\omega^{\mathbf{H}}$ denotes matrix exponential with the usual definition, that is,

$$\omega^{\mathbf{H}} := e^{\ln(\omega)\mathbf{H}} = \sum_{i=0}^{\infty} \frac{(\ln(\omega)\mathbf{H})^n}{n!}; \quad \forall \omega \in \mathbb{R}_+.$$

Remark 1.1. This definition is a special case of the more general concept of what is called operator-self-similar random fields, for example, studied in [7]. In the general case, the vectors $\boldsymbol{\alpha}$ and hence $\omega^{\boldsymbol{\alpha}}$ are replaced by a matrix $\mathbf{E}$ and its matrix exponential $\omega^{\mathbf{E}}$, respectively. Evidently in this more general setting, the Shur product $\omega^{\boldsymbol{\alpha}} \circ \boldsymbol{t}$ should be replaced by the usual matrix multiplication $\omega^{\mathbf{E}} \boldsymbol{t}$. This justifies us calling Property $\mathfrak{A}_3$ as 'diagonal' self-similarity.

Remark 1.2. For zero-mean Gaussian fields, Equation (2) in Property $\mathfrak{A}_3$ is equivalent to the following equation

$$\mathbb{E}\big(\mathbf{X}_{s \circ \omega^{\boldsymbol{\alpha}}} \mathbf{X}_{t \circ \omega^{\boldsymbol{\alpha}}}^{\dagger}\big) = \omega^{\mathbf{H}} \mathbb{E}\big(\mathbf{X}_s \mathbf{X}_t^{\dagger}\big) \omega^{\mathbf{H}^{\dagger}}, \quad \forall s, t \in \mathbb{R}^N,$$

where X_t is considered as a $d \times 1$ matrix as above, and $\mathbf{H}^{\dagger}$ denotes the transpose of matrix $\mathbf{H}$. For more on matrix exponential see, for example, [4], Chapter 2.

An important special case of Property $\mathfrak{A}_3$ is the following condition.

Property $\mathfrak{A}_3^{\circ}$ (Two-sided diagonal self-similarity). There exist $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$ and $(H_1, \ldots, H_d) \in \mathbb{R}_+^d$ such that for every $\omega > 0$ we have

$$\mathbb{E}\big[X_{s \circ \omega^{\boldsymbol{\alpha}}}^{(i)} X_{t \circ \omega^{\boldsymbol{\alpha}}}^{(j)}\big] = \omega^{H_i + H_j} \mathbb{E}\big[X_s^{(i)} X_t^{(j)}\big], \quad \forall i, j \in \{1, \ldots, d\}, \forall s, t \in \mathbb{R}^N, \tag{3}$$

where $\omega^{\boldsymbol{\alpha}} := (\omega^{\alpha_1}, \ldots, \omega^{\alpha_1})$.

Remark 1.3. For a zero-mean Gaussian field X_t, Property $\mathfrak{A}_3^{\circ}$ is satisfied if and only if Property $\mathfrak{A}_3$ is satisfied with $\mathbf{H} = \text{diag}(H_1, H_2, \ldots, H_d)$, that is, the diagonal matrix whose diagonal entries are $\omega^{H_1}, \ldots, \omega^{H_d}$ (respecting the order) and is zero elsewhere. This is true because we have

$$\omega^{\text{diag}(H_1, H_2, \ldots, H_d)} = \text{diag}\big(\omega^{H_1}, \ldots, \omega^{H_d}\big).$$

Remark 1.4. A very important random field that satisfies both Properties $\mathfrak{A}_2$ and $\mathfrak{A}_3^{\circ}$ is the multi-parameter fractional Brownian motion, that is, the centered Gaussian field with stationary increments characterised by $\mathbb{E}[(X_s - X_t)^2] = |s - t|^{2H}$ for every $s, t \in \mathbb{R}^N$, where $H \in (0, 1]$ is the Hurst parameter of the Gaussian field. Furthermore, the centered Gaussian (N, d)-field consisting of d independent multi-parameter fractional Brownian motions each with its own Hurst parameter H_i satisfies also $\mathfrak{A}_2$ and $\mathfrak{A}_3$, hence falls in the scope of this paper as well.

Remark 1.5. Let $c_1, \ldots, c_N \in \mathbb{R}_+$, $p_1, \ldots, p_N \in (0, 2]$ and $H \in (0, 1]$. Consider the $(N, 1)$-Gaussian field that we call 'anisotropic fractional Brownian motion', that is, the $\mathbb{R}^N$-indexed centered Gaussian field X_s with stationary increments given by

$$\mathbb{E}\big[(X_s - X_t)^2\big] = \phi(s - t),$$

where

$$\phi(s) = \left(\sum_{i=1}^N c_i |s_i|^{p_i} \right)^{2H}.$$

This Gaussian field satisfies both Properties $\mathfrak{A}_2$, and $\mathfrak{A}_3^\circ$ with $\tilde{\boldsymbol{\alpha}} := (\frac{1}{p_1}, \ldots, \frac{1}{p_N})$ and $H_1 := H$.

For other interesting examples of operator-self-similar random processes and fields with stationary increments, see e.g. [8].

In Section 2 we gather all the main results of this paper. In Section 3, we discuss the pointwise versus functional definitions of local times which are relevant to our work. In Section 4, we state the relation between the exponential decay rate of the probability tail of local times and the exponential growth rate of their moments. Sections 5 and 6 contain the technical proofs.

2. Main results

In this section, we give some technical definitions and state our results. The proofs will come in the subsequent sections.

For every $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}_+^N$, we define the $\boldsymbol{\alpha}$-length as follows

$$\|t\|_{\boldsymbol{\alpha}} := \sum_{i=1}^N |t_i|^{1/\alpha_i} : \quad \forall t = (t_1, \ldots, t_N) \in \mathbb{R}^N. \tag{4}$$

It is evident that $\|t\|_{\boldsymbol{\alpha}}$ defines a translation invariant topology on $\mathbb{R}_+^N$. Moreover, if $\forall i = 1, \ldots, N : \alpha_i \geq 1$ then $\|t\|_{\boldsymbol{\alpha}}$ defines a translation invariant metric on $\mathbb{R}_+^N$ which we call the $\boldsymbol{\alpha}$-distance. Nevertheless, it is not a norm except for the special case where all the exponents are equal to 1.

We introduce the following definition which generalizes the idea of Strong Local Nondeterminism to vector-valued Gaussian fields.

Definition 2.1 (Strong Local Nondeterminism). We call a centered Gaussian (N, d)-field X_t strongly locally nondeterministic over a cube $\mathcal{J} \subseteq \mathbb{R}^N$ with scaling vector $\boldsymbol{\xi} := (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}_+^N$ if there exist constants $H > 0$ and $C > 0$ such that for any positive integer n, and any arbitrary vectors $u, t_1, \ldots, t_n \in \mathcal{J}$, we have

$$\mathrm{detCov}[X_u | X_{t_1}, X_{t_2}, \ldots, X_{t_n}] \geq C \min_{0 \leq i \leq n} \|u - t_i\|_{\boldsymbol{\alpha}}^{2H},$$

where $t_0 := \mathbf{0}$, the expression $\det\mathrm{Cov}[X_u \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}]$ denotes the determinant of the conditional covariance matrix of the random vector X_u conditioned on all the random vectors $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$, and finally, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) := H\boldsymbol{\xi}$.

Remark 2.1. The reason why only the normalized vector $\boldsymbol{\xi} = \frac{1}{H}\boldsymbol{\alpha}$ is relevant, is due to the fact that for any $p > 0$ there exist positive constants $c_1, c_2 > 0$ such that for every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$

$$c_1 \left(\sum_{i=1}^{N} |x_i|^{1/\alpha_i} \right)^{2H} \leq \left(\sum_{i=1}^{N} |x_i|^{p/\alpha_i} \right)^{\frac{2H}{p}} \leq c_2 \left(\sum_{i=1}^{N} |x_i|^{1/\alpha_i} \right)^{2H}.$$

In fact we have the following proposition.

Proposition 2.1. *Let $f : \mathbb{R}^N \to \mathbb{R}^{\geq 0}$ be a continuous function such that $f(x) = 0$ if and only if $x = \mathbf{0}$, and for some vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}_+^N$ and $H > 0$, we have $f(x \circ \omega^{\boldsymbol{\alpha}}) = \omega^H f(x)$ for every $x \in \mathbb{R}^N$ and $\omega > 0$. Then there exist constants $c_1, c_2 > 0$ such that for every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$*

$$c_1 \left(\sum_{i=1}^{N} |x_i|^{1/\alpha_i} \right)^{H} \leq f(x) \leq c_2 \left(\sum_{i=1}^{N} |x_i|^{1/\alpha_i} \right)^{H}.$$

Proof. In Section 5. $\qquad\qquad\square$

Remark 2.2. Let X_t be a diagonally self-similar centered Gaussian (N, d)-field that satisfies Property $\mathfrak{A}_3$ with matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$ and vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$. If X_t is strongly locally nondeterministic with the scaling vector $\boldsymbol{\xi} := (\xi_1, \xi_2, \ldots, \xi_N)$, then it is easy to verify that $\boldsymbol{\xi} = \frac{1}{\mathrm{tr}(\mathbf{H})}\boldsymbol{\alpha}$. In other words, for diagonally self-similar centered Gaussian (N, d)-fields, the strong local nondeterminism can be satisfied only with a unique scaling vector.

Proposition 2.2. *Let X_t be a diagonally self-similar centered Gaussian (N, d)-field with stationary increments, i.e. it satisfies Properties $\mathfrak{A}_0$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ with some matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$ and vector $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$. Let β be a positive real number. If the kernel $(\mathrm{K}_n^X)^{\beta}$ is integrable over the cube $(\mathcal{I}^N)^n$ for some integer n, then the following inequality has to hold true*

$$\sum_{i=1}^{N} \alpha_i > \beta \, \mathrm{tr}(\mathbf{H}).$$

Proof. In Section 5. $\qquad\qquad\square$

Lemma 2.3. *Let X_t be a centered Gaussian (N, d)-field which is strongly locally nondeterministic over $\mathcal{I}^N$ with scaling vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)$ and constant C_0. Then for any positive real number β such that $\beta < \sum_{i=1}^{N} \xi_i$, and any positive integer n, the kernel $(\mathrm{K}_n^X)^{\beta}$ is integrable over*

the cube $(\mathcal{I}^N)^n$, *and*

$$\int_{(\mathcal{I}^N)^n} \left(\mathrm{K}_n^X(t_1, \ldots, t_n) \right)^{\beta} \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq c^n (n!)^{\frac{\beta}{\sum_{i=1}^N \xi_i}},$$

where c is a constant that depends only on C_0, N, $\boldsymbol{\alpha}$, β, H *and* d.

Proof. In Section 5. □

Theorem 2.4. *Let* X_t *be a centered Gaussian* (N, d)-*field that is strongly locally nondeterministic over* $\mathcal{I}^N$ *with scaling vector* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}_+^N$ *such that* $1 < \sum_{i=1}^N \xi_i$. *Then the local times* $Z_{\boldsymbol{x}} := L_{\boldsymbol{x}}(X, \mathcal{I}^N)$ *of the random field* X_t *exist at every point* $\boldsymbol{x} \in \mathbb{R}^d$, *and*

$$\mathbb{E}\left(Z_{\boldsymbol{x}}^n \right) \leq c^n (n!)^{\frac{1}{\sum_{i=1}^N \xi_i}}$$

for some constant c which does not depend on n.

Proof. It is immediate from Lemma 2.3. □

Definition 2.2. *Let* $\{X_k(t_k) : t_k \in \mathbb{R}^{N_k}\}_{k=1}^m$ *be a family of m independent Gaussian fields such that for every* $k = 1, \ldots, m$, *the random field* $X_k(t)$ *is a centered Gaussian* (N_k, d)-*field. We define their (m-fold) intersection local time around the origin and over* $\mathcal{I}$ *as the local time of the following* $(\sum_{k=1}^m N_k, (m-1)d)$-*field at* $\boldsymbol{0}$ *and over the cube* $\mathcal{I}^{\sum_{k=1}^m N_k}$ *(if it exists)*

$$\left(X_1(t_1) - X_2(t_2), X_2(t_2) - X_3(t_3), \ldots, X_{m-1}(t_{m-1}) - X_m(t_m) \right).$$

Theorem 2.5. *Let m be a positive integer, and* $\{X_k(t_k) : t_k \in \mathbb{R}^{N_k}\}_{k=1}^m$ *be a family of m independent Gaussian fields such that for every* $k = 1, \ldots, m$, *the random field* $X_k(t_k)$ *is a centered Gaussian* (N_k, d)-*field that is strongly locally nondeterministic with scaling vector* $\boldsymbol{\xi}_k = (\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,N_k}) \in \mathbb{R}_+^{N_k}$. *If* $\sum_{k=1}^m \sum_{i=1}^{N_k} \xi_{k,i} > m - 1$, *then the m-fold intersection local time of the family* $\{X_k(\cdot)\}_{k=1}^m$ *over the interval* $\mathcal{I}$ *exists. Moreover, denoting this intersection local time by* $\mathfrak{I}_X$, *and defining* $\tilde{\xi}_k := \sum_{i=1}^{N_k} \xi_{k,i}$, *for any arbitrary sequence of positive numbers* $q_1, q_2, \ldots, q_m$ *such that* $\sum_{k=1}^m q_k = m - 1$, *and such that* $0 \leq q_k \leq 1$ *and* $q_k < \tilde{\xi}_k$ *(for every* $k = 1, \ldots, m$), *we have*

$$\mathbb{E}\left((\mathfrak{I}_X)^n \right) \leq c^n (n!)^{\sum_{k=1}^m \frac{q_k}{\tilde{\xi}_k}},$$

where c is a positive constant that does not depend on n.

