

# Rigid stationary determinantal processes in non-Archimedean fields

YANQI QIU

CNRS, Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118, Route de Narbonne, F-31062 Toulouse Cedex 9, France. E-mail: yqi.qiu@gmail.com

Let  $F$  be a non-discrete non-Archimedean local field. For any subset  $S \subset F$  with finite Haar measure, there is a stationary determinantal point process on  $F$  with correlation kernel  $\widehat{\mathbb{1}}_S(x - y)$ , where  $\widehat{\mathbb{1}}_S$  is the Fourier transform of the indicator function  $\mathbb{1}_S$ . In this note, we give a geometrical condition on the subset  $S$ , such that the associated determinantal point process is rigid in the sense of Ghosh and Peres. Our geometrical condition is very different from the Euclidean case.

*Keywords:* non-Archimedean local field; rigidity; stationary determinantal point processes

## 1. Main result

Let  $F$  be a non-discrete non-Archimedean local field. Write  $\mathcal{O}_F$  for its valuation ring with maximal ideal  $\mathcal{M}_F$ . Let  $q := p^e$  be the number of elements of the finite residue field  $\mathcal{O}_F/\mathcal{M}_F$ , where  $p$  is a prime number and  $e \in \mathbb{N}$ . Fix the standard norm  $|\cdot|$  on  $F$ . Let  $m$  be the Haar measure on  $F$  normalized such that  $m(\mathcal{O}_F) = 1$ .

By a random point process  $\mathcal{X}$  on  $F$ , we mean a random *locally finite* subset of  $F$ . The main objects under consideration in this note are stationary determinantal point processes on  $F$ . For the background on determinantal point processes, the reader is referred to [8–10,13] and references therein.

**Definition 1.1 (Ghosh and Peres [5,7]).** A random point process  $\mathcal{X}$  on  $F$  is number rigid, if for any bounded open subset  $B \subset F$ , the number  $\#(\mathcal{X} \cap B)$  of particles of the random point process  $\mathcal{X}$  inside  $B$ , is almost surely determined by  $\mathcal{X} \setminus B$ .

We refer to [1,2,4,6,11] for further references on the number-rigidity property for determinantal point processes.

Let  $S \subset F$  be a measurable subset such that  $0 < m(S) < \infty$ . By Macchi–Soshnikov theorem, we may introduce a determinantal point process on  $F$ , denoted by  $\mathcal{X}_S$ , whose correlation kernel is given by

$$K_S(x, y) = \widehat{\mathbb{1}}_S(x - y) \quad (x, y \in F),$$

where  $\widehat{\mathbb{1}}_S$  is the Fourier transform of the indicator function  $\mathbb{1}_S$  of the set  $S$ , see Section 2.1 for the precise definition of the Fourier transform in the non-Archimedean setting. The translation-invariance of the kernel  $K_S(x, y)$  implies that the random point process  $\mathcal{X}_S$  is stationary, that is, the probability distribution of  $\mathcal{X}_S$  is invariant under translations.

Our main result is the following theorem.

**Theorem 1.2.** *Assume that  $S \subset F$  is a measurable subset such that  $0 < m(S) < \infty$  and*

$$\int_{B(0, q^{-n})} m(S \setminus (S + y)) \, dm(y) = O(q^{-n}) \quad \text{as } n \rightarrow \infty, \tag{1}$$

where  $\int_{B(0, q^{-n})} dm$  is the normalized integration. Then the determinantal point process induced by the kernel  $\widehat{\mathbb{1}}_S(x - y)$  is number rigid.

The geometrical condition (1) can be replaced by an analytic condition on the  $L^2$ -decay of the Fourier transform  $\widehat{\mathbb{1}}_S$ .

**Theorem 1.3.** *Assume that  $S \subset F$  is a measurable subset such that  $0 < m(S) < \infty$  and*

$$\int_{F \setminus B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, dm = O(q^{-n}) \quad \text{as } n \rightarrow \infty. \tag{2}$$

Then the determinantal point process induced by the kernel  $\widehat{\mathbb{1}}_S(x - y)$  is number rigid.

We note that in Euclidean case, the *stationary* determinantal point processes that are known to be number rigid are

- [5] the Dyson sine process or slightly more generally, the determinantal point processes on  $\mathbb{R}$  with a correlation kernel

$$\widehat{\mathbb{1}}_S(x - y) \quad (x, y \in \mathbb{R}),$$

where  $S$  is a *finite union of intervals*;

- [7] the Ginibre point process.

But the Ginibre point process is induced by the kernel

$$e^{-\frac{|z|^2 + |w|^2}{2} + z\bar{w}} \quad (z, w \in \mathbb{C}),$$

which is not given by the Fourier transform of any function on  $\mathbb{C} \simeq \mathbb{R}^2$ . Therefore, in Euclidean case, only in dimension one do we know examples of *stationary* determinantal point processes whose correlation kernels are convolution kernels. It has also been mentioned in [2] that the existing methods do not seem to produce number rigid stationary determinantal point processes in  $\mathbb{R}^d$  with  $d \geq 2$ .

