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Random walk approximation of BSDEs with Hölder continuous terminal condition

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In this paper, we consider the random walk approximation of the solution of a Markovian BSDE whose terminal condition is a locally Hölder continuous function of the Brownian motion. We state the rate of the L_2-convergence of the approximated solution to the true one. The proof relies in part on growth and smoothness properties of the solution u of the associated PDE. Here we improve existing results by showing some properties of the second derivative of u in space.

Keywords: backward stochastic differential equations; numerical scheme; random walk approximation; speed of convergence

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying the standard Brownian motion $B = (B_t)_{t\geq 0}$ and assume $(\mathcal{F}_t)_{t\geq 0}$ is the augmented natural filtration. We consider the following backward stochastic differential equation (BSDE for short)

$$Y_s = g(B_T) + \int_s^T f(r, B_r, Y_r, Z_r)\,dr - \int_s^T Z_r\,dB_r, \quad 0 \leq s \leq T, \tag{1}$$

where f is Lipschitz continuous and g is a locally α-Hölder continuous and polynomially bounded function (see (3)). In this paper, we are interested in the L_2-convergence of the numerical approximation of (1) by using a random walk. First, results dealing with the numerical approximation of BSDEs date back to the late 1990s. Bally (see [2]) was the first to consider this problem by introducing random discretization, namely the jump times of a Poisson process. In his Ph.D. thesis, Chevance (see [17]) proposed the following discretization

$$y_k = \mathbb{E}\big(y_{k+1} + hf(y_{k+1})|\mathcal{F}_k^n\big), \quad k = n-1, \dots, 0, n \in \mathbb{N}^*$$

and proved the convergence of $(Y_t^n)_t := (y_{[t/h]})_t$ to Y. At the same time, Coquet, Mackevičius and Mémin [18] proved the convergence of Y^n by using convergence of filtrations, still in the case of a generator independent from z. The general case (f depends on z, terminal condition $\xi \in L_2$) has been studied by Briand, Delyon and Mémin (see [7]). In that paper the authors define an approximated solution (Y^n, Z^n) based on random walk and prove weak convergence to

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(Y, Z) using convergence of filtrations. We also refer to [27,29–31] for other numerical methods for BSDEs which use a random walk approach. The rate of convergence of this method was left as an open problem.

Introducing instead of random walk an approach based on the dynamic programming equation, Bouchard and Touzi in [6] and Zhang in [36] managed to establish a rate of convergence. However, to be fully implementable, this algorithm requires to have a good approximation of its associated conditional expectation. For this, various methods have been developed (see [13, 20,25]). Forward methods have also been introduced to approximate (1): a branching diffusion method (see [26]), a multilevel Picard approximation (see [34]) and Wiener chaos expansion (see [9]). Many extensions of (1) have also been considered: high order schemes (see [10,11]), schemes for reflected BSDEs (see [3,15]), for fully-coupled BSDEs (see [4,21]), for quadratic BSDEs (see [14]), for BSDEs with jumps (see [23]) and for McKean–Vlasov BSDEs (see [1,12, 16]).

From a numerical point of view, the random walk is of course not competitive with recent methods listed above. We emphasize that the aim of this paper is to give the convergence rate of the initial method based on random walk, which, to the best of our knowledge, has not been done so far.

As in [7], let us introduce the following approximation of B, based on a random walk:

$$B_t^n = \sqrt{h} \sum_{i=1}^{[t/h]} \varepsilon_i, \quad 0 \le t \le T,$$

where $h = \frac{T}{n}$ ($n \in \mathbb{N}^*$) and $(\varepsilon_i)_{i=1,2,\dots}$ is a sequence of i.i.d. Rademacher random variables. Consider the following approximated solution (Y^n, Z^n) of (Y, Z)

$$Y_{t_k}^n = g\big(B_T^n\big) + h \sum_{m=k}^{n-1} f\big(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n\big) - \sqrt{h} \sum_{m=k}^{n-1} Z_{t_m}^n \varepsilon_{m+1}, \quad 0 \le k \le n-1. \tag{2}$$

The main result of our paper gives the rate of convergence in L_2-norm of $Y_v^n - Y_v$ and $Z_v^n - Z_v$ for each $v \in [0, T)$ (see Theorem 3.1). Basically, we get that the L_2-norm of the error on Y is of order $h^{\frac{\alpha}{4}}$ and the L_2-norm of the error on Z is of order $\frac{h^{\frac{\alpha}{4}}}{\sqrt{T-v}}$. The proof of this result is based on several ingredients. In particular, we need some estimates on the bound of the first and second derivatives of the solution of the PDE associated to the BSDE (1). We establish these bounds in the case of a forward backward SDE (FBSDE for short) whose terminal condition satisfies the Hölder continuity condition (3). This result extends Zhang [37], Theorem 3.2.

The rest of the paper is organized as follows. Section 2 introduces notations, assumptions and the representation for Z and Z^n based on the Malliavin weights. Section 3 states the rate of convergence of the error on Y and Z in L_2-norm, which is the main result of the paper. Section 4 presents numerical simulations and Section 5 recalls some properties of Malliavin weights, of the regularity of solutions to FBSDEs with a locally Hölder continuous terminal condition function and states some properties of the solutions to the PDEs associated to these FBSDEs.

2. Preliminaries

This section is dedicated to notations, assumptions and the representation of Z and Z^n using the Malliavin weights.

Notation.

- $\mathcal{G}_k := \sigma(\varepsilon_i : 1 \le i \le k)$ and $\mathcal{G}_0 = \{\emptyset, \Omega\}$. The associated discrete-time random walk $(B^n_{t_k})_{k=0}^n$ is $(\mathcal{G}_k)_{k=0}^n$-adapted.
- $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{P})}$ for $p \ge 1$ and for $p = 2$ simply $\|\cdot\|$.

Assumption 2.1.

- g is locally Hölder continuous with order $\alpha \in (0,1]$ and polynomially bounded ($p_0 \ge 0$, $C_g > 0$) in the following sense

$$\forall (x,y) \in \mathbb{R}^2, \quad \big|g(x) - g(y)\big| \le C_g\big(1 + |x|^{p_0} + |y|^{p_0}\big)|x-y|^\alpha. \tag{3}$$

- The function $[0,T] \times \mathbb{R}^3 : (t,x,y,z) \mapsto f(t,x,y,z)$ satisfies

$$\big|f(t,x,y,z) - f\big(t',x',y',z'\big)\big| \le L_f\big(\sqrt{t-t'} + \big|x-x'\big| + \big|y-y'\big| + \big|z-z'\big|\big). \tag{4}$$

Notice that (3) implies

$$\big|g(x)\big| \le K\big(1 + |x|^{p_0+1}\big) =: \Psi(x). \tag{5}$$

In the rest of the paper, the study of the error $(Y^n - Y, Z^n - Z)$ will either rely on (2) or on its integral version:

$$Y^n_s = g\big(B^n_T\big) + \int_{(s,T]} f\big(r, B^n_{r-}, Y^n_{r-}, Z^n_{r-}\big)\, d\big[B^n, B^n\big]_r - \int_{(s,T]} Z^n_{r-}\, dB^n_r, \quad 0 \le s \le T, \tag{6}$$

where the backward equation (6) arises from (2) by setting $Y^n_r := Y^n_{t_m}$ and $Z^n_r := Z^n_{t_m}$ for $r \in [t_m, t_{m+1})$. For n large enough, (6) has a unique solution (Y^n, Z^n), and $(Y^n_{t_m}, Z^n_{t_m})_{m=0}^{n-1}$ is adapted to the filtration $(\mathcal{G}_m)_{m=0}^{n-1}$. Let us now introduce the Malliavin representations for Z and Z^n. They are the cornerstone of our study of the error on Z.

2.1. Representations for Z and Z^n

We will use the representation (see Ma and Zhang [28], Theorem 4.2)

$$Z_t = \mathbb{E}_t\bigg(g(B_T)N^t_T + \int_t^T f(s, B_s, Y_s, Z_s)N^t_s\, ds\bigg), \quad 0 \le t \le T, \tag{7}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$, and for all $s \in (t,T]$ we have

$$N^t_s := \frac{B_s - B_t}{s - t}.$$

Lemma 2.2. *Suppose that Assumption 2.1 holds. Then the process Z^n given by (6) has the representation*

$$Z^n_{t_k} = \mathbb{E}_k\left(g\left(B^n_T\right)\frac{B^n_{t_n} - B^n_{t_k}}{t_n - t_k}\right) + \mathbb{E}_k\left(h\sum_{m=k+1}^{n-1} f\left(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\right)\frac{B^n_{t_m} - B^n_{t_k}}{t_m - t_k}\right) \tag{8}$$

for $k = 0, 1, \dots, n-1$, where $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{G}_k]$.

Proof. We multiply equation (2) by ε_{k+1} and take the conditional expectation with respect to $\mathcal{G}_k$. Since $(Y^n_{t_k}, Z^n_{t_k})$ is $\mathcal{G}_k$-measurable, it holds for $0 \le k \le n-1$ that

$$\begin{aligned}
&\mathbb{E}_k\left(Y^n_{t_k}\varepsilon_{k+1}\right) \\
&\quad = \mathbb{E}_k\left(g\left(B^n_T\right)\varepsilon_{k+1}\right) + h\mathbb{E}_k\left(\sum_{m=k}^{n-1} f\left(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\right)\varepsilon_{k+1}\right) - \sqrt{h}\mathbb{E}_k\left(\sum_{m=k}^{n-1} Z^n_{t_m}\varepsilon_{m+1}\varepsilon_{k+1}\right) \\
&\quad = \sqrt{h}\mathbb{E}_k\left(g\left(B^n_T\right)\frac{B^n_{t_n} - B^n_{t_k}}{t_n - t_k}\right) + h^{3/2}\sum_{m=k+1}^{n-1}\mathbb{E}_k\left(f\left(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\right)\frac{B^n_{t_m} - B^n_{t_k}}{t_m - t_k}\right) \\
&\qquad - \sqrt{h}Z^n_{t_k},
\end{aligned} \tag{9}$$

where the l.h.s. is equal to zero. Indeed, for $m \ge k+1$, we have

$$\mathbb{E}_k\left(Z^n_{t_m}\varepsilon_{m+1}\varepsilon_{k+1}h\right) = \mathbb{E}_k\left(Z^n_{t_m}\varepsilon_{k+1}\mathbb{E}_m\varepsilon_{m+1}\right) = 0,$$

and for $m = k$ it holds $\mathbb{E}_k(Z^n_{t_k}\varepsilon^2_{k+1}) = Z^n_{t_k}$. Moreover, the fact that $B^n_T = \sqrt{h}\sum_{m=0}^{n-1}\varepsilon_{m+1}$, where $(\varepsilon_m)_{m=1,2\dots}$ are i.i.d., yields

$$\begin{aligned}
\mathbb{E}_k\left(g\left(B^n_T\right)\varepsilon_{k+1}\right) &= \mathbb{E}_k\left(g\left(B^n_T\right)\sum_{m=k}^{n-1}\frac{\varepsilon_{k+1}}{n-k}\right) = \mathbb{E}_k\left(g\left(B^n_T\right)\sum_{m=k}^{n-1}\frac{\varepsilon_{m+1}}{n-k}\right) \\
&= \sqrt{h}\mathbb{E}_k\left(g\left(B^n_T\right)\frac{B^n_{t_n} - B^n_{t_k}}{t_n - t_k}\right).
\end{aligned}$$

Similarly, for $m \ge k+1$, we get (using [7], Proposition 5.1, where it is stated that both $Y^n_{t_m}$ and $Z^n_{t_m}$ can be represented as functions of t_m and $B^n_{t_m}$)

$$\mathbb{E}_k\left(f\left(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\right)\varepsilon_{k+1}\right) = \sqrt{h}\mathbb{E}_k\left(f\left(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\right)\frac{B^n_{t_m} - B^n_{t_k}}{t_m - t_k}\right).$$

It remains to divide (9) by $\sqrt{h}$ and rearrange. □

3. Main result

This section is devoted to the main result of the paper: the rate of the L_2-convergence of (Y^n, Z^n) to (Y, Z). The proof will rely on the fact that the random walk B^n can be constructed from the Brownian motion B by Skorohod embedding. Let $\tau_0 := 0$ and define

$$\tau_k := \inf\{t > \tau_{k-1} : |B_t - B_{\tau_{k-1}}| = \sqrt{h}\}, \quad k \geq 1.$$

Then $(B_{\tau_k} - B_{\tau_{k-1}})_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with

$$\mathbb{P}(B_{\tau_k} - B_{\tau_{k-1}} = \pm\sqrt{h}) = \frac{1}{2},$$

which means that $\sqrt{h}\varepsilon_k \overset{d}{=} B_{\tau_k} - B_{\tau_{k-1}}$. We will use this random walk for our approximation, that is, we will require

$$B^n_t = \sum_{k=1}^{[t/h]} (B_{\tau_k} - B_{\tau_{k-1}}), \quad 0 \leq t \leq T. \tag{10}$$

Properties satisfied by τ_k and B_{τ_k} are stated in Lemma A.1. We will denote by $\mathbb{E}_{\tau_k}$ the conditional expectation w.r.t. $\mathcal{F}_{\tau_k}$.