Proof. In Section 5. □

Theorem 2.6. *Suppose* X_t *is a centered Gaussian* (N, d)-*field satisfying Properties* $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, *and* $\mathfrak{A}_3$ *with some matrix* $\mathbf{H} \in \mathbb{R}^{d \times d}$ *and vector* $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$ *such that* α_i's *are mutually*

rational, that is, $\frac{\alpha_i}{\alpha_j} \in \mathbb{Q}$ for every i and j. Then the following limits exist in $\mathbb{R}_+ \cup \{+\infty\}$, and are strictly positive

$$\lim_{x \to +\infty} \frac{-\log \mathbb{P}(Z > x)}{x^{\frac{1}{\lambda}}} \quad and \quad \lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda},$$

where $Z := L_0(X, [0,1]^N)$ and $\lambda := \frac{\mathrm{tr}(\mathbf{H})}{\sum_{k=1}^N \alpha_k}$. Moreover, if X_t is also strongly locally nondeterministic over $\mathcal{I}^N$, then the above limits will be finite.

Proof. In Section 6. □

Remark 2.3. One should note that although Properties $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ guarantee the convergence of the sequence $\{\frac{(\mathbb{E}(Z^n))^{\frac{1}{n}}}{n^\lambda}\}_n$, they do not imply the finiteness of the limit. Probably the simplest example would be the centered Gaussian $(2,1)$-field $X(s,t)$ $(s,t \in \mathbb{R})$ characterized by $X(0,0) = 0$ and $\mathbb{E}(X(s_1, t_1) - X(s_2, t_2))^2 = (t_1 - s_1)^{2H} + (t_2 - s_2)^{2H}$, where $H \in (0,1)$. It clearly satisfies all Properties $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ with the self-similarity scaling $(\boldsymbol{\alpha}, H)$, where $\boldsymbol{\alpha} := (1,1)$. So by Theorem 2.6, we know that $\{\frac{\sqrt[n]{\mathbb{E}(L_0^n)}}{n^{H/2}}\}_n$ converges, where L_0 is the local time of X around the origin on the square $\mathcal{I}^2$. On the other hand, one can easily verify that X is equivalent to $\{B_1(s) - B_2(t) : (s,t) \in \mathbb{R}^2\}$, where B_1 and B_2 are two independent fractional Brownian motions of Hurst parameter H. So L_0, that is, the local time of X around the origin, is the same as $\mathfrak{I}_B$, i.e., the intersection local time of two independent fractional Brownian motions with the same Hurst parameter. By [1], Theorem 2.4, we know that $\{\frac{\sqrt[n]{\mathbb{E}(\mathfrak{I}_B^n)}}{n^H}\}_n$ converges to a strictly positive finite constant. This shows that the right growth exponent of $\sqrt[n]{\mathbb{E}(L_0^n)}$ is n^H.

Corollary 2.1. *Let m be a positive integer, and $\{X_k(t_k) : t_k \in \mathbb{R}^{N_k}\}_{k=1}^m$ be a family of m independent Gaussian fields such that for every $k = 1, \ldots, m$, the random field $X_k(t)$ is a centered Gaussian (N_k, d)-field satisfying Properties $\mathfrak{A}_0$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ with the self-similarity scaling matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$ and scaling vector $\boldsymbol{\alpha}_k = (\alpha_{k,1}, \ldots, \alpha_{k,N_k}) \in \mathbb{R}_+^{N_k}$. If for every positive integer n and every $k = 1, \ldots, m$, the kernel $(\mathrm{K}_n^{X_k}(t_1, \ldots, t_n))^{\frac{m-1}{m}}$ is integrable over $(\mathcal{I}^{N_k})^n$, then the m-fold intersection local time of $\{X_k\}_{k=1}^m$ on the interval $[0,1]$ exists, and if moreover, every pair of $\alpha_{k,i}$ and $\alpha_{l,j}$ are mutually rational, that is, $\frac{\alpha_{k,i}}{\alpha_{l,j}} \in \mathbb{Q}$, then denoting the m-fold intersection local time of $\{X_k\}_k$ by $\mathfrak{I}_m$, the following limits exists*

$$\lim_{y \to +\infty} \frac{-\log \mathbb{P}(\mathfrak{I}_m > y)}{y^{\frac{1}{\gamma}}} \quad and \quad \lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}((\mathfrak{I}_m)^n)}}{n^\gamma},$$

where $\gamma := \frac{(m-1)\,\mathrm{tr}(\mathbf{H})}{\sum_{k=1}^m \sum_{i=1}^{N_k} \alpha_{k,i}}$.

Proof. In Section 6. □

Remark 2.4. Although this theorem affirms the convergence of $\{\frac{\sqrt[n]{\mathbb{E}((\Im_m)^n)}}{n^\gamma}\}_n$, it does not guarantee that $\gamma = \frac{(m-1)\operatorname{tr}(\mathbf{H})}{\sum_{i=1}^m \sum_{k=1}^{N_i} \alpha_{i,k}}$ is the right exponent for the growth of $\sqrt[n]{\mathbb{E}((\Im_m)^n)}$. For that, we would also need the finiteness of the above limit. In Corollary 2.3, we saw some examples where the right growth exponent is larger than the exponent given by this theorem (H instead of $H/2$).

3. Definition of local times: Pointwise versus functional

Let X be an (N, d)-random field over a cube $\mathcal{C}(\boldsymbol{p}, \boldsymbol{T})$ in $\mathbb{R}^N$. Let μ be the occupation measure of X, i.e. for every Borelian subset $\mathcal{B} \subseteq \mathbb{R}^d$ we have

$$\mu_X(\mathcal{B}) = \lambda_N\big(\{\boldsymbol{t} \in \mathcal{C}(\boldsymbol{p}, \boldsymbol{T}); X_{\boldsymbol{t}} \in \mathcal{B}\}\big),$$

where λ_N denotes the Lebesgue measure on $\mathbb{R}^N$.

Definition 3.1 (Functional definition of local time). If the occupation measure of X is almost-surely absolutely continuous with respect to the Lebegue measure on $\mathbb{R}^d$, i.e. when $\mu_X \ll \lambda_d$, its Radon–Nikodym derivative is called the local time (or occupation density) of X over $\mathcal{C}(\boldsymbol{p}, \boldsymbol{T})$, and we denote it by we denote this function by $\bar{L}_X$.

Clearly, this definition does not provide a unique pointwise definition for the local time but a set of functions that are equal to each other almost surely. It is also clear that for any positive (or bounded) measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\int_{\mathcal{C}(\boldsymbol{p},\boldsymbol{T})} f(X_{\boldsymbol{t}}) \, d\boldsymbol{t} = \int_{\mathbb{R}^d} f(\boldsymbol{x}) \bar{L}_X(\boldsymbol{x}) \, d\boldsymbol{x}.$$

A sufficient condition for the existence of the local time as defined above, is the following

$$\int_{\mathcal{C}(\boldsymbol{p},\boldsymbol{T})} \int_{\mathcal{C}(\boldsymbol{p},\boldsymbol{T})} \frac{1}{\sqrt{\det \mathrm{Cov}(X_{\boldsymbol{t}} - X_{\boldsymbol{s}})}} \, d\boldsymbol{t} \, d\boldsymbol{s} < +\infty; \tag{5}$$

see, for example, [9] or [3]. By Corollary A.7, we have

$$\det \mathrm{Cov}[X_{\boldsymbol{t}} X_{\boldsymbol{s}}] \leq \det \mathrm{Cov}(X_{\boldsymbol{t}}) \det \mathrm{Cov}(X_{\boldsymbol{t}} - X_{\boldsymbol{s}}).$$

So it is clear that conditions $\mathfrak{A}_1$ and $\mathfrak{A}_0$ imply Equation (5). So assuming these two conditions, we have the existence of the local times, both in the pointwise and functional definitions. Moreover, for every $\boldsymbol{x} \in \mathbb{R}^d$ we have

$$\int_{\mathcal{B}_\varepsilon(\boldsymbol{x})} \bar{L}_X(\boldsymbol{y}) \, d\boldsymbol{y} = \int_{\mathcal{C}(\boldsymbol{p},\boldsymbol{T})} \mathbf{1}_{\|X_{\boldsymbol{t}}-\boldsymbol{x}\|<\varepsilon}(\boldsymbol{t}) \, d\boldsymbol{t},$$

hence

$$\lim_{\varepsilon \to 0} \frac{1}{\mathrm{Vol}(\mathcal{B}_\varepsilon(\boldsymbol{x}))} \int_{\mathcal{B}_\varepsilon(\boldsymbol{x})} \bar{L}_X(\boldsymbol{y}) \, d\boldsymbol{y} = L_{\boldsymbol{x}}\big(X; \mathcal{C}(\boldsymbol{p}, \boldsymbol{T})\big),$$

which means that irrespective of the chosen version of $\bar{L}_X$, its pointwise local mean equals the local time defined in the pointwise manner.

4. Formulation in moments growth rate

The following theorem is a special case of Kasahara's Tauberian theorem [5], which relates the probability tail behavior to the moments asymptotic behavior.

Theorem 4.1 (Kasahara [5]). *For any positive random variable Y, any positive number λ, and any $A \in (0, +\infty]$, the limit*

$$\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Y^n)}}{n^\lambda}$$

exists and equals A if and only if the limit

$$\lim_{x \to +\infty} \frac{-\log \mathbb{P}(Y > x)}{x^{\frac{1}{\lambda}}}$$

exists and equals $\dfrac{\lambda}{eA^{\frac{1}{\lambda}}}$.

We aim to prove that for any centered Gaussian (N, d)-field X satisfying conditions $\mathfrak{A}_1$, $\mathfrak{A}_0$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$, the following limit exists

$$\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda},$$

where $Z := L_0(X, \mathcal{I}^N)$ and $\lambda := \frac{\mathrm{tr}(\mathbf{H})}{\sum_{k=1}^{N} \alpha_k}$. Clearly this along with the above theorem proves the existence of the following limit with $\lambda := \frac{\mathrm{tr}(\mathbf{H})}{\sum_{k=1}^{N} \alpha_k}$.

$$\lim_{x \to +\infty} x^{-\frac{1}{\lambda}} \log \mathbb{P}\big[L_0\big(X, \mathcal{I}^N\big) > x\big].$$

We have the following proposition on the existence of the local time of zero-mean Gaussian fields at the origin and its moments. In the proof we use some arguments of [9].

Proposition 4.2. *For any Gaussian field X satisfying condition $\mathfrak{A}_1$, the local times $Z_x := L_x(X, \mathcal{I}^N)$ exist for every $x \in \mathbb{R}^d$, and we have*

$$\mathbb{E}\big(Z_x^n\big) \leq \int_{\prod_{i=1}^{n} \mathcal{I}^N} \mathrm{K}_n^X(t_1, \ldots, t_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n : \quad \forall x \in \mathbb{R}^d,$$

and

$$\mathbb{E}\big(Z_0^n\big) = \int_{\prod_{i=1}^{n} \mathcal{I}^N} \mathrm{K}_n^X(t_1, \ldots, t_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

where $\prod_{i=1}^{n} \mathcal{I}^N$ denotes the n-times Cartesian product $\mathcal{I}^N \times \cdots \times \mathcal{I}^N$, and $\mathrm{K}_n^X(t_1, \ldots, t_n)$ is as defined in Equation (1).