While in non-Archimedean setting, there exist trivial examples of stationary determinantal point processes: for example, in the case of  $p$ -adic number field  $\mathbb{Q}_p$ , for any  $n \in \mathbb{Z}$ , by the identity

$$p^{-n} \mathbb{1}_{p^{-n}\mathbb{Z}_p} = \widehat{\mathbb{1}}_{p^n\mathbb{Z}_p},$$

we know that the kernel

$$p^{-n} \mathbb{1}_{p^{-n}\mathbb{Z}_p}(x - y) = \widehat{\mathbb{1}}_{p^n\mathbb{Z}_p}(x - y) \quad (x, y \in \mathbb{Q}_p) \tag{3}$$

induces a determinantal point process on  $\mathbb{Q}_p$ . This determinantal point process can be trivially verified to be number rigid since for any pair  $x, y \in \mathbb{Q}_p$  such that  $|x - y|_p > p^{-n}$ , we have

$$\widehat{\mathbb{1}}_{p^n \mathbb{Z}_p}(x - y) = 0$$

and this implies that the determinantal point process induced by the kernel (3) is the union of countably many independent copies of determinantal point processes on the translates of  $p^{-n} \mathbb{Z}_p$ , each of them has exactly one particle. By the same reason, if  $S \subset F$  is a *union of finitely many balls*, then the determinantal point process induced by the kernel  $\widehat{\mathbb{1}}_S(x - y)$  is trivially number rigid. Of course, in these trivial examples, the sets  $S$  satisfy trivially the condition (1).

Theorem 1.2 produces non-trivial examples of Borel subsets  $S \subset F$  such that the associated stationary determinantal point processes are number rigid, see Section 4.2 for such examples. Since any finite dimensional vector space over  $F$  can be seen as a finite extension of the field  $F$ , which is again a non-Archimedean local field, Theorem 1.2 produces non-trivial examples of stationary determinantal point processes on any finite dimensional vector space over  $F$ .

Recall that any open set of the real line  $\mathbb{R}$  is a countable union of intervals, inspired by our result in non-Archimedean situation, it is natural to ask the following.

**Question 1.4.** Let  $S \subset \mathbb{R}$  be an *open* subset with finite positive Lebesgue measure. Is the determinantal point process on  $\mathbb{R}$  induced by the correlation kernel

$$\widehat{\mathbb{1}}_S(x - y) \quad (x, y \in \mathbb{R})$$

number rigid?

## 2. Preliminaries

### 2.1. Notation

We recall some basic notion in the theory of local fields. Let  $F$  be a non-discrete non-Archimedean local field. The classification of local fields (see Ramakrishnan and Valenza [12], Theorem 4-12) implies that  $F$  is isomorphic to one of the following fields:

- a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers for some prime  $p$ ;
- the field of formal Laurent series over a finite field.

Let  $|\cdot|$  denote the *absolute value* on  $F$  and let  $d(\cdot, \cdot)$  denote the metric on  $F$  defined by  $d(x, y) = |x - y|$ . The set  $\mathcal{O}_F = \{x \in F : |x| \leq 1\}$  forms a subring of  $F$  and is called the ring of integers or the valuation ring of  $F$ . The subset  $\mathcal{M}_F = \{x \in F : |x| < 1\}$  is the unique maximal ideal of the integer-ring  $\mathcal{O}_F$ . The quotient  $\mathcal{O}_F/\mathcal{M}_F$  is a finite field with cardinality

$$\#(\mathcal{O}_F/\mathcal{M}_F) = q = p^e, \quad p \text{ is prime and } e \in \mathbb{N}.$$

By fixing any element  $\varpi \in F$  with  $|\varpi| = q^{-1}$ , we have

$$\mathcal{M}_F = \varpi \mathcal{O}_F.$$

Denote by  $\widehat{F}$  the Pontryagin dual of the additive group  $F$ . Elements in  $\widehat{F}$  are called characters of  $F$ . Throughout the note, we fix a non-trivial character  $\chi \in \widehat{F}$  such that

$$\chi|_{\mathcal{O}_F} \equiv 1 \text{ and } \chi \text{ is not constant on } \varpi^{-1}\mathcal{O}_F. \quad (4)$$

For any  $y \in F$ , define a character  $\chi_y \in \widehat{F}$  by  $\chi_y(x) = \chi(yx)$ , then the map  $y \mapsto \chi_y$  defines a group isomorphism from  $F$  to  $\widehat{F}$ .

