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# Deviation inequalities for random polytopes in arbitrary convex bodies

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We prove an exponential deviation inequality for the convex hull of a finite sample of i.i.d. random points with a density supported on an arbitrary convex body in $\mathbb{R}^d$, $d \geq 2$. When the density is uniform, our result yields rate optimal upper bounds for all the moments of the missing volume of the convex hull, uniformly over all convex bodies of $\mathbb{R}^d$: We make no restrictions on their volume, location in the space or smoothness of their boundary. For general densities, the only restriction we make is that the density is bounded from above, even though we believe this restriction is not necessary. However, the density can have any decay to zero near the boundary of its support. After extending an identity due to Efron, we also prove upper bounds for the moments of the number of vertices of the random polytope. Surprisingly, these bounds do not depend on the underlying density and we prove that the growth rates that we obtain are tight in a certain sense. Our results are non asymptotic and hold uniformly over all convex bodies.

*Keywords:* convex body; convex hull; covering number; density support estimation; deviation inequality; random polytope

## 1. Introduction

Probabilistic properties of random polytopes have been studied extensively in the literature in the last fifty years. Consider a collection of i.i.d. uniform random points in a convex body $K$ in $\mathbb{R}^d$. Their convex hull is a random polytope whose volume and number of vertices have been first analyzed in the seminal work of Rényi and Sulanke [19,20]. They derived the asymptotics of the expected volume in the case $d = 2$, when $K$ is either a polygon, with a given number of vertices, or a convex set with smooth boundary. More recently, considerable efforts were devoted to understanding the behavior of the expected volume. Several particular cases were investigated: For instance, when $K$ is a $d$-dimensional simple polytope[1] [1], a $d$-dimensional polytope [2] or a $d$-dimensional Euclidean ball [6]. Groemer [9] (see also the references therein) proved that if $K$ has volume one, the expected volume of the random polytope is minimum when $K$ is an ellipsoid. Bárány and Larman [3] showed that if $K$ has volume one, then one minus the expected volume has the same asymptotic behavior as the volume of the $(1/n)$-wet part of $K$, defined as the union of all caps of $K$ (a cap being the intersection of $K$ with a half space) of volume at most $1/n$. Here, $n$ is the number of uniform random points in $K$. This remarkable result reduces the initial probabilistic problem to computation of such a deterministic volume. This purely analytical problem was then extensively studied. When $K$ has a smooth

[1] A $d$-dimensional simple polytope is a convex polytope such that each of its vertices is adjacent to exactly $d$ edges.

boundary, a key point was the introduction of the affine surface area, see [22,23], and the volume of the $(1/n)$-wet part is of the order $n^{-2/(d+1)}$. When $K$ is a polytope, it is of a much smaller order, namely, $(\ln n)^{d-1}/n$ [3]. The expected volume is actually maximal when $K$ is a simple polytope [3]. As a conclusion, the expectation of the volume is now very well-understood, when the underlying distribution is uniform. Much less is known about its higher moments and the tails of its distribution. Asymptotic results such as central limit theorems are available [16,17,24], with recent developments in the high dimensional setup (see [10,26] and the references therein), but in this work, we focus on finite sample bounds.

Using a jackknife inequality for symmetric functions, Reitzner [18] proved that if the boundary of $K$ is smooth, the variance of the volume is bounded from above by $n^{-(d+3)/(d+1)}$, and he conjectured that this is the right order of magnitude for the variance. In addition, he proved that the second moment of the missing volume (i.e., the volume of $K$ minus the volume of the random polytope) is asymptotically of the order $n^{-4/(d+1)}$, with explicit constants that depend on the affine surface area of $K$. Using martingale inequalities, Vu [27] obtained deviation inequalities for arbitrary convex bodies of volume one, involving quantities such as the volume of the wet part, and derived precise deviation inequalities in the two important cases when $K$ is a polytope or has a smooth boundary. However, these inequalities involve constants which depend on $K$ but are not explicit. As a consequence, upper bounds on the moments of the missing volume are proved, again with non explicit constants depending on $K$: Let $K$ have volume one and $V_n$ stand for the missing volume, then there exist positive constants $\alpha$, $c$ and $\epsilon_0$ such that

$$\mathbb{P}\big[\big|V_n - \mathbb{E}[V_n]\big| \geq \sqrt{\lambda v}\big] \leq 2e^{-\lambda/4} + e^{-c\epsilon n},$$
$$\forall \epsilon \in \big(\alpha(\ln n)/n, \epsilon_0\big], \lambda \in \big(0, n\big|K(\epsilon)\big|\big], \tag{1}$$

where $v = 36ng(\epsilon)^2|K(\epsilon)|$, $g(\epsilon) = \sup\{|F| : F \text{ star-shaped} \subseteq K(\epsilon)\}$, and $K(\epsilon)$ is the $\epsilon$-wet part of $K$ defined in [3]. Moreover, if $K$ has a smooth boundary and volume one, there exist positive constants $c$ and $\alpha$, which depend on $K$, such that for any $\lambda \in (0, (\alpha/4)n^{-\frac{(d-1)(d+3)}{(d+1)(3d+5)}}]$, the following holds [27]:

$$\mathbb{P}\Big[\big|V_n - \mathbb{E}[V_n]\big| \geq \sqrt{\alpha\lambda n^{-\frac{d+3}{d+1}}}\Big] \leq 2\exp(-\lambda/4) + \exp\big(-cn^{\frac{d-1}{3d+5}}\big). \tag{2}$$