Proof. We prove the proposition for $x = 0$. The proof for the general x is similar. For the rest of the proof, we denote Z_0 simply by Z. Let $\| \cdot \|$ be some arbitrary norm on $\mathbb{R}^d$ and define $Z_\varepsilon := \int_{\mathcal{I}^N} \frac{1}{V_\varepsilon} \mathbf{1}_{\{\|X_t\| < \varepsilon\}}(t) \, dt$, where V_ε denotes the volume of the d-dimensional ball $\{x \in \mathbb{R}^d; \|x\| < \varepsilon\}$, and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. First, we show that $\{Z_\varepsilon\}$ is cauchy in $\mathrm{L}^n(\Omega, \mathbb{P})$. Indeed, let $\mathfrak{S}$ be the set of all possible functions from $\{1, \ldots, n\}$ into $\{0, 1\}$. For any function $\sigma \in \mathfrak{S}$, we define ξ_i^σ to be equal to ε if $\sigma(i) = 0$ and be equal to δ if $\sigma(i) = 1$. It is then easy to verify the following equality

$$\mathbb{E}\big[(Z_\varepsilon - Z_\delta)^n\big] = \int_{\prod_{i=1}^{k} I^N} \sum_{\sigma \in \mathfrak{S}} \frac{(-1)^{\sum_{i=1}^{n} \sigma(i)}}{\prod_{i=1}^{n} V_{\xi_i^\sigma}} \mathbb{P}\left(\bigcap_{i=1}^{n} \{\|X_{t_i}\| < \xi_i^\sigma\}\right) dt_1 \cdots dt_n.$$

As the variables $\{X_t\}_t$ are jointly normal with mean zero, for every $\sigma \in \mathfrak{S}$ and each fixed $t_1, \ldots, t_n$, we have

$$\frac{1}{\prod_{i=1}^{n} V_{\xi_i^\sigma}} \mathbb{P}\big(\|X_{t_1}\| < \xi_1^\sigma, \ldots, \|X_{t_n}\| < \xi_n^\sigma\big) \leq \mathrm{K}_n^X(t_1, \ldots, t_n)$$

and

$$\frac{1}{\prod_{i=1}^{n} V_{\xi_i^\sigma}} \mathbb{P}\big(\|X_{t_1}\| < \xi_1^\sigma, \ldots, \|X_{t_n}\| < \xi_n^\sigma\big) \xrightarrow{\varepsilon, \delta \downarrow 0} \mathrm{K}_n^X(t_1, \ldots, t_n).$$

Noting that $\sum_{\sigma \in \mathfrak{S}} (-1)^{\sum_{i=1}^{n} \sigma(i)} = (1-1)^n = 0$, by dominated convergence and Property $\mathfrak{A}_1$ we have

$$\mathbb{E}\big[(Z_\varepsilon - Z_\delta)^n\big] \xrightarrow{\varepsilon, \delta \downarrow 0} 0,$$

which proves that Z_ε is $\mathrm{L}^n(\Omega, \mathbb{P})$-Cauchy whose limit is noting but Z. Similarly one can also show that

$$\mathbb{E}\big(Z^n\big) = \lim_{\varepsilon \to 0} \mathbb{E}\big(Z_\varepsilon^n\big) = \int_{\prod_{i=1}^{n} \mathcal{I}^N} \mathrm{K}_n^X(t_1, \ldots, t_n) \, dt_1 \cdots dt_n. \qquad \square$$

5. Upper bound

In this section, we prove Propositions 2.1, 2.2, Lemma 2.3, and finally Proposition 2.5.

Proof of Proposition 2.1. First, we normalize the vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and H so that for every i we have $\alpha_i \geq 1$. This is possible because $f(x \circ \omega^{\boldsymbol{\alpha}/p}) = \omega^{H/p} f(x)$ for every $p > 0$ as well. This turns the $\boldsymbol{\alpha}$-distance $d(x, y) := \|x - y\|_{\boldsymbol{\alpha}}$ into a translation-invariant metric on $\mathbb{R}^N$.

  (i) We denote the ℓ^2-norm on $\mathbb{R}^N$ by $\| \cdot \|_2$, and take the standard definition for the $\boldsymbol{\alpha}$ and ℓ^2-balls centered at the origin of radius $r > 0$ as $B_{\boldsymbol{\alpha}}(\mathbf{0}, r) := \{x \in \mathbb{R}^N : \|x\|_{\boldsymbol{\alpha}} < r\}$ and $B_2(\mathbf{0}, r) := \{x \in \mathbb{R}^N : \|x\|_2 < r\}$. It is easy to verify that every $\boldsymbol{\alpha}$-ball centered at the origin contains a ℓ^2-ball

centered at the origin, and vice versa. This shows that the $\boldsymbol{\alpha}$-metric induces the same topology on $\mathbb{R}^N$ as the standard topology induced by the ℓ^2-norm. In particular it means that the function f is also continuous under the $\boldsymbol{\alpha}$-metric. So there exists $\varepsilon > 0$ such that for every $\boldsymbol{x}$ we have $|f(\boldsymbol{x})| < 1$ if $\|\boldsymbol{x}\|_{\boldsymbol{\alpha}} < \varepsilon$. Now for any $\boldsymbol{x} \in \mathbb{R}^N$, choose $\omega > 0$ such that $\|\boldsymbol{x} \circ \omega^{\boldsymbol{\alpha}}\|_{\boldsymbol{\alpha}} = \omega \|\boldsymbol{x}\|_{\boldsymbol{\alpha}} = \varepsilon/2$. It follows that $|f(\boldsymbol{x} \circ \omega^{\boldsymbol{\alpha}})| < 1$ which means $|f(\boldsymbol{x})| < \frac{2^H}{\varepsilon^H} \|\boldsymbol{x}\|_{\boldsymbol{\alpha}}^H$.

(ii) As the set $\mathcal{S}_{\boldsymbol{\alpha}} := \{\boldsymbol{x} \in \mathbb{R}^N : \|\boldsymbol{x}\|_{\boldsymbol{\alpha}} = 1\}$ is compact, the image of the continuous function f achieves a minimum over $\mathcal{S}_{\boldsymbol{\alpha}}$ which is strictly positive. In other words, there exists $\delta > 0$ such that for every $\boldsymbol{x} \in \mathcal{S}_{\boldsymbol{\alpha}}$ we have $f(\boldsymbol{x}) \geq \delta$. Now for a general $\boldsymbol{x} \in \mathbb{R}^N$, we can choose $\omega > 0$ such that $\|\boldsymbol{x} \circ \omega^{\boldsymbol{\alpha}}\|_{\boldsymbol{\alpha}} = 1$. So we have $|f(\boldsymbol{x} \circ \omega^{\boldsymbol{\alpha}})| \geq \delta$, hence $|f(\boldsymbol{x})| \geq \delta \|\boldsymbol{x}\|_{\boldsymbol{\alpha}}^H$. $\square$

Proof of Proposition 2.2. We prove it for the case of $n = 1$. The proof for larger n's is similar.

Let the surface $\mathcal{S}_{\boldsymbol{\alpha}}^+ \subset \mathbb{R}^N$ be defined as follows

$$\mathcal{S}_{\boldsymbol{\alpha}}^+ := \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}_+^N : \sum_{i=1}^{N} \sqrt[\alpha_i]{x_i} = 1 \right\}.$$

By diagonal self-similarity (Property $\mathfrak{A}_3$) and noting Remark 1.2, for any $\boldsymbol{t} \in \mathbb{R}^N$ and any $\omega > 0$ we have

$$\mathrm{detCov}(X_{\omega^{\boldsymbol{\alpha}} \circ \boldsymbol{t}}) = \det(\omega^{\mathbf{H}}) \, \mathrm{detCov}(X_{\boldsymbol{t}}) \det(\omega^{\mathbf{H}^{\dagger}}) = \omega^{2 \, \mathrm{tr}(\mathbf{H})} \, \mathrm{detCov}(X_{\boldsymbol{t}}),$$

where we used the fact that the determinant of the exponential of a matrix equals the exponential of its trace, that is, $\det(e^{\mathbf{H}}) = e^{\mathrm{tr}(\mathbf{H})}$ (see, e.g., [4], Chapter 2). Suppose $s \mapsto \boldsymbol{\sigma}_s$ is an arbitrary parametrization of the surface $\mathcal{S}_{\boldsymbol{\alpha}}^+$, where $\boldsymbol{\sigma}_s = (\sigma_1(s), \ldots, \sigma_N(s))$ and $\boldsymbol{s} = (s_1, \ldots, s_{N-1})$. Then using the change of variables $(\omega, \boldsymbol{s})$ with $\boldsymbol{t} = \omega^{\boldsymbol{\alpha}} \circ \boldsymbol{\sigma}_s$, we have

$$\int_{[0,1]^N} \left(K_1(\boldsymbol{t}) \right)^{\beta} d\boldsymbol{t} \geq \int_0^1 \int_{\sigma^{-1}(\mathcal{S}_{\boldsymbol{\alpha}}^+)} \omega^{\sum_{i=1}^{N} \alpha_i - 1} \left(K_1(\omega^{\boldsymbol{\alpha}} \circ \boldsymbol{\sigma}_s) \right)^{\beta} J_{\sigma}(s) \, ds \, d\omega$$

$$= \int_0^1 \frac{\omega^{\sum_{i=1}^{N} \alpha_i - 1}}{\omega^{\beta \, \mathrm{tr}(\mathbf{H})}} \, d\omega \int_{\sigma^{-1}(\mathcal{S}_{\boldsymbol{\alpha}}^+)} \left(K_1(\boldsymbol{\sigma}_s) \right)^{\beta} J_{\sigma}(s) \, ds,$$

where $J_{\sigma}(s)$ is the absolute value of the following determinant

$$\begin{vmatrix} \alpha_1 \sigma_1 & \frac{\partial \sigma_1}{\partial s_1} & \frac{\partial \sigma_1}{\partial s_2} & \cdots & \frac{\partial \sigma_1}{\partial s_{N-1}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_N \sigma_N & \frac{\partial \sigma_N}{\partial s_1} & \frac{\partial \sigma_N}{\partial s_2} & \cdots & \frac{\partial \sigma_N}{\partial s_{N-1}} \end{vmatrix}. \tag{6}$$

This implies that if $(K_1(\boldsymbol{t}))^{\beta}$ is integrable over the cube $[0, 1]^N$, then the following inequality has to hold true

$$\sum_{i=1}^{N} \alpha_i > \beta \, \mathrm{tr}(\mathbf{H}). \qquad \square$$

Now we turn to the proof of Lemma 2.3. First, we need a definition.

Definition 5.1. Consider $n \in \mathbb{N}$, and a sequence of vectors $t_1, \ldots, t_n \in \mathbb{R}^N$, and let d be a metric on $\mathbb{R}^N$. We say that the sequence is 'narrowing' with respect to the metric d, if for any $k = 1, \ldots, n-1$ we have

$$d(t_{k+1}, t_k) = \min_{i=1,\ldots,k} d(t_{k+1}, t_i).$$

Remark 5.1. For any fixed metric d, every finite subset $\mathcal{F} \subset \mathbb{R}^N$ of n points can be arranged in such a way that the ordered sequence is narrowing with respect to the metric d. Indeed, the procedure is simple: start with an arbitrary point in $\mathcal{F}$ and label it t_n. Having chosen $t_n, t_{n-1}, \ldots, t_k$, choose among the remaining points, that is, from $\mathcal{F} \setminus \{t_n, t_{n-1}, \ldots, t_k\}$, the one which is the closest to t_k and label it t_{k-1}.

We will need a theorem from [10] which relates our problem to the nearest neighbor strategy for solving the travelling salesman problem. We proceed with some preliminary definitions from [10].

For a given metric d on $\mathbb{R}^N$, the diameter of a subset $\mathcal{C} \subseteq \mathbb{R}^N$ is defined as

$$D_d(\mathcal{C}) := \sup_{x, y \in \mathcal{C}} d(x, y).$$

A family $\mathcal{P} = \{\mathcal{C}_l\}_{l=1}^P$ of subsets of $\mathbb{R}^N$ is called a covering of subset $\mathcal{A} \subset \mathbb{R}^N$ if $\mathcal{A} \subseteq \bigcup_{l=1}^P \mathcal{C}_l$. The diameter of a covering $\mathcal{P}$, denoted by $D_d(\mathcal{P})$ is defined as the maximum diameter of its elements, that is,

$$D_d(\mathcal{P}) = \max_{l=1,\ldots,P} D_d(\mathcal{C}_l).$$

For any n distinct points in $\mathbb{R}^N$, an arrangement (ordering) $t_1, t_2, \ldots, t_n$ of the points is a 'nearest neighbor tour' if it satisfies the following property

$$d(t_{k+1}, t_k) = \min_{i=k+1,\ldots,n} d(t_k, t_i): \quad \forall k = 1, \ldots, n-1.$$

For every nearest neighbor tour $\mathfrak{T}$ of n points, one defines its (loop) length as

$$\mathbb{L}_d(\mathfrak{T}) := \sum_{k=1}^n d(t_{k+1}, t_k),$$

where $t_{n+1} := \mathbf{0}$. We are interested in finding a general upper bound on this length when all the points of the tour are required to lie inside the unit cube $\mathcal{I}^N$. Indeed, the 'worst case length' of n-point nearest neighbor (NN) tours over a subset $\mathcal{A} \subset \mathbb{R}^N$ is defined as follows

$$\mathbb{L}_d(n; \mathcal{A}) := \sup_{\substack{\mathcal{F} \subset \mathcal{A} \\ |\mathcal{F}|=n}} \max_{\mathfrak{T} \in \text{NN}(\mathcal{F})} \mathbb{L}_d(\mathfrak{T}),$$

where $|\mathcal{F}|$ denotes the cardinality of the set $\mathcal{F}$, and $\mathrm{NN}(\mathcal{F})$ is the set of all possible nearest neighbor tours of the points of $\mathcal{F}$. Then we have the following theorem from [10].

