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A central limit theorem for the realised covariation of a bivariate Brownian semistationary process

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This article presents a weak law of large numbers and a central limit theorem for the scaled realised covariation of a bivariate Brownian semistationary process. The novelty of our results lies in the fact that we derive the suitable asymptotic theory both in a multivariate setting and outside the classical semimartingale framework. The proofs rely heavily on recent developments in Malliavin calculus.

Keywords: bivariate Brownian semistationary process; central limit theorem; fourth moment theorem; high frequency data; moving average process; multivariate setting; stable convergence

1. Introduction

Within the realm of stochastic processes that fail to be a semimartingale, the recent literature has devoted particular attention to the Brownian semistationary ($\mathcal{BSS}$) process, a process that has originally been used in the context of turbulence modelling in [8], but has been subsequently employed as a price process in energy markets in [2]. The $\mathcal{BSS}$ process in its most basic form can be written as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \, dW_s,$$

for a deterministic kernel function g, a stochastic volatility process σ and a Brownian motion W. [34] proved that $\mathcal{BSS}$ processes have conditional full support and thus may be used as a price model in financial markets with transaction costs. Also, $\mathcal{BSS}$ processes can be used in the context of option pricing, through the modelling of *rough volatility* (see [24] and [14]). In this context, [15] present a hybrid simulation scheme used in Monte Carlo option pricing.

Its spreading use in applications has led to many theoretical questions, some of which have only recently obtained an answer.

Still, the stochastic-analytic properties of the Brownian semistationary process are not yet completely understood. The univariate case has been studied in detail, and in particular, numerous papers have been published that deal with its asymptotic theory of multipower variation.

The theory of multipower variation for semimartingales was first introduced in [10] and expanded in several subsequent papers (see [7,11,12,27,29,31,36]). One of the main applications of multipower variation is the construction of robust estimators that allow to disentangle the impact of the jump risk from the stochastic volatility risk in the price of financial assets.

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Outside the semimartingale class a general theory seems to be impossible to achieve and results have to be proved for the particular collection of processes under consideration. For the univariate $\mathcal{BSS}$ process, one can see, for example, [4] with their study of multipower variation through Malliavin calculus and the more recent paper [5] which deals with the multipower variation of higher order differences of the $\mathcal{BSS}$ process in order to estimate its smoothness.

In the present paper, we define and work with the bivariate Brownian semistationary process. The introduction of a second dimension greatly increases the complexity, but also allows for novel possibilities in terms of modelling dependence. Given the importance in practical applications of the Brownian semistationary process, the first natural result in the multivariate theory must be a limit theorem allowing inference to be performed on the dependence between two components.

In the semimartingale case, inference on the dependence can be performed through the quadratic covariation between two processes. Applying the same ideas to this setting immediately poses the question of whether the quadratic covariation can be successfully defined between two $\mathcal{BSS}$ processes. There are very few results in the literature concerning quadratic covariation between two non semimartingales. As an example, [23] deal with this problem, but they only consider $[X, F(X)]$, where X is a semimartingale and F is an absolutely continuous function with square integrable derivative. In this case $F(X)$ is not necessarily a semimartingale, while X always is.

We instead propose the study of $[Y^{(1)}, Y^{(2)}]$, when both $Y^{(1)}$ and $Y^{(2)}$ are $\mathcal{BSS}$ processes and are not semimartingales. Hence, the aim is to show convergence of an appropriately scaled version of the following realised covariation process:

$$\sum_{i=1}^{\lfloor nt \rfloor} \bigl(Y^{(1)}_{\frac{i}{n}} - Y^{(1)}_{\frac{i-1}{n}}\bigr)\bigl(Y^{(2)}_{\frac{i}{n}} - Y^{(2)}_{\frac{i-1}{n}}\bigr). \tag{1}$$

A weak law of large numbers in such a setting has recently been obtained in [25]. Here, we tackle the arguably more difficult case of deriving a suitable central limit theorem. Central limit theorems for processes are results which are usually hard to prove, and techniques to prove them vary from case to case. The most celebrated result of this kind is Donsker's theorem, which states that an appropriately scaled, symmetric random walk converges weakly to Brownian motion (a standard reference is [17]). The high frequency limits of semimartingales are typically processes with a mixed Gaussian distributions, and these central limit theorem results are typically stronger than the standard ones that only state weak convergence in the Skorokhod space, in order for statistical inference to be performed in a feasible way. They instead involve *stable convergence* of processes, which involves proving weak convergence in an extended sample space, where typically a new Brownian motion lives, which is independent from the original processes. We will see that such results can also be obtained in our more general non-semimartingale setting.

The methods we use in our proofs rely heavily on the powerful *Fourth Moment Theorem* which was proven in [33]. Their theory was developed by combining Stein's method with Malliavin calculus. The most comprehensive reference on the subject is the monograph [32].

The outline of the remainder of this article is as follows. Section 2 introduces the notation and defines the bivariate Gaussian core and the bivariate Brownian semistationary process. Moreover, we formulate assumptions which ensure that we are outside the semimartingale setting (since the corresponding theory is well-known in the semimartingale framework). The main contributions of our article can be found in Sections 3 and 4, where we state the central limit theorems for a suitably scaled version of the realised covariation of a Gaussian core and a Brownian semistationary process, respectively. Section 5 concludes. The proof of the central limit theorem in the case of the Gaussian core is presented in Section 6, and in the case of a Brownian semistationary process in Section 7. A brief self-contained summary of the key concepts of Malliavin calculus and the celebrated Fourth Moment Theorem needed for proving our results and some of the proofs of our new results are relegated to the supplemental article [26].

2. The setting

Throughout this article we denote by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a filtered, complete probability space and by $\mathcal{B}(\mathbb{R})$ the class of Borel subsets of $\mathbb{R}$ and we consider a finite time horizon $[0, T]$ for some $T > 0$. We will assume that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ supports two independent $\mathcal{F}_t$-Brownian measures $W^{(1)}$, $W^{(2)}$ on $\mathbb{R}$, for which we briefly recall the definition.

Definition 2.1 (Brownian measure). An $\mathcal{F}_t$-adapted Brownian measure $W\colon \Omega \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a Gaussian stochastic measure such that, if $A \in \mathcal{B}(\mathbb{R})$ with $\mathbb{E}[(W(A))^2] < \infty$, then $W(A) \sim N(0, \mathrm{Leb}(A))$, where Leb is the Lebesgue measure. Moreover, if $A \subseteq [t, +\infty)$, then $W(A)$ is independent of $\mathcal{F}_t$.

Let us first define the so-called bivariate Gaussian core, which is in fact a bivariate Gaussian moving average process with correlated components.

Definition 2.2 (The Gaussian core). Consider two Brownian measures $W^{(1)}$ and $W^{(2)}$ adapted to $\mathcal{F}_t$ with $dW_t^{(1)}\,dW_t^{(2)} = \rho\,dt$, for $\rho \in [-1, 1]$. Further take two nonnegative deterministic functions $g^{(1)}, g^{(2)} \in L^2((0, \infty))$ which are continuous on $\mathbb{R} \setminus \{0\}$. Define, for $j \in \{1, 2\}$,

$$G_t^{(j)} := \int_{-\infty}^{t} g^{(j)}(t-s)\,dW_s^{(j)}.$$

Then the vector process $(\mathbf{G}_t)_{t\geq 0} = (G_t^{(1)}, G_t^{(2)})_{t\geq 0}^{\top}$ is called the (bivariate) Gaussian core.

If we add stochastic volatility to the Gaussian core, then we obtain a bivariate Brownian semistationary ($\mathcal{BSS}$) process defined as follows.

Definition 2.3 (Bivariate Brownian semistationary process). Consider two Brownian measures $W^{(1)}$ and $W^{(2)}$ adapted to $\mathcal{F}_t$ with $dW_t^{(1)}\,dW_t^{(2)} = \rho\,dt$, for $\rho \in [-1, 1]$. Further take two nonnegative deterministic functions $g^{(1)}, g^{(2)} \in L^2((0, \infty))$ which are continuous on $\mathbb{R} \setminus \{0\}$. Let further $\sigma^{(1)}, \sigma^{(2)}$ be càdlàg, $\mathcal{F}_t$-adapted stochastic processes and assume that for $j \in \{1, 2\}$, and for all $t \in [0, T]$: $\int_{-\infty}^{t} g^{(j)2}(t-s)\sigma_s^{(j)2}\,ds < \infty$. Define, for $j \in \{1, 2\}$,

$$Y_t^{(j)} := \int_{-\infty}^{t} g^{(j)}(t-s)\sigma_s^{(j)}\,dW_s^{(j)}.$$

Then the vector process $(\mathbf{Y}_t)_{t\geq 0} = (Y_t^{(1)}, Y_t^{(2)})_{t\geq 0}^\top$ is called a bivariate Brownian semistationary process.

2.1. Technical assumptions

Let us now introduce a few working assumptions. Most of them are standard and already appear in similar forms in the literature, for example, in [20].

2.1.1. *(Non-) semimartingale conditions*

As mentioned in the introduction, we are exclusively interested in the non-semimartingale setting since the corresponding asymptotic theory for semimartingales is well established in the literature, see, for example, [9,35]. It turns out that the (non-) semimartingale property of $G^{(j)}$ or $Y^{(j)}$ (for $j = 1, 2$) depends on the properties of the functions $g^{(j)}$.

Let us for a moment suppress the superscripts and write $G_t = \int_{-\infty}^t g(t-s)\, dW_s$ for a univariate Gaussian core. Consider the filtration $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$ which is the smallest filtration with respect to which W is an adapted Brownian measure and recall the classical result due to [30].

Theorem 2.4 (Knight). *The process $(G_t)_{t\geq 0}$ is an $\mathcal{F}_t^{W,\infty}$-semimartingale if and only if there exists $h \in L^2(\mathbb{R})$ and $\alpha \in \mathbb{R}$ such that: $g(t) = \alpha + \int_0^t h(s)\, ds$.*

In the case of a univariate Brownian semistationary ($\mathcal{BSS}$) process given by

$$Y_t = \int_{-\infty}^t g(t-s)\sigma_s\, dW_s, \tag{2}$$

[8] derived the following sufficient conditions for a $\mathcal{BSS}$ process Y to be a semimartingale:

Theorem 2.5. *Under the assumptions that* (i) *g is absolutely continuous and $g' \in L^2((0,\infty))$*, (ii) $\lim_{x\to 0^+} g(x) =: g(0^+) < \infty$, (iii) *the process $g'(-\cdot)\sigma.$ is square integrable*, *then Y_t defined as in (2) is an $\mathcal{F}_t^{W,\infty}$-semimartingale. In this case, Y_t admits the decomposition: $Y_t = g(0^+)W_t + \int_0^t dl[\int_{-\infty}^l g'(l-s)\sigma_s\, dW_s]$.*

Let us now return to the bivariate case and formulate conditions which ensure that the bivariate processes $\mathbf{G}, \mathbf{Y}$ are *not* semimartingales. This can be achieved by relaxing the first two assumptions in Theorem 2.5 since both assumptions are necessary for $G^{(j)}$ to belong to the semimartingale class (see [13]) for $j = 1, 2$.