This inequality yields upper bounds on the variance and on the $q$-th moment of the missing volume, respectively of orders $n^{-(d+3)/(d+1)}$ and $n^{-2q/(d+1)}$, for $q > 0$, for a *smooth convex body $K$ of volume one*, up to constant factors that depend on $K$ in an unknown way. Note that these two inequalities proved by Vu remain true when $K$ has any positive volume, if $|V_n - \mathbb{E}[V_n]|$ is replaced by $|V_n - \mathbb{E}[V_n]|/|K|$, where $|K|$ is the volume of $K$. In our paper, we do not assume that the underlying distribution is uniform on $K$. We prove deviation inequalities and moment inequalities for a weighted missing volume, for general densities supported on the convex body $K$. In the uniform case, our results yield a deviation inequality which, unlike (1) and (2), which hold for a very small range of $\lambda$, captures the whole tail of the distribution of $V_n$. Our inequality is uniform over all convex bodies $K$, no matter their volume and boundary structure, and our constants do not depend on $K$. Our approach is based on a very simple covering number argument and is not restricted to the uniform distribution, which, to the best of our knowledge, makes our deviation inequalities completely new.

In addition, we derive moment inequalities for the number of vertices of the random polytope. In the uniform case, we prove that the rates in our upper bounds are tight, uniformly on all convex bodies. As a consequence, we also prove that the growth of the moments of the number of vertices is the highest when the underlying density is uniform.

## 2. Notation and statement of the problem

Let $d \geq 2$ be an integer. We denote by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^d$, $\rho$ the Euclidean distance in $\mathbb{R}^d$, $B_d$ the unit Euclidean ball with center 0, and $\kappa_d$ its volume.

If $G \subseteq \mathbb{R}^d$ and $\epsilon > 0$, we denote by $G^\epsilon = \{x \in \mathbb{R}^d : \rho(x, G) \leq \epsilon\}$ the closed $\epsilon$-neighborhood of $G$. Here, $\rho(x, G) = \inf_{y \in G} \rho(x, y)$. If $G$ is measurable, we denote by $|G|$ its volume.

The symmetric difference between two sets $G_1$ and $G_2$ is denoted by $G_1 \triangle G_2$ and their Hausdorff distance is denoted and defined as $d_H(G_1, G_2) = \inf\{\epsilon > 0 : G_1 \subseteq G_2^\epsilon, G_2 \subseteq G_1^\epsilon\}$. We denote by $\mathcal{K}_d$ the class of all convex bodies in $\mathbb{R}^d$, that is, the convex and compact sets with nonempty interior, and by $\mathcal{K}_d^1$ the collection of all those included in $B_d$. The convex hull of $n$ i.i.d. random points $X_1, \ldots, X_n$ is denoted by $\hat{K}_n$. If $X_1, \ldots, X_n$ have a density $f$ with respect to the Lebesgue measure in $\mathbb{R}^d$, we denote by $\mathbb{P}_f$ their joint probability measure and by $\mathbb{E}_f$ the corresponding expectation operator (we omit the dependency in $n$ unless stated otherwise). If $f$ is the uniform density on a convex body $K$, we rather use the notation $\mathbb{P}_K$ and $\mathbb{E}_K$. In general, when the density $f$ of $X_1$ is supported on a convex body $K$, we denote by $d_f(K, \hat{K}_n) = \int_{K \setminus \hat{K}_n} f(x)\,\mathrm{d}x$ and by $V_n$ the missing volume of $\hat{K}_n$, that is, $V_n = |K| - |\hat{K}_n|$. The integral $d_f(K, \hat{K}_n)$ can be interpreted as the weighted missing volume. We are interested in deviation inequalities for $Z$, where $Z$ is either $d_f(K, \hat{K}_n)$ or $V_n$, that is, in bounding from above $\mathbb{P}_K[Z > \epsilon]$, for $\epsilon > 0$. We are also interested in upper bounds for the moments $\mathbb{E}_K[Z^q]$, $q > 0$. Our main result is stated in Section 3: We prove a deviation inequality for the weighted missing volume, and we investigate a special class of densities, satisfying the so called *margin condition*, for which we are also able to control the unweighted missing volume. In Section 4, we investigate the moments of the number of vertices of the random polytope, with no restrictions on $K$ and on the underlying density on $K$, as long as it is bounded from above. Finally, in Section 5, we focus on the uniform case, and we derive a deviation inequality for the missing volume, and prove that the rates of the subsequent moment inequalities are tight. Last section is devoted to some proofs.

## 3. Deviation inequality for the weighted missing volume of random polytopes

Our main result is the following theorem.

**Theorem 1.** *Let $n \geq 1$. Let $K \in \mathcal{K}_d^1$, $f$ be a density supported on $K$ and $X_1, \ldots, X_n$ be i.i.d. random points with density $f$. Assume that $f \leq M$ almost everywhere, for some positive number $M$.*

*Then, there exist positive constants $C_1$ and $C_2$ that depend on $d$ only, such that the following holds.*

$$\mathbb{P}_f\big[n\big(d_f(K,\hat{K}_n)-C_1(M+1)n^{-2/(d+1)}\big)>x\big]\le C_2e^{-x},\quad \forall x\ge 0.$$

**Proof.** This proof is inspired by Theorem 1 in [12], which derives an upper bound for the risk of a convex hull type estimator of a convex function. It is based on an upper bound of the covering number of $\mathcal{K}_d^1$, proven by [4]. For $\delta>0$, a $\delta$-net of $\mathcal{K}_d^1$ for the Hausdorff distance is a finite subset $\mathcal{N}_\delta$ of $\mathcal{K}_d^1$ such that for all $G\in\mathcal{K}_d^1$, there exists $G^*\in\mathcal{N}_\delta$ with $d_H(G,G^*)\le\delta$. The covering number of $\mathcal{K}_d^1$ for the Hausdorff distance is the function that maps $\delta>0$ to the mimimum cardinality of a $\delta$-net of $\mathcal{K}_d^1$ for the Hausdorff distance. The following lemma is proven in [4].

