*Bernoulli* **23**(4B), 2017, 3508–3536
DOI: 10.3150/16-BEJ854

# Asymptotic expansions and hazard rates for compound and first-passage distributions

RONALD W. BUTLER

*Department of Statistical Science, Southern Methodist University, Dallas, TX 75275, USA.*
*E-mail: rbutler@smu.edu*

A general theory which provides asymptotic tail expansions for density, survival, and hazard rate functions is developed for both absolutely continuous and integer-valued distributions. The expansions make use of Tauberian theorems which apply to moment generating functions (MGFs) with boundary singularities that are of gamma-type or log-type. Standard Tauberian theorems from Feller [*An Introduction to Probability Theory and Its Applications* **II** (1971) Wiley] can provide a limited theory but these theorems do not suffice in providing a complete theory as they are not capable of explaining tail behaviour for compound distributions and other complicated distributions which arise in stochastic modelling settings. Obtaining such a complete theory for absolutely continuous distributions requires introducing new "Ikehara" conditions based upon Tauberian theorems whose development and application have been largely confined to analytic number theory. For integer-valued distributions, a complete theory is developed by applying Darboux's theorem used in analytic combinatorics. Characterizations of asymptotic hazard rates for both absolutely continuous and integer-valued distributions are developed in conjunction with these expansions. The main applications include the ruin distribution in the Cramér–Lundberg and Sparre Andersen models, more general classes of compound distributions, and first-passage distributions in finite-state semi-Markov processes. Such first-passage distributions are shown to have exponential-like/geometric-like tails which mimic the behaviour of first-passage distributions in Markov processes even though the holding-time MGFs involved with such semi-Markov processes are typically not rational.

*Keywords:* asymptotic hazard rate; compound distribution; Cramér–Lundberg approximation; Darboux's theorem; first-passage distribution; Ikehara–Delange theorem; Ikehara–Wiener theorem; semi-Markov process; Sparre Andersen model; Tauberian theory

## 1. Introduction

Hazard rate functions, and the density/mass and survival functions used in their computation, are fundamental tools used in probability, survival analysis, and reliability. Within the context of the stochastic models commonly used in these fields, such functions can be difficult to compute since the distribution under consideration may only be specified in terms of its moment generating function (MGF). In such cases, saddlepoint approximations can facilitate the computations, however our aim here is to rather explore asymptotic expansions for all three of these functions. Indeed, a general asymptotic theory for hazard functions has never been formulated in the literature and this is one of our main goals. More generally, the goals of this paper are to formulate an asymptotic theory for all three functions and to develop the theory so it may be applied to the compound distributions and first-passage time distributions commonly dealt with in survival analysis, risk theory, and semi-Markov processes. Our development of such an asymptotic theory relies on using Tauberian theorems, however the standard theorems in Feller

1350-7265 

[20], Chapter XIII.5, for density and mass functions using "Feller" conditions do not apply to these more complicated compound and first-passage distributions. More inclusive conditions that apply to these distributions are needed in both continuous- and integer-time settings. For the continuous setting, we formulate new "Ikehara" conditions by introducing Tauberian theorems that have been extensively used in analytic number theory, but which have not been previously used (to the authors knowledge) in applied probability. Likewise, in the lattice setting, we introduce very weak Darboux conditions, based on using Darboux's theorem from analytic combinatorics, which apply to compound and first-passage distributions in integer time.

Asymptotic hazard rates are characterized quite generally and are shown to exist under the Ikehara/Darboux conditions needed for tail expansions of density/mass and survival functions. For absolutely continuous distributions, this rate is shown to be $b \geq 0$, the right edge of the convergence region for the associated MGF. For integer-valued distributions, the asymptotic hazard rate is $1 - e^{-b}$.

The most compelling reason for considering Ikehara/Darboux conditions rather than Feller conditions is that they are capable of justifying tail expansions for the infinite mixture/convolution distributions associated with compound distributions, first-passage distributions, and other complicated distributions that occur in stochastic modelling. Among the compound distributions with geometric-like weights, we first consider the ruin distribution in the Cramér–Lundberg and Sparre Andersen models and obtain new expansions for ruin densities and alternative derivations for well-known survival expansions. Ikehara/Darboux conditions also justify expansions for more general compound distributions with negative-binomial-like weights and compound distributions with multivariate weights associated with multiple classes of claim distributions. We show that distributions of the latter type include first-passage distributions in finite-state semi-Markov processes, and this leads to new tail expansions for their density/mass and survival functions which are exponential-like/geometric-like. Thus, first-passage distributions in semi-Markov processes have the same tails as would occur in the more restrictive class of Markov processes and this happens with holding times which are not of phase-type and which do not have rational MGFs. Such exponential/geometric tail expansions reinforce the insensitivity property discussed by Tijms [27], Section 5.4, in which semi-Markov processes mimic the behaviour of Markov processes asymptotically.