Assumption 2.1. For $j \in \{1, 2\}$, we assume that $g^{(j)} : \mathbb{R} \to \mathbb{R}^+$ are nonnegative functions and continuous, except possibly at $x = 0$. Also, $g^{(j)}(x) = 0$ for $x < 0$ and $g^{(j)} \in L^2((0, +\infty))$. We further ask that $g^{(j)}$ be differentiable everywhere with derivative $(g^{(j)})' \in L^2((\varepsilon, \infty))$ for all $\varepsilon > 0$ and $((g^{(j)})')^2$ non-increasing in $[b^{(j)}, \infty)$, for some $b^{(j)} > 0$. Moreover, we assume that, for any $t > 0$:

$$\int_1^\infty \big(g^{(i)}(s)\big)^2 \big(\sigma_{t-s}^{(i)}\big)^2 \, ds < \infty.$$

In the following we will set $b = \max\{b^{(1)}, b^{(2)}\}$, then $(g^{(j)})' \in L^2((b,\infty))$ and $((g^{(j)})')^2$ is non-increasing in $[b,\infty)$ for $j = 1, 2$.

It is important to note that we are not imposing that $(g^{(j)})' \in L^2((0,\infty))$ in order to be able to exclude the semimartingale case.

2.1.2. *Technical assumptions for the cross-correlations*

We need some additional technical assumptions to control the terms arising in the covariation between the two components of the bivariate Gaussian core and the bivariate $\mathcal{BSS}$ process. Such assumptions will be formulated in terms of slowly varying functions, for which we briefly recall the definition, see e.g. [18].

Definition 2.6 (Slowly and regularly varying function). A measurable function $L\colon (0,\infty) \to (0,\infty)$ is called *slowly varying at infinity* if, for all $\lambda > 0$ we have that $\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1$. A function $g\colon (0,\infty) \to (0,\infty)$ is called *regularly varying at infinity* if, for x large enough, it can be written as: $g(x) = x^\delta L(x)$, for a slowly varying function L. The parameter δ is called the *index of regular variation*. Finally, a measurable function $L\colon (0,\infty) \to (0,\infty)$ is called *slowly varying at zero* (resp. *regularly varying at zero*) if $x \to L(\frac{1}{x})$ is slowly varying (resp. regularly varying) at infinity.

For $i, j \in \{1, 2\}$, we write $\rho_{i,j} = \rho$ for $i \neq j$ and $\rho_{i,j} = 1$ for $i = j$. Also, let us introduce the functions mapping $\mathbb{R}^+$ into $\mathbb{R}^+$, with $i, j \in \{1, 2\}$:

$$\bar{R}^{(i,j)}(t) := \mathbb{E}\big[\big(G_t^{(j)} - G_0^{(i)}\big)^2\big] = \|g^{(i)}\|_{L^2}^2 + \|g^{(j)}\|_{L^2}^2 - 2\mathbb{E}\big[G_0^{(i)} G_t^{(j)}\big]. \tag{3}$$

We note that we can write $\bar{R}^{(i,j)}(t) = C_{i,j} + 2\rho_{i,j} \int_0^\infty (g^{(j)}(x) - g^{(j)}(x+t)) g^{(i)}(x)\, dx$, where $C_{i,j} := \|g^{(i)}\|_{L^2}^2 + \|g^{(j)}\|_{L^2}^2 - 2\rho_{i,j} \int_0^\infty g^{(i)}(x) g^{(j)}(x)\, dx$, where in particular $C_{i,i} = 0$. This enables us to formulate our next assumption.

Assumption 2.2. For all $t > 0$, there exist functions $L_0^{(i,j)}(t)$ and $L_2^{(i,j)}(t)$ which are continuous on $(0,\infty)$ and slowly varying at zero, such that

$$\begin{aligned} \bar{R}^{(i,j)}(t) &= C_{i,j} + \rho_{i,j} t^{\delta^{(i)}+\delta^{(j)}+1} L_0^{(i,j)}(t), \qquad \text{for } i, j \in \{1,2\}, \quad \text{and} \\ \frac{1}{2}\big(\bar{R}^{(i,j)}\big)''(t) &= \rho_{i,j} t^{\delta^{(i)}+\delta^{(j)}-1} L_2^{(i,j)}(t), \qquad \text{for } i, j \in \{1,2\}, \end{aligned} \tag{4}$$

where $\delta^{(1)}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. Also, if we denote $\tilde{L}_0^{(i,j)}(t) := \sqrt{L_0^{(i,i)}(t) L_0^{(j,j)}(t)}$, we ask that the functions $L_0^{(i,j)}(t)$ and $L_2^{(i,j)}(t)$ are such that, for all $\lambda > 0$, there exists a $H^{(i,j)} \in \mathbb{R}$ such that:

$$\lim_{t \to 0+} \frac{L_0^{(i,j)}(\lambda t)}{\tilde{L}_0^{(i,j)}(t)} = H^{(i,j)} < \infty, \tag{5}$$

and that there exists $d \in (0, 1)$, such that:

$$\limsup_{x \to 0^+} \sup_{y \in (x, x^d)} \left| \frac{L_2^{(i,j)}(y)}{\tilde{L}_0^{(i,j)}(x)} \right| < \infty. \tag{6}$$

In this situation, the restriction $\delta^{(j)} \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ ensures that the process leaves the semimartingale class.

Remark 2.7. A consequence of Assumption 2.2 is that: $\sqrt{\bar{R}^{(i,i)}(t)\bar{R}^{(j,j)}(t)} = t^{\delta^{(i)}+\delta^{(j)}+1} \times \tilde{L}_0^{(i,j)}(t)$, where $\tilde{L}_0^{(i,j)}(t)$ is again a slowly varying function at zero which is continuous on $(0, \infty)$.

Example 2.8. In the univariate case, condition (4) reads (suppressing superscripts):

$$\bar{R}(t) = t^{2\delta+1} L_0(t). \tag{7}$$

The so-called Gamma kernel given by $g(x) = e^{-\lambda x} x^{\delta} 1_{\{x>0\}}$, for $\lambda > 0, \delta > -\frac{1}{2}$, has attracted attention in applications (both to turbulence and finance), see, for instance, the review paper by [1]. In the case when $\delta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, g satisfies Assumptions 2.1 and condition (4), see [4].

Example 2.9. Condition (5) in Assumption 2.2 is satisfied if $\lim_{t \to 0+} L_0^{(i,j)}(t) = M^{(i,j)} < \infty$ and $\lim_{t \to 0+} \sqrt{L_0^{(i)}(t) L_0^{(j)}(t)} = N^{(i,j)} < \infty$ with $\frac{M^{(i,j)}}{N^{(i,j)}} = H^{(i,j)}$. In the case when $i = j$, we have $H^{i,j} = 1$, so condition (5) is satisfied.

As a consequence of Assumption 2.2, we highlight a fact that will be particularly useful for our purposes.

Lemma 2.10.

Define

$$c(x) := \int_0^x g^{(1)}(s) g^{(2)}(s)\, ds + \int_0^\infty \big(g^{(1)}(s+x) - g^{(1)}(s)\big)\big(g^{(2)}(s+x) - g^{(2)}(s)\big)\, ds.$$

If Assumption 2.2 holds, then it is possible to show that:

$$c(x) = x^{\delta^{(1)}+\delta^{(2)}+1} L_4^{(1,2)}(x), \tag{8}$$

where $L_4^{(1,2)}$ *is a continuous function on* $(0,\infty)$ *which is slowly varying at zero, and* $\delta^{(1)},\delta^{(2)}\in(-\frac{1}{2},\frac{1}{2})\setminus\{0\}$. *Moreover, there exists a constant* $|H|<\infty$ *such that*

$$\lim_{x\to 0+}\frac{L_4^{(1,2)}(x)}{\tilde{L}_0^{(1,2)}(x)}=H. \tag{9}$$

More precisely, $H=\frac{1}{2}(H^{(1,2)}+H^{(2,1)})$.

Example 2.11 (Gamma kernel). If the kernel function is the Gamma kernel $g^{(i)}(s)=s^{\delta^{(i)}}e^{-\lambda^{(i)}s}1_{\{s\geq 0\}}$, for $\lambda^{(i)}>0$, $\delta^{(i)}\in(-\frac{1}{2},\frac{1}{2})\setminus\{0\}$, and similarly for $g^{(j)}$, then one can show directly that Lemma 2.10 holds, and give an explicit expression for the constant H:

$$H=\left(-\frac{\Gamma(\delta^{(1)}+1)\Gamma(-1-\delta^{(1)}-\delta^{(2)})}{\Gamma(-\delta^{(1)})}-\frac{\Gamma(\delta^{(2)}+1)\Gamma(-1-\delta^{(1)}-\delta^{(2)})}{\Gamma(-\delta^{(2)})}\right)$$
$$\times 2^{1+\delta^{(1)}+\delta^{(2)}}\sqrt{\frac{\Gamma(\frac{3}{2}+\delta^{(i)})\Gamma(\frac{3}{2}+\delta^{(j)})}{\Gamma(\frac{1}{2}-\delta^{(i)})\Gamma(\frac{1}{2}-\delta^{(j)})}}.$$

A proof of this result can be found in the supplemental article [26].

2.2. Discrete observations and scaling factor

While the stochastic processes we are going to consider are defined in continuous time, we work under the assumption that we only observe them discretely which is the case of practical relevance. Moreover, our asymptotic results rely on so-called *in-fill asymptotics* where the time interval is fixed, but we sample more and more frequently. This is in contrast to the, in time series more widely used, concept of *long span asymptotics* where the stepsize between observations stays constant, but the number of observations grows, meaning that a bigger and bigger time interval is considered in the asymptotic case.

Suppose that we sample our processes discretely along successive partitions of $[0,T]$. A partition Π_n of $[0,T]$ will be a collection of times $0=t_0<\cdots<t_i<t_{i+1}<\cdots<t_n=T$, where, for simplicity, we assume that the partition is equally spaced. The mesh of the partition will therefore be $\Delta_n=\frac{1}{n}$ and we have $\lim_{n\to\infty}\Delta_n=0$.

We will use the following notation for (high-frequent) increments of the stochastic processes we are considering: For instance, for the process $G^{(j)}$, we denote its increment by $\Delta_i^n G^{(j)}:=G_{i\Delta_n}^{(j)}-G_{(i-1)\Delta_n}^{(j)}$, for $j=1,2$. A straightforward computation shows that the increments can be represented as

$$\begin{aligned}\Delta_i^n G^{(j)}=&\int_{-\infty}^{(i-1)\Delta_n}\big(g^{(j)}(i\Delta_n-s)-g^{(j)}\big((i-1)\Delta_n-s\big)\big)\,dW_s^{(j)}\\&+\int_{(i-1)\Delta_n}^{i\Delta_n}g^{(j)}(i\Delta_n-s)\,dW_s^{(j)}.\end{aligned} \tag{10}$$

We define the *realised covariation* as

$$\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n G^{(1)} \Delta_i^n G^{(2)}, \qquad \text{for } n \geq 1, t \in [0, T].$$

We know that in the case when $\mathbf{G}$ is a semimartingale, then

$$\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n G^{(1)} \Delta_i^n G^{(2)} \stackrel{\text{u.c.p.}}{\to} \left[G^{(1)}, G^{(2)}\right]_t, \qquad \text{as } n \to \infty,$$

where the convergence is uniform on compacts in probability (u.c.p.) and the limiting process is the quadratic covariation. However, outside the semimartingale framework, the quadratic covariation does not necessarily exist. [25] recently considered the non-semimartingale case and showed that, under suitable assumptions, the (possibly scaled) realised covariation converges u.c.p. to an appropriate limit which can be viewed as the correlation between the two non-semimartingale components. In the present work, we would like to go a step further and prove a central limit theorem associated with the scaled realised covariation. In order to do so, we need to define the suitable scaling factor. It turns out that the following choice is appropriate. For $j \in \{1, 2\}$, set

$$\tau_n^{(j)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n G^{(j)}\right)^2\right]} = \sqrt{\int_0^\infty \left(g^{(j)}(s + \Delta_n) - g^{(j)}(s)\right)^2 ds + \int_0^{\Delta_n} \left(g^{(j)}(s)\right)^2 ds}. \quad (11)$$

The scaled realised covariation of the Gaussian core is then given by

$$\sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}.$$

Our aim is now to derive a central limit theorem for the suitably centred and scaled realised covariation of the Gaussian core. As soon as we have that result, we will generalise it to the case when the underlying bivariate process is a bivariate Brownian semistationary process and, hence, also accounts for stochastic volatility in each component.