**Lemma 1.** *The covering number of $\mathcal{K}_d^1$ for the Hausdorff distance is not larger than $c_1e^{\delta^{-(d-1)/2}}$, for all $\delta>0$, where $c_1>0$ depends on $d$ only.*

Our next lemma shows that the Nikodym distance (i.e., the volume of the symmetric difference) between two sets in $\mathcal{K}_d^1$ is dominated by their Hausdorff distance. The proof is deferred to the Appendix.

**Lemma 2.** *There exists a positive constant $\alpha_1$ which depends on $d$ only, such that*

$$\big|G\triangle G'\big|\le\alpha_1 d_H\big(G,G'\big),\quad \forall G,G'\in\mathcal{K}_d^1.$$

Let $\delta=n^{-2/(d+1)}$ and $\{K_1,\dots,K_N\}$ be a $\delta$-net of $\mathcal{K}_d^1$, where $N$ is a positive integer satisfying $N\le c_1e^{\delta^{-(d-1)/2}}$, cf. Lemma 1. Let $J\le N$ be such that $d_H(\hat{K}_n,K_J)\le\delta$. By Lemma 2, this implies that $|K_J\setminus\hat{K}_n|\le|\hat{K}_n\triangle K_J|\le\alpha_1\delta$ and hence, since $f$ is nonnegative,

$$d_f(K,\hat{K}_n)\le\int_{K\setminus K_J}f+\int_{K_J\setminus\hat{K}_n}f\le\int_{K\setminus K_J}f+\alpha_1M\delta.$$

In addition, since $d_H(\hat{K}_n,K_J)\le\delta$, it is true that $\hat{K}_n\subseteq K_J^\delta$, yielding $X_i\in K_J^\delta$, for all $i=1,\dots,n$. Therefore, for all $\varepsilon\in(0,1)$,

$$\begin{aligned}\mathbb{P}_f\big[d_f(K,\hat{K}_n)>\varepsilon\big]&\le\mathbb{P}_f\big[d_f(K,K_J)>\varepsilon-\alpha_1M\delta\big]\\&\le\mathbb{P}_f\big[\exists j\in\{1,\dots,N\},d_f(K,K_j)>\varepsilon-\alpha_1M\delta\\&\quad\text{ and }X_i\in K_j^\delta,\forall i=1,\dots,n\big]\\&\le\sum_{j\in I_{\varepsilon-\alpha_1M\delta}}\mathbb{P}_f\big[X_1\in K_j^\delta\big]^n\\&=\sum_{j\in I_{\varepsilon-\alpha_1M\delta}}\Big(\int_{K_j^\delta}f\Big)^n,\end{aligned}\tag{3}$$

where we used the union bound, and for $\eta \in \mathbb{R}$, we denoted by $I_\eta = \{j \in \{1, \ldots, N\} : d_f(K, K_j) > \eta\}$. Note that

$$\int_{K_j^\delta} f = \int_{K_j} f + \int_{K_j^\delta \setminus K_j} f \le 1 - d_f(K, K_j) + \left|K_j^\delta \setminus K_j\right| M. \tag{4}$$

By Lemma 2, the last term is bounded from above by $\alpha_1 M \delta$ and (4) entails, if $j \in I_{\varepsilon - \alpha_1 M \delta}$,

$$\int_{K_j^\delta} f \le 1 - \varepsilon + 2\alpha_1 M \delta.$$

Hence, (3) becomes

$$\begin{aligned}
\mathbb{P}_f\left[d_f(K, \hat{K}_n) > \varepsilon\right] &\le N(1 - \varepsilon + 2\alpha_1 M \delta)^n \\
&\le c_1 \exp\left(-n(\varepsilon - 2\alpha_1 M \delta) + \delta^{-(d-1)/2}\right) \\
&= c_1 \exp\left(-n\left(\varepsilon - (2\alpha_1 M + 1)\delta\right)\right).
\end{aligned} \tag{5}$$

Note that since $d_f(K, \hat{K}_n) \le 1$ almost surely, (5) actually holds for all $\varepsilon > 0$ (we have assumed $\varepsilon \in (0, 1)$ so far). This ends the proof by taking $\varepsilon$ of the form $\frac{x}{n} + (2\alpha_1 M + 1)\delta$. □

As a consequence of Theorem 1, we get upper bounds for all the moments of $d_f(K, \hat{K}_n)$.

**Corollary 1.** *Let the assumptions of Theorem 1 hold. Then, for all $q > 0$, there exists $A_q > 0$ that depends on $q$ and $d$ only such that*

$$\mathbb{E}_f\left[d_f(K, \hat{K}_n)^q\right] \le A_q (M + 1)^q n^{-2q/(d+1)}.$$

**Proof.** The proof is based on an application of Fubini's theorem. Namely, if $Z$ is a nonnegative random variable and $q > 0$, then

$$\mathbb{E}\left[Z^q\right] = q \int_0^\infty t^{q-1} \mathbb{P}[Z > t]\,\mathrm{d}t.$$