Let  $m$  be the Haar measure on the additive group  $F$  normalized such that  $m(\mathcal{O}_F) = 1$ . Given any function  $f \in L^1(F, m)$ , its *Fourier transform* is defined by

$$\widehat{f}(y) = \int_F f(x)\chi(xy) dm(x), \quad y \in F.$$

## 2.2. Determinantal point processes on $F$

We say a random point process  $\mathcal{X}$  is a determinantal point process induced by a correlation kernel  $K : F \times F \rightarrow \mathbb{C}$ , if for any positive integer  $n$  and for any compactly supported bounded measurable function  $\varphi : F^n \rightarrow \mathbb{C}$ , we have

$$\begin{aligned} & \mathbb{E} \left( \sum_{x_1, \dots, x_n \in \mathcal{X}}^* \varphi(x_1, \dots, x_n) \right) \\ &= \int_{F^k} \varphi(y_1, \dots, y_n) \det(K(y_i, y_j))_{1 \leq i, j \leq n} dm(y_1) \cdots dm(y_n), \end{aligned} \quad (5)$$

where  $\sum^*$  denotes the sum over all ordered  $n$ -tuples of *distinct* points  $(x_1, \dots, x_n) \in \mathcal{X}^n$ .

## 3. Kolmogorov minimality

Let  $\Gamma = F/\mathcal{O}_F$  the quotient additive group. Then  $\Gamma$  is a discrete countable group. Elements in  $\Gamma$  will be denoted either by  $\gamma \in \Gamma$  or  $[x] \in \Gamma$  with  $x \in F$ . We equip  $\Gamma$  with an absolute value, denoted again by  $|\cdot|$  and defined by

$$|[x]| := \begin{cases} |x|, & x \notin \mathcal{O}_F, \\ 0, & x \in \mathcal{O}_F. \end{cases}$$

Note that the group  $\Gamma$  is identified naturally with the Pontryagin dual of the additive group  $\mathcal{O}_F$  by the following well-defined pairing

$$\begin{aligned} \Gamma \times \mathcal{O}_F &\rightarrow \mathbb{C}, \\ ([x], h) &\mapsto \chi(xh), \end{aligned}$$

where  $\chi$  is the fixed character of  $F$  satisfying (4).

A  $\Gamma$ -indexed family  $(X_\gamma)_{\gamma \in \Gamma}$  of  $\mathbb{C}$ -valued random variables defined on a common probability space  $(\Omega, \mathbb{P})$  is called a  $\Gamma$ -indexed stochastic process. It is called (weakly)  $\Gamma$ -stationary, if  $X_\gamma \in$

$L^2(\Omega, \mathbb{P})$  for all  $\gamma \in \Gamma$  and for any  $\gamma, \gamma', \alpha \in \Gamma$ , we have

$$\mathbb{E}(X_\gamma) = \mathbb{E}(X_{\gamma'}) \quad \text{and} \quad \text{Cov}(X_\gamma, X_{\gamma'}) = \text{Cov}(X_{\gamma+\alpha}, X_{\gamma'+\alpha}),$$

where  $\mathbb{E}(X_\gamma)$  denotes the expectation of  $X_\gamma$  and  $\text{Cov}(X_\gamma, X_{\gamma'})$  denotes the covariance between  $X_\gamma$  and  $X_{\gamma'}$  defined by

$$\text{Cov}(X_\gamma, X_{\gamma'}) := \mathbb{E}((X_\gamma - \mathbb{E}(X_\gamma))(\overline{X_{\gamma'}} - \mathbb{E}(\overline{X_{\gamma'}}))).$$

The Bochner theorem for positive definite functions on locally compact groups implies that given any weakly  $\Gamma$ -stationary stochastic process  $X = (X_\gamma)_{\gamma \in \Gamma}$ , there exists a unique measure  $\mu_X$  on  $\mathcal{O}_F$ , called the spectral measure of  $X$ , such that

$$\int_{\mathcal{O}_F} \gamma(x) \overline{\gamma'(x)} \mu_X(dx) = \text{Cov}(X_\gamma, X_{\gamma'}). \quad (6)$$

**Definition 3.1.** A weakly  $\Gamma$ -stationary stochastic process  $X = (X_\gamma)_{\gamma \in \Gamma}$  defined on a probability space  $(\Omega, \mathbb{P})$  is called Kolmogorov minimal, if

$$X_0 - \mathbb{E}(X_0) \in \overline{\text{span}}^{L^2(\Omega, \mathbb{P})} \{X_\gamma - \mathbb{E}(X_\gamma) : \gamma \in \Gamma \setminus \{0\}\}. \quad (7)$$

**Proposition 3.2.** Let  $X = (X_\gamma)_{\gamma \in \Gamma}$  be a  $\Gamma$ -stationary stochastic process. Assume that

- (1)  $\sum_{\gamma \in \Gamma} \text{Cov}(X_\gamma, X_0) = 0$ ;
- (2)  $\sum_{|\gamma| > q^n} |\text{Cov}(X_\gamma, X_0)| = O(q^{-n})$ , as  $n \rightarrow \infty$ .