The key component for proving the two central limit theorems is the so-called Fourth Moment Theorem, see [32]. The supplemental article [26] gives a very brief self-contained introduction to Malliavin calculus and reviews the Fourth Moment Theorem.

3. A central limit theorem for the realised covariation of the Gaussian core

This section focusses on the Gaussian core $\mathbf{G}$ as defined in Definition 2.2; we will use the notation from Section 2.2 and from the supplemental article [26] in the following.

Since $\mathbf{G}$ is a Gaussian process, we can apply the Hilbert-space techniques depicted above, using the Hilbert space of L^2-Gaussian variables. To this end, let $\mathcal{H}$ be the Hilbert space generated by the random variables given by the scaled increments of the Gaussian core:

$$\left(\frac{\Delta_i^n G^{(j)}}{\tau_n^{(j)}}\right)_{n\geq 1, 1\leq i\leq \lfloor nt\rfloor, j\in\{1,2\}},$$

equipped with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ induced by $L^2(\Omega,\mathcal{F},\mathbb{P})$, i.e., for $X, Y\in\mathcal{H}$, we have $\langle X, Y\rangle_{\mathcal{H}} = \mathbb{E}[XY]$.

Denoting by I_d the multiple integral of order d, acting on $\mathcal{H}^{\odot d}$, with values in $L^2(\Omega)$, we can write:

$$\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} = I_1\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\right), \qquad \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} = I_1\left(\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

Recall the definition of the symmetrisation of the tensor product: $x \mathbin{\widetilde{\otimes}} y := \frac{1}{2}(x\otimes y + y\otimes x)$. Using the product formula (2.20) in [26], the product of two multiple integrals becomes:

$$\begin{aligned}\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} &= I_1\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\right) I_1\left(\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) = \sum_{r=0}^{1} r!\binom{1}{r}\binom{1}{r} I_{2-2r}\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \mathbin{\widetilde{\otimes}}_r \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right)\\ &= I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \mathbin{\widetilde{\otimes}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) + \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right].\end{aligned}$$

Rearranging, this yields:

$$\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right] = I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \mathbin{\widetilde{\otimes}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

Let us hence define the function $f\colon L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}$ given by $f(X,Y) = XY - \mathbb{E}[XY]$, and the process:

$$Z_t^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor} f\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}, \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor} I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \mathbin{\widetilde{\otimes}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

3.1. A uniform bound for the covariance

We can now formulate a uniform bound for the covariance term $r_{i,j}^{(n)}(k) := \mathbb{E}[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(j)}}]$, for $i, j\in\{1,2\}$.

Theorem 3.1. *Let* $\varepsilon > 0$, *with* $\varepsilon < 1-\delta^{(i)}-\delta^{(j)}$, *for* $i, j\in\{1,2\}$. *Define*:

$$r_{i,j}(k) := (k-1)^{\delta^{(i)}+\delta^{(j)}+\varepsilon-1}, \qquad \textit{if } k > 1,$$

and $r_{i,j}(0) = r_{i,j}(1) = 1$. *Under Assumption 2.2, there exists a positive constant* $C < \infty$ *and a natural number* $n_0(\varepsilon)$ *such that*:

$$\left|r_{i,j}^{(n)}(k)\right| \le C r_{i,j}(k), \qquad \text{for } k \ge 0, \tag{12}$$

for all $n \ge n_0(\varepsilon)$. *Moreover, define* $\rho_\vartheta^{(i,j)}(0) = \rho H$ *for* $i \ne j$ *and* $\rho_\vartheta^{(i,j)}(0) = 1$ *for* $i = j$, *and for any* $i, j \in \{1, 2\}$ *set*

$$\rho_\vartheta^{(i,j)}(k) = \frac{1}{2}\rho_{i,j} H^{(i,j)}\big((k-1)^\vartheta - 2k^\vartheta + (k+1)^\vartheta\big), \qquad \text{for } k \ge 1. \tag{13}$$

Then it holds that:

$$\lim_{n\to\infty} r_{i,j}^{(n)}(k) = \rho_{\delta^{(i)}+\delta^{(j)}+1}^{(i,j)}(k), \qquad \text{for all } k \ge 0,\ i, j \in \{1, 2\}. \tag{14}$$

3.2. Convergence of the finite dimensional distributions of the Gaussian core

In order to look at the convergence of the finite-dimensional distributions, let $\{a_k\}, \{b_k\}$ be two increasing sequences of positive real numbers, with $a_k < b_k < a_{k+1}$, and consider, for any $d \in \mathbb{N}$ the vector:

$$\big(Z_{b_1}^n - Z_{a_1}^n, \dots, Z_{b_d}^n - Z_{a_d}^n\big)^\top,$$

whose generic $k-$th component is:

$$\frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k\rfloor+1}^{\lfloor nb_k\rfloor} I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) = I_2\left(\frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k\rfloor+1}^{\lfloor nb_k\rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

Theorem 3.2 (Convergence of the finite dimensional distributions). *Take a Gaussian core as defined in Definition 2.2. Let Assumption 2.2 be satisfied and suppose that* $\delta^{(1)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$, $\delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$. *Consider* $f: L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ *given by* $f(X, Y) = XY - \mathbb{E}[XY]$, *and the process*:

$$Z_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} f\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}, \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

Let $\{a_k\}, \{b_k\}$ *be two increasing sequences of positive real numbers, with* $a_k < b_k < a_{k+1}$, *and consider, for any* $d \in \mathbb{N}$ *the vector*:

$$\mathbf{Z}_t^n := \big(Z_{b_1}^n - Z_{a_1}^n, \dots, Z_{b_d}^n - Z_{a_d}^n\big)^\top = (F_{1,n} \dots, F_{d,n})^\top.$$

Then $\mathbf{Z}_t^n \Rightarrow \boldsymbol{N} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{C})$, *where* $C_{i,j} = \lim_{n\to\infty} \mathbb{E}[F_{i,n} F_{j,n}]$, $1 \le i, j \le d$. *Finally, the matrix* $\mathbf{C}$ *is diagonal, and the general* j*-th diagonal element is equal to* $C(1,1)(b_j - a_j)$, *with*

$$C(1,1) := 2\sum_{k=1}^{\infty}\big(\rho_{2\delta^{(1)}}^{(1,1)}(k)\rho_{2\delta^{(2)}}^{(2,2)}(k) + \big(\rho_{\delta^{(1)}+\delta^{(2)}}^{(1,2)}(k)\big)^2\big) + \big(1 + \rho^2 H^2\big) < \infty. \tag{15}$$

In order to compute $C(1,1)$, we remark that the definition of the terms of the form $\rho_{\vartheta}^{(i,j)}(k)$ was given in equation (13).

The series in (15) converges absolutely, thanks to Theorem 3.1, as it is bounded by:

$$4\sum_{k=1}^{\infty}(k-1)^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-2},$$

which converges if and only if $2\delta^{(1)} + 2\delta^{(2)} + 2\varepsilon - 2 < -1 \iff \delta^{(1)} + \delta^{(2)} + \varepsilon < \frac{1}{2}$, which is implied by our assumption that $\delta^{(1)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$, $\delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$.

3.3. Tightness of the law of the realised covariation for the Gaussian core

As customary when proving weak convergence, we also need a tightness result for the law of the realised covariation process. This turns out to be a lot simpler than the convergence of the finite dimensional distributions.

Theorem 3.3 (Tightness). *Let the assumptions as in Theorem 3.2 hold. For all* $n \in \mathbb{N}$, *let* $\mathbb{P}^n$ *be the law of the process*:

$$Z_{\cdot}^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor n\cdot\rfloor} f\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}, \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor n\cdot\rfloor} I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right),$$

on the Skorokhod space $\mathcal{D}[0,T]$. *Then, the sequence* $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ *is tight.*

3.4. The central limit theorem for the Gaussian core

With Theorem 3.2 and 3.3 at our disposal, it is immediate to prove the fundamental theorem stating weak convergence of the realised covariation of the Gaussian core.

Theorem 3.4 (Weak Convergence of the Gaussian Core). *With the same setting and assumptions of Theorem 3.2, we obtain*:

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right]\right)\right)_{t\in[0,T]} \Rightarrow (\sqrt{\beta}B_t)_{t\in[0,T]}, \tag{16}$$

where B_t is a Brownian motion independent of the processes $G^{(1)}$, $G^{(2)}$, $\beta = C(1,1)$ from (15) *and the convergence is in the Skorokhod space $D[0,T]$ equipped with the Skorokhod topology.*

Remark 3.5. We remark that the above central limit theorem (Theorem 3.4) can be formulated under more general conditions which do not require the particular integral representation we are working with throughout the paper. That is, consider a bivariate Gaussian stationary process $\mathbf{G} = (G^{(1)}, G^{(2)})^\top$. Define for $i, j \in \{1,2\}$ and $t > 0$: $\bar{R}^{(i,j)}(t) := \mathbb{E}[(G_t^{(j)} - G_0^{(i)})^2]$. Suppose that Assumption 2.2 is satisfied in this setting, where $\rho_{i,j} \in [-1,1]$ is linked to the correlation between $G^{(i)}$ and $G^{(j)}$ and is such that $\rho_{i,j} = 1$ for $i = j$ and $\rho_{i,j} = 0$ if $G^{(i)}$ and $G^{(j)}$ are uncorrelated, also $C_{ij} = \mathbb{E}(G_t^{(j)2}) + \mathbb{E}(G_0^{(i)2}) - 2\mathbb{E}(G_0^{(j)} G_0^{(i)})$.

Then Theorem 3.4 holds for $\beta = C(1,1)$, where in the definition of $C(1,1)$ the parameter ρ is replaced by $\rho_{1,2}$. The proof of this extended result is contained in the supplementary material to this article, see [26], Section 4.3.

4. A central limit theorem for the realised covariation of the Brownian semistationary process

The weak convergence result for the Gaussian core obtained in the previous section is the cornerstone needed to obtain the general central limit theorem for a Brownian semistationary process $\mathbf{Y}$, which includes stochastic volatility in each component, recall Definition 2.3.

We will need two additional assumptions:

Assumption 4.1. For $k \in \{1,2\}$, we require that $\sigma^{(k)}$ has bounded moments of order two, that is: $\sup_{t \in (-\infty,T]} \mathbb{E}[(\sigma_t^{(k)})^2] < \infty$.

Example 4.1. Assumption 4.1 is easily satisfied in many cases of interests, for example, if the stochastic volatility processes are second-order stationary.