The remainder of the paper is organised as follows. Section 2 highlights the main results of the paper and discusses their implications for saddlepoint methods and statistical inference. Section 3 develops expansions for absolutely continuous distributions under Ikehara conditions, and Section 4 considers the analogous results for integer-valued mass functions under Darboux conditions. Section 5 considers finite mixture and convolution applications, and Section 6 discusses compound distributions including the Cramér–Lundberg and Sparre Andersen models. Expansions for first-passage times of semi-Markov processes are in Section 7. Asymptotic theory when $b$ is a logarithmic singularity is presented in Section 8.

## 2. Notation and discussion of main results

Let random variable $X$ have the distribution of interest with MGF $\mathcal{M}(s) = E(e^{sX})$ defined on $\{s \in C : E(e^{sX}) < \infty\}$. Thus, for example, all distributions on $(0, \infty)$ have MGFs which are defined at least on $\{s \in C : \mathrm{Re}(s) \leq 0\}$.

There are four interrelated goals to be achieved in this paper. The first goal is to provide a characterization for the asymptotic hazard rate of $X$. For absolutely continuous $X$, Theorem 1 (Section 3) shows that the liminf of the average cumulative hazard rate is $b \in [0, \infty]$, defined as the right edge of the convergence region of the associated MGF. By considering the Cesàro limit rather than the actual hazard limit, this liminf holds without any further conditions on the distribution. If the limiting hazard is known to exist, then $b$ is this limit (Corollary 1). For integer-valued $X$ with hazard sequence $\{h_n\}$, Theorem 2 (Section 4) shows the liminf for the average of $\{-\ln(1-h_n)\}$ is $b \in [0, \infty]$; thus, a limiting hazard, if it exists, must be $1 - e^{-b}$.

Determining sufficient conditions for the existence of a limiting hazard rate motivates the second goal which is to develop asymptotic expansions for the density and survival function of $X$ to establish such existence. These expansions and the conditions for them depend on the nature of the singularity $b$ for $\mathcal{M}$. When singularity $b > 0$ is gamma-like, so that the MGF is $\mathcal{M}(s) = O(b-s)^{-w}$ as $s \uparrow b$ for $w > 0$, then standard expansions for absolutely continuous $X$ are given by

$$f(t) \sim \frac{g(b)}{\Gamma(w)} t^{w-1} e^{-bt} \quad \text{and} \quad S(t) \sim \frac{1}{b} f(t), \qquad t \to \infty, \tag{2.1}$$

where $g(b) = \lim_{s \uparrow b} (b-s)^w \mathcal{M}(s) > 0$. These expansions can be established subject to “Feller” conditions (Section 3) and are justified by using the Hardy–Littlewood–Karamata Tauberian theorem and its extensions from Feller [20], Chapter XIII.5. Such Feller conditions, however, only apply to simple settings and cannot be verified in the more complicated stochastic modelling settings in which $X$ is a compound distribution or a first-passage time for a semi-Markov process. Accommodating these more complicated settings requires establishing (2.1) under some new more inclusive “Ikehara” conditions which we provide in Proposition 1 (Section 3). Such Ikehara conditions are justified by introducing two Tauberian theorems used exclusively in the field of analytic number theory: the Ikehara–Wiener and Ikehara–Delange theorems, where the former theorem is the main tool for proving the prime number theorem. Thus our two main contributions in developing expansions of the type given in (2.1) are: (i) to replace the restrictive Feller conditions with the new more inclusive Ikehara conditions of Proposition 1, and (ii) to verify that the Ikehara conditions are satisfied for the more complicated compound distributions in stochastic models.

For the setting in which $b$ is a logarithmic singularity, we also propose some Ikehara conditions in Proposition 4 (Section 8) to establish the existence of an asymptotic hazard rate and to justify somewhat different expansions for $f(t)$ and $S(t)$ as $t \to \infty$.