Then  $X$  is Kolmogorov minimal.

The proof of Proposition 3.2 is based on the following Lemma 3.3 due to Kolmogorov. The proof of Lemma 3.3 is similar to the proof of the same result for  $\mathbb{Z}^d$ -indexed stochastic processes and the reader is referred to [2], Lemma 2.1.

**Lemma 3.3 (The Kolmogorov criterion).** Let  $X = (X_\gamma)_{\gamma \in \Gamma}$  be a weakly  $\Gamma$ -stationary stochastic process such that  $\mathbb{E}(X_\gamma) = \mathbb{E}(X_0) = 0$ . Assume that the spectral measure  $\mu_X$  of  $X$  has the Lebesgue decomposition:

$$\mu_X = \mu_X^a + \mu_X^s,$$

where  $\mu_X^a$  and  $\mu_X^s$  are the absolutely continuous and the singular parts of  $\mu_X$  respectively with respect to  $\mathfrak{m}$ . Then the least  $L^2(\Omega, \mathbb{P})$ -distance between  $X_0$  and the space  $\overline{\text{span}}^{L^2} \{X_\gamma : \gamma \in \Gamma \setminus \{0\}\}$  is given by

$$\text{dist}(X_0, \overline{\text{span}}^{L^2} \{X_\gamma : \gamma \in \Gamma \setminus \{0\}\}) = \left( \int_{\mathcal{O}_F} \left( \frac{d\mu_X^a}{d\mathfrak{m}}(x) \right)^{-1} d\mathfrak{m}(x) \right)^{-1/2},$$

where the right-hand side is to be interpreted as zero if

$$\int_{\mathcal{O}_F} \left( \frac{d\mu_X^a}{d\mathfrak{m}}(x) \right)^{-1} d\mathfrak{m}(x) = \infty.$$

**Lemma 3.4.** Let  $(c_\gamma)_{\gamma \in \Gamma}$  be a  $\Gamma$ -indexed sequence of complex numbers such that

$$\sum_{|\gamma| > q^n} |c_\gamma| = O(q^{-n}), \quad \text{as } n \rightarrow \infty.$$

Then there exists a constant  $C > 0$ , such that the function  $f : \mathcal{O}_F \rightarrow \mathbb{C}$  defined by the formula

$$f(x) = \sum_{\gamma \in \Gamma} c_\gamma \gamma(x)$$

satisfies

$$|f(x+h) - f(x)| \leq C|h|, \quad \text{for any } x, h \in \mathcal{O}_F.$$

**Proof.** Let  $h \in \mathcal{O}_F$  be such that  $|h| = q^{-n}$ . Notice that if  $|\gamma| \leq q^n$ , then  $\gamma(h) = 1$ . Therefore, we have

$$f(x+h) - f(x) = \sum_{\gamma \in \Gamma} c_\gamma (\gamma(h) - 1) \gamma(x) = \sum_{\gamma \in \Gamma, |\gamma| > q^n} c_\gamma (\gamma(h) - 1) \gamma(x).$$

It follows that

$$|f(x+h) - f(x)| \leq \sum_{|\gamma| > q^n} c_\gamma |\gamma(h) - 1| \leq 2 \sum_{|\gamma| > q^n} c_\gamma = O(q^{-n}) = O(|h|). \quad \square$$

**Lemma 3.5.**  $\int_{\mathcal{O}_F} |h|^{-1} dm(h) = \infty$ .

**Proof.** Denote  $S(0, q^{-n}) := \varpi^n \mathcal{O}_F \setminus \varpi^{n+1} \mathcal{O}_F$ , then we have the partition

$$\mathcal{O}_F = \bigsqcup_{n=0}^{\infty} S(0, q^{-n}).$$

Note that

$$m(S(0, q^{-n})) = m(\varpi^n \mathcal{O}_F) - m(\varpi^{n+1} \mathcal{O}_F) = q^{-n-1}(q-1).$$

Since any  $h \in S(0, q^{-n})$  has the absolute value  $|h| = q^{-n}$ , we have

$$\int_{\mathcal{O}_F} |h|^{-1} dm(h) = \sum_{n=0}^{\infty} q^n m(S(0, q^{-n})) = \infty. \quad \square$$

**Proof of Proposition 3.2.** The spectral measure of  $X = (X_\gamma)_{\gamma \in \Gamma}$  is given by

$$\mu_X(dx) = \sum_{\gamma \in \Gamma} \text{Cov}(X_\gamma, X_0) \gamma(x) m(dx).$$

Indeed, one can easily check that the measure defined on the right-hand side satisfies the equation (6). By the uniqueness of the spectral measure, it must be  $\mu_X$ . Note that here, the spectral measure  $\mu_X$  has no singular part with respect to  $\mathfrak{m}$ .