Assumption 4.2. The stochastic volatility process $\sigma^{(1)}$ (resp. $\sigma^{(2)}$) has $\alpha^{(1)}$-Hölder (resp. $\alpha^{(2)}$) continuous sample paths, for $\alpha^{(1)} \in (\frac{1}{2}, 1)$. Furthermore, both the kernel functions $g^{(1)}$ and $g^{(2)}$ satisfy the following property: For $j \in \{1,2\}$, write:

$$\pi_n^{(j)}(A) := \frac{\int_A (g^{(j)}(x + \Delta_n) - g^{(j)}(x))^2 \, ds}{\int_0^\infty (g^{(j)}(x + \Delta_n) - g^{(j)}(x))^2 \, ds},$$

and note that $\pi_n^{(j)}$ are probability measures. We ask that there exists a constant $\lambda^{(j)} < -1$ such that for any $\varepsilon_n = O(n^{-\kappa})$, it holds that:

$$\pi_n^{(j)}\big((\varepsilon_n, \infty)\big) = O\big(n^{\lambda^{(j)}(1-\kappa)}\big).$$

4.1. The central limit theorem

We are now in the position to formulate our key result: the central limit theorem for the suitably centred and scaled realised covariation of a bivariate Brownian semistationary process. We remark that the notation $\stackrel{st.}{\Rightarrow}$ is used for *stable convergence in law*, whose definition and basic properties are reviewed in the supplemental article [26].

Theorem 4.2 (Central limit theorem). *Let $\mathcal{G}$ be the sigma algebra generated by the Gaussian core* $\mathbf{G}$, *and let $\sigma^{(1)}$ and $\sigma^{(2)}$ be $\mathcal{G}-$measurable. For the bivariate $\mathcal{BSS}$ process, provided that Assumptions 2.1, 2.2, 4.1 and 4.2 are satisfied with $\delta^{(1)}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$, the following $\mathcal{G}$-stable convergence holds*:

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} - \sqrt{n} \mathbb{E} \left[\frac{\Delta_1^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_1^n G^{(2)}}{\tau_n^{(2)}} \right] \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} \, ds \right)_{t \in [0,T]} \stackrel{st.}{\underset{n \to \infty}{\Longrightarrow}} \left(\sqrt{\beta} \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} \, dB_s \right)_{t \in [0,T]}, \tag{17}$$

in the Skorokhod space $\mathcal{D}[0, T]$, where $\beta = C(1, 1)$, see equation (15). Also, B is Brownian motion, independent of $\mathcal{F}$ and defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

We note that the central limit theorem implies a weak law of large numbers, which we present next, cf. also [25].

Proposition 4.3. *Assume that the conditions of Theorem 4.2 hold. Then*

$$\frac{\Delta_n}{c(\Delta_n)} \sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \stackrel{\mathbb{P}}{\to} \rho \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} \, ds, \qquad \text{as } n \to \infty.$$

So Theorem 4.2 implies a weak law of large numbers. It is to be stressed though, that the law of large numbers can be formulated in a more general way, modulo some different assumptions on the volatility processes. We refer to the discussion in [25] for the details. In particular, for the weak law of large numbers to hold, we do not need the restriction that $\delta^{(1)}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$, but we can have the whole range $\delta^{(1)}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. On the other hand, we remark that the weak law of large numbers formulated in [25] required the kernel functions to be decreasing, and we do not have such a restriction for the central limit theorem.

5. Conclusion

In this article, we have employed techniques that were successfully used in the univariate case for the power, multipower, and bipower variation of the $\mathcal{BSS}$ process and of Gaussian processes, (as appearing in [3,4,6,20]) to show a central limit theorem for the realised covariation of the bivariate Gaussian core and the $\mathcal{BSS}$ process.

This result, apart from being interesting from a purely mathematical point of view, can be viewed as the starting point of the use of multivariate $\mathcal{BSS}$ processes in stochastic modelling. The central limit theorem unlocks inference on the dependence parameter for the multivariate $\mathcal{BSS}$ process. There are still parts of such a multivariate theory that need to be developed in the future. For instance, one interesting aspect would be to allow for the correlation coefficient to be stochastic. Another direction of future research would include extending our results from the realised covariation to more general functionals, obtaining a fully multidimensional theory of multipower variation of the $\mathcal{BSS}$ process. Also, one could investigate whether similar results can be obtained for other forms of volatility modulated Gaussian processes outside the semimartingale setting.

6. Proofs for the Gaussian core

The proofs of Lemma 2.10 and Example 2.11 are relegated to the supplemental article [26].

6.1. Proof of Theorem 3.1

The uniform bound on the covariances $r_{i,j}^{(n)}(k)$ that we prove on Theorem 3.1 is a fundamental analytical result that allows us to sit within the reach of some powerful results of Malliavin calculus. In this section, we give the proof of that theorem. Let us start off with an elementary result.

Lemma 6.1. *For a C^2 function u, and $h > 0$:*

$$u(x+h) - 2u(x) + u(x-h) = h^2 u''(\zeta),$$

where $\zeta \in (x-h, x+h)$.

The proof of Lemma 6.1 is given in the supplemental article [26]. We have now the tools to tackle the proof of Theorem 3.1.

Proof of Theorem 3.1. The objective in the section is to show that we can bound:

$$\left|r_{i,j}^{(n)}(k)\right| \leq r_{i,j}(k), \tag{18}$$

uniformly in n, for all choices of i, j. In order to do so, recall the functions mapping $\mathbb{R}^+$ into $\mathbb{R}^+$, with $i, j \in \{1, 2\}$: $\bar{R}^{(i,j)}(t) := \mathbb{E}[(G_t^{(j)} - G_0^{(i)})^2]$. We need to show that this function is well defined. More generally, note that for the Gaussian core, we have for any $u \in \mathbb{R}$:

$$\begin{aligned}\mathbb{E}\big[\big(G_{u+t}^{(j)} - G_u^{(i)}\big)^2\big] &= \int_0^\infty \big(g^{(j)}(y)\big)^2\,dy + \int_0^\infty \big(g^{(i)}(y)\big)^2\,dy - 2\int_0^\infty g^{(i)}(y)g^{(j)}(y+t)\rho_{i,j}\,dy\\ &= \|g^{(i)}\|_{L^2}^2 + \|g^{(j)}\|_{L^2}^2 - 2\mathbb{E}\big[G_0^{(i)}G_t^{(j)}\big],\end{aligned}$$

which is indeed a function of t only. It is straightforward to find the connection between $r_{i,j}^{(n)}(k)$ and $\bar{R}^{(i,j)}(k)$, when $k \in \mathbb{N}$:

$$\begin{aligned} r_{i,j}^{(n)}(k) &= \mathbb{E}\left[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(j)}}\right] \\ &= \frac{1}{\tau_n^{(i)}\tau_n^{(j)}}\left(-\bar{R}^{(i,j)}\left(\frac{k}{n}\right) + \frac{1}{2}\bar{R}^{(i,j)}\left(\frac{k-1}{n}\right) + \frac{1}{2}\bar{R}^{(i,j)}\left(\frac{k+1}{n}\right)\right) \qquad (19) \\ &= \frac{1}{2n^2\tau_n^{(i)}\tau_n^{(j)}}\left(\bar{R}^{(i,j)}\right)''\left(\frac{k}{n} + \frac{\vartheta_k^n}{n}\right), \qquad (20) \end{aligned}$$

for some $|\vartheta_k^n| < 1$, thanks to the elementary result stated in Lemma 6.1.

The connection between $r_{i,j}^{(n)}$ and $\bar{R}^{(i,j)}(t)$ was derived in (19) and (20):

$$r_{i,j}^{(n)}(k) = \frac{-2\bar{R}^{(i,j)}(\frac{k}{n}) + \bar{R}^{(i,j)}(\frac{k+1}{n}) + \bar{R}^{(i,j)}(\frac{k-1}{n})}{2\sqrt{\bar{R}^{(i,i)}(\frac{1}{n})\bar{R}^{(j,j)}(\frac{1}{n})}} = \frac{1}{2n^2\tau_n^{(i)}\tau_n^{(j)}}\left(\bar{R}^{(i,j)}\right)''\left(\frac{k}{n} + \frac{\vartheta_k^n}{n}\right), \qquad (21)$$

as well as:

$$\tau_n^{(i)} = \sqrt{\bar{R}^{(i,i)}\left(\frac{1}{n}\right)} = \sqrt{\mathbb{E}\left[\left(G_{\frac{1}{n}}^{(i)} - G_0^{(i)}\right)^2\right]} = \left(\frac{1}{n}\right)^{\frac{1}{2}(2\delta^{(i)}+1)} \sqrt{L_0^{(i)}\left(\frac{1}{n}\right)}.$$

Let us now show the uniform bound (12) and the limit result for the case when $k \in \mathbb{N}$. For $k \in \mathbb{N}$, we go back to the second equality in (21), and deduce that:

$$\left|r_{i,j}^{(n)}(k)\right| = \left|\rho_{i,j} \frac{(k+\vartheta_k^n)^{\delta^{(i)}+\delta^{(j)}-1} L_2^{(i,j)}(\frac{k}{n} + \frac{\vartheta_k^n}{n})}{\tilde{L}_0^{(i,j)}(\frac{1}{n})}\right| \le \left|\frac{(k+\vartheta_k^n)^{\delta^{(i)}+\delta^{(j)}-1} L_2^{(i,j)}(\frac{k}{n} + \frac{\vartheta_k^n}{n})}{\tilde{L}_0^{(i,j)}(\frac{1}{n})}\right|.$$

Note that for $k > 1$, we deduce from $\vartheta_k^n \in (-1, 1)$ and $\delta^{(i)} + \delta^{(j)} - 1 < -\varepsilon < 0$ that

$$\left(k+\vartheta_k^n\right)^{\delta^{(i)}+\delta^{(j)}-1} \le (k-1)^{\delta^{(i)}+\delta^{(j)}-1}.$$

Now, if $2 \le k < \lfloor n^{1-d} \rfloor$, then $\frac{k}{n} + \frac{\vartheta_k^n}{n} \in (\frac{2}{n} + \frac{\vartheta_k^n}{n}, \frac{\lfloor n^{1-d}\rfloor - 1}{n} + \frac{\vartheta_k^n}{n}) \subset (\frac{1}{n}, \frac{\lfloor n^{1-d}\rfloor}{n}) \subset (\frac{1}{n}, \frac{1}{n^d})$ and hence, the bound (6) in Assumption 2.2 applies and we obtain that

$$\frac{L_2^{(i,j)}(\frac{k}{n} + \frac{\vartheta_k^n}{n})}{\tilde{L}_0^{(i,j)}(\frac{1}{n})}$$

is bounded close to the origin for n big enough.

If instead $\lfloor n^{1-d} \rfloor \leq k \leq n$, then, for all $\varepsilon > 0$, and any $\delta < \varepsilon(1-d)$ there exists a constant $C(\delta) > 0$ such that

$$\begin{aligned} \left| r_{i,j}^{(n)}(k+1) \right| &= \left| \frac{(k+1+\vartheta_{k+1}^n)^{\delta^{(i)}+\delta^{(j)}-1} L_2^{(i,j)}(\frac{k+1}{n} + \frac{\vartheta_{k+1}^n}{n})}{\tilde{L}_0^{(i,j)}(\frac{1}{n})} \right| \\ &\leq C(\delta) \frac{k^{\delta^{(i)}+\delta^{(j)}-1+\varepsilon-\delta}}{n^{\varepsilon(1-d)-\delta}} \frac{1}{\tilde{L}_0^{(i,j)}(\frac{1}{n})}. \end{aligned} \tag{22}$$

We used the fact that for any $\delta, t > 0$, there exists a constant C depending on δ (and t) only such that $|L_2(x)| \leq C(\delta) x^{-\delta}$, in a neighborhood $x \in (0, t]$.