Theorem 3.1. *Let Assumption 2.1 hold. If B^n satisfies (10) then we have (for sufficiently large n) that*

$$\mathbb{E}\big|Y_v - Y^n_v\big|^2 \leq C_0 h^{\frac{\alpha}{2}} \quad \text{for } v \in [0, T),$$

$$\mathbb{E}\big|Z_v - Z^n_v\big|^2 \leq C_0 \frac{h^{\frac{\alpha}{2}}}{T - t_k} + C_1 \frac{h^{\frac{\alpha}{2}}}{(T - v)^{1-\frac{\alpha}{2}}} \mathbf{1}_{v \neq t_k} \quad \text{for } v \in [t_k, t_{k+1}), k = 0, \dots, n-1,$$

where we have the dependencies $C_0 = C(T, p_0, L_f, C_g, C^y_{5.3}, C^z_{5.3}, K_f, c_{5.4}, \alpha)$, $C_1 = C(T, p_0, C^z_{5.3}, \alpha)$ and $K_f := \sup_{0 \leq t \leq T} |f(t, 0, 0, 0)|$.

Remark 3.2. Theorem 3.1 implies that

$$\sup_{v \in [0,T)} \mathbb{E}\big|Y_v - Y^n_v\big|^2 \leq C_0 h^{\frac{\alpha}{2}} \quad \text{and} \quad \mathbb{E}\int_0^T \big|Z_v - Z^n_v\big|^2 dv \leq C(C_0, C_1, \beta) h^{\beta} \quad \text{for } \beta \in \left(0, \frac{\alpha}{2}\right).$$

Proof of Theorem 3.1. Let $u : [0, T) \times \mathbb{R} \to \mathbb{R}$ be the solution of the PDE associated to (1). Since by Theorem 5.4

$$Y_s = u(s, B_s), \qquad Z_s = u_x(s, B_s), \quad a.s.$$

we introduce

$$F(s, x) := f\big(s, x, u(s, x), u_x(s, x)\big),$$

so that $F(s, B_s) = f(s, B_s, Y_s, Z_s)$. We first give some properties satisfied by F.

Lemma 3.3. *If Assumption 2.1 holds then F is a Lipschitz continuous and polynomially bounded function in x:*

$$\big|F(t,x_1)-F(t,x_2)\big| \le C\big(T,L_f,c^{2,3}_{5.4}\big)\big(1+|x_1|^{p_0+1}+|x_2|^{p_0+1}\big)\frac{|x_1-x_2|}{(T-t)^{1-\frac{\alpha}{2}}},$$

$$\big|F(t,x)\big| \le C\big(T,L_f,c^{1,2}_{5.4},K_f\big)\frac{\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}},$$

where $\Psi(x)$ is given in (5).

Proof of Lemma 3.3. Thanks to the mean value theorem and Theorem 5.4-(ii-c) and (iii-b) we have for $x_1,x_2\in\mathbb{R}$ that there exist $\xi_1,\xi_2\in[\min\{x_1,x_2\},\max\{x_1,x_2\}]$ such that

$$\begin{aligned}\big|F(t,x_1)-F(t,x_2)\big| &= \big|f\big(t,x_1,u(t,x_1),u_x(t,x_1)\big)-f\big(t,x_2,u(t,x_2),u_x(t,x_2)\big)\big| \\ &\le L_f\big(|x_1-x_2|+\big|u(t,x_1)-u(t,x_2)\big|+\big|u_x(t,x_1)-u_x(t,x_2)\big|\big) \\ &\le L_f\left(1+\frac{c^2_{5.4}\Psi(\xi_1)}{(T-t)^{\frac{1-\alpha}{2}}}+\frac{c^3_{5.4}\Psi(\xi_2)}{(T-t)^{1-\frac{\alpha}{2}}}\right)|x_1-x_2| \\ &\le C\big(T,L_f,c^{2,3}_{5.4}\big)\big(1+|x_1|^{p_0+1}+|x_2|^{p_0+1}\big)\frac{|x_1-x_2|}{(T-t)^{1-\frac{\alpha}{2}}}.\end{aligned}$$

The second inequality can be shown similarly. □

For the estimate of $\mathbb{E}|Y_{t_k}-Y^n_{t_k}|^2$ we will use (1) and (2): Since $Y^n_{t_k}$ is $\mathcal{F}_{\tau_k}$-measurable we have

$$\begin{aligned}\big\|Y_{t_k}-Y^n_{t_k}\big\| &\le \big\|\mathbb{E}_{t_k}g(B_T)-\mathbb{E}_{\tau_k}g\big(B^n_T\big)\big\| \\ &\quad+\left\|\mathbb{E}_{t_k}\int_{t_k}^T f(s,B_s,Y_s,Z_s)\,ds-h\mathbb{E}_{\tau_k}\sum_{m=k}^{n-1}f\big(t_{m+1},B^n_{t_m},Y^n_{t_m},Z^n_{t_m}\big)\right\|.\end{aligned} \tag{11}$$

We frequently express conditional expectations with the help of an independent copy of B denoted by $\tilde{B}$, for example, $\mathbb{E}_t g(B_T)=\tilde{\mathbb{E}}g(B_t+\tilde{B}_{T-t})$.

By (3) and Lemma A.1,

$$\begin{aligned}\big\|\mathbb{E}_{t_k}g(B_T)-\mathbb{E}_{\tau_k}g\big(B^n_T\big)\big\|^2 &= \mathbb{E}\big|\tilde{\mathbb{E}}g(B_{t_k}+\tilde{B}_{T-t_k})-\tilde{\mathbb{E}}g(B_{\tau_k}+\tilde{B}_{\tilde{\tau}_{n-k}})\big|^2 \\ &\le \big(\mathbb{E}\tilde{\mathbb{E}}(\Psi_1)^4\big)^{\frac{1}{2}}\big(\mathbb{E}\tilde{\mathbb{E}}|B_{t_k}-B_{\tau_k}+\tilde{B}_{T-t_k}-\tilde{B}_{\tilde{\tau}_{n-k}}|^{4\alpha}\big)^{\frac{1}{2}} \\ &\le C(C_g,T,p_0)\big(\big(\mathbb{E}|B_{t_k}-B_{\tau_k}|^{4\alpha}\big)^{\frac{1}{2}}+\big(\mathbb{E}|B_{T-t_k}-B_{\tau_{n-k}}|^{4\alpha}\big)^{\frac{1}{2}}\big) \\ &\le C(C_g,T,p_0)h^{\frac{\alpha}{2}},\end{aligned} \tag{12}$$

where $\Psi_1 := C_g(1+|B_{t_k}+\tilde{B}_{T-t_k}|^{p_0}+|B_{\tau_k}+\tilde{B}_{\tilde{\tau}_{n-k}}|^{p_0})$. To estimate the other term in (11), we consider the decomposition

$$\begin{aligned}
&\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) - \mathbb{E}_{\tau_k} f\big(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\big) \\
&\quad = \big(\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) - \mathbb{E}_{t_k} f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m})\big) + \big(\mathbb{E}_{t_k} F(t_m, B_{t_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{\tau_m})\big) \\
&\qquad + \big(\mathbb{E}_{\tau_k} F(t_m, B_{\tau_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{t_m})\big) \\
&\qquad + \big(\mathbb{E}_{\tau_k} f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m}) - \mathbb{E}_{\tau_k} f\big(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\big)\big) \\
&\quad =: D_1(s,m) + D_2(m) + \cdots + D_4(m)
\end{aligned}$$

so that

$$\begin{aligned}
&\left\| \mathbb{E}_{t_k} \int_{t_k}^T f(s, B_s, Y_s, Z_s)\, ds - h\mathbb{E}_{\tau_k} \sum_{m=k}^{n-1} f\big(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\big) \right\| \\
&\quad \le \sum_{m=k}^{n-1} \left(\left\| \int_{t_m}^{t_{m+1}} D_1(s,m)\, ds \right\| + h \sum_{i=2}^{4} \|D_i(m)\| \right).
\end{aligned}$$

For D_1 we have by Theorem 5.3 that

$$\begin{aligned}
\|D_1(s,m)\| &\le L_f\big(\sqrt{s-t_m} + \|B_s - B_{t_m}\| + \|Y_s - Y_{t_m}\| + \|Z_s - Z_{t_m}\|\big) \\
&\le C\big(T, L_f, C^y_{5.3}, C^z_{5.3}, p_0\big)(T-s)^{\frac{\alpha-2}{2}} h^{\frac{1}{2}},
\end{aligned} \tag{13}$$

where the last inequality follows from $\|B_s - B_{t_m}\| = \sqrt{s-t_m} \le h^{\frac{1}{2}}$ for $s \in [t_m, t_{m+1}]$ and

$$\begin{aligned}
&\|Y_s - Y_{t_m}\| + \|Z_s - Z_{t_m}\| \\
&\quad \le \big(\mathbb{E}\Psi(B_{t_m})^2\big)^{\frac{1}{2}} \left(C^y_{5.3} \left(\int_{t_m}^s (T-r)^{\alpha-1}\, dr \right)^{\frac{1}{2}} + C^z_{5.3} \left(\int_{t_m}^s (T-r)^{\alpha-2}\, dr \right)^{\frac{1}{2}} \right) \\
&\quad \le C\big(T, C^y_{5.3}, C^z_{5.3}, p_0\big)\sqrt{s-t_m}\big((T-s)^{\frac{\alpha-1}{2}} + (T-s)^{\frac{\alpha-2}{2}}\big).
\end{aligned}$$

We bound D_2 using Lemma 3.3 and Lemma A.1. Similar to (12) we conclude (setting $\Psi_2 := 1 + |B_{t_k} + \tilde{B}_{t_{m-k}}|^{p_0+1} + |B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{m-k}}|^{p_0+1}$) that

$$\begin{aligned}
\|D_2(m)\| &= \big(\mathbb{E}\big|\mathbb{E}_{t_k} F(t_m, B_{t_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{\tau_m})\big|^2\big)^{\frac{1}{2}} \\
&\le C\big(T, L_f, c^{2,3}_{5.4}\big)\big(\mathbb{E}\tilde{\mathbb{E}}\Psi_2^4\big)^{\frac{1}{4}} \frac{1}{(T-t_m)^{1-\frac{\alpha}{2}}} (t_k h + t_{m-k} h)^{\frac{1}{4}} \\
&\le C\big(T, p_0, L_f, c^{2,3}_{5.4}\big) \frac{1}{(T-t_m)^{1-\frac{\alpha}{2}}} h^{\frac{1}{4}}.
\end{aligned}$$

For D_3 we apply again Lemma 3.3 and Lemma A.1,

$$\begin{aligned}\|D_3(m)\| &\le \|F(t_m, B_{t_m}) - F(t_m, B_{\tau_m})\| \le C\big(T, L_f, c^{2,3}_{5.4}\big)\frac{1}{(T-t_m)^{1-\frac{\alpha}{2}}}\|\Psi_3 |B_{t_m} - B_{\tau_m}|\| \\ &\le C\big(T, p_0, L_f, c^{2,3}_{5.4}\big)\frac{1}{(T-t_m)^{1-\frac{\alpha}{2}}} h^{\frac{1}{4}},\end{aligned}$$

where $\Psi_3 := 1 + |B_{t_m}|^{p_0+1} + |B_{\tau_m}|^{p_0+1}$. For the last term D_4 we get

$$\|D_4(m)\| \le L_f\big(h^{\frac{1}{2}} + \|B_{t_m} - B^n_{t_m}\| + \|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|\big).$$

Finally, using the estimates for the terms $D_1(s,m), D_2(m), \ldots, D_4(m)$ we arrive at

$$\begin{aligned}\|Y_{t_k} - Y^n_{t_k}\| \le C(C_g, T, p_0)h^{\frac{\alpha}{4}} &+ C\big(T, L_f, C^y_{5.3}, C^z_{5.3}, p_0\big)h^{\frac{1}{2}}\int_{t_k}^T (T-s)^{\frac{\alpha-2}{2}}\,ds \\ &+ C\big(T, p_0, L_f, c^{2,3}_{5.4}\big)h^{\frac{1}{4}}\sum_{m=k}^{n-1}\frac{h}{(T-t_m)^{1-\frac{\alpha}{2}}} \\ &+ hL_f\sum_{m=k}^{n-1}\big(\|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|\big) \\ \le\, & C\big(C_g, T, p_0, L_f, c^{2,3}_{5.4}, C^y_{5.3}, C^z_{5.3}\big)h^{\frac{\alpha}{4}} \\ &+ hL_f\sum_{m=k}^{n-1}\big(\|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|\big). \qquad (14)\end{aligned}$$

For $\|Z_{t_k} - Z^n_{t_k}\|$ we exploit the representations (7) and (8) and estimate

$$\begin{aligned}\|Z_{t_k} - Z^n_{t_k}\| \le \frac{1}{T-t_k}&\|\mathbb{E}_{t_k} g(B_T)(B_T - B_{t_k}) - \mathbb{E}_{\tau_k} g(B_{\tau_n})(B_{\tau_n} - B_{\tau_k})\| \\ &+ \Bigg\|\mathbb{E}_{t_k}\bigg(\int_{t_{k+1}}^T f(s, B_s, Y_s, Z_s)\frac{B_s - B_{t_k}}{s-t_k}\,ds\bigg) \\ &- \mathbb{E}_{\tau_k}\bigg(h\sum_{m=k+1}^{n-1} f\big(t_{m+1}, B^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}\big)\frac{B^n_{t_m} - B^n_{t_k}}{t_m - t_k}\bigg)\Bigg\| \\ &+ \bigg\|\mathbb{E}_{t_k}\int_{t_k}^{t_{k+1}} f(s, B_s, Y_s, Z_s)\frac{B_s - B_{t_k}}{s-t_k}\,ds\bigg\|.\end{aligned}$$