Now let

$$f(x) := \frac{d\mu_X}{d\mathfrak{m}}(x) = \sum_{\gamma \in \Gamma} \text{Cov}(X_\gamma, X_0)\gamma(x).$$

The first condition in Proposition 3.2 implies that  $f(0) = 0$ . While the second condition, combined with Lemma 3.4, implies that there exists  $C > 0$ , such that

$$f(h) = |f(h) - f(0)| \leq C|h|.$$

It follows that

$$\int_{\mathcal{O}_F} f(h)^{-1} d\mathfrak{m}(h) \geq C^{-1} \int_{\mathcal{O}_F} |h|^{-1} d\mathfrak{m}(h) = \infty.$$

This combined with Lemma 3.3 implies the desired relation (7).  $\square$

## 4. Determinantal point processes on $F$

### 4.1. Proofs of Theorem 1.2 and Theorem 1.3

For any positive integer  $m > 0$ , fix a set of representatives  $\mathbb{L}_m$  of the quotient additive group  $F/\varpi^{-m}\mathcal{O}_F$ . The cosets of  $\varpi^{-m}\mathcal{O}_F$  are exactly the closed balls  $B(x, q^m)$  with  $x \in \mathbb{L}_m$  and we have

$$F = \bigsqcup_{x \in \mathbb{L}_m} B(x, q^m).$$

In what follows, we assume that  $\mathbb{L}_m$  contains the origin  $0 \in F$ . We will also identify  $\mathbb{L}_m$  with  $F/\varpi^{-m}\mathcal{O}_F$  and equip  $\mathbb{L}_m$  with the same additive group structure as  $F/\varpi^{-m}\mathcal{O}_F$ .

Note that we have a natural group isomorphism between two additive groups:

$$\begin{aligned} \mathbb{L}_m = F/\varpi^{-m}\mathcal{O}_F &\longrightarrow \Gamma = F/\varpi\mathcal{O}_F, \\ x + \varpi^{-m}\mathcal{O}_F &\mapsto \varpi^{m+1}x + \varpi\mathcal{O}_F. \end{aligned} \tag{8}$$

Using the group isomorphism (8), we immediately get a corollary from Proposition 3.2.

**Corollary 4.1.** *Fix any positive integer  $m > 0$ . Let  $X = (X_x)_{x \in \mathbb{L}_m}$  be an  $\mathbb{L}_m$ -stationary stochastic process defined on a probability space  $(\Omega, \mathbb{P})$ . Assume that*

- (1)  $\sum_{x \in \mathbb{L}_m} \text{Cov}(X_x, X_0) = 0$ ;
- (2)  $\sum_{x \in \mathbb{L}_m: |x| > q^n} |\text{Cov}(X_x, X_0)| = O(q^{-n})$ , as  $n \rightarrow \infty$ .

Then  $X$  is Kolmogorov minimal, that is,

$$X_0 - \mathbb{E}(X_0) \in \overline{\text{span}}^{L^2(\Omega, \mathbb{P})} \{X_x - \mathbb{E}(X_x) : x \in \mathbb{L}_m \setminus \{0\}\}.$$

Let  $S \subset F$  be a measurable subset such that  $0 < \mathfrak{m}(S) < \infty$ . Let

$$K_S(x, y) = \widehat{\mathbb{1}}_S(x - y), \quad \text{for } x, y \in F.$$

Then the kernel  $K_S$  induces a determinantal point process on  $F$ . In what follows, let  $\mathcal{X}_S$  be a determinantal point process induced by the correlation kernel  $K_S(x, y)$ . The translation invariance of the kernel

$$K_S(x, y) = K_S(x + z, y + z)$$

implies that the determinantal point process  $\mathcal{X}_S$  is translation-invariant.

For each  $x \in \mathbb{L}_m$ , we set a random variable

$$N_x^{(m)} := \#(\mathcal{X}_S \cap B(x, q^m)). \quad (9)$$

Since the law of  $\mathcal{X}_S$  is invariant under the translations  $x \in \mathbb{L}_m$ , the stochastic process  $(N_x^{(m)})_{x \in \mathbb{L}_m}$  is also  $\mathbb{L}_m$ -stationary.

For simplifying the notation, in what follows, when  $m$  is clear from the context, we will denote  $N_x^{(m)}$  by  $N_x$ .