Theorem 5.1. *Let d be a metric on $\mathbb{R}^N$, and $\mathcal{A} \subset \mathbb{R}^N$. Then for any sequence of coverings $\{\mathcal{P}_m\}_{m=1}^M$ of $\mathcal{A}$ with decreasing diameters, that is, $D_d(\mathcal{P}_m) \geq D_d(\mathcal{P}_{m+1})$ for every $m = 1, \ldots, M - 1$. Then the worst case length of n-point nearest neighbor tours is bounded as follows*

$$\mathbb{L}_d(n; \mathcal{A}) \leq n D_d(\mathcal{P}_M) + \sum_{m=2}^{M} |\mathcal{P}_m| \big(D_d(\mathcal{P}_{m-1}) - D_d(\mathcal{P}_m) \big) + |\mathcal{P}_1| \big(D_d(\mathcal{A}) - D_d(\mathcal{P}_1) \big),$$

where $|\mathcal{P}_m|$ denotes the cardinality of $\mathcal{P}_m$.

We should mention that this theorem has been stated in [10] only for the case where the metric d comes from a norm on $\mathbb{R}^N$. Nevertheless, with a careful examination of their proof one can verify that their arguments work even when d is a general metric on $\mathbb{R}^N$.

Now we are ready to prove the following theorem.

Theorem 5.2. *Let $N \in \mathbb{N}^{\geq 2}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in [1, \infty)^N$. Then the worst case length of nearest neighbor tours of n-point subsets of $[0, 1]^N$ with respect to the metric $\| \cdot \|_{\boldsymbol{\alpha}}$ defined in (4), is bounded as follows*

$$\mathbb{L}_{\boldsymbol{\alpha}}\big(n, [0, 1]^N\big) \leq c_{\boldsymbol{\alpha}} n^{1 - \frac{1}{\sum_{i=1}^N \alpha_i}},$$

where $c_{\boldsymbol{\alpha}}$ is a constant that depends only on $\boldsymbol{\alpha}$.

Proof. For every $m \in \mathbb{N}$, we define the set of points $\mathcal{G}_m \subset [0, 1]^N$ as

$$\mathcal{G}_m := \left\{ \left(\frac{\ell_1}{m^{\alpha_1}}, \frac{\ell_2}{m^{\alpha_2}}, \ldots, \frac{\ell_N}{m^{\alpha_N}} \right) \Big| \ell_i \in \{0, 1, \ldots, \lceil m^{\alpha_i} \rceil - 1\} : \forall i = 1, 2, \ldots, N \right\},$$

where $\lceil \cdot \rceil$ denotes the ceiling function that is, $\lceil x \rceil$ is the smallest integer that is larger than or equal to x. For every $\boldsymbol{p} \in \mathcal{G}_m$, we define the sub-cube $\mathcal{C}(\boldsymbol{p}, m^{-\boldsymbol{\alpha}})$ as usual, that is,

$$\mathcal{C}(\boldsymbol{p}, m^{-\boldsymbol{\alpha}}) := \boldsymbol{p} + \prod_{i=1}^N \left[0, \frac{1}{m^{\alpha_i}} \right].$$

So for any positive integer m we define the following covering of the set $[0, 1]^N$

$$\mathcal{P}_m := \big\{ \mathcal{C}(\boldsymbol{p}, m^{-\boldsymbol{\alpha}}) : \boldsymbol{p} \in \mathcal{G}_m \big\}.$$

We note that for every m we have

$$D(\mathcal{P}_m) = \frac{N}{m} \quad \text{and} \quad |\mathcal{P}_m| = \prod_{i=1}^N \lceil m^{\alpha_i} \rceil.$$

So by Theorem 5.1, for every $M \in \mathbb{N}$ we have

$$\mathbb{L}_{\boldsymbol{\alpha}}\big(n, [0, 1]^N\big) \le \frac{nN}{M} + N \sum_{m=2}^M \bigg(\frac{1}{m-1} - \frac{1}{m}\bigg) \prod_{i=1}^N \lceil m^{\alpha_i} \rceil \le \frac{nN}{M} + N 2^{N+1} \sum_{m=2}^M m^{\sum_{i=1}^N \alpha_i - 2}.$$

Due to the assumption, we have $\sum_{i=1}^N \alpha_i \ge 2$, and hence

$$\mathbb{L}_{\boldsymbol{\alpha}}\big(n, [0, 1]^N\big) \le \frac{nN}{M} + N 2^{N+1} \int_1^M m^{\sum_{i=1}^N \alpha_i - 2} \le \frac{nN}{M} + \frac{N 2^{N+1}}{\sum_{i=1}^N \alpha_i - 1} M^{\sum_{i=1}^N \alpha_i - 1}.$$

If we choose $M := n^{\frac{1}{\sum_{i=1}^N \alpha_i}}$ we get the desired bound. □

So we are ready to prove Lemma 2.3.

Proof of Lemma 2.3. Choose $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}_+^N$ and $H > 0$ such that for every $i = 1, \ldots, N$ we have $\alpha_i \ge 1$ and $\alpha_i = H \xi_i$. First, we prove the theorem for the case of $N \ge 2$. We define $\mathcal{N}_{n,\boldsymbol{\alpha}}$ as the set of all nN-tuples $(t_1, t_2, \ldots, t_n) \in \mathcal{I}^N \times \cdots \times \mathcal{I}^N$ such that the sequence $t_1, t_2, \ldots, t_n$ is narrowing with respect to the $\boldsymbol{\alpha}$-distance defined by $\| \cdot \|_{\boldsymbol{\alpha}}$ in Equation (4). Then by Remark 5.1, and the fact that $\mathrm{K}_n^X(t_1, t_2, \ldots, t_n)$ is symmetric with respect to its arguments, that is, it is permutation-invariant, we have

$$\int_{(\mathcal{I}^N)^n} \big(\mathrm{K}_n^X(t_1, \ldots, t_n)\big)^\beta \, dt_1 \cdots dt_n \le n! \int_{\mathcal{N}_{n,\boldsymbol{\alpha}}} \big(\mathrm{K}_n^X(t_1, \ldots, t_n)\big)^\beta \, dt_1 \cdots dt_n. \tag{7}$$

By strong local nondeterminism, for any narrowing sequence $t_1, t_2, \ldots, t_n$ we have

$$\det\mathrm{Cov}[X_{t_k} | X_{t_1}, X_{t_2}, \ldots, X_{t_{k-1}}] \ge C \min\big\{\|t_k - t_{k-1}\|_{\boldsymbol{\alpha}}^{2H}, \|t_k\|_{\boldsymbol{\alpha}}^{2H}\big\}: \quad \forall k = 2, \ldots, n.$$

So by Proposition A.6 we have

$$\mathrm{K}_n^X(t_1, \ldots, t_n) \le c_1^n \prod_{k=1}^n \frac{1}{\min\{\|t_k - t_{k-1}\|_{\boldsymbol{\alpha}}^H, \|t_k\|_{\boldsymbol{\alpha}}^H\}}, \tag{8}$$

where $c_1 := \frac{1}{\sqrt{C(2\pi)^d}}$.

On the other hand, it is easy to verify that if a sequence $t_1, t_2, \ldots, t_n$ is narrowing, then its reversal, that is, $t_n, t_{n-1}, \ldots, t_1$ satisfies the nearest neighbor property. Hence, by Theorem 5.2, for any $(t_1, t_2, \ldots, t_n) \in \mathcal{N}_{n,\boldsymbol{\alpha}}$, we have

$$\sum_{k=2}^n \|t_k - t_{k-1}\|_{\boldsymbol{\alpha}} \le \mathbb{L}_{\boldsymbol{\alpha}}\big(n; \mathcal{I}^N\big) \le c_{\boldsymbol{\alpha}} n^{1 - \frac{1}{\sum_{i=1}^N \alpha_i}},$$

so

$$\sum_{k=1}^n k^{\frac{1}{\sum_{i=1}^N \alpha_i}} \|t_k - t_{k-1}\|_{\boldsymbol{\alpha}} \le c_2 n, \tag{9}$$

where c_2 is a constant that only depends on N, $\boldsymbol{\alpha}$ and d. So (9) and (8) imply

$$\int_{\mathcal{N}_{n,\boldsymbol{\alpha}}} \left(K_n^X(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)\right)^\beta \, \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n$$

$$\leq e^{c_2 n} \int_{\mathcal{N}_{n,\boldsymbol{\alpha}}} c_1^{n\beta} \prod_{k=1}^n \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \min\{\|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}}, \|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}}\})}{\min\{\|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}}^{\beta H}, \|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}}^{\beta H}\}} \, \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n$$

$$\leq c_3^n \int_{\mathbb{R}^{nN}} \prod_{k=1}^n \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \min\{\|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}}, \|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}}\})}{\min\{\|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}}^{\beta H}, \|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}}^{\beta H}\}} \, \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n$$

$$\leq c_3^n \int_{\mathbb{R}^{nN}} \prod_{k=1}^n \left(\frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{t}_k - \boldsymbol{t}_{k-1}\|_{\boldsymbol{\alpha}}^{\beta H}} + \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{t}_k\|_{\boldsymbol{\alpha}}^{\beta H}} \right) \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n,$$

where $c_3 := e^{c_2} c_1^\beta$. Let $\mathfrak{S}$ be the set of all possible functions from $\{1,\ldots,n\}$ into $\{0,1\}$, and for any function $\vartheta \in \mathfrak{S}$ and any $k \in \{1,\ldots,n\}$, define

$$\boldsymbol{y}_k^\vartheta = \begin{cases} \boldsymbol{t}_k, & \text{if } \vartheta(k) = 0, \\ \boldsymbol{t}_k - \boldsymbol{t}_{k-1}, & \text{if } \vartheta(k) = 1. \end{cases}$$

Then the last inequality can be written as follows

$$\int_{\mathcal{N}_{n,\boldsymbol{\alpha}}} \left(K_n^X(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)\right)^\beta \, \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n$$

$$\leq c_3^n \sum_{\vartheta \in \mathfrak{S}} \int_{\mathbb{R}^{nN}} \prod_{k=1}^n \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{y}_k^\vartheta\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{y}_k^\vartheta\|_{\boldsymbol{\alpha}}^{\beta H}} \, \mathrm{d}\boldsymbol{t}_1 \cdots \mathrm{d}\boldsymbol{t}_n$$

$$= c_3^n \sum_{\vartheta \in \mathfrak{S}} \int_{\mathbb{R}^{nN}} \prod_{k=1}^n \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{y}_k^\vartheta\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{y}_k^\vartheta\|_{\boldsymbol{\alpha}}^{\beta H}} \, \mathrm{d}\boldsymbol{y}_1^\vartheta \cdots \mathrm{d}\boldsymbol{y}_n^\vartheta$$

$$= c_3^n \sum_{\vartheta \in \mathfrak{S}} \prod_{k=1}^n \int_{\mathbb{R}^N} \frac{\exp(-k^{\overline{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{y}\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{y}\|_{\boldsymbol{\alpha}}^{\beta H}} \, \mathrm{d}\boldsymbol{y}. \tag{10}$$

Define the surface $\mathcal{S}_{\boldsymbol{\alpha}} \subset \mathbb{R}^N$ as follows

$$\mathcal{S}_{\boldsymbol{\alpha}} := \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \sum_{i=1}^N \sqrt[\alpha_i]{|x_i|} = 1 \right\},$$

and take an arbitrary parametrization $s \mapsto \boldsymbol{\sigma}_s$ of $\mathcal{S}_{\boldsymbol{\alpha}}$, where $\boldsymbol{\sigma}_s = (\sigma_1(s), \ldots, \sigma_N(s))$ and $s = (s_1, \ldots, s_{N-1})$. Then using the change of variables (ω, s) with $\boldsymbol{y} = r^{\boldsymbol{\alpha}} \circ \boldsymbol{\sigma}_s$, we have

$$\int_{\mathbb{R}^N} \frac{\exp(-k^{\frac{1}{\sum_{i=1}^N \alpha_i}} \|\boldsymbol{y}\|_{\boldsymbol{\alpha}})}{\|\boldsymbol{y}\|_{\boldsymbol{\alpha}}^{\beta H}} \, d\boldsymbol{y} = |\mathcal{S}_{\boldsymbol{\alpha}}^{\circ}| \int_0^{+\infty} \frac{\exp(-rk^{\frac{1}{\sum_{i=1}^N \alpha_i}}) r^{\sum_{i=1}^N \alpha_i}}{r^{\beta H + 1}} \, dr$$

$$= |\mathcal{S}_{\boldsymbol{\alpha}}^{\circ}| \frac{k^{\frac{\beta H}{\sum_{i=1}^N \alpha_i}}}{k} \int_0^{+\infty} \frac{e^{-r}}{r} r^{\sum_{i=1}^N \alpha_i - \beta H} \, dr, \qquad (11)$$

where

$$|\mathcal{S}_{\boldsymbol{\alpha}}^{\circ}| := \int_{\boldsymbol{\sigma}^{-1}(\mathcal{S}_{\boldsymbol{\alpha}})} J_{\sigma}(s) \, ds,$$

and J_{σ} is the absolute value of the determinant given in Equation (6). It is important to note that the right-hand side of (11) is finite only if $\sum_{i=1}^N \alpha_i > \beta H$. So by Equations (10) and (11) we get

$$\int_{\mathcal{N}_{n,\boldsymbol{\alpha}}} \left(K_n^X(t_1, \ldots, t_n)\right)^{\beta} \, dt_1 \cdots dt_n \leq c_4^n \frac{(n!)^{\frac{\beta H}{\sum_{i=1}^N \alpha_i}}}{n!}, \qquad (12)$$

where

$$c_4 := 2c_3 |\mathcal{S}_{\boldsymbol{\alpha}}^{\circ}| \int_0^{+\infty} \frac{e^{-r}}{r} r^{\sum_{i=1}^N \alpha_i - \beta H}.$$

So finally, plugging Equation (12) into Equation (7) we get

$$\int_{(\mathcal{I}^N)^n} \left(K_n^X(t_1, \ldots, t_n)\right)^{\beta} \, dt_1 \cdots dt_n \leq c_4^n (n!)^{\frac{\beta H}{\sum_{i=1}^N \alpha_i}}.$$

The proof for the case of $N = 1$ is similar to the above proof, except for the fact that instead of arranging t_i's based on the narrowing property, we order them by the natural ordering on $\mathbb{R}$. The rest of the proof remains basically the same. $\qquad\qquad\square$

Proof of Theorem 2.5. Let us define $\tilde{d} := (m-1)d$, $\tilde{N} := \sum_{i=1}^m N_i$, and the following vector

$$\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}} := \left(X_1(t_1) - X_2(t_2), X_2(t_2) - X_3(t_3), \ldots, X_{m-1}(t_{m-1}) - X_m(t_m)\right) \in \mathbb{R}^{\tilde{d}}.$$

Evidently, $\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}}$ is a centered Gaussian $(\tilde{N}, \tilde{d})$-field.