A similar situation occurs when developing mass and survival function expansions for integer-valued $X$. In the common setting where the MGF has a gamma-like singularity at $b > 0$, so $\mathcal{M}(s) = O(e^b - e^s)^{-w}$ as $s \uparrow b$ for $w > 0$, then a well-known Tauberian theorem from Feller [20], Chapter XIII.5, establishes a Negative Binomial $(w, e^{-b})$-like tail with expansions

$$p(n) \sim \frac{g(e^b) e^{-bw}}{\Gamma(w)} n^{w-1} e^{-bn} \quad \text{and} \quad S(n) \sim \frac{1}{1 - e^{-b}} p(n), \qquad n \to \infty, \tag{2.2}$$

where $g(e^b) = \lim_{s \uparrow b} (e^b - e^s)^w \mathcal{M}(s) > 0$. Unfortunately, a condition for using this Tauberian theorem is that $\{p(n)\}$ is monotone in $n$, and verifying such a condition is difficult when only

$\mathcal{M}$ is known. Therefore, the expansions in (2.2) are established under the alternative "Darboux" conditions given in Proposition 2 (Section 4) and afforded by using Darboux's theorem derived from the field of analytic combinatorics. These minimal conditions avoid the monotonicity assumption and apply to the more complicated settings in which $X$ has a compound or first-passage time distribution. Comparable results when $b > 0$ is a logarithmic singularity of $\mathcal{M}$ are given in Proposition 5 (Section 8).

In a large number of practical examples, $b > 0$ is a simple pole so $w = 1$. In such examples, factor $g(b)$ in (2.1) is the negative residue of the MGF at $b$ in the continuous case, while $g(e^b)e^{-b}$ is the negative residue of $\mathcal{M}$ at $b$ in the discrete case. For this simple pole setting, survival and density/mass functions of distributions have exponential-like and geometric-like tails.

These expansions may be broaden to apply to both finite mixture distributions and finite convolutions under either Ikehara/Darboux conditions or Feller conditions. Within this context, our new characterization of the asymptotic hazard rate clarifies an assertion in [8] that the overall asymptotic hazard rate is the asymptotic hazard rate associated with the strongest and most enduring component within the mixture. This happens because the mixture convergence region is determined by the strongest component having the smallest non-negative convergence region. The same may be said about convolutions of independent random variables; the strongest addend has MGF whose non-negative convergence region is a proper subset of those for the weaker addends. The main applications for such results include sums and products of independent random variables. For mixture and convolution distributions whose components ostensibly have equal strength and share a common convergence boundary $b > 0$ for their MGFs, we show that the strongest components are those for which the singularity at $b$ attains the highest common order. Applications include sums of i.i.d. random variables.

Our third major goal is to establish these asymptotic expansions in infinite mixture/convolution distributions, such as compound distributions and first-passage distributions in semi-Markov processes, thereby succeeding under Ikehara/Darboux conditions when Feller conditions fail. Examples include density and survival expansions for the ruin amount $R$ in both the Cramér–Lundberg and Sparre Andersen models in Theorem 3 (Section 6.1) and Theorem 4 (Section 6.2). The density expansions are new and have the form $f_R(t) \sim \beta e^{-bt}$, while the survival expansions $S_R(t) \sim \beta e^{-bt}/b$ are well established and have traditionally been proven by using renewal theory as in Feller [20], XII.5. Once the density expansions have been established, however, the survival expansions follow directly from the smoothing of integration. The converse is not true; the density expansion does not follow from the coarsening effect of differentiating the survival expansion. Thus, the new Ikehara conditions stipulate when both the density and survival functions of $R$ admit exponential expansions. Further examples include general compound distributions with negative-binomial-like weights (Theorem 5 and Corollary 6 in Section 6.3), where new density expansions are established to complement the survival expansions of Embrechts *et al.* [18] and Willmot [30]. Additional examples include compound distributions determined from multiple classes of claim distributions (Theorems 6 and 7 in Section 6.3.1), where new expansions for density and survival functions are established under Ikehara/Darboux conditions.