Then, similar to (12), we have for the terminal condition by Lemma A.1 that

$$\begin{aligned}&\big\|\mathbb{E}_{t_k}\big[g(B_T)(B_T - B_{t_k})\big] - \mathbb{E}_{\tau_k}\big[g(B_{\tau_n})(B_{\tau_n} - B_{\tau_k})\big]\big\| \\ &\quad = \big\|\tilde{\mathbb{E}}\big[g(B_{t_k} + \tilde{B}_{T-t_k}) - g(B_{t_k})\big](\tilde{B}_{T-t_k} - \tilde{B}_{\tilde{\tau}_{n-k}})\end{aligned}$$

$$+\tilde{\mathbb{E}}\big[g(B_{t_k}+\tilde{B}_{T-t_k})-g(B_{\tau_k}+\tilde{B}_{\tilde{\tau}_{n-k}})\big]\tilde{B}_{\tilde{\tau}_{n-k}}\|$$
$$\leq C(C_g,T,p_0)h^{\frac{1}{4}}(T-t_k)^{\frac{\alpha}{2}+\frac{1}{4}}+C(C_g,T,p_0)h^{\frac{\alpha}{4}}(T-t_k)^{\frac{1}{2}}\leq C(C_g,T,p_0)h^{\frac{\alpha}{4}}(T-t_k)^{\frac{1}{2}}.$$

Here we have used that $\tilde{\mathbb{E}}[g(B_{t_k})(\tilde{B}_{T-t_k}-\tilde{B}_{\tilde{\tau}_{n-k}})]=0$. The term $\tilde{\mathbb{E}}[g(B_{t_k}+\tilde{B}_{T-t_k})-g(B_{t_k})](\tilde{B}_{T-t_k}-\tilde{B}_{\tilde{\tau}_{n-k}})$ provides us with the factor $(T-t_k)^{\frac{\alpha}{2}}((T-t_k)h)^{\frac{1}{4}}$. For the next term of the estimate of $\|Z_{t_k}-Z^n_{t_k}\|$ we use for $s\in[t_m,t_{m+1})$, where $m\geq k+1$, the decomposition

$$\begin{aligned}
&\frac{\mathbb{E}_{t_k}f(s,B_s,Y_s,Z_s)(B_s-B_{t_k})}{s-t_k}-\frac{\mathbb{E}_{\tau_k}f(t_{m+1},B^n_{t_m},Y^n_{t_m},Z^n_{t_m})(B^n_{t_m}-B^n_{t_k})}{t_m-t_k}\\
&\quad=\frac{\mathbb{E}_{t_k}f(s,B_s,Y_s,Z_s)(B_s-B_{t_k})}{s-t_k}-\frac{\mathbb{E}_{t_k}f(t_m,B_{t_m},Y_{t_m},Z_{t_m})(B_{t_m}-B_{t_k})}{t_m-t_k}\\
&\qquad+\frac{\mathbb{E}_{t_k}F(t_m,B_{t_m})(B_{t_m}-B_{t_k})}{t_m-t_k}-\frac{\mathbb{E}_{\tau_k}F(t_m,B_{\tau_m})(B_{\tau_m}-B_{\tau_k})}{t_m-t_k}\\
&\qquad+\mathbb{E}_{\tau_k}\left[\big[F(t_m,B_{\tau_m})-F(t_m,B_{t_m})\big]\frac{B_{\tau_m}-B_{\tau_k}}{t_m-t_k}\right]\\
&\qquad+\mathbb{E}_{\tau_k}\left[\big[f(t_m,B_{t_m},Y_{t_m},Z_{t_m})-f\big(t_{m+1},B^n_{t_m},Y^n_{t_m},Z^n_{t_m}\big)\big]\frac{B^n_{t_m}-B^n_{t_k}}{t_m-t_k}\right]\\
&\quad=:T_1(s,m)+T_2(m)+\cdots+T_4(m).
\end{aligned}$$

Then by the conditional Hölder inequality and by (13) as well as by Lemma 3.3 we have

$$\begin{aligned}
\big\|T_1(s,m)\big\|&\leq\big\|D_1(s,m)\big\|\frac{\|B_s-B_{t_k}\|}{s-t_k}+\big\|f(t_m,B_{t_m},Y_{t_m},Z_{t_m})\big\|\left\|\frac{B_s-B_{t_k}}{s-t_k}-\frac{B_{t_m}-B_{t_k}}{t_m-t_k}\right\|\\
&\leq C\big(T,L_f,C^y_{5.3},C^z_{5.3},p_0\big)(T-s)^{\frac{\alpha-2}{2}}\frac{h^{\frac{1}{2}}}{\sqrt{s-t_k}}\\
&\quad+C\big(T,L_f,c^{1,2}_{5.4},K_f\big)\frac{(\mathbb{E}\Psi(B_{t_m})^2)^{\frac{1}{2}}}{(T-t_m)^{\frac{1-\alpha}{2}}}\\
&\quad\times\left(\frac{\|B_s-B_{t_m}\|}{s-t_k}+\|B_{t_m}-B_{t_k}\|\left|\frac{1}{s-t_k}-\frac{1}{t_m-t_k}\right|\right)\\
&\leq C\big(T,L_f,K_f,C^y_{5.3},C^z_{5.3},c^{1,2}_{5.4},p_0\big)(T-s)^{\frac{\alpha-2}{2}}\frac{h^{\frac{1}{4}}}{(s-t_k)^{\frac{3}{4}}}.
\end{aligned}$$

Indeed,

$$\frac{\|B_s-B_{t_m}\|}{s-t_k}+\|B_{t_m}-B_{t_k}\|\left|\frac{1}{s-t_k}-\frac{1}{t_m-t_k}\right|\leq\frac{\sqrt{s-t_m}}{s-t_k}+\frac{\sqrt{t_m-t_k}(s-t_m)}{(s-t_k)(t_m-t_k)}\leq C\frac{h^{\frac{1}{4}}}{(s-t_k)^{\frac{3}{4}}},$$

where the last inequality follows from $s - t_m \leq t_{m+1} - t_m = h$ and $h \leq t_m - t_k \leq s - t_k$. We estimate T_2 with the help of Lemma 3.3 and Lemma A.1 as follows:

$$\|T_2(m)\| \leq \|\widehat{D}_2(m)\| \frac{\|B_{t_m} - B_{t_k}\|}{t_m - t_k} + \|F(t_m, B_{\tau_m})\| \frac{\|B_{t_{m-k}} - B_{\tau_{m-k}}\|}{t_m - t_k}$$
$$\leq C(T, p_0, L_f, K_f, c_{5.4}) \frac{1}{(T - t_m)^{1-\frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{3}{4}}}.$$

Here $\widehat{D}_2(m) := (\tilde{\mathbb{E}}|F(t_m, B_{t_k} + \tilde{B}_{t_{m-k}}) - F(t_m, B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{m-k}})|^2)^{\frac{1}{2}}$ which can be estimated as $D_2(m)$. For T_3 the conditional Hölder inequality and Lemma A.1 yield

$$\|T_3(m)\| \leq \|\widehat{D}_3(m)\| \left\| \frac{B_{\tau_m} - B_{\tau_k}}{t_m - t_k} \right\| \leq C\big(T, p_0, L_f, c_{5.4}^{2,3}\big) \frac{1}{(T - t_m)^{1-\frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{1}{2}}},$$

where $\widehat{D}_3(m) := F(t_m, B_{\tau_m}) - F(t_m, B_{t_m})$ is estimated as $D_3(m)$. Finally,

$$\|T_4(m)\| \leq L_f\big(h^{\frac{1}{2}} + \|B_{t_m} - B_{t_m}^n\| + \|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|\big) \frac{1}{\sqrt{t_m - t_k}}.$$

For the estimate of $\|\mathbb{E}_{t_k} \int_{t_k}^{t_{k+1}} f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k} ds\|$ one notices that by the conditional Hölder inequality,

$$\left\|\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k}\right\| = \left\|\mathbb{E}_{t_k}\left[\big(f(s, B_s, Y_s, Z_s) - f(s, B_{t_k}, Y_{t_k}, Z_{t_k})\big) \frac{B_s - B_{t_k}}{s - t_k}\right]\right\|$$
$$\leq \|f(s, B_s, Y_s, Z_s) - f(s, B_{t_k}, Y_{t_k}, Z_{t_k})\| \frac{1}{\sqrt{s - t_k}}$$
$$\leq C\big(T, L_f, C_{5.3}^y, C_{5.3}^z, p_0\big)(T - s)^{\frac{\alpha-2}{2}} \frac{h^{\frac{1}{2}}}{\sqrt{s - t_k}},$$

where the last inequality follows in the same way as in (13). Consequently, we have

$$\|Z_{t_k} - Z_{t_k}^n\| \leq \frac{C(C_g, T, p_0)}{(T - t_k)^{\frac{1}{2}}} h^{\frac{\alpha}{4}}$$
$$+ C\big(T, L_f, K_f, C_{5.3}^y, C_{5.3}^z, c_{5.4}^{1,2}, p_0\big) \int_{t_k}^T \frac{ds}{(T - s)^{1-\frac{\alpha}{2}} (s - t_k)^{\frac{3}{4}}} h^{\frac{1}{4}}$$
$$+ C(T, p_0, L_f, K_f, c_{5.4}) h \sum_{m=k+1}^{n-1} \frac{1}{(T - t_m)^{1-\frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{3}{4}}}$$
$$+ L_f h \sum_{m=k+1}^{n-1} \big(\|B_{t_m} - B_{t_m}^n\| + \|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|\big) \frac{1}{\sqrt{t_{m-k}}}.$$

Lemma A.2 enables to bound the second and third term of the r.h.s. by $C\frac{h^{\frac{1}{4}}}{(T-t_k)^{\frac{3}{4}-\frac{\alpha}{2}}}B(\frac{\alpha}{2},\frac{1}{4})$, which is bounded by $C\frac{h^{\frac{\alpha}{4}}}{(T-t_k)^{\frac{1}{2}-\frac{\alpha}{4}}}$. Thus, we get

$$\left\|Z_{t_k}-Z^n_{t_k}\right\| \le \frac{C_0 h^{\frac{\alpha}{4}}}{(T-t_k)^{\frac{1}{2}}} + L_f h \sum_{m=k+1}^{n-1} \left(\left\|Y_{t_m}-Y^n_{t_m}\right\| + \left\|Z_{t_m}-Z^n_{t_m}\right\|\right) \frac{1}{\sqrt{t_{m-k}}}.$$

Then we use (14) and the above estimate to get

$$\left\|Y_{t_k}-Y^n_{t_k}\right\| + \left\|Z_{t_k}-Z^n_{t_k}\right\| \le \frac{C_0 h^{\frac{\alpha}{4}}}{(T-t_k)^{\frac{1}{2}}} + C(L_f) h \sum_{m=k+1}^{n-1} \left(\left\|Y_{t_m}-Y^n_{t_m}\right\| + \left\|Z_{t_m}-Z^n_{t_m}\right\|\right) \frac{1}{\sqrt{t_{m-k}}}.$$

If this inequality is iterated, one gets a shape where the Gronwall lemma applies. Indeed, setting $a_m := (\|Y_{t_m}-Y^n_{t_m}\| + \|Z_{t_m}-Z^n_{t_m}\|)$ one has to consider the double sum

$$\sum_{m=k+1}^{n-1}\left(\sum_{l=m+1}^{n-1} a_l \frac{h}{\sqrt{t_{l-m}}}\right)\frac{h}{\sqrt{t_{m-k}}} = h\sum_{l=k+1}^{n-1}\left(\sum_{m=k+1}^{l-1}\frac{h}{\sqrt{t_{m-k}}\sqrt{t_{l-m}}}\right) a_l \le Ch\sum_{l=k+1}^{n-1} a_l.$$

Consequently,

$$\left\|Y_{t_k}-Y^n_{t_k}\right\| + \left\|Z_{t_k}-Z^n_{t_k}\right\| \le \frac{C_0 h^{\frac{\alpha}{4}}}{(T-t_k)^{\frac{1}{2}}}$$

which gives the bound on the error on Z. Moreover, (14) yields

$$\left\|Y_{t_k}-Y^n_{t_k}\right\| \le C_0 h^{\frac{\alpha}{4}}.$$

If $v \in [t_k, t_{k+1})$, we have by Theorem 5.3 that

$$\left\|Y_v-Y^n_v\right\| \le \|Y_v-Y_{t_k}\| + \left\|Y_{t_k}-Y^n_{t_k}\right\| \le C\big(C^y_{5.3}, T, p_0\big)\left(\int_{t_k}^{v}(T-r)^{\alpha-1}\,dr\right)^{\frac{1}{2}} + \left\|Y_{t_k}-Y^n_{t_k}\right\|,$$

$$\begin{aligned}\left\|Z_v-Z^n_v\right\| &\le \|Z_v-Z_{t_k}\| + \left\|Z_{t_k}-Z^n_{t_k}\right\| \\ &\le C\big(C^z_{5.3}, T, p_0\big)\left(\int_{t_k}^{v}(T-r)^{\alpha-2}\,dr\right)^{\frac{1}{2}} + \left\|Z_{t_k}-Z^n_{t_k}\right\|,\end{aligned}$$

where

$$\int_{t_k}^{v}(T-r)^{\alpha-1}\,dr \le \frac{1}{\alpha}(v-t_k)^{\alpha} \le \frac{1}{\alpha}h^{\alpha}$$

and

$$\int_{t_k}^{v} (T-r)^{\alpha-2}\,dr \le \frac{1}{(T-v)^{1-\frac{\alpha}{2}}} \int_{t_k}^{v} (T-r)^{\frac{\alpha}{2}-1}\,dr \le \frac{1}{(T-v)^{1-\frac{\alpha}{2}}} \frac{2}{\alpha} (v-t_k)^{\frac{\alpha}{2}}$$
$$\le \frac{2}{\alpha} \frac{h^{\frac{\alpha}{2}}}{(T-v)^{1-\frac{\alpha}{2}}}. \qquad \square$$

4. Numerical simulations

This section deals with the algorithm used to compute $(Y^n_{t_k}, Z^n_{t_k})_{k=0,\dots,n}$ and numerical experiments for different terminal conditions and drivers. In the first three cases the exact solution is available and we are able to compute the error $(Y^n - Y, Z^n - Z)$ in L_2-norm. In the last two cases the exact solution is unknown, therefore we plot the evolution of (Y^n, Z^n) w.r.t. n.