Let $Z = d_f(K, \hat{K}_n)$ and let $\delta = C_1(M + 1)n^{-2/(d+1)}$. Then,

$$\begin{aligned}
\mathbb{E}_f\left[Z^q\right] &= q \int_0^\infty t^{q-1} \mathbb{P}[Z > t]\,\mathrm{d}t \\
&= q \int_0^\delta t^{q-1} \mathbb{P}[Z > t]\,\mathrm{d}t + q \int_\delta^\infty t^{q-1} \mathbb{P}[Z > t]\,\mathrm{d}t \\
&\le \delta^q + q \int_0^\infty (t + \delta)^{q-1} \mathbb{P}[Z > t + \delta]\,\mathrm{d}t \\
&= \delta^q + \frac{q}{n} \int_0^\infty \left(\frac{x}{n} + \delta\right)^{q-1} \mathbb{P}[Z > x/n + \delta]\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le \delta^q + \frac{C_2 q}{n}\int_0^\infty \left(\frac{x}{n}+\delta\right)^{q-1} e^{-x}\,\mathrm{d}x \\
&\le \delta^q + \frac{C_2 q \max(1,2^{q-2})}{n}\int_0^\infty 2^{q-2}\left(\frac{x^{q-1}}{n^{q-1}}+\delta^{q-1}\right)e^{-x}\,\mathrm{d}x \\
&\le a_q\delta^q,
\end{aligned}
$$

for some constant $a_q$ that depends on $d$ and $q$ only. □

The inequalities that we have obtained for $d_f(K,\hat{K}_n)$ yield guarantees for the missing volume $|K\setminus\hat{K}_n|$ under some conditions on $f$. An important such condition is called the *margin condition* (see [15,25]). The density $f$ satisfies the margin condition with parameters $\alpha\in(0,\infty]$, $L, t_0>0$ if and only if

$$
\left|\left\{x\in K : f(x)\le t\right\}\right| \le Lt^\alpha,
$$

for all $t\in(0,t_0]$. For example, a margin condition with $\alpha=\infty$ is satisfied by any density $f$ that is almost everywhere bounded away from zero on $K$. Let us give two other important cases where a margin condition is satisfied.

*Slow decay of $f$ near the boundary of $K$:* Assume that $f$ does not decay too fast near the boundary of $K$ (which we denote by $\partial K$). Namely, assume the existence of positive numbers $\rho_0$, $c$ and $\gamma$ such that for all $x\in K$,

$$
f(x)\ge c\min\bigl(\rho_0,\rho(x,\partial K)\bigr)^\gamma. \tag{6}
$$

Then, $f$ satisfies the margin condition with $t_0=c\rho_0^\gamma$, $L=\frac{\tau}{c^{1/\gamma}}$ and $\alpha=1/\gamma$, where $\tau$ is any number that is no smaller than the surface area of $K$ (e.g., take $\tau$ to be the surface area of the unit ball, if $K\in\mathcal{K}_d^1$).

*Projection of higher dimensional convex bodies:* Let $D>d$ be an integer and $K_0\in\mathcal{K}_D^1$. Let $Y_1,\dots,Y_n$ be i.i.d. uniform random points in $K_0$. Identify $\mathbb{R}^d$ with a linear subspace of $\mathbb{R}^D$. Let $K$ be the orthogonal projection of $K_0$ onto $\mathbb{R}^d$ and let $X_i$ be the orthogonal projection of $Y_i$ onto $\mathbb{R}^d$, for $i=1,\dots,n$. Assume that $K_0$ satisfies the $r$-rolling ball condition, where $r>0$: Namely, assume that for all $x\in\partial K_0$, there exists $a\in K_0$ with $x\in B_D(a,r)\subseteq K_0$. Then, we have the following lemma, whose proof is deferred to the Appendix.

**Lemma 3.** *The density $f$ of the $X_i$'s satisfies (6), with $\rho_0=r$, $c=r^{(D-d)/2}\kappa_D\kappa_{D-d}$ and $\gamma=(D-d)/2$.*

Hence, as we already saw in the previous example, $f$ satisfies the margin condition with $\alpha=2/(D-d)$.

The following lemma gives a (deterministic) control of $d_f(K,\hat{K}_n)$ on the missing volume. For completeness of the presentation, we provide its proof in the Appendix (see also Proposition 1 in [25]).

**Lemma 4.** *Let $f$ satisfy the margin condition with parameters $\alpha$, $L$, $t_0$ and let $K'$ be a convex set included in $K$ with $d_f(K, K') \le t_0^{\alpha+1}$. Then,*

$$\left|K \setminus K'\right| \le (L+1)d_f\left(K, K'\right)^{\alpha/(\alpha+1)}.$$

**Proof.** For all $t \in (0, t_0]$,

$$\begin{aligned}
\left|K \setminus K'\right| &= \int_K \mathbb{1}_{x \notin K'}\,\mathrm{d}x \\
&= \int_{K_f(t)} \mathbb{1}_{x \notin K'}\,\mathrm{d}x + \int_{K \setminus K_f(t)} \mathbb{1}_{x \notin K'}\,\mathrm{d}x \\
&\le \left|K_f(t)\right| + \frac{1}{t}\int_{K \setminus K_f(t)} f(x)\mathbb{1}_{x \notin K'}\,\mathrm{d}x \\
&\le Lt^\alpha + \frac{1}{t} d_f\left(K, K'\right) \qquad (7)
\end{aligned}$$

where $K_f(t) = \{x \in K : f(x) \le t\}$. If $d_f(K, K') \le t_0^{\alpha+1}$, take $t = d_f(K, K')^{\alpha+1}$ in (7). $\square$

If $f$ satisfies a margin condition, the deviation and moment inequalities that we have for $d_f(K, \hat{K}_n)$ transfer to the missing volume, as shown in the next two results.