Our fourth and perhaps most important goal is to extend Cramér–Lundberg-type expansions for density/mass and survival functions so they apply to the broad class of first-passage distributions in general finite-state semi-Markov processes in continuous and integer time. To do this, we first characterise such first-passage distributions as compound distributions determined from multiple classes of claim distributions with multivariate weights as just mentioned; see Proposition 3 (Section 7). This, along with some Ikehara conditions in continuous time, justifies new expansions for first-passage density and survival functions of the form $f(t) \sim \beta e^{-bt}$ and $S(t) \sim \beta e^{-bt}/b$ as given in Theorem 8 (Section 7). Here, $b = b(\mathcal{M}) > 0$ denotes the asymptotic failure rate of the first-passage distribution with MGF $\mathcal{M}$ and $\beta = \beta(\mathcal{M}, b)$ is the negative residue of $\mathcal{M}$ at $b$ given explicitly in (7.4). In integer time, first-passage times under minimal Darboux conditions admit geometric-like mass and survival expansions as specified in Theorem 9 (Section 7). Had Feller conditions been used, justification for the $p(n)$ expansion would have required the assumption that $\{p(n)\}$ is monotone in $n$. The importance of these expansions should not be understated because the great majority of failure time distributions in applied probability may be formulated as such first-passage times. For example, the ruin distribution in the Cramér–Lundberg and Sparre Andersen models is such a first-passage distribution for the semi-Markov process described in Example 9 (Section 7).

## 2.1. Implications of results

From Theorems 8 and 9, one may conclude that first-passage time distributions in semi-Markov processes admit the same exponential-like and geometric-like tail expansions that are known to occur for the class of Markov processes. Furthermore, the dominant rate is given by the asymptotic hazard rate $b$ or $1 - e^{-b}$. These findings are the most important results derived by using the general asymptotic theory, and obtaining such results was the original motivation in addressing the whole subject. From the many numerical examples in Butler [9,10], Chapter 13, it had already been made clear that first-passage hazard rates approach an asymptote of height $b$; see the plots of hazard rate functions computed from saddlepoint methods in Butler [9,10], Chapter 13. What Theorems 8 and 9 now provide is the theoretical underpinning for the asymptotes in these plots and an explanation for the exponential appearance of the accompanying saddlepoint density and survival plots.

Establishing exponential/geometric tails for such first-passage distributions has important statistical implications for estimating tail probabilities from such distributions using passage-time data. Butler and Bronson [12,13] developed nonparametric bootstrap methods for estimating such probabilities using saddlepoint methods based upon an estimate $\hat{\mathcal{M}}(s)$ for the first-passage MGF. Now, however, rather than estimating $S(t)$ nonparametrically from $\hat{\mathcal{M}}(s)$, expansion estimate $\hat{\beta} e^{-\hat{b}t}/\hat{b}$ can be used instead, where $\hat{b} = b(\hat{\mathcal{M}})$ and $\hat{\beta} = \beta(\hat{\mathcal{M}}, \hat{b})$ are estimates based on $\hat{\mathcal{M}}$. In the context of the Cramér–Lundberg approximation, Chung [16] has shown in his Ph.D. dissertation that this is indeed better. Starting with the true MGF $\mathcal{M}$, he first showed that expansion $\beta e^{-bt}/b$ is typically more accurate than the Lugananni–Rice saddlepoint approximation for $S(t)$ in the upper quartile of the distribution. Through simulation, he also showed that survival estimate $\hat{\beta} e^{-\hat{b}t}/\hat{b}$ typically has smaller relative error in the upper quartile than a fully nonparametric survival estimate based on $\hat{\mathcal{M}}$ using the methods in Butler and Bronson [12,13].

Another important reason for creating a widely applicable theory for expanding density/mass and survival functions under Ikehara/Darboux conditions is to provide very simple general conditions under which saddlepoint approximations for density/mass and survival functions achieve uniform tail accuracy. Existing conditions in Jensen [22], Sections 6.3–6.4, stipulate that distributions must satisfy relatively complicated conditions related to his method of proof which can be difficult to verify. The much simpler Ikehara conditions of Proposition 1 suffice when tails are gamma-like, and the author has recently shown (in new unpublished work) that saddlepoint approximations for density and survival functions achieve limiting relative error given by Stirling's approximation for $\Gamma(w)$. For lattice distributions, the same results hold for saddlepoint approximations of mass and survival functions under minimal Darboux conditions. Insofar as Ikehara/Darboux conditions ensure that such expansions apply to compound distributions and first-passage distributions in semi-Markov processes, then such uniform tail accuracy also carries over when saddlepoint methods are used to approximate such distributions. Thus, this work generalises and simplifies the uniformity results derived in Jensen [22], Chapter 7, for compound distributions and extends the uniformity results to first-passage distributions.