4.1. Simulation of $(\tau_1, \dots, \tau_n)$ and B^n

In order to simulate $(\tau_1, \dots, \tau_n)$, we use the fact that

$$\tau_0 = 0 \quad \text{and} \quad \forall k \ge 1, \quad \tau_k = \tau_{k-1} + \sigma_k,$$

where $(\sigma_k)_{1\le n}$ is an i.i.d. sequence whose common law σ represents the first exit time of the Brownian motion B of the interval $[-\sqrt{h}, \sqrt{h}]$,

$$\sigma := \inf\{t > 0 : |B_t| = \sqrt{h}\}.$$

From the book of Borodin and Salminen [5], we have that the Laplace transform of σ is given by $\mathbb{E}(e^{-\lambda\sigma}) = \frac{1}{\cosh(\sqrt{2\lambda h})}$.

Let F denote the cumulative distribution function of σ. It holds $\mathbb{E}(e^{-\lambda\sigma}) = \lambda \hat{F}(\lambda)$, where $\hat{F}$ is the Laplace transform of F. Then, to obtain F, it remains to inverse numerically its Laplace transform. Once we have F, we simulate the sequence $(\sigma_k)_{1\le k\le n}$ by following the steps of Algorithm 1.

Algorithm 1 Simulation of the sequence $(\tau_1, \dots, \tau_n)$

Simulate one vector with uniform law $(U_1, \dots, U_n)$

$\tau_0 = 0$
for $k = 1 : n$ **do**
  Compute $\sigma_k := F^{-1}(U_k)$
  Define $\tau_k = \tau_{k-1} + \sigma_k$
end for

4.2. Simulation of B^n

In order to get the trajectory $B^n_{t_1}, \dots, B^n_{t_n}$ ($B^n_{t_0} = 0$), we simulate an i.i.d. Bernoulli sequence $(\xi_k)_{1\le k\le n}$ that is, $\mathbb{P}(\xi_k = \pm 1) = \frac{1}{2}$. Then

$$B^n_{t_{k+1}} = \begin{cases} B^n_{t_k} + \sqrt{h} & \text{if } \xi_k = 1, \\ B^n_{t_k} - \sqrt{h} & \text{otherwise.} \end{cases} \tag{15}$$

4.3. Simulation of (Y^n, Z^n)

Since B^n is built using the random walk (15), it can be represented by a recombining binomial tree. Both $(Y^n_{t_k})_{0\le k\le n}$ and $(Z^n_{t_k})_{0\le k\le n-1}$ can then also be represented as a recombining binomial tree. Since $Y^n_{t_n} = g(B^n_{t_n})$, we solve backward in time the BSDE by following these equalities, ensuing from (2) ($Y^n_{t_k}$ has been replaced by $Y^n_{t_{k+1}}$ in the generator term, but the error induced by this modification is smaller than the ones we consider)

$$\begin{cases} Z^n_{t_k} = \dfrac{1}{\sqrt{h}}\mathbb{E}_{\tau_k}\big(Y^n_{t_{k+1}}\varepsilon_{k+1}\big), \\ Y^n_{t_k} = \mathbb{E}_{\tau_k}\big(Y^n_{t_{k+1}} + hf\big(t_{k+1}, B^n_{t_k}, Y^n_{t_{k+1}}, Z^n_{t_k}\big)\big). \end{cases} \tag{16}$$

Practically, we compute (Y^n, Z^n) backward in time in the following way. From (15) it is easy to see that $B^n_{t_k}$ has $k+1$ different values. We store all values of B^n in the upper triangular $(n+1)\times(n+1)$ matrix $\mathtt{B}$ defining $\mathtt{B}[i,k] = \sqrt{h}(k-2i)$ (i and k vary from 0 to n). Similarly we store the values of Y^n and Z^n in the upper triangular $(n+1)\times(n+1)$ matrices $\mathtt{Y}$ and $\mathtt{Z}$ setting $\mathtt{Y}[i,k] = Y^n_{t_k}(B^n_{t_k} = \mathtt{B}[i,k])$ and $\mathtt{Z}[i,k] = Z^n_{t_k}(B^n_{t_k} = \mathtt{B}[i,k])$ (i and k vary from 0 to n). Algorithm 2 describes the computation of the conditional expectations appearing in (16). Note that the computation of the conditional expectation is very simple: to get for instance the values of $\mathbb{E}_{\tau_k}(Y^n_{t_{k+1}})$, we compute $\frac{1}{2}(\mathtt{Y}[i,k+1] + \mathtt{Y}[i+1,k+1])$ for $i = 0, \dots, k+1$.

Algorithm 2 Computation of (Z^n, Y^n)

$\mathtt{Y}[:,n] = g(\mathtt{B}[:,n])$, $\mathtt{Z}[:,n] = 0$
for $k = n-1 : 0$ **do**
  for $i = 0 : (k+1)$ **do**
    Compute $\mathtt{Z}[i,k] = \frac{1}{2\sqrt{h}} * (\mathtt{Y}[i,k+1] - \mathtt{Y}[i+1,k+1])$
    Compute $\mathtt{Y}[i,k] = \frac{1}{2}(\mathtt{Y}[i,k+1] + hf(t[k+1], \mathtt{B}[i,k], \mathtt{Y}[i,k+1], \mathtt{Z}[i,k])$
      $+ \mathtt{Y}[i+1,k+1] + hf(t[k+1], \mathtt{B}[i,k], \mathtt{Y}[i+1,k+1], \mathtt{Z}[i,k]))$
  end for
end for

4.4. Study of the error $\mathbb{E}|Y^n_{t_k} - Y_{t_k}|^2$ and $\mathbb{E}|Z^n_{t_k} - Z_{t_k}|^2$

In this subsection, we assume that we are able to compute the exact solution (Y, Z). We want to study numerically the convergence in n of $\mathbb{E}|Y^n_{t_k} - Y_{t_k}|^2$ and $\mathbb{E}|Z^n_{t_k} - Z_{t_k}|^2$, where (Y, Z) solves (1) and (Y^n, Z^n) solves (6). To do so, we approximate the error $\mathbb{E}|A^n_{t_k} - A_{t_k}|^2$ ($A = Y$ or $A = Z$) by Monte Carlo:

$$\mathbb{E}\big|A^n_{t_k} - A_{t_k}\big|^2 \sim \frac{1}{M}\sum_{m=1}^{M}\big|A^{n,m}_{t_k} - A^m_{t_k}\big|^2 := E_A \tag{17}$$

1. For each Monte Carlo simulation, we pick at random one sequence $(\xi_1, \ldots, \xi_n)$ (which gives the value of $(B^n_{t_1}, \ldots, B^n_{t_n})$) and one sequence $(\tau_1, \ldots, \tau_n)$.
2. From the sequence $(\xi_1, \ldots, \xi_n)$ we get the trajectory of Y^n, including $Y^n_{t_k}$.
3. From the sequence $(B_{\tau_1}, \ldots, B_{\tau_n})$ (which is equal to $(B^n_{t_1}, \ldots, B^n_{t_n})$), we compute B_{t_k} by using the Brownian bridge method. We deduce (Y_{t_k}, Z_{t_k}) as functions of B_{t_k}.

In the following experiments, we plot the logarithm of the errors E_Y and E_Z (defined in (17)) w.r.t. $\log(n)$. From Theorem 3.1, we get that $\log(E_Y)$ and $\log(E_Z)$ decrease as $-\frac{\alpha}{2}\log(n)$. By using a linear regression, we compute the slope of the line solving the least square problem and compare it to $-\frac{\alpha}{2}$.

4.5. Numerical experiment

4.5.1. *Case $g(x) = e^{T+x}$ and $f(y, z) = y + z$*

We consider the BSDE with terminal condition $g(x) = e^{T+x}$ and driver $f(y, z) = y + z$. In this case, we know that $Y_t = e^{T+B_t+\frac{5}{2}(T-t)}$. We run $M = 20{,}000$ Monte Carlo simulations. We fix $T = 1$.

Figure 1 represents log(error on Y) and log(error on Z) at time $t = 0.5$ (the error is defined by (17)) with respect to $\log(n)$, when n varies between 10 to 70 with step 10. For the Y case, the slope ensuing from the linear regression is -0.53. Even though $g(x) = e^{T+x}$ does not satisfy (3), g is locally Lipschitz continuous, and the outcome seems to be consistent with Theorem 3.1 for $\alpha = 1$. For the Z case, we get the slope -0.61.

4.5.2. *Case $g(x) = x^2$ and $f(y, z) = y + z$*

In that case, we know that $Y_t = e^{T-t}((B_t - (T-t))^2 + T - t)$ and $Z_t = 2e^{T-t}(B_t - (T-t))$. We run $M = 20{,}000$ Monte Carlo simulations. We fix $T = 1$.

Figure 2 represents log(error on Y) and log(error on Z) with respect to $\log(n)$ at time $t = 0.5$ when n varies between 10 to 100 with step 10 and from 200 to 500 with step 100. The slope of the linear regression for Y (resp. for Z) is -0.465 (resp. -0.48). The results are then consistent with Theorem 3.1.

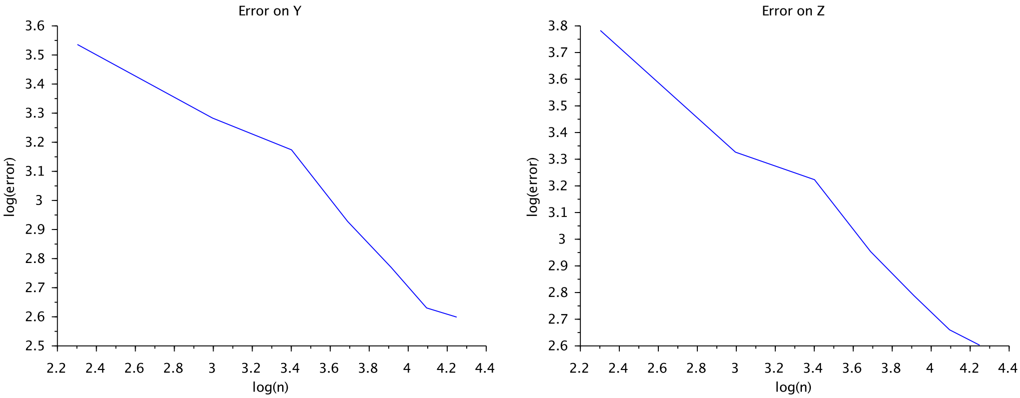

Figure 1. log(error on Y) and log(error on Z) at time $t = 0.5$ w.r.t. $\log(n) - f(y, z) = y + z - g(x) = e^{1+x}$.

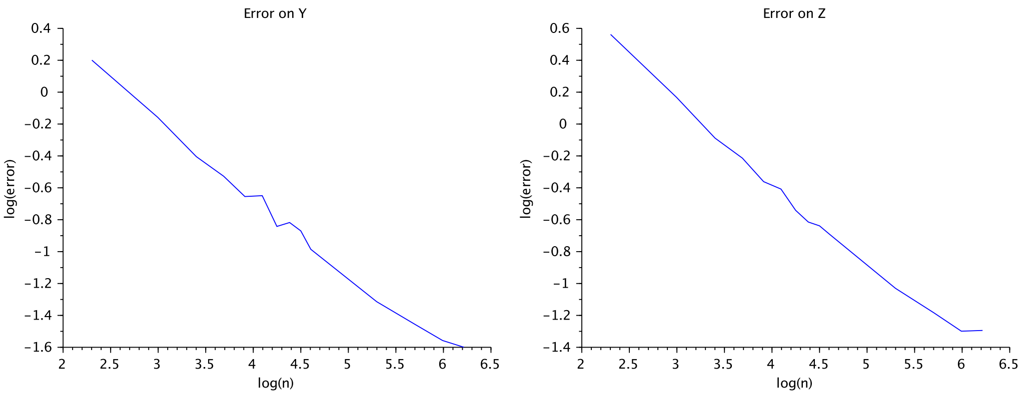

Figure 2. log(error on Y) (on the left) and log(error on Z) (on the right) at time $t = 0.5$ as a function of $\log(n) - f(y, z) = y + z - g(x) = x^2$.