**Theorem 2.** *Let $f$ satisfy the margin condition with parameters $\alpha$, $L$ and $t_0$ and assume that $f \le M$ for some positive number $M$. Then, there exist a positive integer $n_0$ that depends on $d$, $t_0$ and $M$ and positive constants $C_3$ and $C_4$ that depend on $d$, $\alpha$ and $L$ such that, for all $n \ge n_0$ and for all $x \ge 0$,*

$$\mathbb{P}_f\left[n^{\frac{\alpha}{\alpha+1}}\left(|K \setminus \hat{K}_n| - C_3(M+1)^{\alpha/(\alpha+1)} n^{-\frac{2\alpha}{(\alpha+1)(d+1)}}\right) > x\right] \le C_4 e^{-x^{(\alpha+1)/\alpha}} + C_4 e^{-nt_0/2}.$$

Here, $C_3 = C_1^{\alpha/(\alpha+1)}$ and $C_4 = C_2$, where $C_1$ and $C_2$ are the constants appearing in Theorem 1 and $n_0$ is the first integer $n$ that satisfies $C_1(M+1)n^{-2/(d+1)} \le t_0/2$.

**Proof.** For $\varepsilon > 0$, write

$$\begin{aligned}
&\mathbb{P}_f\left[|K \setminus \hat{K}_n| > \varepsilon\right] \\
&\quad = \mathbb{P}_f\left[|K \setminus \hat{K}_n| > \varepsilon, d_f(K, \hat{K}_n) \le t_0\right] + \mathbb{P}_f\left[|K \setminus \hat{K}_n| > \varepsilon, d_f(K, \hat{K}_n) > t_0\right] \\
&\quad \le \mathbb{P}_f\left[d_f(K, \hat{K}_n) > \varepsilon^{(\alpha+1)/\alpha}\right] + \mathbb{P}_f\left[d_f(K, \hat{K}_n) > t_0\right] \qquad (8)
\end{aligned}$$

and apply Theorem 1 with $\varepsilon = C_1^{\alpha/(\alpha+1)}(M+1)^{\alpha/(\alpha+1)} n^{\frac{-2\alpha}{(\alpha+1)(d+1)}} + xn^{\frac{-\alpha}{\alpha+1}}$ to get the desired result. $\square$

As a consequence of the deviation inequality of Theorem 2, we get the following moment inequalities.

**Corollary 2.** *Recall the notation and assumptions of Theorem 2. Then, for all $q > 0$, there exists a positive constant $A'_q$ that depends on $d$, $\alpha$, $L$, $t_0$ and $q$ only, such that*

$$\mathbb{E}_f\big[|K \setminus \hat{K}_n|^q\big] \le A'_q (M+1)^{\frac{\alpha q}{\alpha+1}} n^{-\frac{2\alpha q}{(\alpha+1)(d+1)}}, \quad \forall n \ge n_0.$$

**Proof.** The proof is based on the same argument as in the proof of Corollary 1 and is omitted. □

It is easy to see that the constants $C_3$ and $C_4$ in Theorem 2 are bounded, as functions of $\alpha$. Hence, when $\alpha = \infty$, which includes the case of the uniform distribution on $K$, the rate obtained in Theorem 2 coincides with that obtained in Section 5. As a byproduct, the deviation inequality given in Theorem 4 below, for the missing volume, still holds, with different constants, for any density that is bounded away from zero and infinity (see Remark 2 in Section 5).

# 4. Moment inequalities for the number of vertices of random polytopes

In this section, we are interested in the number of vertices of random polytopes. Let $\mu$ be any probability measure in $\mathbb{R}^d$. Let $X_1, X_2, \ldots$ be i.i.d. realizations of $\mu$ and $\hat{K}_n$ be the convex hull of the first $n$ of them, for $n \ge 1$. Denote by $\mathcal{V}_n$ the set of vertices of $\hat{K}_n$ and $R_n$ its cardinality, that is, the number of vertices of $\hat{K}_n$. Efron [7] proved a simple but elegant identity, which relates the expected missing mass $\mathbb{E}[1-\mu(\hat{K}_n)]$ to the expected number of vertices $\mathbb{E}[R_{n+1}]$ of $\hat{K}_{n+1}$. Namely, one has

$$\mathbb{E}\big[1-\mu(\hat{K}_n)\big] = \frac{\mathbb{E}[R_{n+1}]}{n+1}, \quad \forall n \in \mathbb{N}^*. \tag{9}$$

In the case when $\mu$ is the uniform probability measure on a convex body $K$, extensions of this identity to higher moments of $|K \setminus \hat{K}_n|$ can be found in [5]. Here, we prove the following inequalities that hold for any distribution $\mu$. The proof is deferred to the Appendix.

**Lemma 5.** *For all positive integers $q$,*

$$\mathbb{E}\left[\prod_{j=0}^{q-1} \frac{R_{n+q}-j}{n+q-j}\right] \le \mathbb{E}\big[\big(1-\mu(\hat{K}_n)\big)^q\big].$$

If $\mu$ has a bounded density $f$ with respect to the Lebesgue measure and is supported on a convex body $K$, combining Corollary 3 and Corollary 5 yields the following inequality:

$$\mathbb{E}_f\big[R_n(R_n-1)\ldots(R_n-q+1)\big] \le A_q (M+1)^q n^{\frac{q(d-1)}{d+1}}, \quad \forall n \in \mathbb{N}^*, \forall q \in \mathbb{N}^*,$$

where $A_q$ is the same constant as in Corollary 3 and $M$ is an almost everywhere upper bound of $f$. Since the polynomial $x^q$ is a linear combination of the polynomials $x(x-1)\ldots(x-k+1)$, $0 \le k \le q$, we get the following result.