Observe now that $M^{(i,j)}(n) := \frac{1}{\tilde{L}_0^{(i,j)}(\frac{1}{n})}$ is a slowly varying function at ∞. Indeed, for any $\lambda > 0$:

$$\lim_{n\to\infty} \frac{M^{(i,j)}(\lambda n)}{M^{(i,j)}(n)} = \lim_{n\to\infty} \frac{\tilde{L}_0^{(i,j)}(\frac{1}{\lambda n})}{\tilde{L}_0^{(i,j)}(\frac{1}{n})} = 1.$$

But since $M^{(i,j)}$ is slowly varying, there exists a constant $\tilde{C}$ such that, by Potter's bound:

$$M(n) \leq \tilde{C} n^{-\varepsilon(1-d)+\delta} \quad \iff \quad \frac{1}{n^{\varepsilon(1-d)-\delta} \tilde{L}_0^{(i,j)}(\frac{1}{n})} \leq \tilde{C},$$

that gives us

$$\left| r_{i,j}^{(n)}(k+1) \right| \leq \tilde{C} C(\delta) k^{\delta^{(i)}+\delta^{(j)}-1+\varepsilon}.$$

As δ is arbitrary, set $C_1 = \tilde{C} C(\delta)$ for any $\delta > 0$.

Next, let us prove the limit result. To this end, we will use the first equality in (21) to show the convergence in (14). Using the expression (4) from Assumptions 2.2, we get for $k \in \mathbb{N}$:

$$\begin{aligned} r_{i,j}^{(n)}(k) = \rho_{i,j} \Bigg(&-2k^{\delta^{(i)}+\delta^{(j)}+1} L_0^{(i,j)}\left(\frac{k}{n}\right) + (k-1)^{\delta^{(i)}+\delta^{(j)}+1} L_0^{(i,j)}\left(\frac{k-1}{n}\right) \\ &+ (k+1)^{\delta^{(i)}+\delta^{(j)}+1} L_0^{(i,j)}\left(\frac{k+1}{n}\right) \Bigg) \Big/ \left(2\tilde{L}_0^{(i,j)}\left(\frac{1}{n}\right) \right). \end{aligned} \tag{23}$$

Because of (5), we get in the limit:

$$\lim_{n\to\infty} r_{i,j}^{(n)}(k) = \rho_{i,j} H^{(i,j)} \frac{(-2k^{\delta^{(i)}+\delta^{(j)}+1} + (k-1)^{\delta^{(i)}+\delta^{(j)}+1} + (k+1)^{\delta^{(i)}+\delta^{(j)}+1})}{2}.$$

Let us now consider the case when $k = 0$. We need to show that $\lim_{n\to\infty} r_{i,j}^{(n)}(0) = \rho H$ for $i \neq j$ and $\lim_{n\to\infty} r_{i,j}^{(n)}(0) = 1$ for $i = j$. First, suppose that $i = j$. Then $r_{i,j}^{(n)}(0) = 1$, and hence

$\lim_{n\to\infty} r^{(n)}_{i,j}(0) = 1$. Next, assume that $i \neq j$. Then

$$r^{(n)}_{i,j}(0) = \rho \frac{\zeta_n}{\xi_n},$$

where

$$\begin{aligned}\zeta_n &= \int_0^{\Delta_n} g^{(1)}(s) g^{(2)}(s)\, ds + \int_0^{\infty} \big(g^{(1)}(s+\Delta_n) - g^{(1)}(s)\big)\big(g^{(2)}(s+\Delta_n) - g^{(2)}(s)\big)\, ds,\\ \xi_n &= \bigg[\bigg(\int_0^{\Delta_n} \big(g^{(1)}(s)\big)^2\, ds + \int_0^{\infty} \big(g^{(1)}(s+\Delta_n) - g^{(1)}(s)\big)^2\, ds\bigg)\\ &\quad \cdot \bigg(\int_0^{\Delta_n} \big(g^{(2)}(s)\big)^2\, ds + \int_0^{\infty} \big(g^{(2)}(s+\Delta_n) - g^{(2)}(s)\big)^2\, ds\bigg)\bigg]^{1/2}.\end{aligned} \tag{24}$$

Using Lemma 2.10, we get:

$$\zeta_n = \Delta_n^{\delta^{(1)}+\delta^{(2)}+1} L_4^{(1,2)}(\Delta_n).$$

Also, Remark 2.7 implies that

$$\xi_n = \Delta_n^{\delta^{(1)}+\delta^{(2)}+1} \sqrt{L_0^{(1)}(\Delta_n) L_0^{(2)}(\Delta_n)} = \Delta_n^{\delta^{(1)}+\delta^{(2)}+1} \tilde{L}_0^{(1,2)}(\Delta_n).$$

Then equation (9) in Lemma 2.10 ensures that $\lim_{n\to\infty} \zeta_n/\xi_n = H$ and hence $\lim_{n\to\infty} r^n_{i,j}(0) = \rho H$ for $i \neq j$. Finally, we remark that since $r^{(n)}_{i,j}(0)$ converges, there exists a positive constant C_2 such that $|r^{(n)}_{i,j}(0)| \leq C_2$ for all $n \in \mathbb{N}$. So, we can conclude that (12) holds with $C = \max\{1, C_1, C_2\}$. □

6.2. Limiting covariance

Our strategy for proving the central limit theorem for the Gaussian core relies on the Fourth Moment Theorem reviewed in Theorem 2.21 in the supplemental article [26], which gives us the fundamental tool for proving convergence in distribution to a Gaussian variable in this setting.

In order to be able to apply the Fourth Moment Theorem to prove the central limit theorem later on, we must first compute the limiting covariance: that is, we need to compute $\lim_{n\to\infty} \mathbb{E}[I_2(f_{r,n}) I_2(f_{s,n})]$, where:

$$f_{r,n} := \frac{1}{\sqrt{n}} \sum_{i=\lfloor na_r \rfloor+1}^{\lfloor nb_r \rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}.$$

We start with the case $r \neq s$:

$$\begin{aligned}
&\mathbb{E}\big[I_2(f_{r,n})I_2(f_{s,n})\big] \\
&\quad = 2\langle f_{r,n}, f_{s,n}\rangle_{\mathcal{H}^{\otimes 2}} \\
&\quad = 2\left\langle \frac{1}{\sqrt{n}} \sum_{i=\lfloor na_r\rfloor+1}^{\lfloor nb_r\rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}, \frac{1}{\sqrt{n}} \sum_{j=\lfloor na_s\rfloor+1}^{\lfloor nb_s\rfloor} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right\rangle_{\mathcal{H}^{\otimes 2}}.
\end{aligned} \tag{25}$$

Without loss of generality, we will choose $r = 1, s = 2, a_1 = 0, b_1 = a_2 = 1, b_2 = 2$, obtaining:

$$\frac{2}{n}\left\langle \sum_{i=1}^{n} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}, \sum_{j=n+1}^{2n} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right\rangle_{\mathcal{H}^{\otimes 2}}. \tag{26}$$

Now, let $k = j - i$. Also recall the definition $r_{a,b}^{(n)}(k) := \mathbb{E}[\frac{\Delta_1^n G^{(a)}}{\tau_n^{(a)}} \frac{\Delta_{1+k}^n G^{(b)}}{\tau_n^{(b)}}]$. Then, the single scalar product equals:

$$\begin{aligned}
&\left\langle \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}, \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \right\rangle_{\mathcal{H}^{\otimes 2}} \\
&\quad = \frac{1}{4}\left\langle \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} + \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}, \right. \\
&\qquad \left. \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} + \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}} \otimes \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \right\rangle_{\mathcal{H}^{\otimes 2}} \\
&\qquad + \frac{1}{4}\mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}}\right]\mathbb{E}\left[\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}}\right] \\
&\qquad + \frac{1}{4}\mathbb{E}\left[\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}}\right]\mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}}\right] \\
&\quad = \frac{1}{2} r_{1,1}^{(n)}(k) r_{2,2}^{(n)}(k) + \frac{1}{2} r_{2,1}^{(n)}(k) r_{1,2}^{(n)}(k).
\end{aligned}$$

Thus, we have that expression (26) becomes:

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} k\big(r_{1,1}^{(n)}(k) r_{2,2}^{(n)}(k) + r_{2,1}^{(n)}(k) r_{1,2}^{(n)}(k)\big) \\
&\quad + \frac{1}{n}\sum_{k=n+1}^{2n-1} (2n-k)\big(r_{1,1}^{(n)}(k) r_{2,2}^{(n)}(k) + r_{2,1}^{(n)}(k) r_{1,2}^{(n)}(k)\big).
\end{aligned} \tag{27}$$

By Cesaro's theorem, if:

$$\lim_{k\to\infty} k\big(r_{1,1}^{(n)}(k)r_{2,2}^{(n)}(k)+r_{2,1}^{(n)}(k)r_{1,2}^{(n)}(k)\big)=0, \tag{28}$$

then the first sum in (27) will converge to zero. Theorem 3.1 gives us:

$$\big|r_{1,1}^{(n)}(k)r_{2,2}^{(n)}(k)+r_{2,1}^{(n)}(k)r_{1,2}^{(n)}(k)\big| \le 2(k-1)^{2(\delta^{(1)}+\delta^{(2)})+2\varepsilon-2}. \tag{29}$$

Hence, we have the limit in (28) provided that

$$\begin{aligned} 2\big(\delta^{(1)}+\delta^{(2)}\big)+2\varepsilon-2<-1 \quad &\iff \quad \big(\delta^{(1)}+\delta^{(2)}\big)+\varepsilon-1<-\frac{1}{2} \\ &\iff \quad \varepsilon<\frac{1}{2}-\big(\delta^{(1)}+\delta^{(2)}\big), \end{aligned}$$

which, in order for $\varepsilon>0$ to hold, implies that we must ask:

$$\delta^{(1)}+\delta^{(2)}<\frac{1}{2}. \tag{30}$$

Applying Theorem 3.1 again shows that the absolute value of the second sum in (27) can be bounded by:

$$\begin{aligned} &\frac{1}{n}\sum_{k=n+1}^{2n-1}(2n-k)2(k-1)^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-2} \\ &\quad = 4\sum_{k=n}^{2n-2}k^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-2}-\frac{2}{n}\sum_{k=n}^{2n-2}k^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-1}-\frac{2}{n}\sum_{k=n}^{2n-2}k^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-2} \\ &\quad \le 4\sum_{k=n}^{2n-2}k^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-2}+\frac{4}{n}\sum_{k=n}^{2n-2}k^{2\delta^{(1)}+2\delta^{(2)}+2\varepsilon-1}. \end{aligned}$$

The first sum goes to zero whenever the summand is summable, thus we get $\delta^{(1)}+\delta^{(2)}<1$, which is clearly satisfied under condition (30). For the second sum, we have in particular that $k<2n \iff \frac{1}{n}<\frac{2}{k}$, so we can write:

$$\frac{4}{n}\sum_{k=n}^{2n-2}k^{2\delta^{(i)}+2\delta^{(j)}+2\varepsilon-1}<8\sum_{k=n}^{2n-2}k^{2\delta^{(i)}+2\delta^{(j)}+2\varepsilon-2}.$$

Condition (30) again ensures convergence to zero.