**Lemma 4.2.** *For any measurable subset  $S \subset F$  such that  $0 < \mathfrak{m}(S) < \infty$ , we have*

$$\sum_{x \in \mathbb{L}_m} \text{Cov}(N_x, N_0) = 0. \quad (10)$$

**Proof.** By taking  $n = 1$  in the formula (5), for any  $x \in \mathbb{L}_m$ , we have

$$\mathbb{E}(N_x) = \mathbb{E}(N_0) = \mathbb{E}\left(\sum_{u \in \mathcal{X}_S} \mathbb{1}_{B(0, q^m)}(u)\right) = \int_{B(0, q^m)} \widehat{\mathbb{1}}_S(0) \, \text{d}\mathfrak{m}(z) = q^m \mathfrak{m}(S).$$

We can write

$$\begin{aligned} \mathbb{E}(N_0^2) &= \mathbb{E}\left(\sum_{u, v \in \mathcal{X}_S} \mathbb{1}_{B(0, q^m)}(u) \mathbb{1}_{B(0, q^m)}(v)\right) \\ &= \underbrace{\mathbb{E}\left(\sum_{u \in \mathcal{X}_S} \mathbb{1}_{B(0, q^m)}(u)\right)}_{\text{denoted by } I} + \underbrace{\mathbb{E}\left(\sum_{u, v \in \mathcal{X}_S, u \neq v} \mathbb{1}_{B(0, q^m)}(u) \mathbb{1}_{B(0, q^m)}(v)\right)}_{\text{denoted by } II}. \end{aligned}$$

The first term  $I$  has already been shown to be  $q^m \mathfrak{m}(S)$ . To compute the second term, we take  $n = 2$  in (5) and get

$$\begin{aligned} II &= \iint_{B(0, q^m)^2} (\widehat{\mathbb{1}}_S(0)^2 - |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2) \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2) \\ &= q^{2m} \mathfrak{m}(S)^2 - \iint_{B(0, q^m)^2} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2). \end{aligned}$$

Hence,

$$\begin{aligned} \text{Cov}(N_0, N_0) &= \text{Var}(N_0, N_0) = \mathbb{E}(N_0^2) - \mathbb{E}(N_0)^2 \\ &= q^m \mathfrak{m}(S) - \iint_{B(0, q^m)^2} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2). \end{aligned}$$

Note that for  $x \in \mathbb{L}_m \setminus \{0\}$ , the two closed balls  $B(0, q^m)$  and  $B(x, q^m)$  are disjoint. Therefore, if  $x \in \mathbb{L}_m \setminus \{0\}$ , we have

$$\begin{aligned} \mathbb{E}(N_0 N_x) &= \mathbb{E}\left(\sum_{u, v \in \mathcal{X}_S} \mathbb{1}_{B(0, q^m)}(u) \mathbb{1}_{B(x, q^m)}(v)\right) \\ &= \mathbb{E}\left(\sum_{u, v \in \mathcal{X}_S, u \neq v} \mathbb{1}_{B(0, q^m)}(u) \mathbb{1}_{B(x, q^m)}(v)\right). \end{aligned}$$

By taking  $n = 2$  in (5), we get

$$\begin{aligned} \mathbb{E}(N_0 N_x) &= \iint_{B(0, q^m) \times B(x, q^m)} (\widehat{\mathbb{1}}_S(0)^2 - |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2) \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2) \\ &= q^{2m} \mathfrak{m}(S)^2 - \iint_{B(0, q^m) \times B(x, q^m)} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2). \end{aligned}$$

Consequently,

$$\text{Cov}(N_x, N_0) = - \iint_{B(0, q^m) \times B(x, q^m)} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2). \quad (11)$$

Finally, by using the equality

$$\begin{aligned} &\sum_{x \in \mathbb{L}_m} \iint_{B(0, q^m) \times B(x, q^m)} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2) \\ &= \iint_{B(0, q^m) \times F} |\widehat{\mathbb{1}}_S(z_1 - z_2)|^2 \mathfrak{d}\mathfrak{m}(z_1) \mathfrak{d}\mathfrak{m}(z_2) \\ &= \mathfrak{m}(B(0, q^m)) \int_F |\widehat{\mathbb{1}}_S|^2 \mathfrak{d}\mathfrak{m} = q^m \mathfrak{m}(S), \end{aligned}$$

we obtain the desired equality (10).  $\square$

**Lemma 4.3.** *Let  $n$  be an integer such that  $n > m$ . We have*

$$\sum_{x \in \mathbb{L}_m; |x| > q^n} |\text{Cov}(N_x, N_0)| = q^m \int_{F \setminus B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m}. \quad (12)$$

**Proof.** Let  $x \in \mathbb{L}_m$ . Then for any  $z \in B(0, q^m)$ , we have

$$z + B(x, q^m) = B(x, q^m).$$

Since  $\mathfrak{m}$  is the Haar measure on  $F$ , the restriction  $\mathfrak{m}|_{B(x, q^m)}$  is invariant under the translation of  $z$ . Therefore, the equality (11) can be written as

$$\text{Cov}(N_x, N_0) = -q^m \int_{B(x, q^m)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m}. \quad (13)$$

It follows that

$$\begin{aligned} \sum_{x \in \mathbb{L}_m; |x| > q^n} |\text{Cov}(N_x, N_0)| &= \sum_{x \in \mathbb{L}_m; |x| > q^n} q^m \int_{B(x, q^m)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m} \\ &= q^m \int_F |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m} - \sum_{x \in \mathbb{L}_m; |x| \leq q^n} q^m \int_{B(x, q^m)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m}. \end{aligned}$$