4.5.3. *Case $g(x) = \sqrt{|x|}$ and $f(y, z) = y + z$*

In that case, we know that $Y_t = e^{\frac{T-t}{2}} \tilde{\mathbb{E}}(\sqrt{|\tilde{B}_{T-t} + B_t|} e^{\tilde{B}_{T-t}})$. We run $M = 20{,}000$ Monte Carlo simulations. We fix $T = 1$.

Figure 3 represents log(error on Y) at time $t = 0.5$ and at time $t = 0$ with respect to $\log(n)$ when n varies between 10 to 100 with step 10 and from 200 to 500 by step 100. The slope of the linear regression is -0.51 (resp. -1.7) when $t = 0.5$ (resp. when $t = 0$). Here we notice that the modulus of the slope we get is larger than $\frac{1}{4}$, the upper bound obtained in that case in Theorem 3.1.

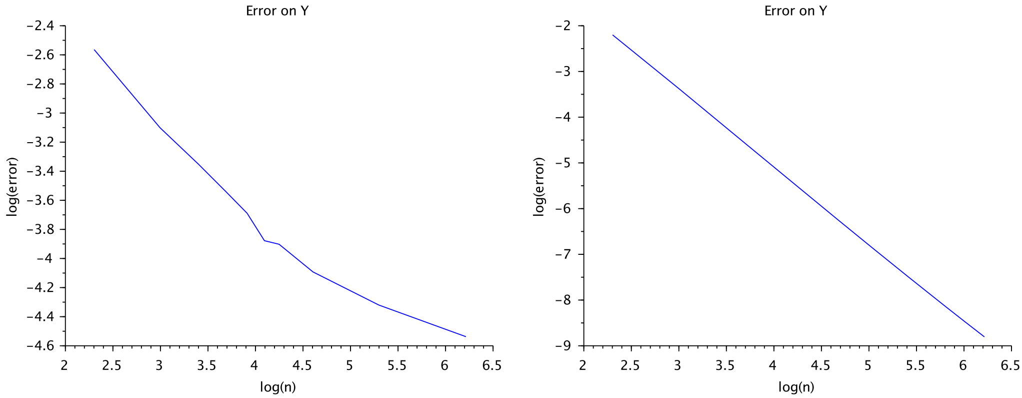

Figure 3. log(error on Y) at time $t = 0.5$ (on the left) and at time $t = 0$ (on the right) as a function of $\log(n)$ – $f(y,z) = y + z$ – $g(x) = \sqrt{|x|}$.

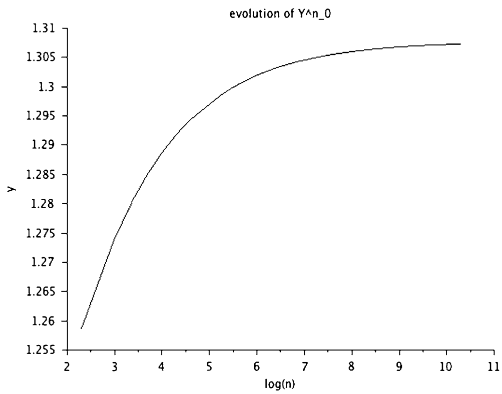

Figure 4. Evolution of Y_0^n as a function of $\log(n)$ – $f(y,z) = \cos(y)$ – $g(x) = |x|^{1/4}$.

4.5.4. *Case* $g(x) = |x|^{1/4}$ *and* $f(y,z) = \cos(y)$

There is no explicit solution. We fix $T = 1$ and plot the evolution of Y_0^n for different values of n where n varies from 10 to 100 by step 10, then from 100 to 1000 by step 100, from 2000 to 10,000 by step 1000 and the last three values are 15,000, 20,000 and 30,000. We notice a slow convergence (in n) of Y_0^n which can be expected to happen in view of Theorem 3.1.

4.5.5. *Case* $g(x) = x^2$ *and* $f(y,z) = \cos(y) + \sin(z)$

There is no explicit solution. We fix $T = 1$ and plot the evolution of Y_0^n and Z_0^n for different values of n (see Figure 5), where n varies from 10 to 100 by step 10. The convergence of Y_0^n and Z_0^n in n is quite fast.

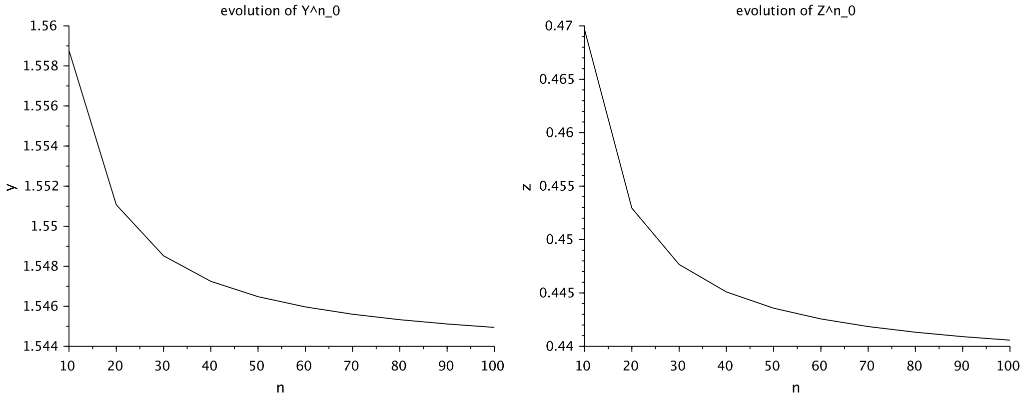

Figure 5. Evolution of Y_0^n (on the left) and Z_0^n (on the right) as a function of n – $f(y,z) = \cos(y) + \sin(z)$ – $g(x) = x^2$.

5. Some properties of solutions to PDEs and BSDEs

In the following, we recall and prove results for FBSDEs with a general forward process, even though we apply them in the present paper only for the case where the forward process is just the Brownian motion. Restricting ourselves to the case of Brownian motion would not shorten the proofs considerably. Let us consider the following SDE started in (t, x),

$$X_s^{t,x} = x + \int_t^s b\big(r, X_r^{t,x}\big)\, dr + \int_t^s \sigma\big(r, X_r^{t,x}\big)\, dB_r, \quad 0 \le t \le s \le T, \tag{18}$$

where b and σ satisfy the following.

Assumption 5.1.

1. $b, \sigma \in C_b^{0,2}([0,T] \times \mathbb{R})$, in the sense that the derivatives of order $k = 0, 1, 2$ w.r.t. the space variable are continuous and bounded on $[0,T] \times \mathbb{R}$,
2. the first and second derivatives of b and σ w.r.t. the space variable are assumed to be γ-Hölder continuous (for some $\gamma \in (0,1]$, w.r.t. the parabolic metric $d((x,t),(x',t')) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}}$ on all compact subsets of $[0,T] \times \mathbb{R}$,
3. b, σ are $\frac{1}{2}$-Hölder continuous in time, uniformly in space,
4. $\sigma(t,x) \ge \delta > 0$ for all (t,x).

5.1. Malliavin weights

In this section, we recall the Malliavin weights and their properties from [22], Section 1.1 and Remark 3.

Lemma 5.2. *Let $H:\mathbb{R}\to\mathbb{R}$ be a polynomially bounded Borel function. If Assumption 5.1 holds and $X^{t,x}$ is given by (18), then setting*

$$G(t,x) := \mathbb{E}H\big(X_R^{t,x}\big)$$

implies that $G\in C^{1,2}([0,R)\times\mathbb{R})$. Especially it holds for $0\le t\le r<R\le T$ that

$$\partial_x G\big(r,X_r^{t,x}\big) = \mathbb{E}\big[H\big(X_R^{t,x}\big)N_R^{r,1,(t,x)}|\mathcal{F}_r^t\big], \quad \textit{and} \quad \partial_x^2 G\big(r,X_r^{t,x}\big) = \mathbb{E}\big[H\big(X_R^{t,x}\big)N_R^{r,2,(t,x)}|\mathcal{F}_r^t\big],$$

where $(\mathcal{F}_r^t)_{r\in[t,T]}$ is the augmented natural filtration of $(B_r^{t,0})_{r\in[t,T]}$,

$$N_R^{r,1,(t,x)} = \frac{1}{R-r}\int_r^R \frac{\nabla X_s^{t,x}}{\sigma(s,X_s^{t,x})\nabla X_r^{t,x}}\,dB_s \quad \textit{and}$$

$$N_R^{r,2,(t,x)} = \frac{N_R^{\rho,1,(t,x)}\nabla X_R^{t,x} N_\rho^{r,1,(t,x)} + \nabla N_R^{\rho,1,(t,x)}}{\nabla X_r^{t,x}},$$

with $\rho := \frac{r+R}{2}$. Moreover, for $q\in(0,\infty)$ it holds a.s.

$$\big(\mathbb{E}\big[\big|N_R^{r,i,(t,x)}\big|^q|\mathcal{F}_r^t\big]\big)^{\frac{1}{q}} \le \frac{\kappa_q}{(R-r)^{\frac{i}{2}}}, \tag{19}$$

and $\mathbb{E}[N_R^{r,i,(t,x)}|\mathcal{F}_r^t]=0$ a.s. for $i=1,2$. Finally, we have

$$\big\|\partial_x G\big(r,X_r^{t,x}\big)\big\|_{L_p(\mathbb{P})} \le \kappa_q \frac{\|H(X_R^{t,x}) - \mathbb{E}[H(X_R^{t,x})|\mathcal{F}_r^t]\|_{L_p(\mathbb{P})}}{\sqrt{R-r}}$$

and

$$\big\|\partial_x^2 G\big(r,X_r^{t,x}\big)\big\|_{L_p(\mathbb{P})} \le \kappa_q \frac{\|H(X_R^{t,x}) - \mathbb{E}[H(X_R^{t,x})|\mathcal{F}_r^t]\|_{L_p(\mathbb{P})}}{R-r}$$

for $1<q,p<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$.

5.2. Regularity of solutions to BSDEs

Let us now consider the FBSDE

$$Y_s^{t,x} = g\big(X_T^{t,x}\big) + \int_s^T f\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,dr - \int_s^T Z_r^{t,x}\,dB_r, \quad 0\le t\le s\le T, \tag{20}$$

where $X^{t,x}$ is the process satisfying (18). The following result is taken from [22], Theorem 1. We reformulate it here for the simple situation where we need it. On the other hand, we will use $\mathbb{P}_{t,x}$ and are interested in an estimate for all $(t,x)\in[0,T)\times\mathbb{R}$.

Theorem 5.3. *Let Assumption 2.1 and 5.1 hold. Then for any $p \in [2, \infty)$ the following assertions are true.*

(i) *There exists a constant $C^y_{5.3} > 0$ such that for $0 \le t < s < T$ and $x \in \mathbb{R}$,*

$$\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \le C^y_{5.3}\Psi(x)\left(\int_t^s (T-r)^{\alpha-1}\,dr\right)^{\frac{1}{2}}.$$

(ii) *There exists a constant $C^z_{5.3} > 0$ such that for $0 \le t < s < T$ and $x \in \mathbb{R}$,*

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \le C^z_{5.3}\Psi(x)\left(\int_t^s (T-r)^{\alpha-2}\,dr\right)^{\frac{1}{2}}.$$

The constants $C^y_{5.3}$ and $C^z_{5.3}$ depend on K_f, L_f, C_g, $c^{1,2}_{5.4}$, T, p_0, b, σ, κ_q and p.

Proof of Theorem 5.3. (i) First, we follow the step [22], Theorem 1, proof of $(C2_l) \implies (C3_l)$. We conclude from the linear growth $|f(r,x,y,z)| \le L_f(|x|+|y|+|z|) + K_f$ and from the Burkholder–Davis–Gundy inequality with constant $a_p > 0$ that

$$\begin{aligned}
&\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \\
&\quad = \left\| \int_t^s f(r, X_r, Y_r, Z_r)\,dr - \int_t^s Z_r\,dB_r \right\|_{L_p(\mathbb{P}_{t,x})} \\
&\quad \le K_f(s-t) + L_f \int_t^s \|X_r\|_{L_p(\mathbb{P}_{t,x})} + \|Y_r\|_{L_p(\mathbb{P}_{t,x})} \\
&\qquad + \|Z_r\|_{L_p(\mathbb{P}_{t,x})}\,dr + a_p\left(\int_t^s \|Z_r\|^2_{L_p(\mathbb{P}_{t,x})}\,dr\right)^{\frac{1}{2}}.
\end{aligned}$$

We then use (i) and (ii) of Theorem 5.4 below to get

$$\begin{aligned}
&\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \\
&\quad \le K_f(s-t) \\
&\qquad + C\big(T, L_f, c^{1,2}_{5.4}, p, b, \sigma, p_0\big)\Psi(x)\left[\int_t^s \big(1 + (T-r)^{\frac{\alpha-1}{2}}\big)\,dr + \left(\int_t^s (T-r)^{\alpha-1}\,dr\right)^{\frac{1}{2}}\right].
\end{aligned}$$

(ii) Here one can follow [22], Theorem 1, proof of $(C4_l) \implies (C1_l)$.