6.3. Limiting variance

Now we consider the case when $r=s$ in (4) in Theorem 2.21 in the supplemental article [26], as we have to find the limiting variance. Again, take, by simplicity, $r=s=1, a_1=0, b_1=1$, and this time, $k = |i - j|$:

$$\begin{aligned}\mathbb{E}\big[I_2(f_{1,n})I_2(f_{1,n})\big] &= 2\|f_{1,n}\|^2_{\mathcal{H}^{\otimes 2}}\\ &= 2\left\langle \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}, \frac{1}{\sqrt{n}}\sum_{j=1}^n \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \tilde{\otimes} \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}}\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \big(r_{1,1}^{(n)}(k)r_{2,2}^{(n)}(k) + r_{2,1}^{(n)}(k)r_{1,2}^{(n)}(k)\big).\end{aligned} \tag{31}$$

Now write $r_{1,1}^{(n)}(k)r_{2,2}^{(n)}(k)+r_{2,1}^{(n)}(k)r_{1,2}^{(n)}(k) = p_n(|i-j|)$ (note that, if $j < i$, $r_{a,b}^{(n)}(j-i) = r_{b,a}^{(n)}(i-j)$), so that:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n p_n\big(|i-j|\big) &= \frac{2}{n}\sum_{i=1}^n\sum_{j=1}^{i-1} p_n(i-j) + \frac{1}{n}\sum_{i=1}^n p_n(0)\\ &= 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)p_n(k) + p_n(0).\end{aligned} \tag{32}$$

Thanks to (14), we see that, for $k \geq 1$:

$$p_n(k) = r_{1,1}^{(n)}(k)r_{2,2}^{(n)}(k) + r_{2,1}^{(n)}(k)r_{1,2}^{(n)}(k) \to \rho_{2\beta^{(1)}}^{(1,1)}(k)\rho_{2\beta^{(2)}}^{(2,2)}(k) + \big(\rho_{\beta^{(1)}+\beta^{(2)}}^{(1,2)}(k)\big)^2,$$

In the case when $k = 0$, we have

$$p_n(0) = 1 + \frac{1}{(\tau_n^{(1)}\tau_n^{(2)})^2}\big(\mathbb{E}\big[\Delta_1^n G^{(1)}\Delta_1^n G^{(2)}\big]\big)^2 1 + \rho^2\left(\frac{\zeta_n}{\xi_n}\right)^2,$$

where ζ_n and ξ_n are defined as in (24). As above, using Assumption 2.2 and Lemma 2.10, ensures that $\lim_{n\to\infty}\zeta_n/\xi_n = H^2$ and hence $\lim_{n\to\infty} p_n(0) = 1+\rho^2 H^2$.

By the bound (12) in Theorem 3.1 and the bounded convergence theorem, (32) converges to

$$\begin{aligned}C(1,1) &:= \lim_{n\to\infty}\mathbb{E}\big[I_2(f_{1,n})I_2(f_{1,n})\big]\\ &= 2\sum_{k=1}^{\infty}\big(\rho_{2\beta^{(1)}}^{(1,1)}(k)\rho_{2\beta^{(2)}}^{(2,2)}(k) + \big(\rho_{\beta^{(1)}+\beta^{(2)}}^{(1,2)}(k)\big)^2\big) + \big(1+\rho^2H^2\big) < \infty.\end{aligned} \tag{33}$$

6.4. Proof of Theorem 3.2

Since the proof of Theorem 3.2 is rather long, we have moved it to the supplemental material [26].

6.5. Proof of Theorem 3.3

Proof of Theorem 3.3.

$$
\begin{aligned}
&\mathbb{E}\big[\big(Z_t^n - Z_s^n\big)^2\big] \\
&= \mathbb{E}\big[\big(Z_{t-s}^n\big)^2\big] = \mathbb{E}\left[\frac{1}{n}\left(I_2\left(\sum_{i=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right)\right)^2\right] \\
&= \frac{1}{n}\left\langle \sum_{i=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}, \sum_{j=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor} \frac{\Delta_j^n G^{(1)}}{\tau_n^{(1)}} \otimes \frac{\Delta_j^n G^{(2)}}{\tau_n^{(2)}}\right\rangle_{\mathcal{H}^{\otimes 2}} \\
&= \frac{1}{2n} \sum_{i=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{j=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor} \left(r_{1,1}^{(n)}\big(|i-j|\big) r_{2,2}^{(n)}\big(|i-j|\big) + \frac{1}{2} r_{2,1}^{(n)}\big(|i-j|\big) r_{1,2}^{(n)}\big(|i-j|\big)\right).
\end{aligned}
$$

Multiplying and dividing by $\lfloor nt\rfloor - \lfloor ns\rfloor$ yields:

$$
\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\left(\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{k=1}^{\lfloor nt\rfloor - \lfloor ns\rfloor - 1}\left(1 - \frac{k}{n}\right) p_n(k) + p_n(0)\right),
$$

thanks to the same arguments as in equation (32). We now know that the quantity in brackets is convergent, hence bounded. Tightness now follows as in the proof of Theorem 7 in [19], invoking the criterion of Theorem 13.5 in [17]. The criterion applies thanks to the *hypercontractivity property* of multiple integrals: inside a fixed Wiener chaos, all $L^q(\Omega)$ norms are equivalent (see [32], Theorem 2.7.2). □

6.6. Proof of Theorem 3.4

Proof of Theorem 3.4. The fact that the finite dimensional distributions of the realised covariation converge to those of Brownian motion is the content of Theorem 3.2: the limiting finite dimensional distributions we had there coincide with those on the right hand side of (16). The fact that the limiting Brownian motion B_t is independent of $G^{(1)}$ and $G^{(2)}$ follows from the fact that

$$
\left(G_{b_k}^{(1)} - G_{a_k}^{(1)}, G_{b_k}^{(2)} - G_{a_k}^{(2)}, \frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k\rfloor + 1}^{\lfloor nb_k\rfloor}\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right]\right)\right)_{n\in\mathbb{N}}
$$

converges to a multivariate Gaussian, and, for all $n \in \mathbb{N}$ the third component is orthogonal to the first two, as it belongs to a different Wiener chaos. Given the tightness result in Theorem 3.3, an application of Theorem 13.1 in [17] allows to conclude. □

7. Proofs for the Brownian semistationary process

7.1. Strategy and outline of the proof

In order to prove the central limit theorem for the bivariate Brownian semistationary process we will introduce a *blocking technique*, see [16], whereby, alongside the original time-grid indexed by n, we introduce a coarser grid with a new index $l \leq n$, and we freeze the volatility processes at the start of each $l-$interval. Heuristically, letting n go to infinity, for a fixed l, allows us invoke the weak convergence of the Gaussian core we have proven in the previous section, as the volatilities are "frozen". A further limit in l gives us the final result where the volatilities are integrated against the limiting Brownian motion.

Let us now show how the blocking technique will be introduced. We define

$$\mu_n := r_{1,2}^{(n)}(0) = \mathbb{E}\left[\frac{\Delta_1^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_1^n G^{(2)}}{\tau_n^{(2)}}\right],$$

which is bounded by 1, and also

$$I_{(l,n)}(j) = \left\{ i \,\middle|\, \frac{i}{n} \in \left(\frac{j-1}{l}, \frac{j}{l} \right] \right\}.$$

We note here that $\#(I_{(l,n)}(j)) \in \{\lfloor \frac{n}{l} \rfloor, \lfloor \frac{n}{l} \rfloor + 1\}$, so that we can write:

$$\#\big(I_{(l,n)}(j)\big) = \frac{n}{l} + e_{(n,l)}(j), \qquad \text{with } e_{(n,l)}(j) \in (-1,1], \text{ for all } 1 \leq l \leq n, j \geq 1. \tag{34}$$

For any $l \leq n$ we have the decomposition:

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} - \sqrt{n}\mu_n \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} \, ds \\
&= \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} - \sigma_{(i-1)\Delta_n}^{(1)} \sigma_{(i-1)\Delta_n}^{(2)} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right)}_{A_t^n} \\
&\quad + \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \sigma_{(i-1)\Delta_n}^{(1)} \sigma_{(i-1)\Delta_n}^{(2)} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor lt \rfloor} \sigma_{(j-1)\Delta l}^{(1)} \sigma_{(j-1)\Delta_l}^{(2)} \sum_{i \in I_{(l,n)}(j)} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}}_{A_t^{\prime n,l}} \\
&\quad + \underbrace{\frac{\sqrt{n}}{l} \mu_n \sum_{j=1}^{\lfloor lt \rfloor} \sigma_{(j-1)\Delta_l}^{(1)} \sigma_{(j-1)\Delta_l}^{(2)} - \frac{1}{\sqrt{n}} \mu_n \sum_{j=1}^{\lfloor nt \rfloor} \sigma_{(j-1)\Delta_n}^{(1)} \sigma_{(j-1)\Delta_n}^{(2)}}_{A_t^{\prime\prime n,l}}
\end{aligned}$$

$$+\underbrace{\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor}\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}\sum_{i\in I_{(l,n)}(j)}\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}-\frac{\sqrt{n}}{l}\mu_n\sum_{j=1}^{\lfloor lt\rfloor}\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}}_{C_t^{n,l}}$$

$$+\underbrace{\frac{1}{\sqrt{n}}\mu_n\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}-\sqrt{n}\mu_n\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds}_{D_t^n}.$$

The term denoted by $C_t^{n,l}$ will give us the stable convergence to a nonzero limit, while the terms $A_t^n, A_t'''^{n,l} := A_t'^{n,l} + A_t''^{n,l}, D_t^n$ will converge to zero (in a way that will be made precise below.)

We will divide the proof into four parts, each one dealing separately with one of the terms.

7.2. Convergence of the term A_t^n

Proposition 7.1. *Assume that the assumptions of Theorem 4.2 hold. Then A_t^n given by*

$$A_t^n=\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}}-\sigma^{(1)}_{(i-1)\Delta_n}\sigma^{(2)}_{(i-1)\Delta_n}\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right)$$

converges to zero uniformly on compacts in probability (*u.c.p.*).

Proof of Proposition 7.1. Let us call:

$$J_t^n=\frac{1}{\sqrt{n}}\left(\frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}}-\sigma^{(1)}_{(i-1)\Delta_n}\sigma^{(2)}_{(i-1)\Delta_n}\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right).$$

For the claim of the proposition to be true, it is sufficient to prove that, for all $t\in[0,T]$, we have: $\lim_{n\to\infty}\mathbb{E}[\sum_{i=1}^{\lfloor nt\rfloor}|J_t^n|]=0$. Indeed, this implies that convergence also holds in probability. For each n, $\sum_{i=1}^{\lfloor nt\rfloor}|J_t^n|$ is increasing with t, and 0 (the limit in probability) is also increasing and it is continuous. This means that we get convergence to 0 $u.c.p.$ in $[0,T]$ (see, for example, (2.2.16) in [28]). This easily implies the required convergence.

Let us use the following notation, for $k\in\{1,2\}$:

$$\Delta_i^n g_s^{(k)} := g^{(k)}(i\Delta_n - s) - g^{(k)}\bigl((i-1)\Delta_n - s\bigr).$$

We write:

$$\begin{aligned} &\frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} \\ &\quad = \frac{1}{\tau_n^{(1)} \tau_n^{(2)}} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\sigma_s^{(1)}\, dW_s^{(1)} + \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(1)} \sigma_s^{(1)}\, dW_s^{(1)} \right) \\ &\qquad \times \left(\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\sigma_s^{(2)}\, dW_s^{(2)} + \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \sigma_s^{(2)}\, dW_s^{(2)} \right). \end{aligned} \tag{35}$$

We also have the corresponding 4 terms for $\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}$. We start by showing that:

$$\begin{aligned} &\frac{1}{\sqrt{n}\tau_n^{(1)} \tau_n^{(2)}} \sum_{i=1}^{\lfloor nt \rfloor} \biggl[\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\sigma_s^{(1)}\, dW_s^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\sigma_s^{(2)}\, dW_s^{(2)} \\ &\quad - \sigma_{(i-1)\Delta_n}^{(1)} \sigma_{(i-1)\Delta_n}^{(2)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\, dW_s^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\, dW_s^{(2)} \biggr] \end{aligned} \tag{36}$$

goes to zero.