**Theorem 3.** *Let $K$ be a convex body and $f$ satisfy the assumptions of Theorem 1. Then, for all positive integers $q$, there exists a positive constant $B_q$ that depends on $d$ and $q$ only such that*

$$\mathbb{E}_f\left[R_n^q\right] \le B_q(M+1)^q n^{\frac{q(d-1)}{d+1}}. \tag{10}$$

**Remark 1.** The boundedness assumption on $f$ in Theorem 3 seems unavoidable, in the following sense: Let $\mu$ be a probability measure that puts positive mass on countably many points in $K$, near its boundary and let $f$ be the density of a regularized version of $\mu$, truncated so it remains supported on $K$. Then, with high probability, $R_n$ can be arbitrarily large.

# 5. The case of uniform distributions

As discussed in the introduction, the case of the uniform distribution on a convex body $K$ is an extremely important case in the stochastic geometry literature. In this section, we use the results proven in the previous sections in order to derive universal inequalities for the convex hull of uniform random points. By universal, we mean uniform for all convex bodies, irrespective of their volume, or facial structure.

**Theorem 4.** *There exist three positive constants $\aleph_1$, $\aleph_2$ and $\aleph_3$, which depend on $d$ only, such that*:

$$\sup_{K\in\mathcal{K}_d} \mathbb{P}_K\left[n\left(\frac{|K\setminus\hat{K}_n|}{|K|} - \aleph_1 n^{-2/(d+1)}\right) > x\right] \le \aleph_2 e^{-\aleph_3 x}, \quad \forall x > 0. \tag{11}$$

Note that the rate $n^{-2/(d+1)}$ is a worst case scenario rate, and it is sharp for smooth convex bodies $K$, but suboptimal for polytopes, for which the decay, in expectation, is of the order of $(\ln n)^{d-1}/n$, with constants depending on $K$ (see, e.g., [3]).

**Proof.** In order to prove Theorem 4, we first state two lemmas, the first of which is about the so called *John's ellipsoid* of a convex body.

**Lemma 6.** *For all $K \in \mathcal{K}_d$, there exist $a \in \mathbb{R}^d$ and an ellipsoid $E$ such that*

$$a + d^{-1}(E - c) \subseteq K \subseteq E. \tag{12}$$

Proof of Lemma 6 can be found in [13] and [11]. If $E$ is an ellipsoid of maximum volume satisfying (12) for some $a \in \mathbb{R}^d$, then $a + d^{-1}(E - c)$ is called *John ellipsoid* of $K$.

Let $K \in \mathcal{K}_d$ and $X_1, \dots, X_n$ be i.i.d. uniform random points in $K$. Let $E$ be an ellipsoid that satisfies (12), and $T$ an affine transformation in $\mathbb{R}^d$ which maps $E$ to the unit ball $B_d$. Note that $TX_1, \dots, TX_n$ are independent and uniformly distributed in $TK$, and their convex hull is $T\hat{K}_n$. Hence, the distribution of $\frac{|K\setminus\hat{K}_n|}{|K|} = \frac{|TK\setminus T\hat{K}_n|}{|TK|}$ is the same as that of $\frac{|TK\setminus\hat{K}_n'|}{|TK|}$, where $\hat{K}_n'$ is the convex hull of $X'_1, \dots, X'_n$, which are i.i.d. uniform random points in $TK$. Therefore, for all $\varepsilon > 0$,

$$\mathbb{P}_K\left[\frac{|K \setminus \hat{K}_n|}{|K|} > \varepsilon\right] = \mathbb{P}_f\left[d_f(TK, \hat{K}_n) > \varepsilon\right], \tag{13}$$

where $f$ is the uniform density on $TK$. The density $f$ is bounded from above by $M = 1/|TK| \leq d^d/|TE| = d^d/\kappa_d$, by definition of $E$ and $T$. Hence, applying Theorem 1 yields the desired result, since all the constants in that theorem depend on $d$ only. □

**Remark 2.** Similar arguments could be used to prove a deviation inequality with the same rate rate $n^{-2/(d+1)}$, for a density $f$ that is nearly uniform on $K$, that is, that satisfies $0 < m \leq f(x) \leq M$ for all $x \in K$, where $m$ and $M$ are positive numbers. Indeed, for all invertible affine transformations $T$, $d_g(TK, \hat{K}'_n)$ and $d_f(K, \hat{K}_n)$ have the same distribution, where $g(y) = |\det T|^{-1} f(T^{-1}y)$, $y \in TK$ and $\hat{K}'_n$ is the convex hull of $n$ i.i.d. points with density $g$. In addition, $d_f(K, \hat{K}_n) \geq \frac{m}{M}\frac{|K\setminus\hat{K}_n|}{|K|}$. Therefore, the same reasoning as in the proof of Theorem 4 yields

$$\mathbb{P}_f\left[n\left(\frac{|K\setminus\hat{K}_n|}{|K|} - \aleph'_1 n^{-2/(d+1)}\right) > x\right] \leq \aleph'_2 e^{-\aleph'_3 x}, \quad \forall x > 0,$$

where $\aleph'_1$, $\aleph'_2$ and $\aleph'_3$ are positive constants that depend only on $d$ and on the ratio $M/m$.

A drawback of Theorem 4 is that it involves constants which depend at least exponentially on the dimension $d$. However, this seems to be the price for getting a uniform deviation inequality on $\mathcal{K}_d$.

The following moment inequalities are a consequence of Theorem 4.

**Corollary 3.** *For every positive number $q$, there exists a positive constant $A_q$, which depends on $d$ and $q$ only, such that*

$$\sup_{K\in\mathcal{K}_d} \mathbb{E}_K\left[\left(\frac{|K\setminus\hat{K}_n|}{|K|}\right)^q\right] \leq A_q n^{-2q/(d+1)}. \tag{14}$$

Note that this corollary could also be derived from Vu's result [27], combined with a result of [8] (see Remark 3 below).