## 3. Absolutely continuous distributions

Suppose $X$ is an absolutely continuous random variable with support $(0, \infty)$, density $f(t)$, and survival function $S(t) = 1 - F(t)$. The hazard and cumulative hazard rate functions are

$$h(t) = f(t)/S(t) \quad \text{and} \quad H(t) = \int_0^t h(x)\, dx = -\ln\{S(t)\}.$$

The associated MGF is defined as $\mathcal{M}(s) = E(e^{sX})$ and converges on the real line for either $s \in (-\infty, b)$ or $(-\infty, b]$ for $b \geq 0$. The limiting average hazard rate is now characterised in terms of its MGF.

**Theorem 1.** *If a non-negative absolutely continuous random variable $X$ has moment generating function $\mathcal{M}(s)$ converging on $(-\infty, b)$ or $(-\infty, b]$ for $b \geq 0$, then*

$$\liminf_{t\to\infty} \frac{H(t)}{t} = b.$$

The theorem can be derived from first principles (see Supplementary Materials [11], Section A.1.1) or by using Theorem 2.4e from Widder [28], page 44.

**Lemma 1 (Widder).** *Suppose Laplace–Stieltjes transform $\mathcal{G}(s) = \int_0^\infty e^{-st}\, dG(t)$ converges on* $\mathrm{Re}(s) > -b < 0$ *for some function $G(t)$ of bounded variation. Then, $G(\infty)$ exists and*

$$-b = \limsup_{t\to\infty} \{t^{-1} \ln|G(\infty) - G(t)|\}.$$

**Proof of Theorem 1.** Let $G(t) = F(t)$ so that $G(\infty) = 1$ and

$$b = -\limsup_{t\to\infty}\{t^{-1} \ln S(t)\} = \liminf_{t\to\infty}\{-t^{-1} \ln S(t)\} = \liminf_{t\to\infty}\{t^{-1} H(t)\}. \qquad \square$$

Theorem 1 generalises to apply to any absolutely continuous random variable $X$ with distribution on $(-\infty, \infty)$. If $X$ has a MGF which converges on $(a, b)$ or $(a, b]$, where $a \leq 0 \leq b$, then $\liminf_{t\to\infty} H(t)/t = b$ as shown in Section A.1.2 of Supplementary Materials. For example, the Cauchy distribution has $a = 0 = b$ and $\lim_{t\to\infty} H(t)/t = 0$.

If the limiting hazard rate exists, as it does for many commonly used distributions, then these liminfs are indeed limits.

**Corollary 1.** *For an absolutely continuous random variable X with support on* $(-\infty, \infty)$, *if* $\lim_{t\to\infty} h(t)$ *exists*, *then*

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{H(t)}{t} = b. \tag{3.1}$$

**Proof.** If $h(t) \to b_0$ then the Cesàro mean $t^{-1}H(t) = t^{-1}\int_0^t h(s)\,ds \to b_0$ as $t \to \infty$. A proof of this follows the same approach as used for sequences. By Theorem 1, this limit must be $b$ so $b_0 = b$ and (3.1) holds. □

While this is the characterization we seek, the presumption that $h(t)$ has a limit is a fact that would not typically be known for a new unfamiliar distribution. Thus, the benefit of the corollary is to eliminate the computation but only if the limit is known to exist. Sufficient conditions are needed to guarantee such a limit and are provided below. The following pathological example provides some guidance for determining what these sufficient conditions need to be. The distribution has a periodic hazard rate function with no limit and has $\liminf_{t\to\infty} h(t)$ different from $\liminf_{t\to\infty} H(t)/t$.

***Example 1.*** The density

$$f(t) = 2/3(1 + \sin t)e^{-t}, \qquad t > 0 \tag{3.2}$$

takes value 0 on the set $\{3\pi/2 + 2\pi k : k = 0, 1, \ldots\}$ so that $\liminf_{t\to\infty} h(t) = 0$. The hazard rate is

$$h(t) = \frac{2(1 + \sin t)}{2 + \cos t + \sin t},$$

a $2\pi$-periodic function that does not have a limit. Its MGF is

$$\mathcal{M}(s) = 2/3\big[(1 - s)^{-1} + \big\{(1 - s)^2 + 1\big\}^{-1}\big], \qquad \text{Re}(s) < 1 = b. \tag{3.3}$$

Direct computation shows that

$$t^{-1}H(t) = 1 + t^{-1}\big(\ln 3 - \ln\big[2\cos^2(t/2) + \big\{\cos(t/2) + \sin(t/2)\big\}^2\big]\big).$$

The coefficient of $t^{-1}$ is bounded so that $\lim_{t\to\infty} H(t)/t = 1$.