Take an arbitrary positive integer n, and consider any family of points $\{t_k^i\}_{i,k}$ where $k = 1, \ldots, m$, $i = 1, \ldots, n$, such that for every i and k we have $t_k^i \in \mathcal{I}^{N_k}$. Note that the superscript i in t_k^i is simply an index and should not be confused with exponent. For any such family of points, and for any $i = 1, \ldots, n$, we define the vector $\tilde{\boldsymbol{t}}_i := (t_1^i, \ldots, t_m^i) \in \mathbb{R}^{\tilde{N}}$. So we are interested in

$$\text{detCov}[\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}_1}, \boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}_2}, \ldots, \boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}_n}],$$

where $\tilde{\boldsymbol{t}}_1, \ldots, \tilde{\boldsymbol{t}}_n \in \mathcal{I}^{\tilde{N}}$.

We first note that the determinant of the covariance matrix of a random vector is invariant under permutations of the entries of the random vector. In other words, for any d dimensional random vector $(Y_1, \ldots, Y_d)$, and σ an arbitrary permutation of the set $\{1, 2, \ldots, d\}$, we have

$$\text{detCov}[Y_{\sigma(1)}, Y_{\sigma(2)}, \ldots, Y_{\sigma(d)}] = \text{detCov}[Y_1, Y_2, \ldots, Y_d].$$

This is true because any permutation of a vector is equivalent to multiplying it with a permutation matrix, and the determinant of any permutation matrix is ± 1. Using this fact, it is easy to verify the following equality

$$\text{detCov}[\boldsymbol{\Delta}_{\tilde{\boldsymbol{i}}_1}, \boldsymbol{\Delta}_{\tilde{\boldsymbol{i}}_2}, \ldots, \boldsymbol{\Delta}_{\tilde{\boldsymbol{i}}_n}] = \text{detCov}[A_1 - A_2, A_2 - A_3, \ldots, A_{m-1} - A_m],$$

where the random vectors $A_1, \ldots, A_m$ are defined as follows

$$A_k := \big(X_k(t_k^1), X_k(t_k^2), \ldots, X_k(t_k^n)\big) \in \mathbb{R}^{nd}; \quad \forall k = 1, \ldots, m.$$

Note that for any positive integer m, any sequence of Gaussian random vectors of the same size $A_1, A_2, \ldots, A_m$, and for any $k = 1, \ldots, m$, we have

$$\text{detCov}[A_1, \ldots, A_m] = \text{detCov}\bigg[A_1, \ldots, A_{k-1}, A_k + \sum_{j \neq k} c_j A_j, A_{k+1}, \ldots, A_m\bigg],$$

where $\sum_{j \neq k} c_j A_j$ is any arbitrary linear combination of all the involved vectors excluding A_k. Using this simple fact we have

$$\text{detCov}[A_1 - A_k, \ldots, A_{k-1} - A_k, A_{k+1} - A_k, \ldots, A_m - A_k]$$
$$= \text{detCov}[A_1 - A_2, A_2 - A_3, \ldots, A_{m-1} - A_m]: \quad \forall k = 1, \ldots, m.$$

So using Proposition A.7, for every k we have

$$\text{detCov}[A_1 - A_2, A_2 - A_3, \ldots, A_{m-1} - A_m] \geq \frac{\text{detCov}[A_1, A_2, \ldots, A_m]}{\text{detCov}[A_k]}. \tag{13}$$

Let $p_k := 1 - q_k$ for every $k = 1, \ldots, m$. By (13), and noting that $\sum_{k=1}^m p_k = 1$, we have

$$\text{detCov}[A_1 - A_2, A_2 - A_3, \ldots, A_{m-1} - A_m] \geq \prod_{k=1}^m \left(\frac{\text{detCov}[A_1, A_2, \ldots, A_m]}{\text{detCov}[A_k]}\right)^{p_k}.$$

Using the independence of A_i's, we get

$$\text{detCov}[A_1 - A_2, A_2 - A_3, \ldots, A_{m-1} - A_m] \geq \prod_{k=1}^m \big(\text{detCov}[A_k]\big)^{\sum_{j \neq k} p_j}.$$

As $\sum_{j \neq k} p_j = q_k$, we come to

$$K_n^{\boldsymbol{\Delta}}(\tilde{\boldsymbol{t}}_1, \ldots, \tilde{\boldsymbol{t}}_n) \leq \prod_{i=1}^m \big(K_n^{X_k}(t_k^1, \ldots, t_k^n)\big)^{q_k}. \tag{14}$$

Applying Lemma 2.3, it is clear that $K_n^{\Delta}(\tilde{t}_1, \ldots, \tilde{t}_n)$ is integrable over $\mathcal{I}^{n\tilde{N}}$ and its integral is bounded from above by $c^n (n!)^{\sum_{k=1}^m \frac{q_k}{\tilde{\xi}_k}}$, where c is a positive constant that does not depend on n. $\square$

6. Existence of the limit

In this section, we will prove that for any centered Gaussian (N, d)-field X_t satisfying conditions $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ (with mutually rational α_i's), the following limit exists

$$\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda},$$

where $Z := L_0(X, \mathcal{I}^N)$ and $\lambda := \frac{\mathrm{tr}(\mathbf{H})}{\sum_{k=1}^N \alpha_k}$, as in the last section. In the sequel, we prove Theorem 2.6 and Corollary 2.1.

We use the standard notation $\lfloor \cdot \rfloor$ for the floor function, i.e., for every $x \in \mathbb{R}$, the expression $\lfloor x \rfloor$ is the largest integer that is smaller than or equal to x.

Lemma 6.1. *For any $\omega \in \mathbb{R}_+$, there exist strictly positive constants r_1 and κ that only depend on N, α_i's and $\mathbf{H}$, such that for every $r > r_1$ we have*

$$\mathbb{E}(Z^{M(r+1)}) \geq \frac{\kappa^M}{r^{M(N+1)}} \left(\frac{M}{\omega^{\sum_{k=1}^N \alpha_k}} \right)^{rM} \omega^{rM\,\mathrm{tr}(\mathbf{H})} \big(\mathbb{E}(Z^r)\big)^M,$$

where $M := \prod_{i=1}^N \lfloor \omega^{\alpha_i} \rfloor$.

Proof. *Step 1*: Let $\mu := M(r + 1)$. Using Proposition 4.2, we have the following probabilistic representation

$$\mathbb{E}(Z^{M(r+1)}) = \mathbb{E}^{\tau}(K_\mu(\tau_1, \ldots, \tau_\mu)), \tag{15}$$

where $\{\tau_i\}_{i=1}^\mu$ is a family independent identically distributed (i.i.d.) random variables, each τ_i being distributed over $[0, 1]^N$, and $\mathbb{E}^{\tau}$ denotes expectation with respect to the family of random variables $\{\tau_i\}_{i=1}^\mu$.

We define the set of points $\mathcal{P} \subset [0, 1]^N$ as

$$\mathcal{P} := \left\{ \left(\frac{i_1}{\omega^{\alpha_1}}, \frac{i_2}{\omega^{\alpha_2}}, \ldots, \frac{i_N}{\omega^{\alpha_N}} \right) \Big| i_k \in \{0, 1, \ldots, \lfloor \omega^{\alpha_k} \rfloor - 1\} : \forall k = 1, 2, \ldots, N \right\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function, as usual. For every $p \in \mathcal{P}$, we define the sub-cube $\mathcal{C}^\circ(p, \omega^{-\alpha})$ as

$$\mathcal{C}^\circ(p, \omega^{-\alpha}) := p + \prod_{k=1}^N \left(0, \frac{1}{\omega^{\alpha_k}} \right).$$

We define the event Ξ as the event on the family $\{\boldsymbol{\tau}_i\}_{i=1}^{M(r+1)}$ such that every sub-cube $\mathcal{C}^\circ(\boldsymbol{p}, \omega^{-\alpha})$, $\boldsymbol{p} \in \mathcal{P}$, contains exactly $r + 1$ points of the set $\{\boldsymbol{\tau}_i\}_{i=1}^{M(r+1)}$. We also define Ω_M^μ as the set of all functions $\theta : \{1, 2, \ldots, M(r + 1)\} \to \mathcal{P}$ such that at every point $\boldsymbol{p} \in \mathcal{P}$, the cardinality of the inverse image of θ equals $r + 1$, that is, $|\theta^{-1}(\boldsymbol{p})| = r + 1$; in other words, every member $\theta \in \Omega_M^\mu$ is a partitioning of the set $\{1, 2, \ldots, M(r + 1)\}$ into M distinct boxes indexed by $\mathcal{P}$ such that every box contains exactly $r + 1$ elements of $\{1, 2, \ldots, M(r + 1)\}$. For every $\theta \in \Omega_M^\mu$, let Ξ_θ be the event that the points of $\{\boldsymbol{\tau}_i\}_{i=1}^{M(r+1)}$ are distributed among the sub-cubes $\{\mathcal{C}^\circ(\boldsymbol{p}, \omega^{-\alpha})\}_{\boldsymbol{p} \in \mathcal{P}}$ according to θ. It is clear that $\Xi = \bigcup_{\theta \in \Omega_M^\mu} \Xi_\theta$. So by Equation (15), we have

$$\mathbb{E}\big(Z^{M(r+1)}\big) \geq \mathbb{E}^\tau\big[K_\mu(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_\mu)\mathbf{1}_\Xi\big] = \sum_{\theta \in \Omega_M^\mu} \mathbb{E}^\tau\big[K_\mu(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_\mu)\mathbf{1}_{\Xi_\theta}\big].$$

For any $\theta \in \Omega_M^\mu$ and $\boldsymbol{p} \in \mathcal{P}$, we use the following notation

$$K_{r+1}(\tau, \theta, \boldsymbol{p}) := K_{r+1}(\boldsymbol{\tau}_{i_1}, \ldots, \boldsymbol{\tau}_{i_{r+1}}),$$

where $i_1, \ldots, i_{r+1}$ denote all the distinct elements of $\theta^{-1}(\boldsymbol{p})$. Using Proposition A.5, for every $\theta \in \Omega_M^\mu$ we have the following inequality

$$K_{M(r+1)}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{M(r+1)}) \geq \prod_{\boldsymbol{p} \in \mathcal{P}} K_{r+1}(\tau, \theta, \boldsymbol{p}).$$

Hence, we obtain

$$\mathbb{E}\big(Z^{M(r+1)}\big) \geq \sum_{\theta \in \Omega_M^\mu} \mathbb{E}^\tau\left[\prod_{\boldsymbol{p} \in \mathcal{P}} K_{r+1}(\tau, \theta, \boldsymbol{p})\mathbf{1}_{\Xi_\theta}\right].$$

For any $\theta \in \Omega_M^\mu$ and $\boldsymbol{p} \in \mathcal{P}$, let $\Xi_\theta^{\boldsymbol{p}}$ denote the event that the points $\boldsymbol{\tau}_{i_1}, \ldots, \boldsymbol{\tau}_{i_{r+1}}$ lie in the sub-cube $\mathcal{C}^\circ(\boldsymbol{p}, \omega^{-\alpha})$, where $\{i_1, \ldots, i_{r+1}\} := \theta^{-1}(\boldsymbol{p})$. It is evident that

$$\mathbf{1}_{\Xi_\theta} = \prod_{\boldsymbol{p} \in \mathcal{P}} \mathbf{1}_{\Xi_\theta^{\boldsymbol{p}}}.$$