This upper bound is tight, as shown in the next corollary.

**Corollary 4.** *For every positive number $q$, there exist positive constants $a_q$ and $A_q$, which depend on $d$ and $q$ only, such that*

$$a_q n^{-2q/(d+1)} \leq \sup_{K\in\mathcal{K}_d} \mathbb{E}_K\left[\left(\frac{|K\setminus\hat{K}_n|}{|K|}\right)^q\right] \leq A_q n^{-2q/(d+1)}.$$

**Proof.** The upper bound is given in Corollary 3. For the lower bound, first note that

$$\begin{aligned}\sup_{K\in\mathcal{K}_d}\mathbb{E}_K\left[\left(\frac{|K\setminus\hat{K}_n|}{|K|}\right)^q\right] &\geq \sup_{K\in\mathcal{K}_d^1}\mathbb{E}_K\left[\left(\frac{|K\setminus\hat{K}_n|}{|K|}\right)^q\right]\\ &\geq \kappa_d^{-q}\sup_{K\in\mathcal{K}_d^1}\mathbb{E}_K\left[|K\setminus\hat{K}_n|^q\right]\\ &\geq a_q n^{-2q/(d+1)},\end{aligned}$$

using [14], Theorem 5.2. □

**Remark 3.** An analogous result to Theorem 4 could be derived from Vu's result (2) combined with an elegant result proven by Giannopoulos and Tsolomitis [8], Theorem 3.6. Let $K \in \mathcal{K}_d^d$ have volume one and $\phi$ be a non decreasing function defined on the positive real line. Then, the expectation $\mathbb{E}_K[\phi(|\hat{K}_n|)]$ is minimized when $K$ is an ellipsoid. The key argument is that this expectation does not increase when $K$ is replaced by its Steiner symmetral with respect to a hyperplane, and performing such a transformation iteratively on $K$ leads to a Euclidean ball at the limit. When $\phi$ is the indicator function of the interval $(x,\infty)$, for $x>0$, Giannopoulos and Tsolomitis' result implies that $\mathbb{P}[V_n \geq x]$ is maximized when $K$ is an ellipsoid. Hence, applying (2) to an ellipsoid of volume one yields a uniform deviation inequality as in Theorem 4. However, the range for $x$ would be much smaller than ours, which allows to capture the whole right tail of $V_n$. Yet, it would still yield similar bounds for the moments of $V_n$, as in Corollary 3.

Let $K \in \mathcal{K}_d$ and $n$ and $q$ be positive integers. Hölder's inequality yields $\mathbb{E}_K[R_n^q] \geq \mathbb{E}_K[R_n]^q$ yielding, by Efron's identity (9),

$$\mathbb{E}_K\left[R_n^q\right] \geq n^q \mathbb{E}_K\left[\frac{|K\setminus\hat{K}_{n-1}|}{|K|}\right]^q.$$

Hence, one gets the following theorem:

**Theorem 5.** *Let $n$ and $q$ be positive integers. Then, for some positive constants $\underline{b}_q$ and $\overline{b}_q$ which depend on $d$ and $q$ only,*

$$\underline{b}_q n^{\frac{q(d-1)}{d+1}} \leq \sup_{K\in\mathcal{K}_d}\mathbb{E}_K\left[R_n^q\right] \leq \overline{b}_q n^{\frac{q(d-1)}{d+1}}.$$

**Proof.** The lower bound is a consequence of Corollary 4, together with Hölder's inequality, as explained above. For the upper bound, let $K \in \mathcal{K}_d$ and let $\tilde{K} = \lambda K$, with $\lambda = |K|^{-1/d}$ be a dilated (or contracted) version of $K$ of volume 1. Then, by affine equivariance, $\mathbb{E}_K[R_n]^q = \mathbb{E}_{\tilde{K}}[R_n]^q$. Hence, Theorem 3, with $f$ being the uniform density on $\tilde{K}$ and $M=1$, yields the desired result. □

Combined with Theorem 3, this theorem has two consequences. First, the rate in the upper bound of Theorem 3 is tight, uniformly on all arbitrary convex bodies and bounded densities.

Second, the uniform case is the worst case, that is, yields the largest possible rate for the expected number of vertices of $\hat{K}_n$. Namely, the following holds. For $K \in \mathcal{K}_d^1$ and $M > 0$, denote by $\mathcal{F}(K, M)$ the collection of all densities that are supported on $K$ and bounded by $M$. For two positive sequences $u_n$ and $v_n$, write $u_n = O(v_n)$ if the ratio $u_n/v_n$ is bounded, independently of $n$.

**Theorem 6.** *For all $M > 0$ and all positive integer $q$,*

$$\sup_{K\in\mathcal{K}_d^1}\sup_{f\in\mathcal{F}(K,M)}\mathbb{E}_f\big[R_n^q\big] = O\Big(\sup_{K\in\mathcal{K}_d^1}\mathbb{E}_K\big[R_n^q\big]\Big).$$

**Remark 4.** We do not know whether the supremum over $K$ could be removed in Theorem 6: We propose the following open question. Is it true that for all $M > 0$ and $K \in \mathcal{K}_d^1$,

$$\sup_{f\in\mathcal{F}(K,M)}\mathbb{E}_f[R_n] = O\big(\mathbb{E}_K[R_n]\big)\ ?$$

# Appendix: Proof of the lemmas

**Proof of Lemma 2.** Let $G \in \mathcal{K}_d$. Steiner formula (see Section 4.1 in [21]) states that there exist positive numbers $v_1(G), \dots, v_d(G)$, such that

$$\big|G^\lambda \backslash G\big| = \sum_{j=1}^d v_j(G)\lambda^j, \quad \lambda \ge 0. \tag{A.1}$$

Besides, the $v_j(G)$, $j = 1, \dots, d$, are increasing functions of $G$. In particular, if $G \in \mathcal{K}_d^1$, then $v_j(G) \le v_j(B_d)$.