Step 1: We first assume additionally that $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable in x, y, and z with uniformly bounded derivatives as it was assumed for [22], Theorem 1. To take the dependency on x into consideration which arises since we use $\mathbb{P}_{t,x}$, it suffices to replace everywhere in the proof in [22] the constant $c_{B^{\Theta}_{p,\infty}}$ by $C(T, C_g, \sigma, b, p, p_0)\Psi(x)$. The constant $C^z_{5.3}$ depends moreover on L_f and κ_q.

Step 2: Now let f be as in Assumption 5.1. In [22], Theorem 1, proof of $(C4_l) \implies (C1_l)$, a linear BSDE is used which describes the behaviour of the process Z minus its counterpart where the generator is identically 0. Here the partial derivatives of f_x, f_y, f_z appear but only their uniform bound is needed in the estimates. Hence, if f satisfies (4), we can use mollifying as explained in (26) below (one may choose $N = \infty$). Since $|\partial_x f^\varepsilon(t,x,y,z)|$, $|\partial_y f^\varepsilon(t,x,y,z)|$ and $|\partial_z f^\varepsilon(t,x,y,z)|$ are bounded by L_f we conclude from Step 1 that for all $\varepsilon > 0$ the process Z^ε corresponding to f^ε satisfies

$$\left\| Z_s^\varepsilon - Z_t^\varepsilon \right\|_{L_p(\mathbb{P}_{t,x})} \le C_{5.3}^z \Psi(x) \left(\int_t^s (T-r)^{\alpha-2}\, dr \right)^{\frac{1}{2}} \tag{21}$$

for $p \ge 2$. Especially, the family $\{|Z_s^\varepsilon - Z_t^\varepsilon|^q : \varepsilon > 0\}$ is then uniformly integrable provided that $q < p$. By an a priori estimate (cf. [8], Lemma 3.1) we have that

$$\mathbb{E} \int_0^T \left| Z_r - Z_r^\varepsilon \right|^2 dr \le C \int_0^T \sup_{x,y,z} \left| f(r,x,y,z) - f^\varepsilon(r,x,y,z) \right|^2 dr \le C\varepsilon^2 T L_f^2.$$

Fubini's theorem implies that there exists a sequence $\varepsilon_m \to 0$ and a measurable set $N \subseteq [0,T]$ of Lebesgue measure zero, such that $\lim_{m\to\infty} \mathbb{E}|Z_r - Z_r^{\varepsilon_m}|^2 = 0$ for all $r \in [0,T] \setminus N$. Consequently, for any $q < p$ and all $t, s \in [0,T] \setminus N$ with $t < s$,

$$\| Z_s - Z_t \|_{L_q(\mathbb{P}_{t,x})} \le C_{5.3}^z \Psi(x) \left(\int_t^s (T-r)^{\alpha-2}\, dr \right)^{\frac{1}{2}}.$$

The assertion follows for all $q \ge 2$ since (21) holds for all $p \in [2,\infty)$. Since by Theorem 5.4(ii) the process Z does have a continuous version, we finally get the assertion for all $t < s$. □

5.3. Properties of the associated PDE

We collect in the theorem below properties of the solution to the PDE which are mainly known. The new part concerns $\partial_x^2 u$. For Lipschitz continuous g, the behaviour of $\partial_x^2 u$ has been studied in [35]. General results related to this topic can be found in [19].

Theorem 5.4. *Consider the FBSDE* (20) *and let Assumptions* 2.1 *and* 5.1 *hold. Then for the solution u of the associated PDE*

$$\begin{cases} u_t(t,x) + \dfrac{\sigma^2(t,x)}{2} u_{xx}(t,x) + b(t,x)u_x(t,x) + f\big(t,x,u(t,x),\sigma(t,x)u_x(t,x)\big) = 0, \\ \quad t \in [0,T), x \in \mathbb{R}, \\ u(T,x) = g(x), \quad x \in \mathbb{R} \end{cases}$$

we have

(i) $Y_t = u(t, X_t)$ *where* $u(t,x) = \mathbb{E}_{t,x}(g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)\,dr)$ *and* $|u(t,x)| \le c^1_{5.4}\Psi(x)$ *with* Ψ *given in* (5), *where* $c^1_{5.4}$ *depends on* C_g, T, p_0, L_f, K_f *and on the bounds and Lipschitz constants of* b *and* σ.

(ii) u_x *exists*,

$$u_x(t,x) = \mathbb{E}_{t,x}\left(g(X_T)N_T^{t,1} + \int_t^T f(r, X_r, Y_r, Z_r)N_r^{t,1}\,dr\right), \tag{22}$$

and

(a) u_x *is continuous in* $[0,T) \times \mathbb{R}$,

(b) $Z_s^{t,x} = u_x(s, X_s^{t,x})\sigma(s, X_s^{t,x})$,

(c) $|u_x(t,x)| \le \frac{c^2_{5.4}\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}$,

where $c^2_{5.4}$ *depends on* C_g, T, p_0, κ_2, L_f, K_f *and on the bounds and Lipschitz constants of* b *and* σ.

(iii) u_{xx} *exists*,

$$\begin{aligned} u_{xx}(t,x) = \mathbb{E}_{t,x}\bigg(& g(X_T)N_T^{t,2} \\ & + \int_t^T \big[f(r, X_r, Y_r, Z_r) - f(r, X_t, Y_t, Z_t)\big]N_r^{t,2}\,dr\bigg), \end{aligned} \tag{23}$$

and

(a) u_{xx} *is continuous in* $[0,T) \times \mathbb{R}$,

(b) $|u_{xx}(t,x)| \le \frac{c^3_{5.4}\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}$,

where $c^3_{5.4}$ *depends on* C_g, T, p_0, κ_2, L_f, $C^y_{5.3}$, $C^z_{5.3}$ *and on the bounds and Lipschitz constants of* b *and* σ.

In the following $c_{5.4}$ *represents* $(c^1_{5.4}, c^2_{5.4}, c^3_{5.4})$ *and* $c^{i,j}_{5.4}$ $(i \ne j)$ *represents* $(c^i_{5.4}, c^j_{5.4})$, $(i,j) \in \{1,2,3\}$.

Proof. (i): This follows from [37], Theorem 3.2.

(ii): From the proof of [37], Theorem 3.2, we get (22). The points (ii)(a) and (b) ensue from [37], Theorem 3.2(i). It remains to prove (c).

Proof of (ii)(c). We show the assertion for a generator not depending on X, since the terms arising from that dependency would be easy to treat. Since $\mathbb{E}_{t,x}(\mathbb{E}_{t,x}(g(X_T))N_T^{t,1}) = 0$ we can subtract it from the right-hand side of (22) and get

$$\partial_x u(t,x) = \mathbb{E}_{t,x}\left(\big[g(X_T) - \mathbb{E}_{t,x}\big(g(X_T)\big)\big]N_T^{t,1} + \int_t^T f(r, Y_r, Z_r)N_r^{t,1}\,dr\right).$$

It holds

$$\mathbb{E}_{t,x}\big|g(X_T)-\mathbb{E}_{t,x}g(X_T)\big|^2=\mathbb{E}_{t,x}\big|g(X_T)-\tilde{\mathbb{E}}g\big(\tilde X_T^{t,X_t}\big)\big|^2\le \mathbb{E}_{t,x}\tilde{\mathbb{E}}\big|g(X_T)-g\big(\tilde X_T^{t,X_t}\big)\big|^2,$$

and thanks to the Cauchy–Schwarz inequality with $\Psi_1=C_g(1+|X_T|^{p_0}+|\tilde X_T^{t,X_t}|^{p_0})$ and equation (3),

$$\begin{aligned}\mathbb{E}_{t,x}\tilde{\mathbb{E}}\big|g(X_T)-g\big(\tilde X_T^{t,X_t}\big)\big|^2&\le \mathbb{E}_{t,x}\tilde{\mathbb{E}}\big(\Psi_1^2\big|X_T-\tilde X_T^{t,X_t}\big|^{2\alpha}\big)\\&\le\big[\mathbb{E}_{t,x}\tilde{\mathbb{E}}\Psi_1^4\big]^{\frac12}\big[\mathbb{E}_{t,x}\tilde{\mathbb{E}}\big|X_T-\tilde X_T^{t,X_t}\big|^{4\alpha}\big]^{\frac12}\\&\le C(C_g,T,p_0,b,\sigma)\Psi^2(x)(T-t)^\alpha.\end{aligned}\tag{24}$$

Relation (19) and the Lipschitz continuity of f imply

$$\begin{aligned}&\big|\partial_x u(t,x)\big|\\&\quad\le\frac{C(C_g,T,p_0,\kappa_2,b,\sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}\\&\qquad+C(L_f,K_f)\mathbb{E}_{t,x}\int_t^T\big(1+\big|u(r,X_r)\big|+\big|\partial_x u(r,X_r)\sigma(r,X_r)\big|\big)\big|N_r^{t,1}\big|\,dr.\end{aligned}\tag{25}$$

Since we have $|g(x)|\le\Psi(x)$, [37], Theorem 3.2(ii), gives $|u(t,x)|\le c\Psi(x)$ and $|\partial_x u(t,x)|\le c\Psi(x)(T-t)^{-1/2}$, where c depends on T, L_f, K_f, κ_2, b, σ and p_0. Hence, inequality (25) becomes

$$\begin{aligned}\big|\partial_x u(t,x)\big|&\le\frac{C(C_g,T,p_0,\kappa_2,b,\sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}\\&\qquad+C(L_f,K_f,c,\sigma)\mathbb{E}_{t,x}\left(\int_t^T\left(1+\Psi(X_r)+\frac{\Psi(X_r)}{(T-r)^{\frac12}}\right)\big|N_r^{t,1}\big|\,dr\right)\\&\le\frac{C(C_g,T,p_0,\kappa_2,b,\sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}\\&\qquad+C(T,L_f,K_f,\kappa_2,b,\sigma,p_0)\int_t^T\frac{\Psi(x)}{(T-r)^{\frac12}(r-t)^{\frac12}}\,dr\\&\le\frac{C(C_g,T,p_0,\kappa_2,L_f,K_f,b,\sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}.\end{aligned}$$

□

(iii): We start with an approximation of g and f by smooth and bounded functions. Let ϕ be a non-negative C^∞ function with support $[-1,1]$, such that $\int_{\mathbb{R}}\phi(u)\,du=1$, and $\varepsilon\in(0,1]$. For $N\in\mathbb{N}$ let $b_N:\mathbb{R}\to[-N-1,N+1]$ be a monotone C^∞ function such that $0\le b_N'(x)\le 1$

and

$$b_N(x) := \begin{cases} N+1, & x > N+2, \\ x, & |x| \le N, \\ -N-1, & x < -N-2. \end{cases}$$

Define

$$g^{\varepsilon,N}(x) = \int_{-1}^{1} \phi(u) g\big(b_N(x) - \varepsilon u\big)\, du$$

and

$$f^{\varepsilon,N}(r,y,z) = \int_{-1}^{1}\int_{-1}^{1} \phi(u)\phi(v) f\big(r, b_N(y) - \varepsilon u, b_N(z) - \varepsilon v\big)\, du\, dv. \tag{26}$$

Lemma 5.5. *$g^{\varepsilon,N}$ and $f^{\varepsilon,N}$ satisfy*

(a) *$\|g^{\varepsilon,N}\|_\infty + \|f^{\varepsilon,N}\|_\infty \le C = C(\varepsilon, N)$ for some $C(\varepsilon, N) > 0$,*
(b) *$g^{\varepsilon,N}$ and $f^{\varepsilon,N}$ are C^∞ functions, with bounded derivatives (the bounds depend on ε and N). Moreover, $f^{\varepsilon,N}$ is a Lipschitz function in y and z, with Lipschitz constant L_f,*
(c) *$g^{\varepsilon,N}$ satisfies (3), uniformly in $\varepsilon \in (0,1)$ and $N \ge 1$,*
(d) *for all $x \in \mathbb{R}$ and $\varepsilon \in [0,1]$, we have $|g^{\varepsilon,N}(x) - g(x)| \le C(C_g)\Psi(x)(\varepsilon^\alpha + \frac{|x|^{\alpha+1}}{N})$,*
(e) *for all $r \in [0,T]$ and for all $(y,z) \in \mathbb{R}^2$, we have*

$$\big|f^{\varepsilon,N}(r,y,z) - f(r,y,z)\big| \le L_f\big(2\varepsilon + \big|b_N(y) - y\big| + \big|b_N(z) - z\big|\big).$$

Proof.

(a) Since g is locally Hölder continuous in the sense of (3), $|g(x)| \le C_g(1 + |x|^{p_0+1})$. Then, we get $|g^{\varepsilon,N}(x)| \le C_g(1 + (N+1+\varepsilon)^{p_0+1})$, and for f being Lipschitz continuous in y and z, uniformly in time, the same type of result applies.
(b) Since ϕ is a C^∞ function and f and g are of polynomial growth, we get the result.
(c) Since g is locally Hölder continuous, we get

$$\begin{aligned} &\big|g^{\varepsilon,N}(x) - g^{\varepsilon,N}(y)\big| \\ &\quad\le \int_{-1}^{1} \big|\phi(u)\big| C_g\big(1 + \big|b_N(x) - \varepsilon u\big|^{p_0} + \big|b_N(y) - \varepsilon u\big|^{p_0}\big)\big|b_N(x) - b_N(y)\big|^\alpha\, du \\ &\quad\le \int_{-1}^{1} C_g\big|\phi(u)\big|\big(1 + \big(|x| + \varepsilon\big)^{p_0} + \big(|y| + \varepsilon\big)^{p_0}\big)|x-y|^\alpha\, du \\ &\quad\le C(C_g)\big(1 + |x|^{p_0} + |y|^{p_0}\big)|x-y|^\alpha. \end{aligned}$$

(d) We have

$$\big|g^{\varepsilon,N}(x) - g(x)\big| = \left|\int_{-1}^{1} \phi(u)\big(g\big(b_N(x) - \varepsilon u\big) - g(x)\big)\, du\right|$$

$$\leq C_g \int_{-1}^{1} |\phi(u)| \big(1 + |b_N(x)|^{p_0} + \varepsilon^{p_0} + |x|^{p_0}\big)\big(|b_N(x) - x|^{\alpha} + \varepsilon^{\alpha}\big)\, du$$
$$\leq C(C_g)\big(1 + |x|^{p_0}\big)\big(\varepsilon^{\alpha} + |x|^{\alpha} \mathbf{1}_{|x|\geq N}\big),$$

and the result follows.