So we get

$$\mathbb{E}\big(Z^{M(r+1)}\big) \geq \sum_{\theta \in \Omega_M^\mu} \prod_{\boldsymbol{p} \in \mathcal{P}} \mathbb{E}^\tau\big[K_{r+1}(\tau, \theta, \boldsymbol{p})\mathbf{1}_{\Xi_\theta^{\boldsymbol{p}}}\big]. \tag{16}$$

Step 2: For any θ and $\boldsymbol{p}$ fixed, let $\{i_1, \ldots, i_{r+1}\} := \theta^{-1}(\boldsymbol{p})$, that is, $\{i_1, \ldots, i_{r+1}\}$ is the set of indices such that $\boldsymbol{\tau}_{i_1}, \ldots, \boldsymbol{\tau}_{i_{r+1}} \in \mathcal{C}^\circ(\boldsymbol{p}, \omega^{-\alpha})$. Using Proposition A.7, we have

$$\mathrm{detCov}(X_{\boldsymbol{\tau}_{i_1}}, \ldots, X_{\boldsymbol{\tau}_{i_{r+1}}})$$
$$\leq \mathrm{detCov}(X_{\boldsymbol{\tau}_{i_{r+1}}})\,\mathrm{detCov}(X_{\boldsymbol{\tau}_{i_1}} - X_{\boldsymbol{\tau}_{i_{r+1}}}, \ldots, X_{\boldsymbol{\tau}_{i_r}} - X_{\boldsymbol{\tau}_{i_{r+1}}}).$$

By Property $\mathfrak{A}_0$ and using Corollaries A.2 and A.5, we have

$$\det\mathrm{Cov}(X_{\tau_{i_{r+1}}}) \leq c_0^d.$$

Using the fact that X has stationary increments, that is, Property $\mathfrak{A}_2$, we have

$$\det\mathrm{Cov}(X_{\tau_{i_1}} - X_{\tau_{i_{r+1}}}, \dots, X_{\tau_{i_r}} - X_{\tau_{i_{r+1}}}) = \det\mathrm{Cov}(X_{\tau_{i_1} - \tau_{i_{r+1}}}, \dots, X_{\tau_{i_r} - \tau_{i_{r+1}}}).$$

So we have

$$
\begin{aligned}
\mathrm{K}_{r+1}(\tau, \theta, \boldsymbol{p}) &= \mathrm{K}_{r+1}(\tau_{i_1}, \dots, \tau_{i_{r+1}}) \\
&\geq (2\pi)^{-\frac{d(r+1)}{2}} c_0^{-\frac{d}{2}} \left(\det\mathrm{Cov}(X_{\tau_{i_1} - \tau_{i_{r+1}}}, \dots, X_{\tau_{i_r} - \tau_{i_{r+1}}}) \right)^{-\frac{1}{2}} \\
&= c_1 \mathrm{K}_r(\tau_{i_1} - \tau_{i_{r+1}}, \dots, \tau_{i_r} - \tau_{i_{r+1}}),
\end{aligned}
$$

where $c_2 := (2\pi c_0)^{-\frac{d}{2}}$. Therefore, we have

$$
\begin{aligned}
\mathbb{E}^{\tau}\big[\mathrm{K}_{r+1}(\tau, \theta, \boldsymbol{p})\mathbf{1}_{\Xi_\theta^p}\big] & \\
&\geq c_1 \int_{t_1, \dots, t_{r+1} \in \mathcal{C}^\circ(\boldsymbol{p}, \omega^{-\boldsymbol{\alpha}})} \mathrm{K}_r(t_1 - t_{r+1}, \dots, t_r - t_{r+1}) \, dt_1 \cdots dt_{r+1} \\
&= c_1 \int_{z \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\boldsymbol{\alpha}})} \int_{t_1, \dots, t_r \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\boldsymbol{\alpha}})} \mathrm{K}_r(t_1 - z, \dots, t_r - z) \, dt_1 \cdots dt_r \, dz, \qquad (17)
\end{aligned}
$$

where we used change of variables in the last line.

For every $z = (z_1, \dots, z_N) \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\boldsymbol{\alpha}})$, we define

$$\zeta_z := \min_k \left\{ \left(\omega^{-\alpha_k} - z_k\right)^{\frac{1}{\alpha_k}} \right\}$$

and

$$\tilde{\boldsymbol{\zeta}}_z := \zeta_z^{\boldsymbol{\alpha}} = \left(\zeta_z^{\alpha_1}, \dots, \zeta_z^{\alpha_N}\right).$$

For every such z, we introduce the new variables $\{s_k\}_{k=1,\dots,r}$ in the following way

$$\tilde{\boldsymbol{\zeta}}_z \circ s_k := t_k - z: \quad \forall k = 1, \dots, r,$$

where $\circ$ as usual, denotes the Schur product of two vectors, that is, the vector formed by entry-wise multiplication of the two vectors. It can be easily verified that for every such z we have

$$
\begin{aligned}
\int_{t_1, \dots, t_r \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\boldsymbol{\alpha}})} &\mathrm{K}_r(t_1 - z, \dots, t_r - z) \, dt_1 \cdots dt_r \\
&\geq \zeta_z^{r(\alpha_1 + \dots + \alpha_N)} \int_{s_1, \dots, s_r \in (0,1)^N} \mathrm{K}_r(\tilde{\boldsymbol{\zeta}}_z \circ s_1, \dots, \tilde{\boldsymbol{\zeta}}_z \circ s_r) \, ds_1 \cdots ds_r. \qquad (18)
\end{aligned}
$$

On the other hand, by diagonal self-similarity (Property $\mathfrak{A}_3$) and noting Remark 1.2, we have

$$\text{Cov}(X_{\tilde{\xi}_z \circ s_1}, \ldots, X_{\tilde{\xi}_z \circ s_r}) = \Lambda_r \, \text{Cov}(X_{s_1}, \ldots, X_{s_r}) \Lambda_r^{\dagger},$$

where $\Lambda_r \in \mathbb{R}^{rN \times rN}$ is the block diagonal matrix consisting of r copies of the matrix $\zeta_z^{\mathbf{H}}$ on its main diagonal, and filled with zero elsewhere, in other words $\Lambda_r = \text{diag}[\zeta_z^{\mathbf{H}}, \ldots, \zeta_z^{\mathbf{H}}]$. Now we notice the fact that the determinant of the exponential of a matrix equals the exponential of its trace, i.e., $\det(e^{\mathbf{A}}) = e^{\text{tr}(\mathbf{A})}$; see, for example, [4], Chapter 2. So we get

$$\det \text{Cov}(X_{\tilde{\xi}_z \circ s_1}, \ldots, X_{\tilde{\xi}_z \circ s_r}) = \zeta_z^{2r \, \text{tr}(\mathbf{H})},$$

which implies that

$$\mathrm{K}_r(\tilde{\xi}_z \circ s_1, \ldots, \tilde{\xi}_z \circ s_r) = \zeta_z^{-r \, \text{tr}(\mathbf{H})} \mathrm{K}_r(s_1, \ldots, s_r). \tag{19}$$

So, by Equations (17), (18), and (19) we have

$$\mathbb{E}^{\tau}\big[\mathrm{K}_{r+1}(\tau, \theta, p)\mathbf{1}_{\Xi_\theta^p}\big] \geq c_1 \mathbb{E}(Z^r) \int_{z \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\alpha})} \zeta_z^{r(\sum_{k=1}^N \alpha_k - \text{tr}(\mathbf{H}))} \, dz, \tag{20}$$

where we used the following equality which is a result of Proposition 4.2

$$\mathbb{E}(Z^r) = \int_{s_1, \ldots, s_r \in (0,1)^N} \mathrm{K}_r(s_1, \ldots, s_r) \, ds_1 \cdots ds_r.$$

Now we define $\{x_k\}_{i=1}^N$ as $x_k := 1 - \omega^{\alpha_k} z_k$. By this change of variables, we have

$$\zeta_z = \frac{1}{\omega} \min_k \big\{(x_k)^{\frac{1}{\alpha_k}}\big\}.$$

So we have

$$\int_{z \in \mathcal{C}^\circ(\mathbf{0}, \omega^{-\alpha})} \zeta_z^{-r(\sum_{k=1}^N \alpha_k - \text{tr}(\mathbf{H}))} \, dz = \omega^{-\sum_{k=1}^N \alpha_k} \omega^{-r(\sum_{k=1}^N \alpha_k - \text{tr}(\mathbf{H}))} J(\alpha, \mathbf{H}, r), \tag{21}$$

where

$$J(\alpha, \mathbf{H}, r) := \int_0^1 \cdots \int_0^1 \Big(\min_k\big\{(x_k)^{\frac{1}{\alpha_k}}\big\}\Big)^{r(\sum_{k=1}^N \alpha_k - \text{tr}(\mathbf{H}))} \, dx_1 \cdots dx_N.$$

This shows that if $\sum_{k=1}^N \alpha_k$ is smaller than $\text{tr}(\mathbf{H})$, then Property $\mathfrak{A}_1$ can not hold. So we assume $\sum_{k=1}^N \alpha_k \geq \text{tr}(\mathbf{H})$.

Denote $\eta := r\left(\sum_{k=1}^{N} \alpha_k - \mathrm{tr}(\mathbf{H})\right)$ and $\alpha_0 := \min\{\alpha_1, \ldots, \alpha_N\}$. One can easily verify that for r larger than $r_1 := \frac{2\alpha_0}{\sum_{k=1}^{N} \alpha_k - \mathrm{tr}(\mathbf{H})}$ (so that $\eta > 2\alpha_0$), we have

$$\int_0^1 \cdots \int_0^1 \left(\min_k\{(x_k)^{\frac{1}{\alpha_k}}\}\right)^{\eta} \mathrm{d}x_1 \cdots \mathrm{d}x_N \geq \int_0^1 \cdots \int_0^1 \left(\min_k\{x_k\}\right)^{\frac{\eta}{\alpha_0}} \mathrm{d}x_1 \cdots \mathrm{d}x_N$$

$$\geq \int_{1-\frac{\alpha_0}{\eta}}^1 \cdots \int_{1-\frac{\alpha_0}{\eta}}^1 \left(\min_k\{x_k\}\right)^{\frac{\eta}{\alpha_0}} \mathrm{d}x_1 \cdots \mathrm{d}x_N$$

$$\geq \left(\frac{\alpha_0}{\eta}\right)^N \left(1 - \frac{\alpha_0}{\eta}\right)^{\frac{\eta}{\alpha_0}} \geq C \left(\frac{\alpha_0}{\eta}\right)^N,$$

where $C > 0$ is global contact. So for r large enough ($r \geq r_1$), we have

$$J(\boldsymbol{\alpha}, \mathbf{H}, r) \geq \frac{c_2}{r^N}, \tag{22}$$

where $c_2 > 0$ is a constant that only depends on α_i's and N. When $\sum_{k=1}^{N} \alpha_k = \mathrm{tr}(\mathbf{H})$, Equation (22) remains valid for every $r \in \mathbb{N}$. So in this case we define r_1 equal to 1.

So by applying Equations (22) and (21) into Equation (20) we get

$$\mathbb{E}^{\boldsymbol{\tau}}\left[\mathrm{K}_{r+1}(\tau, \theta, \boldsymbol{p})\mathbf{1}_{\Xi_{\theta}^p}\right] \geq \frac{c_1 c_2}{r^N \omega^{\sum_{k=1}^{N} \alpha_k}} \omega^{-r\left(\sum_{k=1}^{N} \alpha_k - \mathrm{tr}(\mathbf{H})\right)} \mathbb{E}(Z^r). \tag{23}$$

Step 3: Applying Equation (23) to (16), we get

$$\mathbb{E}(Z^{M(r+1)}) \geq \frac{c_1^M c_2^M \mathfrak{N}_M^{M(r+1)}}{r^{MN} \omega^{M\sum_{k=1}^{N} \alpha_k}} \omega^{-rM\left(\sum_{k=1}^{N} \alpha_k - \mathrm{tr}(\mathbf{H})\right)} \left(\mathbb{E}(Z^r)\right)^M, \tag{24}$$

where $\mathfrak{N}_M^{M(r+1)} := |\Omega_M^{M(r+1)}|$, that is, the cardinality of $\Omega_M^{M(r+1)}$. Using Lemma A.8, we have

$$\mathfrak{N}_M^{M(r+1)} \geq \frac{\kappa_1 \sqrt{M}}{\kappa_2^M \sqrt{(r+1)^M}} M^{M(r+1)},$$

where κ_1 and κ_2 are global constants. So we have

$$\mathbb{E}(Z^{M(r+1)}) \geq \frac{\kappa_1 c_1^M c_2^M \sqrt{M}}{\kappa_2^M r^{MN} \sqrt{(r+1)^M}} \left(\frac{M}{\omega^{\sum_{k=1}^{N} \alpha_k}}\right)^{M(r+1)} \omega^{rM\,\mathrm{tr}(\mathbf{H})} \left(\mathbb{E}(Z^r)\right)^M,$$

which clearly implies the statement of the lemma. $\qquad\square$

Lemma 6.2. *There exists a positive number r_1 that only depends on N, α_i's and $\mathbf{H}$, such that for every $r > r_1$ and any $\omega \in \mathbb{R}_+$ with the property that ω^{α_i}'s are all integer numbers, and for any positive integer q, we have*

$$\frac{(\mathbb{E}(Z^{rM^q(1+o_r)}))^{\frac{1}{rM^q(1+o_r)}}}{(rM^q(1+o_r))^{\lambda}} \geq B_{\omega}(r) \left(\frac{\sqrt[r]{\mathbb{E}(Z^r)}}{r^{\lambda}}\right)^{\frac{1}{1+o_r}}, \tag{25}$$

where $M := \prod_{i=1}^{N} \lfloor \omega^{\alpha_i} \rfloor$, $o_r := \frac{1}{r} \sum_{k=0}^{q-1} \frac{1}{M^k}$, and $B_\omega(r)$ *is a strictly-positive-valued function* $(B_\omega(r) > 0)$ *that depends only on* ω, r, N *and* **H** *such that* $\lim_{r \to +\infty} B_\omega(r) = 1$.