Let $G, G' \in \mathcal{K}_d^1$, and let $\lambda = d_H(G, G')$. Since $G$ and $G'$ are included in the unit ball, $\lambda$ is not greater than its diameter, so $\lambda \le 2$. By definition of the Hausdorff distance, $G \subseteq G'^\lambda$ and $G' \subseteq G^\lambda$. Hence,

$$\begin{aligned}\big|G\triangle G'\big| &= \big|G\backslash G'\big| + \big|G'\backslash G\big| \le \big|G'^\lambda\backslash G'\big| + \big|G^\lambda\backslash G\big| \\ &\le 2\sum_{j=1}^d v_j(B_d)\lambda^j \le \lambda\sum_{j=1}^d v_j(B_d)2^j.\end{aligned}$$

The lemma is proved by setting $\alpha_1 = \sum_{j=1}^d v_j(B_d)2^j$.

Note that since $\delta \le 1$, Steiner formula (A.1) implies, for $G \in \mathcal{K}_d^1$, that

$$\big|G^\delta\backslash G\big| \le \alpha_2\delta, \tag{A.2}$$

where $\alpha_2 = \sum_{j=1}^d v_j(B_d)$. $\square$

**Proof of Lemma 3.** For $x \in K$,

$$f(x) = \frac{\mathsf{Vol}_{D-d}((x+H) \cap K_0)}{\mathsf{Vol}_D(K_0)},$$

where, for all integers $p$, $\mathsf{Vol}_p$ stands for the $p$-dimensional volume and $H$ stands for the orthogonal space of $\mathbb{R}^d$ in $\mathbb{R}^D$. Let $x \in K$ with $t = \rho(x, \partial K) \le r$. Let $x' \in \partial K$ such that $\rho(x, \partial K) = \rho(x, x')$. Let $x_0 \in \partial K_0$ whose orthogonal projection onto $\mathbb{R}^d$ is $x'$. By the $r$-rolling condition, there exists $a \in K_0$ with $x' \in B_D(a, r) \subseteq K_0$. Note that $x' - a \in \mathbb{R}^d$ ($\mathbb{R}^d$ being identified with a subspace of $R^D$, orthogonal to $H$) since the (unique) tangent space to $K_0$ at $x'$ needs to be tangent to $B_D(a, r)$ as well. Therefore,

$$\mathsf{Vol}_{D-d}\big((x+H) \cap K_0\big) \ge \mathsf{Vol}_{D-d}\big((x+H) \cap B_D(a,r)\big)$$

and $(x+H) \cap B_D(a,r)$ is a $(D-d)$-dimensional ball with radius $h$, where $h = \sqrt{2rt - t^2} \ge \sqrt{rt}$. Hence, for all $x \in K$ with $\rho(x, \partial K) \le r$,

$$f(x) \ge \frac{(rt)^{(D-d)/2}\kappa_{D-d}}{\mathsf{Vol}_D(K_0)} \ge (rt)^{(D-d)/2}\kappa_{D-d}\kappa_D,$$

which proves the lemma. □

**Proof of Lemma 5.** For precision's sake, we denote by $\mathbb{P}^{\otimes n}$ the $n$-product of the probability measure $\mu$, that is, the joint probability measure of the random variables $X_1, \dots, X_n$, and by $\mathbb{E}^{\otimes n}$ the corresponding expectation operator. First, note that the expectation $\mathbb{E}^{\otimes n}[(1 - \mu(\hat{K}_n))^q]$ can be rewritten as:

$$\begin{aligned} \mathbb{E}^{\otimes n}\big[\big(1 - \mu(\hat{K}_n)\big)^q\big] &= \mathbb{E}^{\otimes n}\big[\mathbb{P}^{\otimes q}[X_{n+1} \notin \hat{K}_n, \dots, X_{n+q} \notin \hat{K}_n \mid X_1, \dots, X_n]\big] \\ &= \mathbb{P}^{\otimes(n+q)}[X_{n+j} \notin \hat{K}_n, \forall j = 1, \dots, q]. \end{aligned} \tag{A.3}$$

Using the symmetric role of $X_1, \dots, X_{n+q}$, and since the event $\{X_{n+j} \notin \hat{K}_n, \forall j = 1, \dots, q\}$ contains the event $\{X_{n+j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q\}$, (A.3) yields

$$\begin{aligned} &\mathbb{E}^{\otimes n}\big[\big(1 - \mu(\hat{K}_n)\big)^q\big] \\ &\quad\ge \mathbb{P}^{\otimes n+q}[X_{n+j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q] \\ &\quad= \frac{1}{\binom{n+q}{q}} \sum_{1 \le i_1 < \dots < i_q \le n+q} \mathbb{P}^{\otimes n+q}[X_{i_j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q] \\ &\quad= \frac{1}{\binom{n+q}{q}} \mathbb{E}^{\otimes n+q}\Bigg[\sum_{1 \le i_1 < \dots < i_q \le n+q} \mathbb{1}(X_{i_j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q)\Bigg] \end{aligned}$$

$$= \frac{1}{\binom{n+q}{q}} \mathbb{E}^{\otimes n+q}\left[\binom{R_{n+q}}{q}\right]$$

$$= \frac{\mathbb{E}^{\otimes n+q}[R_{n+q}(R_{n+q}-1)\dots(R_{n+q}-q+1)]}{(n+q)(n+q-1)\dots(n+1)},$$

which proves the lemma. □

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