(e) We simply have to apply the Lipschitz property of f to get the result. □

We put now $\varepsilon := \frac{1}{N}$ and write (g^N, f^N) instead of $(g^{\frac{1}{N},N}, f^{\frac{1}{N},N})$ in order to simplify the notation and consider the BSDE

$$Y_t^N = g^N(X_T) + \int_t^T f^N\big(r, Y_r^N, Z_r^N\big)\, dr - \int_t^T Z_r^N\, dB_r.$$

Representation for $\partial_x^2 u^N(t,x)$

By (i) we have that

$$u^N(t,x) = \mathbb{E}_{t,x} g^N\big(X_T^{t,x}\big) + \int_t^T \mathbb{E}_{t,x} f^N\big(r, Y_r^N, Z_r^N\big)\, dr.$$

According to Lemma 5.2 it holds that $\partial_x^2 \mathbb{E}_{t,x} g^N(X_T) = \mathbb{E}_{t,x}[g^N(X_T) N_T^{t,2}]$ and

$$\partial_x^2 \mathbb{E}_{t,x} f^N\big(r, Y_r^N, Z_r^N\big) = \mathbb{E}_{t,x}\big[f^N\big(r, Y_r^N, Z_r^N\big) N_r^{t,2}\big],$$

because

$$f^N\big(r, Y_r^N, Z_r^N\big) = f^N\big(r, u^N(r, X_r), \sigma(r, X_r) u_x^N(r, X_r)\big),$$

and $f^N(r,y,z)$ is continuous and bounded. Moreover, [24], Proposition 4 (or [21], Theorem 2.1) implies that $u^N(r,x)$ is $C^{1,2}$ and it holds that $|u^N(r,x)| + |\partial_x u^N(r,x)| + |\partial_x^2 u^N(r,x)| \leq C^N$ for some $C^N > 0$. Since σ is continuous,

$$(r,x) \mapsto f^N\big(r, u^N(r,x), \sigma(r,x) u_x^N(r,x)\big)$$

is a bounded Borel function. Notice that by Lemma 5.2

$$\mathbb{E}_{t,x}\big[N_r^{t,2}\big] = 0 \quad \text{and} \quad \mathbb{E}_{t,x}\big[\big(N_r^{t,2}\big)^2\big] \leq \frac{\kappa_2^2}{(r-t)^2}, \tag{27}$$

so that

$$\mathbb{E}_{t,x}\big[f^N\big(r, Y_r^N, Z_r^N\big) N_r^{t,2}\big] = \mathbb{E}_{t,x}\big(\big[f^N\big(r, Y_r^N, Z_r^N\big) - f^N\big(r, Y_t^N, Z_t^N\big)\big] N_r^{t,2}\big).$$

Using the Lipschitz continuity of f^N (see Lemma 5.5), the inequality of Cauchy–Schwarz and Theorem 5.3 one can derive the upper bound

$$\begin{aligned}
&\left|\partial_x^2 \mathbb{E}_{t,x} f^N\big(r, Y_r^N, Z_r^N\big)\right| \\
&\quad\le \mathbb{E}_{t,x}\big[\big|f^N\big(r, Y_r^N, Z_r^N\big) - f^N\big(r, Y_t^N, Z_t^N\big)\big|\big|N_r^{t,2}\big|\big] \\
&\quad\le C(L_f, \kappa_2)\big(\mathbb{E}_{t,x}\big(\big|Y_r^N - Y_t^N\big|^2 + \big|Z_r^N - Z_t^N\big|^2\big)\big)^{\frac{1}{2}} \frac{1}{r-t} \\
&\quad\le C\big(L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z\big)\left[\left(\int_t^r (T-s)^{\alpha-1}\,ds\right)^{\frac{1}{2}} + \left(\int_t^r (T-s)^{\alpha-2}\,ds\right)^{\frac{1}{2}}\right]\frac{\Psi(x)}{r-t} \\
&\quad\le C\big(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z\big)\Psi(x)\frac{1}{(T-r)^{1-\frac{\alpha}{2}}(r-t)^{\frac{1}{2}}}. \qquad (28)
\end{aligned}$$

By this we do have an integrable bound for the derivative, and by dominated convergence we get

$$\begin{aligned}
\partial_x^2 \int_t^T \mathbb{E}_{t,x} f^N\big(r, Y_r^N, Z_r^N\big)\,dr &= \int_t^T \partial_x^2 \mathbb{E}_{t,x} f^N\big(r, Y_r^N, Z_r^N\big)\,dr \\
&= \int_t^T \mathbb{E}_{t,x}\big\{\big[f^N\big(r, Y_r^N, Z_r^N\big) - f^N\big(r, Y_t^N Z_t^N\big)\big]N_r^{t,2}\big\}\,dr.
\end{aligned}$$

Hence, we can write (using Fubini's theorem for the integral)

$$\partial_x^2 u^N(t,x) = \mathbb{E}_{t,x}\left(g^N(X_T)N_T^{t,2} + \int_t^T \big[f^N\big(r, Y_r^N, Z_r^N\big) - f^N\big(r, Y_t^N, Z_t^N\big)\big]N_r^{t,2}\,dr\right).$$

Convergence of $\partial_x^2 u^N(t,x)$

Since $\mathbb{E}_{t,x}[\mathbb{E}_{t,x}(g^N(X_T))N_T^{t,2}] = 0$, Cauchy–Schwarz's inequality and the local Hölder continuity of g^N (see Lemma 5.5) give like in (24) that

$$\begin{aligned}
\big|\mathbb{E}_{t,x}\big(g^N(X_T)N_T^{t,2}\big)\big| &= \big|\mathbb{E}_{t,x}\big(\big[g^N(X_T) - \mathbb{E}_{t,x}\big(g^N(X_T)\big)\big]N_T^{t,2}\big)\big| \\
&\le \big(\mathbb{E}_{t,x}\big(\big|g^N(X_T) - \mathbb{E}_{t,x}\big(g^N(X_T)\big)\big|^2\big)\big)^{\frac{1}{2}} \frac{\kappa_2}{T-t} \\
&\le C(C_g, T, p_0, \kappa_2, b, \sigma)\frac{\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}},
\end{aligned}$$

for all $N \in \mathbb{N}$. For the second term we can use the upper bound (28) and Lemma A.2 to get

$$\begin{aligned}
&\mathbb{E}_{t,x}\int_t^T \big|\big[f^N\big(r, Y_r^N, Z_r^N\big) - f^N\big(r, Y_t^N, Z_t^N\big)\big]N_r^{t,2}\big|\,dr \\
&\quad\le C\big(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z\big)\int_t^T \frac{\Psi(x)}{(T-r)^{1-\frac{\alpha}{2}}(r-t)^{\frac{1}{2}}}\,dr,
\end{aligned}$$

$$\le C\big(T, L_f, \kappa_2, C^y_{5.3}, C^z_{5.3}\big)\Psi(x)\frac{B(\frac{\alpha}{2},\frac{1}{2})}{(T-t)^{\frac{1}{2}-\frac{\alpha}{2}}},$$

which implies

$$\big|\partial_x^2 u^N(t,x)\big| \le C\big(C_g, T, L_f, p_0, \kappa_2, C^y_{5.3}, C^z_{5.3}, b, \sigma\big)\frac{\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}. \tag{29}$$

According to [21], Theorem 2.1, $\partial_x^2 u^N(t,x)$ is continuous. Let

$$v(t,x) := \mathbb{E}_{t,x}\bigg(g(X_T)N_T^{t,2} + \int_t^T \big[f(r,Y_r,Z_r) - f(r,Y_t,Z_t)\big]N_r^{t,2}\,dr\bigg).$$

We show that for any $(t,x) \in [0,T)\times\mathbb{R}$ it holds $\partial_x^2 u^N(t,x) \to v(t,x)$ if $N\to\infty$, and that v is continuous on $[0,T)\times\mathbb{R}$. The idea to show continuity of v is as follows: If $(t_n,x_n)\to(t,x)$, then we may assume that we can find a $\delta>0$ such that $x_n\in(x-\delta,x+\delta)$ and $t_n\in(t-\delta,t+\delta)\subseteq[0,T)$ for each sufficiently large n. We consider

$$\begin{aligned}\big|v(t_n,x_n)-v(t,x)\big| &\le \big|v(t_n,x_n)-\partial_x^2u^N(t_n,x_n)\big| + \big|\partial_x^2u^N(t_n,x_n)-\partial_x^2u^N(t,x)\big| \\ &\quad + \big|\partial_x^2u^N(t,x)-v(t,x)\big|.\end{aligned}$$

Since $\partial_x^2u^N$ is continuous, the term $|\partial_x^2u^N(t_n,x_n)-\partial_x^2u^N(t,x)|$ is small for large n. Hence, it suffices to show that $\sup_{s\in(t-\delta,t+\delta),y\in(x-\delta,x+\delta)}|\partial_x^2u^N(s,y)-v(s,y)|$ is small for large N. Let $(s,y)\in(t-\delta,t+\delta)\times(x-\delta,x+\delta)$. It holds

$$\big|\partial_x^2u^N(s,y)-v(s,y)\big| \le \mathbb{E}_{s,y}\big|\big[g^N(X_T)-g(X_T)\big]N_T^{s,2}\big| + \int_s^T D^{\frac{1}{2}}(r,s)\frac{\kappa_2}{r-s}\,dr := D_1 + D_2,$$

where (setting $\|\cdot\|_{\mathbb{P}_{s,y}} := \|\cdot\|_{L_2(\mathbb{P}_{s,y})}$)

$$\begin{aligned}D(r,s) &:= \big\|f^N\big(r,Y_r^N,Z_r^N\big) - f^N\big(r,Y_s^N,Z_s^N\big) - \big[f(r,Y_r,Z_r)-f(r,Y_s,Z_s)\big]\big\|^2_{\mathbb{P}_{s,y}} \\ &\le L_f\big(\big\|Y_r^N-Y_s^N\big\|_{\mathbb{P}_{s,y}} + \big\|Z_r^N-Z_s^N\big\|_{\mathbb{P}_{s,y}} + \|Y_r-Y_s\|_{\mathbb{P}_{s,y}} + \|Z_r-Z_s\|_{\mathbb{P}_{s,y}}\big) \\ &\quad \times \big(\big\|f^N\big(r,Y_r^N,Z_r^N\big)-f(r,Y_r,Z_r)\big\|_{\mathbb{P}_{s,y}} + \big\|f^N\big(r,Y_s^N,Z_s^N\big)-f(r,Y_s,Z_s)\big\|_{\mathbb{P}_{s,y}}\big).\end{aligned}$$

First, let us bound D_1. According to Cauchy–Schwarz's inequality, (31) below and (27) we get

$$D_1 \le \delta_1\sqrt{\mathbb{E}_{s,y}\big(\big|N_T^{s,2}\big|^2\big)} \le \frac{\delta_1\kappa_2}{T-s} \le \frac{\delta_1\kappa_2}{T-t-\delta}.$$

Now let us bound D_2. According to Theorem 5.3 it holds

$$\begin{aligned}D^{\frac{1}{2}}(r,s) &\le C\big(T,L_f,C^y_{5.3},C^z_{5.3}\big)\Psi^{\frac{1}{2}}(y)\frac{(r-s)^{\frac{1}{4}}}{(T-r)^{\frac{1}{2}-\frac{\alpha}{4}}} \\ &\quad \times \big(\big\|f^N\big(r,Y_r^N,Z_r^N\big)-f(r,Y_r,Z_r)\big\|_{\mathbb{P}_{s,y}} + \big\|f^N\big(r,Y_s^N,Z_s^N\big)-f(r,Y_s,Z_s)\big\|_{\mathbb{P}_{s,y}}\big)^{\frac{1}{2}}.\end{aligned}$$

Then, using (32), (34), (35) and Proposition 5.6 below gives

$$D^{\frac{1}{2}}(r,s) \le C\big(T, L_f, \kappa_2, C^y_{5.3}, C^z_{5.3}\big)\Psi(y)\frac{(r-s)^{\frac{1}{4}}}{(T-r)^{\frac{1}{2}-\frac{\alpha}{4}}}\frac{\delta_1}{(T-r)^{\frac{1}{4}}}.$$

Hence, we have shown that

$$\begin{aligned}
D_2 &\le C\big(T, L_f, \kappa_2, C^y_{5.3}, C^z_{5.3}\big)\Psi(y)\int_s^T \frac{\delta_1}{(r-s)^{\frac{3}{4}}(T-r)^{\frac{1}{2}-\frac{\alpha}{4}+\frac{1}{4}}}\,dr \\
&\le C\big(T, L_f, \kappa_2, C^y_{5.3}, C^z_{5.3}\big)\Psi(y)\frac{\delta_1}{(T-s)^{\frac{1}{2}-\frac{\alpha}{4}}}, \\
&\le C\big(T, L_f, \kappa_2, C^y_{5.3}, C^z_{5.3}\big)\Psi(x+\delta)\frac{\delta_1}{(T-t-\delta)^{\frac{1}{2}-\frac{\alpha}{4}}} \\
&\qquad \forall (s,y) \in (t-\delta, t+\delta)\times(x-\delta, x+\delta).
\end{aligned}$$

Consequently, $\sup_{y\in(x-\delta,x+\delta),s\in(t-\delta,t+\delta)} |\partial_x^2 u^N(s,y) - v(s,y)|$ is small for large N, hence v is continuous. Since

$$\partial_x u^N(t,x) - \partial_x u^N(t,y) = \int_y^x \partial_x^2 u^N(t,z)\,dz$$

converges to

$$\partial_x u(t,x) - \partial_x u(t,y) = \int_y^x v(t,z)\,dz,$$

it follows that $\partial_x^2 u(t,x) = v(t,x)$. Then point (iii-a) and (23) are proved. Since $\partial_x^2 u^N$ converges to v for $N \to \infty$, we deduce point (iii-b) from (29). □

Proposition 5.6. *Let Assumptions 5.1 and 2.1 hold. Then for any $(s,y) \in (t-\delta, t+\delta)\times(x-\delta, x+\delta)$ with $t+\delta < T$ and r such that $s \le r < T$ we have*

$$\big\|Y^N_r - Y_r\big\|_{L_2(\mathbb{P}_{s,y})} + \big\|Z^N_r - Z_r\big\|_{L_2(\mathbb{P}_{s,y})} \le \frac{\delta_1}{\sqrt{T-r}},$$

where δ_1 denotes a generic constant which tends to 0 when N tends to $+\infty$.