By applying the partition

$$\bigsqcup_{x \in \mathbb{L}_m; |x| \leq q^n} B(x, q^m) = B(0, q^n),$$

we obtain the desired equality

$$\begin{aligned} \sum_{x \in \mathbb{L}_m; |x| > q^n} |\text{Cov}(N_x, N_0)| &= q^m \int_F |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m} - q^m \int_{B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m} \\ &= q^m \int_{F \setminus B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \text{d}\mathfrak{m}. \quad \square \end{aligned}$$

**Lemma 4.4.** *Let  $n$  be an integer such that  $n > m$ . We have*

$$\sum_{x \in \mathbb{L}_m; |x| > q^n} |\text{Cov}(N_x, N_0)| = q^m \int_{B(0, q^{-n})} \mathfrak{m}(S \setminus (S + y)) \, \text{d}\mathfrak{m}(y). \quad (14)$$

**Proof.** Note that we have (see, e.g., [3], Lemma 3.3)

$$\mathbb{1}_{B(0, q^n)} = q^n \widehat{\mathbb{1}}_{B(0, q^{-n})}$$

and also

$$\widehat{\mathbb{1}}_S \cdot \widehat{\mathbb{1}}_{B(0, q^{-n})} = \mathcal{F}(\mathbb{1}_S * \mathbb{1}_{B(0, q^{-n})}).$$

Here  $\mathcal{F}$  denotes also the Fourier transform. Therefore,

$$\begin{aligned} \int_{B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \mathrm{d}\mathbf{m} &= q^{2n} \int_F |\widehat{\mathbb{1}}_S \cdot \widehat{\mathbb{1}}_{B(0, q^{-n})}|^2 \, \mathrm{d}\mathbf{m} \\ &= q^{2n} \int_F |\mathcal{F}(\mathbb{1}_S * \mathbb{1}_{B(0, q^{-n})})|^2 \, \mathrm{d}\mathbf{m}. \end{aligned}$$

By Parseval's identity

$$\begin{aligned} \int_{B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \mathrm{d}\mathbf{m} &= q^{2n} \int_F |\mathbb{1}_S * \mathbb{1}_{B(0, q^{-n})}|^2 \, \mathrm{d}\mathbf{m} \\ &= q^{2n} \int_F \left[ \int_{B(0, q^{-n})} \mathbb{1}_S(x-y) \, \mathrm{d}\mathbf{m}(y) \int_{B(0, q^{-n})} \mathbb{1}_S(x-y') \, \mathrm{d}\mathbf{m}(y') \right] \, \mathrm{d}\mathbf{m}(x) \\ &= q^{2n} \int_{B(0, q^{-n})} \int_{B(0, q^{-n})} \mathbf{m}((S+y) \cap (S+y')) \, \mathrm{d}\mathbf{m}(y) \, \mathrm{d}\mathbf{m}(y'). \end{aligned}$$

Note that  $B(0, q^{-n})$  is an additive group and for any  $y, y' \in B(0, q^{-n})$ , we have

$$\mathbf{m}((S+y) \cap (S+y')) = \mathbf{m}((S+(y-y')) \cap S).$$

Since the restriction  $\mathbf{m}|_{B(0, q^{-n})}$  is invariant under the translations of  $y' \in B(0, q^{-n})$ , we have

$$\begin{aligned} \int_{B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \mathrm{d}\mathbf{m} &= q^{2n} \mathbf{m}(B(0, q^{-n})) \int_{B(0, q^{-n})} \mathbf{m}((S+y) \cap S) \, \mathrm{d}\mathbf{m}(y) \\ &= \int_{B(0, q^{-n})} \mathbf{m}((S+y) \cap S) \, \mathrm{d}\mathbf{m}(y). \end{aligned}$$

The equality (12) combined with the following equality

$$\int_F |\widehat{\mathbb{1}}_S|^2 \, \mathrm{d}\mathbf{m} = \mathbf{m}(S) = \int_{B(0, q^{-n})} \mathbf{m}(S) \, \mathrm{d}\mathbf{m}(y)$$

yields the desired equality

$$\begin{aligned} \sum_{x \in \mathbb{L}_m; |x| > q^n} |\mathrm{Cov}(N_x, N_0)| &= q^m \int_{F \setminus B(0, q^n)} |\widehat{\mathbb{1}}_S|^2 \, \mathrm{d}\mathbf{m} \\ &= q^m \int_{B(0, q^{-n})} (\mathbf{m}(S) - \mathbf{m}((S+y) \cap S)) \, \mathrm{d}\mathbf{m}(y) \\ &= q^m \int_{B(0, q^{-n})} \mathbf{m}(S \setminus (S+y)) \, \mathrm{d}\mathbf{m}(y). \end{aligned}$$