Proof. By Lemma 6.1, we have

$$\mathbb{E}\big(Z^{M(r+1)}\big) \geq \frac{\kappa^M}{r^{M(N+1)}} \big(\rho \omega^{\mathrm{tr}(\mathbf{H})}\big)^{rM} \big(\mathbb{E}(Z^r)\big)^M,$$

where $M = \prod_{i=1}^{N} \lfloor \omega^{\alpha_i} \rfloor$, and $\rho := \frac{M}{\omega^{\sum_{k=1}^{N} \alpha_k}}$. Reiterating this inequality q times, and using the inequality $M^k r + \sum_{i=1}^{k} M^i \leq M^k(r+2)$, we get

$$\mathbb{E}\big(Z^{rM^q + \sum_{i=1}^{q} M^i}\big) \geq A_{\omega,r,q} \big(\rho \omega^{\mathrm{tr}(\mathbf{H})}\big)^{(qrM^q + \sum_{i=1}^{q-1} iM^{i+1})} \big(\mathbb{E}(Z^r)\big)^{M^q},$$

where

$$A_{\omega,r,q} := \frac{\kappa^{\sum_{i=1}^{q} M^i}}{M^{(N+1)\sum_{i=1}^{q-1} iM^{q-i}} (r+2)^{(N+1)\sum_{i=1}^{q} M^i}}.$$

For $\lambda := \frac{\mathrm{tr}(\mathbf{H})}{\sum_{i=1}^{N} \alpha_i}$, and using the notation $o_r := \frac{1}{r} \sum_{i=0}^{q-1} \frac{1}{M^i}$, we have

$$\frac{(\mathbb{E}(Z^{rM^q + \sum_{i=1}^{q} M^i}))^{\frac{1}{rM^q + \sum_{i=1}^{q} M^i}}}{(rM^q + \sum_{i=1}^{q} M^i)^\lambda} \geq B'_{\omega,r,q} \left(\frac{\sqrt[r]{\mathbb{E}(Z^r)}}{r^\lambda}\right)^{\frac{1}{1+o_r}}, \tag{26}$$

where

$$B'_{\omega,r,q} := (A_{\omega,r,q})^{\frac{1}{rM^q(1+o_r)}} \big(\rho \omega^{\mathrm{tr}(\mathbf{H})}\big)^{\frac{qrM^q + \sum_{i=1}^{q-1} iM^{i+1}}{rM^q + \sum_{i=1}^{q} M^i}} M^{-q\lambda} r^{-\lambda \frac{o_r}{1+o_r}} (1 + o_r)^{-\lambda}.$$

Using the inequalities $\sum_{i=0}^{+\infty} x^i = \frac{1}{(1-x)}$ and $\sum_{i=1}^{+\infty} i x^{i-1} = \frac{1}{(1-x)^2}$, and noting that $M \geq 2$, we can easily verify that as r goes to $+\infty$, the function o_r converges to zero uniformly in q and ω, and

$$M^{q-1} \leq \sum_{i=1}^{q-1} i M^{q-i} \leq 4M^{q-1} \quad \text{and} \quad M^q \leq \sum_{i=1}^{q} i M^i \leq 2M^q. \tag{27}$$

Using these inequalities, we can easily show that

$$(A_{\omega,r,q})^{\frac{1}{rM^q(1+o_r)}} \geq A_r, \tag{28}$$

where $A_r > 0$ is only a function of r and N such that $\lim_{r \to +\infty} A_r = 1$. It is also easy to verify that

$$q \sum_{i=1}^{q} M^i - \sum_{i=1}^{q-1} i M^{i+1} = M^q \sum_{i=1}^{q} \frac{i}{M^{i-1}},$$

and hence

$$q - \frac{qrM^q + \sum_{i=1}^{q-1} iM^{i+1}}{rM^q + \sum_{i=1}^{q} M^i} = \frac{\sum_{i=1}^{q} \frac{i}{M^{i-1}}}{r(1+o_r)} \leq \frac{1}{r(1+o_r)(1-\frac{1}{M})^2}. \tag{29}$$

Noting that under the assumptions of the lemma, ρ is equal to 1, and using Equations (28) and (29), we obtain

$$B'_{\omega,r,q} \geq B_\omega(r),$$

where $B_\omega(r)$ is a strictly-positive-valued function that only depends on N, r, ω, and **H** such that $\lim_{r \to +\infty} B_\omega(r) = 1$. This completes the proof. $\qquad\square$

Lemma 6.3. *Let X_t be an (N,d)-Gaussian random field satisfying Properties $\mathfrak{A}_1$, $\mathfrak{A}_0$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ with self-similarity vector $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_N)$ such that for every i and j, the quotient α_i/α_j is a rational number. Then the following limit exists*

$$\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda}, \tag{30}$$

where $Z := L_{\mathbf{0}}(X, [0,1]^N)$ and $\lambda := \frac{\operatorname{tr}(\mathbf{H})}{\sum_{k=1}^{N} \alpha_k}$.

Proof. Define

$$\overline{\ell} := \limsup_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda} \quad \text{and} \quad \underline{\ell} := \liminf_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda}.$$

Consider any positive real number ℓ that is strictly less than $\overline{\ell}$. Let ℓ_1 and ℓ_2 be real numbers satisfying $\ell < \ell_1 < \ell_2 < \overline{\ell}$.

Step 1: As for every i and j, the quotient α_i/α_j is a rational number, we can find a real number $\alpha > 0$ such that for every i, the quotient $\frac{\alpha_i}{\alpha}$ is an integer. Now choose ω_1 and ω_2 such that $\omega_1^\alpha = 2$ and $\omega_2^\alpha = 3$. Clearly, in this case all $\omega_j^{\alpha_i}$'s are integer-valued for every $j = 1, 2$ and $i = 1, \ldots, N$, hence we may apply Lemma 6.2. Also note that in this case, there exists a positive integer m_0 such that $M_1 := \prod_{i=1}^{N} \lfloor \omega_1^{\alpha_i} \rfloor = 2^{m_0}$ and $M_2 := \prod_{i=1}^{N} \lfloor \omega_2^{\alpha_i} \rfloor = 3^{m_0}$. Let r be any integer larger than r_1, and p and q be two arbitrary positive integers. Applying Equation (25) first with ω_1 and p, and then repeating it with ω_2 and q, we get

$$\frac{(\mathbb{E}(Z^{\Phi_{r,p,q}}))^{\frac{1}{\Phi_{r,p,q}}}}{(\Phi_{r,p,q})^\lambda} \geq B_{\omega_2}(R)\big(B_{\omega_1}(r)\big)^{\frac{1}{1+o_R}} \left(\frac{\sqrt[r]{\mathbb{E}(Z^r)}}{r^\lambda} \right)^{\frac{1}{(1+o_r)(1+\bar{o}_R)}}, \tag{31}$$

where $R := rM_1^p + \sum_{k=1}^{p} M_1^k$, $o_r := \frac{1}{r} \sum_{k=0}^{p-1} \frac{1}{M_1^k}$, $\bar{o}_R := \frac{1}{R} \sum_{k=0}^{q-1} \frac{1}{M_2^k}$, and

$$\Phi_{r,p,q} := RM_2^q + \sum_{k=1}^{q} M_2^k = r2^{m_0 p} 3^{m_0 q}(1+o_R)(1+o_r).$$

We note that B_{ω_1} and B_{ω_2} converge to one uniformly in p and q, and o_r and o_R converge to zero uniformly in p and q.

Step 2: Choose $\varepsilon > 0$ such that $(1 + \varepsilon) < \frac{\ell_1}{\ell}$. Clearly, there exists $r_2 > 0$ such that for every $R, r \geq r_2$ we have

$$(1 + o_r)(1 + \bar{o}_R) < 1 + \varepsilon \quad \text{and} \quad B_{\omega_2}(R)\left(B_{\omega_1}(r)\right)^{\frac{1}{1+o_R}} \ell_2^{\frac{1}{(1+o_r)(1+o_R)}} > \ell_1.$$

By the definition of lim sup, there exists an integer $r > \max\{r_1, r_2\}$ such that $\frac{\sqrt[r]{\mathbb{E}(Z^r)}}{r^\lambda} > \ell_2$. Now we apply this r to Equation (31), along with any arbitrary integers p and q. Noting that $R = rM_1^p + \sum_{k=1}^p M_2^k > r > r_2$, we have

$$\frac{(\mathbb{E}(Z^{\Phi_{r,p,q}}))^{\frac{1}{\Phi_{r,p,q}}}}{(\Phi_{r,p,q})^\lambda} \geq \ell_1; \quad \forall p, q \in \mathbb{N}. \tag{32}$$

We also have

$$r2^{m_0 p} 3^{m_0 q} \leq \Phi_{r,p,q} = r2^{m_0 p} 3^{m_0 q}(1 + o_R)(1 + o_r) \leq r2^{m_0 p} 3^{m_0 q}(1 + \varepsilon). \tag{33}$$

As $\log_2 3$ is not a rational number, by Dirichlet's approximation theorem (Theorem A.9) there exist $p_0, q_0 \in \mathbb{N}$ such that

$$0 < |p_0 - q_0 \log_2 3| < \frac{1}{m_0} \log_2\left(\frac{\ell_1}{\ell(1 + \varepsilon)}\right). \tag{34}$$

We proceed with the assumption that $p_0 > q_0 \log_2 3$; when $p_0 < q_0 \log_2 3$, the proof is similar. So by Equation (34) we have

$$1 < v := \frac{2^{m_0 p_0}}{3^{m_0 q_0}} < \frac{\ell_1}{\ell(1 + \varepsilon)}. \tag{35}$$

We choose $k_0 \in \mathbb{N}$ such that $v^{k_0} > 3$, and define $n_1 := r(1 + \varepsilon)3^{m_0 q_0 k_0}$. Take any arbitrary integer $n \geq n_1$, and define the following

$$q_1 := \max\{k; n \geq r(1 + \varepsilon)3^{m_0 k}\} \quad \text{and} \quad k_1 := \max\{k; n \geq r(1 + \varepsilon)3^{m_0 q_1} v^k\}.$$

As we have

$$3^{m_0 q_1} v^{k_1} = 3^{m_0(q_1 - k_1 q_0)} 2^{m_0 k_1 q_0},$$

hence

$$r(1 + \varepsilon)3^{m_0(q_1 - k_1 q_0)} 2^{m_0 k_1 q_0} \leq n < vr(1 + \varepsilon)3^{m_0(q_1 - k_1 q_0)} 2^{m_0 k_1 q_0}. \tag{36}$$

We note that $q = q_1 - k_1 q_0 \geq 0$, because

(I) as $n > n_1$, by definition $q_1 \geq q_0 k_0$, and
(II) $k_1 \leq k_0$, otherwise if $k_0 < k_1$, then $v^{k_1} > 3$, and hence $n > r(1 + \varepsilon)3^{m_0(q_1 + 1)}$ which is in contradiction with the definition of q_1.

So we can apply Equation (32) to $q = q_1 - k_1 q_0$ and $p = k_1 q_0$. By Equations (33) and (36), we get

$$\Phi_{r,p,q} \leq n \leq \nu(1 + \varepsilon)\Phi_{r,p,q}. \tag{37}$$

Step 3: As $\Phi_{r,p,q} \leq n$, by Hölder's inequality we have

$$\left(\mathbb{E}\left(Z^{\Phi_{r,p,q}}\right)\right)^{\frac{1}{\Phi_{r,p,q}}} \leq \left(\mathbb{E}\left(Z^n\right)\right)^{\frac{1}{n}}.$$

Hence, by Equation (37) we have

$$\left(\nu(1 + \varepsilon)\right)^{-\lambda} \frac{\left(\mathbb{E}(Z^{\Phi_{r,p,q}})\right)^{\frac{1}{\Phi_{r,p,q}}}}{(\Phi_{r,p,q})^{\lambda}} \leq \frac{\left(\mathbb{E}(Z^n)\right)^{\frac{1}{n}}}{n^{\lambda}}. \tag{38}$$

But by (35), we have

$$\frac{\ell}{\ell_1} \leq \left(\nu(1 + \varepsilon)\right)^{-1} \leq \left(\nu(1 + \varepsilon)\right)^{-\lambda}.$$

So by Equations (38) and (32) we finally get

$$\frac{\left(\mathbb{E}(Z^n)\right)^{\frac{1}{n}}}{n^{\lambda}} \geq \ell.$$

This means that $\underline{\ell}$ is larger than or equal to ℓ. As this is true for any positive number ℓ that is strictly less than $\overline{\ell}$, this implies $\overline{\ell} = \underline{\ell}$; in other words the limit in (30) exists. $\qquad\square$

Proof of Theorem 2.6. Lemma 6.3 guarantees the convergence of $\{\frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^{\lambda}}\}_n$, so Theorem 4.1 can be applied if we show that the limit is strictly positive. This can indeed be easily verified applying Lemma 6.2 with some arbitrary $r > r_1$ and $\omega \in \mathbb{R}_+$ such that ω^{α_i}'s are all integers, and then letting q converge to $+\infty$. Clearly the left-hand side of Equation (25) converges to $\lim_{n \to +\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^{\lambda}}$ whereas the right-hand side is strictly positive and independent of q. $\qquad\square$

Now we can easily prove Corollary 2.1.