Proof. Let here $\|\cdot\|$ stand for $\|\cdot\|_{L_2(\mathbb{P}_{s,y})}$. We will use for the Y differences the inequality

$$\big\|Y^N_r - Y_r\big\| \le \big\|g^N(X_T) - g(X_T)\big\| + \int_r^T \big\|f^N\big(w, Y^N_w, Z^N_w\big) - f(w, Y_w, Z_w)\big\|\,dw.$$

For the Z differences, we get by (22) and (ii-b)

$$\begin{aligned}
&\|Z_r^N - Z_r\| \\
&\quad \le C(\sigma)\bigg(\big\|\mathbb{E}_r\big(g^N(X_T) - g(X_T)\big)N_T^{r,1}\big\| \\
&\qquad + \bigg\|\mathbb{E}_r \int_r^T \big(f^N\big(w, Y_w^N, Z_w^N\big) - f(w, Y_w, Z_w)\big)N_w^{r,1}\, dw\bigg\|\bigg) \\
&\quad \le C(\kappa_2, \sigma)\bigg(\frac{\|g^N(X_T) - g(X_T)\|}{\sqrt{T-r}} + \int_r^T \big\|f^N\big(w, Y_w^N, Z_w^N\big) - f(w, Y_w, Z_w)\big\| \frac{1}{\sqrt{w-r}}\, dw\bigg).
\end{aligned}$$

Let $S(r) := \|Y_r^N - Y_r\| + \|Z_r^N - Z_r\|$. Using the inequality $(1 + \frac{1}{\sqrt{w-r}}) \le C(T)\frac{1}{\sqrt{w-r}}$ for $r < w \le T$ gives

$$\begin{aligned}
S(r) \le C(T, \kappa_2, \sigma)\bigg(&\big\|g^N(X_T) - g(X_T)\big\| \frac{1}{\sqrt{T-r}} \\
&+ \int_r^T \big\|f^N\big(w, Y_w^N, Z_w^N\big) - f(w, Y_w, Z_w)\big\| \frac{1}{\sqrt{w-r}}\, dw\bigg).
\end{aligned} \tag{30}$$

Let us bound $\|g^N(X_T) - g(X_T)\|$. By Lemma 5.5, we get the estimate

$$\begin{aligned}
\mathbb{E}_{s,y}\big|g^N(X_T) - g(X_T)\big|^2 &\le C(C_g)\mathbb{E}_{s,y}\big(\Psi(X_T)^4\big)^{\frac{1}{2}}\bigg(\mathbb{E}_{s,y}\bigg(\frac{1}{N^\alpha} + \frac{|X_T|^{\alpha+1}}{N}\bigg)^4\bigg)^{\frac{1}{2}} \\
&\le C(C_g, T, b, \sigma, p_0)\Psi(y)^2\bigg(\frac{1}{N^{2\alpha}} + \frac{|y|^{2\alpha+2}}{N^2}\bigg) \\
&\le C(C_g, T, b, \sigma, p_0)\Psi(x+\delta)^2\bigg(\frac{1}{N^{2\alpha}} + \frac{|x+\delta|^{2\alpha+2}}{N^2}\bigg) \le \delta_1^2,
\end{aligned} \tag{31}$$

for any arbitrarily small $\delta_1 > 0$, provided that N is sufficiently large. Let us now bound $\|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\|$. Using again Lemma 5.5 yields to

$$\begin{aligned}
&\big\|f^N\big(w, Y_w^N, Z_w^N\big) - f(w, Y_w, Z_w)\big\| \\
&\quad \le \big\|f^N\big(w, Y_w^N, Z_w^N\big) - f^N(w, Y_w, Z_w)\big\| + \big\|f^N(w, Y_w, Z_w) - f(w, Y_w, Z_w)\big\| \\
&\quad \le L_f\bigg(\big\|Y_w^N - Y_w\big\| + \big\|Z_w^N - Z_w\big\| + \frac{2}{N} + \big\|b_N(Y_w) - Y_w\big\| + \big\|b_N(Z_w) - Z_w\big\|\bigg).
\end{aligned} \tag{32}$$

Then, plugging (31) and (32) into (30) gives

$$S(r) \le \frac{C(T,\kappa_2,\sigma)\delta_1}{\sqrt{T-r}} + C(T,\kappa_2,\sigma)L_f \int_r^T \frac{S(w)}{\sqrt{w-r}}dw$$
$$+ C(T,\kappa_2,\sigma)L_f \int_r^T \frac{\frac{1}{N} + \|b_N(Y_w)-Y_w\| + \|b_N(Z_w)-Z_w\|}{\sqrt{w-r}}dw. \tag{33}$$

To estimate $\|b_N(Z_w)-Z_w\|$ we use $Z_w = \sigma(w,X_w)u_x(w,X_w)$ and choose a small $a>0$ such that $\beta := \frac{(2+a)(1-\alpha)}{2} < 1$. Then

$$\left\|b_N(Z_w)-Z_w\right\|^2 = \mathbb{E}_{s,y}\left|b_N(Z_w)-Z_w\right|^2 \mathbf{1}_{|Z_w|\ge N}$$
$$\le \frac{\mathbb{E}_{s,y}|Z_w|^{2+a}}{N^a} = \frac{\mathbb{E}_{s,y}|\sigma(w,X_w)u_x(w,X_w)|^{2+a}}{N^a}.$$

Using Theorem 5.4(ii-c) yields

$$\mathbb{E}_{s,y}\left|b_N(Z_w)-Z_w\right|^2 \le \frac{C(c^2_{5.4},\sigma)\mathbb{E}_{s,y}\Psi(X_w)^{(2+a)}}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}N^a} \le \frac{C(T,p_0,c^2_{5.4},\sigma,b)\Psi(y)^{(2+a)}}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}N^a}$$
$$\le \frac{\delta_1}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}}, \quad \forall (s,y) \in (t-\delta,t+\delta)\times(x-\delta,x+\delta). \tag{34}$$

Similarly,

$$\mathbb{E}_{s,y}\left|b_N(Y_w)-Y_w\right|^2 \le \frac{C(T,p_0,c^1_{5.4},b,\sigma)\Psi(y)^{(2+a)}}{N^a}$$
$$\le \delta_1, \quad \forall (s,y) \in (t-\delta,t+\delta)\times(x-\delta,x+\delta). \tag{35}$$

Plugging (34) and (35) into (33) gives

$$S(r) \le \frac{C(T,\kappa_2)\delta_1}{\sqrt{T-r}} + C(T,\kappa_2)L_f \int_r^T \frac{S(w)}{\sqrt{w-r}}dw$$
$$+ C(T,\kappa_2)L_f \int_r^T \frac{\frac{1}{N}+\delta_1}{\sqrt{w-r}} + \frac{\delta_1}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}\sqrt{w-r}}dw$$
$$\le C(T,\kappa_2,L_f)\left(\frac{\delta_1}{\sqrt{T-r}} + \int_r^T \frac{S(w)}{\sqrt{w-r}}dw\right),$$

where the last inequality comes from Lemma A.2 ($\beta < 1$). If we iterate this inequality by replacing $S(w)$ with its estimate and then change the oder of integration we get by Lemma A.2 that

$$S(r) \le C(T,\kappa_2,L_f)$$

$$\times \left(\frac{\delta_1}{\sqrt{T-r}} + \int_r^T \frac{\delta_1}{\sqrt{T-w}} \frac{1}{\sqrt{w-r}} dw + \int_r^T \int_w^T \frac{S(v)}{\sqrt{v-w}\sqrt{w-r}} dv dw \right)$$
$$\leq C(T, \kappa_2, L_f) \left(\frac{\delta_1}{\sqrt{T-r}} + \delta_1 B\left(\frac{1}{2}, \frac{1}{2}\right) + B\left(\frac{1}{2}, \frac{1}{2}\right) \int_r^T S(v) dv \right).$$

It remains to apply Gronwall's lemma to see that $S(r) \leq \frac{C(T,\kappa_2,L_f)\delta_1}{\sqrt{T-r}}$. Since $C(T, \kappa_2, L_f)\delta_1$ becomes arbitrarily small for N large, we will slightly abuse the notation and write $S(r) \leq \frac{\delta_1}{\sqrt{T-r}}$. □

Appendix: Technical results and estimates

Lemma A.1. *For all* $0 \leq k \leq m \leq n$ *and* $p > 0$, *it holds for* $h = \frac{T}{n}$ *that*

(i) $\mathbb{E}\tau_k = kh$,
(ii) $\mathbb{E}|\tau_1|^p \leq C(p)h^p$,
(iii) $\mathbb{E}|B_{\tau_m} - B_{\tau_k}|^2 = t_m - t_k$,
(iv) $\mathbb{E}|B_{\tau_k} - B_{t_k}|^{2p} \leq C(p)\mathbb{E}|\tau_k - t_k|^p \leq C(p)(t_k h)^{\frac{p}{2}}$.

Proof. The strong Markov property of the Brownian motion implies that $(\tau_i - \tau_{i-1})_{i=1}^{\infty}$ is an i.i.d. sequence. According to [33], Proposition 11.1(iii), we have that $\mathbb{E}\tau_1 = \frac{T}{n}$, and (i) follows. Item (ii) follows by [33], Proposition 11.1(iv), and Jensen's inequality. To prove item (iii), recall that $(B_{\tau_i} - B_{\tau_{i-1}})_{i=1}^{\infty}$ is a centered i.i.d. sequence with $\mathbb{E}(B_{\tau_i} - B_{\tau_{i-1}})^2 = h$, $i \geq 1$. (iv): The BDG inequality implies that for each $p > 0$,

$$\mathbb{E}|B_{\tau_k} - B_{t_k}|^p = \mathbb{E}\left| \int_0^{\tau_k \vee t_k} \left(\mathbf{1}_{[0,\tau_k]}(r) - \mathbf{1}_{[0,t_k]}(r)\right) dB_r \right|^p$$
$$\leq C(p)\mathbb{E}\left(\int_0^{\tau_k \vee t_k} \mathbf{1}_{[0,\tau_k]\Delta[0,t_k]}(r)\, dr \right)^{p/2} = \mathbb{E}|\tau_k - t_k|^{p/2}.$$

To prove the second inequality of (iv), a generalization of [33], Proposition 11.1(iv), we first assume that $p \geq 1$. Let us rewrite $\tau_k - t_k = \sum_{i=1}^k \eta_i$ where $(\eta_i)_{1 \leq i \leq k}$ is an i.i.d. centered sequence of random variables distributed as $\tau_1 - h$. Burkholder's and Hölder's inequalities, and finally item (ii) yield

$$\mathbb{E}|\tau_k - t_k|^p \leq C(p)\mathbb{E}\left(\sum_{i=1}^k \eta_i^2 \right)^{\frac{p}{2}} \leq k^{\frac{p}{2}-1} \sum_{i=1}^k \mathbb{E}\left(\eta_i^p\right) \leq C(p)(t_k h)^{\frac{p}{2}},$$

which proves the claim for $p \geq 1$. The case $p < 1$ follows from this result by Jensen's inequality. □

Lemma A.2. *For all* $t \in [0, T)$ *and for all* $\alpha < 1$, $\beta < 1$ *we have*

$$\int_t^T \frac{1}{(T-r)^{\alpha}(r-t)^{\beta}} dr = \frac{1}{(T-t)^{\alpha+\beta-1}} B(1-\alpha, 1-\beta),$$

where B *denotes the beta function.*

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