□

**Proof of Theorem 1.2.** In Definition 1.1, the subset  $B$  ranges over all bounded open subsets. It is easy to see that without changing the definition of number rigidity, we can let  $B$  only range over all the closed balls  $B(0, q^m)$  with  $m \in \mathbb{N}$ . In our notation (9), for each  $m \in \mathbb{N}$ , the number of particles of our determinantal point process  $\mathcal{X}_S$  inside  $B(0, q^m)$  is denoted by  $N_0^{(m)}$ , by Corollary 4.1, Lemma 4.2 and Lemma 4.4, the assumption (1) implies

$$N_0^{(m)} - \mathbb{E}(N_0^{(m)}) \in \overline{\text{span}}^{L^2(\Omega, \mathbb{P})} \{N_x^{(m)} - \mathbb{E}(N_x^{(m)}) : x \in \mathbb{L}_m \setminus \{0\}\}.$$

Thus, the random variable  $N_0^{(m)}$  is measurable with respect to the completion of the  $\sigma$ -algebra generated by the family  $\{N_x^{(m)} : x \in \mathbb{L}_m \setminus \{0\}\}$ . Therefore,  $N_0^{(m)}$  is almost surely determined by  $\mathcal{X}_S \setminus B(0, q^m)$ . Since  $m$  is arbitrary, we complete the proof of the number rigidity of  $\mathcal{X}_S$ .  $\square$

**Proof of Theorem 1.3.** In the proof of Theorem 1.2, replacing Lemma 4.4 by Lemma 4.3, we obtain Theorem 1.3.  $\square$

## 4.2. Examples

Let us now concentrate in the case where  $S \subset F$  is an open subset with  $0 < m(S) < \infty$ . It is easy to see that  $S$  has a unique decomposition

$$S = \bigsqcup_{k=1}^N B(x_k, q^{-n_k}), \quad (15)$$

such that  $B(x_k, q^{-n_k})$  (with  $n_k \in \mathbb{Z}$ ) is the *largest* closed ball in  $S$  containing  $x_k \in S$ . Here  $N \in \mathbb{N} \cup \{\infty\}$ .

In the decomposition (15) of  $S$ , we may assume that

$$l_0(S) = n_1 \leq n_2 \leq \dots,$$

where  $l_0(S)$  is the smallest integer in the sequence  $(n_k)_{k=1}^N$ . For any  $l \geq l_0(S)$ , we define the multiplicity of  $l$  in the sequence  $(n_k)_{k=1}^N$  by

$$m_l(S) := \#\{k : n_k = l\}. \quad (16)$$

For any given  $y \in F$  such that

$$|y| = q^{-l} \quad \text{with } l \geq l_0(S),$$

it is easy to see that all those sub-balls  $B(x_k, q^{-n_k})$  in the decomposition (15) whose radius is not smaller than  $q^{-l}$ , we have

$$B(x_k, q^{-n_k}) + y = B(x_k, q^{-n_k}).$$

Therefore, by recalling the definition (16) for the multiplicity, we have

$$m(S \setminus (S + y)) \leq \sum_{a=l+1}^{\infty} m_a(S) \cdot m(B(0, q^{-a})) = \sum_{a=l+1}^{\infty} m_a(S) q^{-a}. \quad (17)$$

On the other hand, for all those sub-balls  $B(x_k, q^{-n_k})$  in the decomposition (15) whose radius is strictly smaller than  $q^{-l}$ , we have the following proposition.

**Proposition 4.5.** *If  $M(S) := \sup_l m_l(S) < \infty$ , then the open subset  $S$  satisfies the condition (1).*

**Proof.** For any given  $|y| = q^{-l}$  with  $l \geq l_0(S)$ , the inequality (17) implies that

$$m(S \setminus (S + y)) \leq M(S) \sum_{a=l+1}^{\infty} q^{-a} = M(S) \frac{q^{-l}}{q-1}.$$

Therefore, for any  $n \geq l_0(S)$ , we have

$$\begin{aligned} & \int_{B(0, q^{-n})} m(S \setminus (S + y)) \, dm(y) \\ & \leq q^n \sum_{l=n}^{\infty} \int_{\{y \in F : |y|=q^{-l}\}} m(S \setminus (S + y)) \, dm(y) \\ & \leq q^n M(S) \sum_{l=n}^{\infty} \frac{q^{-l}}{q-1} m(\{y \in F : |y|=q^{-l}\}) \\ & = q^n M(S) \sum_{l=n}^{\infty} q^{-2l+1} = M(S) \frac{q^3}{q^2-1} q^{-n}. \end{aligned}$$

This implies that  $S$  satisfies the desired condition (1). □

## Acknowledgements

The author is supported by the grant IDEX UNITI – ANR-11-IDEX-0002-02, financed by Programme “Investissements d’Avenir” of the Government of the French Republic managed by the French National Research Agency. During an earlier stage of this research, he was also partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015–2019 Polish MNiSW fund.

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*Received February 2017*