Adding and subtracting

$$\sigma_{(i-1)\Delta_n}^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\, dW_s^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\sigma_s^{(2)}\, dW_s^{(2)},$$

we get:

$$\begin{aligned} &\frac{1}{\sqrt{n}\tau_n^{(1)} \tau_n^{(2)}} \sum_{i=1}^{\lfloor nt \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\sigma_s^{(2)}\, dW_s^{(2)} \\ &\quad \times \left[\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\bigl(\sigma_s^{(1)} - \sigma_{(i-1)\Delta_n}^{(1)}\bigr)\, dW_s^{(1)} \right] \\ &\quad + \frac{1}{\sqrt{n}\tau_n^{(1)} \tau_n^{(2)}} \sum_{i=1}^{\lfloor nt \rfloor} \sigma_{(i-1)\Delta_n}^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\, dW_s^{(1)} \\ &\quad \times \left[\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\bigl(\sigma_s^{(2)} - \sigma_{(i-1)\Delta_n}^{(2)}\bigr)\, dW_s^{(2)} \right]. \end{aligned} \tag{37}$$

We can now start to prove the L^1 convergence that we need. To prove it, we will invoke some results as appearing in [4]. In particular, we will use by-products of the proof of Theorem 4 of that paper. Before we can apply the conclusions of the theorem, we need to verify that we satisfy its assumption called CLT, as stated on pages 1167–1168. This is easily done by combining Theorem 3.1 with Assumption 4.2.

If we take the first term of (37), we need to show convergence of:

$$\frac{1}{\sqrt{n}\tau_n^{(1)}\tau_n^{(2)}}\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}\bigg|\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n-s)\sigma_s^{(2)}\,dW_s^{(2)} \times\bigg[\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n-s)\big(\sigma_s^{(1)}-\sigma_{(i-1)\Delta_n}^{(1)}\big)\,dW_s^{(1)}\bigg]\bigg|.$$

By Cauchy–Schwarz $\mathbb{E}[|XY|]\le\sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}$. Now:

$$\sqrt{\int_{(i-1)\Delta_n}^{i\Delta_n}\big(g^{(2)}(i\Delta_n-s)\big)^2\mathbb{E}\big[\sigma_s^{(2)}\big]\,ds}=\sqrt{\int_0^{\Delta_n}\big(g^{(2)}(s)\big)^2\mathbb{E}\big[\big(\sigma_{i\Delta_n-s}^{(2)}\big)^2\big]\,ds},$$

since σ is càdlàg, it is bounded on compact intervals, so we get the bound:

$$K\frac{\sqrt{\int_0^{\Delta_n}(g^{(2)}(s))^2\,ds}}{\tau_n^{(2)}} \times\frac{1}{\sqrt{n}\tau_n^{(1)}}\sum_{i=1}^{\lfloor nt\rfloor}\sqrt{\mathbb{E}\bigg[\bigg(\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n-s)\big(\sigma_s^{(1)}-\sigma_{(i-1)\Delta_n-s}^{(1)}\big)\,dW_s^{(1)}\bigg)^2\bigg]},$$

for some constant K. Now, the first term is bounded by K, and the second one goes to zero since $\sigma^{(1)}$ is Hölder continuous in mean square, as implied by Assumption 4.2.

We can repeat the reasoning for the second term of (37). Let's take another term now:

$$\frac{1}{\sqrt{n}\tau_n^{(1)}\tau_n^{(2)}}\sum_{i=1}^{\lfloor nt\rfloor}\bigg[\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(1)}\sigma_s^{(1)}\,dW_s^{(1)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(2)}\sigma_s^{(2)}\,dW_s^{(2)} -\sigma_{(i-1)\Delta_n}^{(1)}\sigma_{(i-1)\Delta_n}^{(2)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(1)}\,dW_s^{(1)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(2)}\,dW_s^{(2)}\bigg]. \tag{38}$$

Adding and subtracting $\sigma_{(i-1)\Delta_n}^{(1)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(1)}\,dW_s^{(1)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(2)}\sigma_s^{(2)}\,dW_s^{(2)}$, we get as the first term:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\underbrace{\frac{\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(2)}\sigma_s^{(2)}\,dW_s^{(2)}}{\tau_n^{(2)}}}_{(1)} \times\underbrace{\frac{[\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(1)}\sigma_s^{(1)}\,dW_s^{(1)}-\sigma_{(i-1)\Delta_n}^{(1)}\int_{-\infty}^{(i-1)\Delta_n}\Delta_i^n g_s^{(1)}\,dW_s^{(1)}]}{\tau_n^{(1)}}}_{(2)}.$$

We can use the same arguments as above. The only difference is the expectation of (1) over the infinite interval:

$$\frac{\sqrt{\mathbb{E}[(\int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \sigma_s^{(2)} \, dW_s^{(2)})^2]}}{\tau_n^{(2)}} = \frac{\sqrt{\int_0^\infty (g^{(2)}(s+\Delta_n) - g^{(2)}(s))^2 \mathbb{E}[(\sigma_{(i-1)\Delta_n - s}^{(2)})^2] \, ds}}{\tau_n^{(2)}}.$$

Assumption 4.1 allows to conclude that this quantity is bounded. The remaining term (2) is equal to the sum $B_i^{n,\varepsilon_n^{(1)}} + \sum_{j=1}^{l} C_i^{n,\varepsilon_n^{(j)},\varepsilon_n^{(j+1)}}$ from the proof of Theorem 4 in [4] and goes to zero in L^2 by the same arguments.

Now we consider the cross term

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \biggl(\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s) \sigma_s^{(1)} \, dW_s^{(1)} \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \sigma_s^{(2)} \, dW_s^{(2)} \\ &\quad - \sigma_{(i-1)\Delta_n}^{(1)} \sigma_{(i-1)\Delta_n}^{(2)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s) \, dW_s^{(1)} \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \, dW_s^{(2)} \biggr). \end{aligned} \tag{39}$$

We add and subtract: $\sigma_{(i-1)\Delta_n}^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s) \, dW_s^{(1)} \int_{-\infty}^{t} \Delta_i^n g_s^{(2)} \sigma_s^{(2)} \, dW_s^{(2)}$.

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \sigma_s^{(2)} \, dW_s^{(2)} \biggl(\int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s) \bigl(\sigma_s^{(1)} - \sigma_{(i-1)\Delta_n}^{(1)} \bigr) \, dW_s^{(1)} \biggr) \\ &\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \sigma_{(i-1)\Delta_n}^{(1)} \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s) \, dW_s^{(1)} \\ &\quad \times \biggl(\int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g_s^{(2)} \bigl(\sigma_s^{(2)} - \sigma_{(i-1)\Delta_n}^{(2)} \bigr) \, dW_s^{(2)} \biggr). \end{aligned} \tag{40}$$

We can proceed exactly as above, and convergence to zero is proved. □

7.3. Convergence of the term $A_t^{\prime\prime\prime n,l} = A_t^{\prime n,l} + A_t^{\prime\prime n,l}$

It is worth mentioning at this point that proofs that terms similar to the one we called $A_t^{\prime\prime\prime n,l}$ converge to zero in the univariate case have had a tormented history in the literature. Indeed, a mistake appeared in the proof of a similar result in [22] in the context of power variation for integral processes. The application of the mean value theorem on page 724 of that paper is invalid.

The mistake was not simple to correct. Years later, the paper [21] was published, which highlighted the techniques from fractional integration that were needed to correct the proof. As it turns out, in our multivariate setting it is sufficient to invoke that univariate result to obtain the required convergence. This section contains the details of the proof.

Proposition 7.2. *Assume that the assumptions of Theorem 4.2 hold. Then*

$$\mathbb{P}-\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]}\left|A_t^{'''n,l}\right|=0.$$

Proof of Proposition 7.2. We need to set the following notation:

$$\xi_{i,m}=\frac{1}{\sqrt{m}}\left(\frac{\Delta_i^m G^{(1)}}{\tau_m^{(1)}}\frac{\Delta_1^m G^{(2)}}{\tau_m^{(2)}}-\mathbb{E}\left[\frac{\Delta_i^m G^{(1)}}{\tau_m^{(1)}}\frac{\Delta_1^m G^{(2)}}{\tau_m^{(2)}}\right]\right),$$

and $f(t_i)=\sigma^{(1)}_{(i-1)\Delta_n}\sigma^{(2)}_{(i-1)\Delta_n}$. We will be using Remark 1.1 in the paper [21]. We know that:

$$\sum_{i=1}^{\lfloor mt\rfloor}\xi_{i,m}\Rightarrow\sqrt{\beta}W_t.$$

Convergence (4) in the paper reads:

$$\mathbb{P}-\lim_{n\to\infty}\limsup_{m\to\infty}\sup_{t\in[0,T]}\left|\sum_{j=1}^{\lfloor nt\rfloor}\sum_{i\in I_n(j)}\big(f(t_i)-f(u_{j-1})\big)\xi_{i,m}\right|=0, \tag{41}$$

which in our setting and with our notation becomes:

$$\begin{aligned}\mathbb{P}-\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]}&\left|\sum_{j=1}^{\lfloor lt\rfloor+1}\sum_{i\in I_{(l,n)}(j)}\big(\sigma^{(1)}_{(i-1)\Delta_n}\sigma^{(2)}_{(i-1)\Delta_n}-\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(1)}_{(j-1)\Delta_l}\big)\right.\\ &\left.\times\frac{1}{\sqrt{n}}\underbrace{\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}-\mu_n\right)}_{(1)}\right|=0.\end{aligned}$$

Expanding the bracket in (1) above, the first term gives us exactly term $A_t^{'n,l}$. The second term from the bracket (1) is:

$$\begin{aligned}&\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}\sum_{i\in I_{(l,n)}(j)}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}-\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}\sum_{i\in I_{(l,n)}(j)}\sigma^{(1)}_{(i-1)\Delta_n}\sigma^{(2)}_{(i-1)\Delta_n}\\ &=\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}\#\big(I_{(l,n)}(j)\big)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}-\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}.\end{aligned}$$

If we use (34), we get:

$$\begin{aligned}&\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}\left(\frac{n}{l}+e_{(n,l)}(j)\right)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}-\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\\&\quad=A_t^{\prime\prime n,l}+\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}.\end{aligned}$$

We can then conclude that (41) implies:

$$\mathbb{P}-\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]}\left|A_t^{\prime n,l}+A_t^{\prime\prime n,l}+\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\right|=0.$$

Now, we can write:

$$\begin{aligned}\left|A_t^{\prime\prime\prime n,l}\right|&=\left|A_t^{\prime n,l}+A_t^{\prime\prime n,l}+\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\right.\\&\qquad\left.-\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\right|\\&\le\left|A_t^{\prime n,l}+A_t^{\prime\prime n,l}+\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\right|\\&\qquad+\left|\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}e_{(n,l)}(j)\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}\right|,\end{aligned}$$

but since $|e_{(n,l)}(j)|\le 1$, the last term goes to zero a.s. for any fixed l, uniformly for t in $[0,T]$, so we can conclude that:

$$\mathbb{P}-\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]}\left|A_t^{\prime\prime\prime n,l}\right|=0.$$ □

7.4. Convergence of the term $C_t^{n,l}$

The term $C_t^{n,l}$ is the one that will give us the stable convergence we seek.