Proof of Corollary 2.1. We define the Gaussian field $\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}}$

$$\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}} := \left(X_1(\boldsymbol{t_1}) - X_2(\boldsymbol{t_2}), X_2(\boldsymbol{t_2}) - X_3(\boldsymbol{t_3}), \ldots, X_{m-1}(\boldsymbol{t_{m-1}}) - X_m(\boldsymbol{t_m})\right),$$

where $\tilde{\boldsymbol{t}} := (\tilde{\boldsymbol{t}}_1, \ldots, \tilde{\boldsymbol{t}}_n)$ and $\tilde{\boldsymbol{t}}_1 \in \mathcal{I}^{\tilde{N}_k}$ for every $k = 1, \ldots, m$.

It is evident that $\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}}$ is a a centered Gaussian $(\tilde{N}, \tilde{d})$-field, where $\tilde{d} := (m - 1)d$ and $\tilde{N} := \sum_{k=1}^{m} N_k$.

The proof of the existence of the local time of $\boldsymbol{\Delta}_{\tilde{\boldsymbol{t}}}$ around $\boldsymbol{0}$ over the cube $\mathcal{I}^{\tilde{N}}$ and the finiteness of all its moments, is similar to the proof of Theorem 2.5. Indeed, using Equation (14) with

$q_i = \frac{m-1}{m}$ $(p_i = \frac{1}{m})$ for every $i = 1, \ldots, m$, we obtain

$$\mathrm{K}_n^{\boldsymbol{\Delta}}(\tilde{\boldsymbol{t}}_1, \ldots, \tilde{\boldsymbol{t}}_n) \leq \prod_{k=1}^{m} \big(\mathrm{K}_n^{X_k}(\boldsymbol{t}_k^1, \ldots, \boldsymbol{t}_k^n)\big)^{\frac{m-1}{m}}.$$

So the Gaussian field $\boldsymbol{\Delta}_{\tilde{t}}$ satisfies Property $\mathfrak{A}_1$.

As all the random fields X_k, $k = 1, \ldots, m$, satisfy Properties $\mathfrak{A}_0$ and $\mathfrak{A}_2$, so does the Gaussian field $\boldsymbol{\Delta}_{\tilde{t}}$. As every X_k is diagonally self-similar (i.e., it satisfies Property $\mathfrak{A}_3$) with scaling vector $\boldsymbol{\alpha}_k := (\alpha_{k,1}, \ldots, \alpha_{k,N_k}) \in \mathbb{R}_+^{N_k}$ and scaling matrix $\mathbf{H} \in \mathbb{R}^{d \times d}$, it can be easily verified that $\boldsymbol{\Delta}_{\tilde{t}}$ is also diagonally self-similar with the scaling vector $\tilde{\boldsymbol{\alpha}} \in \mathbb{R}_+^{\tilde{N}}$ constructed by adjoining all the vectors $\boldsymbol{\alpha}_k$ together, that is, $\tilde{\boldsymbol{\alpha}} := (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m)$ and with the scaling matrix $\tilde{\mathbf{H}} \in \mathbb{R}^{(m-1)d \times (m-1)d}$ which is a block diagonal matrix containing $m-1$ copies of $\mathbf{H}$ on its main diagonal and zero elsewhere; in other words, $\tilde{\mathbf{H}} := \mathrm{diag}(\mathbf{H}, \mathbf{H}, \ldots, \mathbf{H})$. Clearly in this case we have $\mathrm{tr}(\tilde{\mathbf{H}}) = (m-1)\,\mathrm{tr}\,\mathbf{H}$. Now the desired conclusion is evident applying Theorem 2.6. $\qquad\square$

Appendix

Lemma A.4. *Let $\mathcal{H}$ be a Gaussian Hilbert space, that is, for any $n \in \mathbb{Z}^+$, and any elements $X_1, \ldots, X_n \in \mathcal{H}$, the set $\{X_i\}_{i=1}^n$ is a family of jointly Gaussian zero-mean random variables, and $\mathcal{H}$ forms a Hilbert space with respect to the inner product $\langle X, Y \rangle := \mathbb{E}(XY)$. Let $\mathcal{G}$ be a subspace of $\mathcal{H}$, and X be an element of $\mathcal{H}$. Then we have*

$$\mathrm{var}(X|\mathcal{G}) = \big\| Q_{\mathcal{G}}(X) \big\|^2,$$

where $Q_{\mathcal{G}}(X) := X - P_{\mathcal{G}}(X)$, and $P_{\mathcal{G}}(X)$ is the orthogonal projection of X over the subspace $\mathcal{G}$.

Proof. By definition, we have

$$\mathrm{var}(X|\mathcal{G}) = \mathbb{E}\big[\big(X - \mathbb{E}(X|\mathcal{G})\big)^2 \big| \mathcal{G}\big].$$

Replacing X by $P_{\mathcal{G}}(X) + Q_{\mathcal{G}}(X)$ on the right-hand side of the above equation, and noting that $Q_{\mathcal{G}}(X)$ is independent of $\mathcal{G}$, we can easily derive the desired result. $\qquad\square$

Corollary A.1. *An immediate implication of the previous lemma is the following inequality*

$$\mathrm{var}(X|\mathcal{G}) \leq \mathrm{var}(X).$$

Lemma A.5. *Let $\mathcal{Y}$ be an arbitrary inner-product space. Then for any $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n \in \mathcal{Y}$, $n \in \mathbb{N}$, we have*

$$\det \begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_n \end{bmatrix} = \|\boldsymbol{y}_1\|^2 \prod_{k=2}^{n} \big\| Q_{\langle \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k-1} \rangle}(\boldsymbol{y}_k) \big\|^2,$$

where $\langle \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k-1} \rangle$ is the subspace generated by $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k-1}\}$.

Proof. We assume that $\{y_i\}_{i=1}^n$ are linearly independent, because otherwise, the equality is trivially true. By orthogonal decomposition of each y_k over the subspace $\langle y_1, \ldots, y_{k-1} \rangle$, we can obtain the sequences $\{f_i\}_{i=1}^n$, $\{\eta_i\}_{i=1}^n$, and $\{p_i\}_{i=1}^n$ such that for $\forall k = 1, \ldots, n$, we have $y_k = \eta_k f_k + p_k$ where $\eta_k \in \mathbb{R}_+$, $p_k \in \langle y_1, \ldots, y_{k-1} \rangle$, $f_k \perp \langle y_1, \ldots, y_{k-1} \rangle$, and $\|f_i\| = 1$. In fact for each $k = 2, \ldots, n$, $p_k = P_{\langle y_1, \ldots, y_{k-1} \rangle}(y_k)$ and $\eta_k f_k = Q_{\langle y_1, \ldots, y_{k-1} \rangle}(y_k)$. Let $f : \mathbb{R}^n \to \mathcal{Y}$ be the linear isometry such that $f(e_i) = f_i$ for every $i = 1, \ldots, n$, where $\{e_i\}_{i=1}^n$ is the standard basis for the Euclidean space $\mathbb{R}^n$, that is, e_i is the *column vector* that is 1 in the i-th entry and 0 elsewhere. For each $i = 1, \ldots, n$, define $x_i \in \mathbb{R}^n$ as the inverse image of y_i, that is, $f(x_i) = y_i$. As f is an isometry, we have

$$\det \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \det \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \left(\det \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \right)^2.$$

Again due to the fact that f is an isometry, for every k we have $x_k \in \langle e_1, \ldots, e_k \rangle$, i.e., the matrix $[x_1 \cdots x_n]$ is upper triangular with η_i's on its diagonal. So we have

$$\det \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \prod_{i=1}^n \eta_i.$$

But $\eta_1 = \|y_1\|$, and $\eta_i = \|Q_{\langle y_1, \ldots, y_{i-1} \rangle}(y_i)\|$ for every $i = 2, \ldots, n$. So the proof is complete. $\square$

Corollary A.2. *Suppose* $\{X_i\}_{i=1}^n$ *is a family of jointly Gaussian random variables. Using Lemma* A.4*, we obtain the following formula*

$$\det \mathrm{Cov}(X_1, \ldots, X_n) = \mathrm{var}(X_1) \prod_{k=2}^n \mathrm{var}(X_k | X_1, \ldots, X_{k-1}).$$

Using Lemma A.5, we can easily verify the following two propositions.

Proposition A.6. *Suppose that* $\{Y_i^j ; i = 1, \ldots, n, j = 1, \ldots, m_i\}$ *is family of jointly Gaussian random variables, where* $m_i \in \mathbb{N}$ *for every* $i = 1, \ldots, n$*. Also, for each* $i = 1, \ldots, n$*, let* $Y_i := (Y_i^1, \ldots, Y_i^{m_i})$*. Then we have*

$$\det\mathrm{Cov}(Y_1, \ldots, Y_n) = \det\mathrm{Cov}(Y_1) \prod_{k=2}^n \det\mathrm{Cov}(Y_k | Y_1, \ldots, Y_{k-1}) \leq \prod_{i=1}^n \det\mathrm{Cov}(Y_i),$$

where $\det\mathrm{Cov}(Y_k | Y_1, \ldots, Y_{k-1})$ *is the determinant of the conditional covariance matrix of* Y_k *conditioned on the random vectors* $Y_1, \ldots, Y_{k-1}$*.*

Proposition A.7. *Let $m \in \mathbb{N}$, and consider any family of n jointly Gaussian random vectors of size m, that is, $\mathbf{Y}_i := (Y_i^1, \ldots, Y_i^m)$ for every i. Then we have*

$$\mathrm{detCov}(\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$$
$$\leq \mathrm{detCov}(\mathbf{Y}_k) \, \mathrm{detCov}(\mathbf{Y}_1 - \mathbf{Y}_k, \ldots, \mathbf{Y}_{k-1} - \mathbf{Y}_k, \mathbf{Y}_{k+1} - \mathbf{Y}_k, \ldots, \mathbf{Y}_n - \mathbf{Y}_k).$$

We have the following lemma which can be proved by elementary probability and then Stirling's approximation, that is, the fact that $\kappa_1 \leq \frac{n!}{(\frac{n}{e})^n \sqrt{n}} \leq \kappa_2$, where κ_1 and κ_2 are strictly positive global constants.

Lemma A.8. *Let $n, m \in \mathbb{N}$, and suppose that we have nm distinct balls and n distinct baskets. Let $\mathfrak{N}_n^{nm}$ be the number of different ways one can distribute the balls among the baskets such that each basket contains exactly m balls. In other words, $\mathfrak{N}_n^{nm}$ is the cardinality of Ω_n^{mn} where Ω_n^{mn} is the set of all functions $\sigma : \{1, 2, \ldots, mn\} \to \{1, 2, \ldots, n\}$ such that for every $p \in \{1, 2, \ldots, n\}$, the cardinality of the inverse image of σ equals $m + 1$, that is, $|\sigma^{-1}(\boldsymbol{p})| = m$. We have*

$$\mathfrak{N}_n^{nm} = \frac{(nm)!}{(m!)^n} \geq \frac{\kappa_1 \sqrt{n}}{\kappa_2^n \sqrt{m^n}} n^{nm},$$

where κ_1 and κ_2 are strictly positive global constants.

The following theorem, also known as Dirichlet's theorem on Diophantine approximation, is a direct application of Pigeonhole Principle which itself was first used by Dirichlet [2]. For completeness, we provide the proof.

Theorem A.9 (Dirichlet's approximation theorem). *For any real number α and any positive integer n, there exist integers p and q such that $1 \leq q \leq n$ and $|p - q\alpha| \leq \frac{1}{n}$.*

Proof. Consider the numbers $\alpha - \lfloor \alpha \rfloor, 2\alpha - \lfloor 2\alpha \rfloor, \ldots, n\alpha - \lfloor n\alpha \rfloor$, and the intervals $[\frac{i}{n}, \frac{i+1}{n})$, for $i = 0, \ldots, n - 1$. Either one of the numbers falls into the first interval $[0, \frac{1}{n})$, or otherwise there will an interval that contains more than one point. In either case, we can find the desired p and q. $\qquad\square$

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