Proposition 7.3. *Assume that the assumptions of Theorem 4.2 hold. Then*

$$\left(G_t^{(1)},G_t^{(2)},\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^nG^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^nG^{(2)}}{\tau_n^{(2)}}-\mu_n\right)\right)_{t\in[0,T]}$$

converges weakly to

$$\big(G_t^{(1)}, G_t^{(2)}, \sqrt{\beta} B_t\big)_{t\in[0,T]}.$$

Proof of Proposition 7.3. We split the proof into two parts: First, we prove tightness and then convergence of the finite dimensional distributions.

Tightness: Theorem 13.2 in [17] gives two necessary and sufficient conditions for a sequence of measures $\mathbb{P}_n$ to be tight. Our probability measures $\mathbb{P}_n$ live in $\mathcal{D}([0,T];\mathbb{R}^3)$, the space of càdlàg functions with values in $\mathbb{R}^3$, equipped with the Skorokhod topology. The norm in this space is defined as:

$$\|f\|_{\mathcal{D}([0,T];\mathbb{R}^3)} = \sup_{t\in[0,T]} \|f\|_{\mathbb{R}^3},$$

and hence the two conditions in the theorem only depend on the norm in $\mathbb{R}^3$. It is then sufficient to show them component-wise. The first two components trivially satisfy them, as the sequences reduce to only one measure per component. The fact that the third component satisfies them both is a consequence of Theorem 3.3.

Convergence of the finite dimensional distributions: We need to show that for any choice of positive numbers $a_k < b_k$, $k \in \{1,\dots,D\}$, the sequence of matrix variables:

$$\left(G_{b_k}^{(1)} - G_{a_k}^{(1)}, G_{b_k}^{(2)} - G_{a_k}^{(2)}, \frac{1}{\sqrt{n}} \sum_{i=\lfloor na_k\rfloor+1}^{\lfloor nb_k\rfloor} \left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mu_n\right)\right)_{1\le k\le D}$$

converges in law, as $n\to\infty$, to:

$$\big(G_{b_k}^{(1)} - G_{a_k}^{(1)}, G_{b_k}^{(2)} - G_{a_k}^{(2)}, \sqrt{\beta}(B_{b_k} - B_{a_k})\big)_{1\le k\le D}. \tag{42}$$

This we know already, as pointed out in the proof of Theorem 3.4, as marginal convergence of sequence of variables within fixed Wiener chaoses implies joint convergence. The first two components lie in the first chaos, the third one lies in the second chaos. The statement of Theorem 3.4 allows to conclude. □

Proposition 7.4. *Assume that the assumptions of Theorem 4.2 hold. Then C_t^n converges stably in law to $\sqrt{\beta}\int_0^t \sigma_s^{(1)}\sigma_s^{(2)}\,dB_s$ in the Skorokhod space $\mathcal{D}[0,T]$, where $\beta = C(1,1)$, see equation (15), and where first $n\to\infty$ for fixed l and then $l\to\infty$. Also, B is Brownian motion, independent of $\mathcal{F}$ and defined on an extension of the filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$.*

Proof of Proposition 7.4. The joint weak convergence in (42) paired with the asymptotic independence of the limit B and $G^{(1)}, G^{(2)}$ and an application of Proposition 3.3 in [26] ensure that:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} \left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mu_n\right) \Rightarrow \sqrt{\beta} B_t \qquad \text{(mixing)}.$$

Applying the continuous mapping Theorem 3.2 in [26] with the sigma-algebra $\mathcal{G}$, $\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}$ as the measurable variable σ, $\frac{1}{\sqrt{n}}\sum_{i\in I_{(l,n)}(j)}(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mu_n)$ as Y_n and $g(x,y)=xy$, since $Y_n \overset{st.}{\Rightarrow} \sqrt{\beta}(B_{j\Delta_l} - B_{(j-1)\Delta_l})$, we have the following $\mathcal{G}$-stable convergence for fixed l as $n\to\infty$:

$$\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}\frac{1}{\sqrt{n}}\sum_{i\in I_{(l,n)}(j)}\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mu_n\right)$$
$$\overset{st.}{\Rightarrow} \sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}\sqrt{\beta}(B_{j\Delta_l} - B_{(j-1)\Delta_l}).$$

Finally, we have that

$$\mathbb{P}-\lim_{l\to\infty}\sum_{j=1}^{\lfloor lt\rfloor}\sigma^{(1)}_{(j-1)\Delta_l}\sigma^{(2)}_{(j-1)\Delta_l}\sqrt{\beta}(B_{j\Delta_l} - B_{(j-1)\Delta_l}) = \sqrt{\beta}\int_0^t \sigma_s^{(1)}\sigma_s^{(2)}\,dB_s,$$

because the integrand is càdlàg. Modulo another term of the form $\frac{\mu_n}{\sqrt{n}}\sum_{j=1}^{\lfloor lt\rfloor+1}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}$, which goes to zero a.s. as $n\to\infty$, we have proven stable convergence of the term $C_t^{n,l}$ in our decomposition. □

7.5. Convergence of the term D_t^n

Proposition 7.5. *Assume that the assumptions of Theorem 4.2 hold. Then* $\sup_{t\in[0,T]}|D_t^n|\to 0$ *almost surely.*

Proof of Proposition 7.5. Note that D_t^n is given by

$$\frac{1}{\sqrt{n}}\mu_n\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n} - \sqrt{n}\mu_n\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds.$$

Recall that $\alpha^{(i)}$ denotes the Hölder continuity index of $\sigma^{(i)}$. Rewriting the integral:

$$\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds = \sum_{j=1}^{\lfloor nt\rfloor}\int_{(j-1)\Delta_n}^{j\Delta_n}\sigma_s^{(1)}\sigma_s^{(2)}\,ds + \int_{\lfloor nt\rfloor\Delta_n}^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds,$$

and using the mean value theorem, we get:

$$\begin{aligned}
|D_t^n| &\le \frac{1}{\sqrt{n}}\mu_n\left(\sum_{j=1}^{\lfloor nt\rfloor}\left|\sigma^{(1)}_{(j-1)\Delta_n}\sigma^{(2)}_{(j-1)\Delta_n}-\sigma^{(1)}_{s_j}\sigma^{(2)}_{s_j}\right|\right)+\frac{1}{\sqrt{n}}\mu_n\left\|\sigma^{(1)}_{s_j}\sigma^{(2)}_{s_j}\right\|_\infty \\
&\le \frac{1}{\sqrt{n}}\mu_n\left(\sum_{j=1}^{\lfloor nt\rfloor}\left|(j-1)\Delta_n-s_j\right|^{\min(\alpha^{(1)},\alpha^{(2)})}\left|\sigma^{(1)}_{(j-1)\Delta_n}+\sigma^{(2)}_{s_j}\right|\right)+\frac{1}{\sqrt{n}}\mu_n\left\|\sigma^{(1)}_{s_j}\sigma^{(2)}_{s_j}\right\|_\infty \\
&\le C\frac{1}{\sqrt{n}}\mu_n\Delta_n^{\min(\alpha^{(1)},\alpha^{(2)})}nTl+\frac{1}{\sqrt{n}}\mu_n\left\|\sigma^{(1)}_{s_j}\sigma^{(2)}_{s_j}\right\|_\infty \\
&= C\sqrt{n}\mu_n\Delta_n^{\min(\alpha^{(1)},\alpha^{(2)})}T+\frac{1}{\sqrt{n}}\mu_n\left\|\sigma^{(1)}_{s_j}\sigma^{(2)}_{s_j}\right\|_\infty.
\end{aligned}$$

Hence, $\sup_{t\in[0,T]}|D_t^n|\to 0$ almost surely, since $\min(\alpha^{(1)},\alpha^{(2)})>\frac{1}{2}$. □

7.6. Proofs of Theorem 4.2 and Proposition 4.3

Proof of Theorem 4.2. The statement of Theorem 4.2 is a consequence of Propositions 7.1, 7.2, 7.4, 7.5, noting that they imply that, for any $\varepsilon>0$,

$$\lim_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]}\left|A_t^n+A_t'^{n,l}+A_t''^{n,l}+D_t^n\right|\ge\varepsilon\Big)=0.$$

It is now sufficient to apply Theorem 3.2 in [17] to conclude. □

Finally, we provide the proof of the weak law of large numbers.

Proof of Proposition 4.3. We note that, for each fixed $t\in[0,T]$, (17) implies that:

$$\left\{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^nY^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^nY^{(2)}}{\tau_n^{(2)}}-\mathbb{E}\left[\frac{\Delta_1^nG^{(1)}}{\tau_n^{(1)}}\frac{\Delta_1^nG^{(2)}}{\tau_n^{(2)}}\right]\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds\right)\right\}_{n\in\mathbb{N}}$$

converges weakly, hence, by Prohorov's theorem, it is a tight sequence. It then follows that:

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^nY^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^nY^{(2)}}{\tau_n^{(2)}}-\mathbb{E}\left[\frac{\Delta_1^nG^{(1)}}{\tau_n^{(1)}}\frac{\Delta_1^nG^{(2)}}{\tau_n^{(2)}}\right]\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\,ds\xrightarrow{\mathbb{P}}0.$$

Now:

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\Delta_1^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_1^n G^{(2)}}{\tau_n^{(2)}}\right] \\
&\quad = \frac{\int_0^{\Delta_n} g^{(1)}(s)g^{(2)}(s)\rho\,ds + \int_0^\infty (g^{(1)}(s+\Delta_n)-g^{(1)}(s))(g^{(2)}(s+\Delta_n)-g^{(2)}(s))\rho\,ds}{\tau_n^{(1)}\tau_n^{(2)}} \\
&\quad = \rho\frac{c(\Delta_n)}{\tau_n^{(1)}\tau_n^{(2)}}.
\end{aligned}
$$

Hence,

$$
\Delta_n \sum_{i=1}^{\lfloor nt\rfloor} \frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} - \rho\frac{c(\Delta_n)}{\tau_n^{(1)}\tau_n^{(2)}}\int_0^t \sigma_s^{(1)}\sigma_s^{(2)}\,ds \xrightarrow{\mathbb{P}} 0,
$$

which is equivalent to

$$
\Delta_n \sum_{i=1}^{\lfloor nt\rfloor} \Delta_i^n Y^{(1)}\Delta_i^n Y^{(2)} - \rho c(\Delta_n)\int_0^t \sigma_s^{(1)}\sigma_s^{(2)}\,ds \xrightarrow{\mathbb{P}} 0,
$$

or indeed to

$$
\frac{\Delta_n}{c(\Delta_n)} \sum_{i=1}^{\lfloor nt\rfloor} \Delta_i^n Y^{(1)}\Delta_i^n Y^{(2)} \xrightarrow{\mathbb{P}} \rho\int_0^t \sigma_s^{(1)}\sigma_s^{(2)}\,ds. \qquad \square
$$

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Supplementary Material

Supplement to "A central limit theorem for the realised covariation of a bivariate Brownian semistationary process" (DOI: 10.3150/18-BEJ1052SUPP; .pdf). We collect technical details and proofs in the supplementary article, which should be read in conjunction with the present paper.

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