The lack of a limiting hazard rate in this example can be explained by $\mathcal{M}$ not having a dominant pole on the boundary $\{s \in C : \text{Re}(s) = 1\}$ of its convergence region. From (3.3), we see that it has three simple poles $\{1, 1 \pm i\}$, which all vie for dominance of the hazard rate, and this leads to the periodic behaviour of $h(t)$. To ensure the existence of a limiting hazard function, we exclude such MGFs by stipulating some new Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$ in the next result.

**Proposition 1.** *Let $X$ have an absolutely continuous distribution $F(t)$ with support $(0, \infty)$ and associated moment generating function $\mathcal{M}(s)$ that converges on the complex half-plane $\{s \in C : \mathrm{Re}(s) < b\}$ for $b > 0$. Let $X$ also satisfy the Ikehara conditions as given below. Then* $\lim_{t\to\infty} h(t) = b$,

$$f(t) \sim \frac{g(b)}{\Gamma(w)} t^{w-1} e^{-bt} \quad \text{and} \quad S(t) \sim \frac{1}{b} f(t) \tag{3.4}$$

*as $t \to \infty$, where $g(b) = \lim_{s\uparrow b}(b - s)^w \mathcal{M}(s)$.*

**Ikehara conditions:** *$X$ (or $F$ or $\mathcal{M}$) satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$ where*:

($\mathfrak{I}_{\mathcal{M}}$) *$b$ is a dominant singularity in that the analytic continuation of $\mathcal{M}$ may be expressed as*

$$\mathcal{M}(s) = g(s)(b - s)^{-w} + h(s), \tag{3.5}$$

*where $w > 0$, $g$ and $h$ are analytic on $\{s \in C : \mathrm{Re}(s) \le b\}$, and $g(b) \ne 0$; and*

($\mathfrak{I}_{\mathrm{UND}}$) *there exists an $\varepsilon > 0$ such that the $(b+\varepsilon)$-tilted improper density $f_{b+\varepsilon}(t) := \exp\{(b+\varepsilon)t\} f(t)$ is non-decreasing for $t > A$, for some $A$.*

If $w$ is not a positive integer, then Ikehara condition $\mathfrak{I}_{\mathcal{M}}$ has the multi-function factor $(b - s)^{-w}$ which assumes principal branch values that are real-valued for $s < b$ and makes use of a branch cut along $[b, \infty]$.

These results state that gamma-like MGFs have densities with gamma-like tails. While such conclusions are not new, the Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$ for making such conclusions are new to the field of probability. A proof of Proposition 1 is given in Section B.1.3 and follows from two Tauberian theorems that have mostly been used in analytic number theory. In the case of a simple pole ($w = 1$) at $b$, this includes the Ikehara–Wiener theorem, given in Theorem B1 of Section B.1.1, which is well known as the primary tool for proving the prime number theorem; see Widder [28], pages 233–236, for its use in the proof. Other versions of this theorem are also described in, for example, Chandrasekharan [15], page 124, Doetsch [17], page 524, or Korevaar [23], Theorem 4.2, page 124. For other cases in which $0 < w \ne 1$, the proof uses the lesser known Ikehara–Delange theorem as stated in Theorem B2 of Section B.1.2 and given in Narkiewicz [24], Theorem 3.9, page 119.

Proposition 1 also holds if Ikehara conditions are replaced with the following Feller conditions, which are those needed to use results based on the Hardy–Littlewood–Karamata theorem in Feller [20], Section XIII.5; see Section A.2.1 for a proof.

**Feller conditions:** *$X$ satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{\mathrm{UM}}$ where*:

($\mathfrak{F}_{\mathcal{M}}$) *For real $s$, $M(s) \sim g(s)(b - s)^{-w}$ as $s \uparrow b$ for $w > 0$, and $g$ is left-continuous at $b$ with $g(b) > 0$; and*

($\mathfrak{F}_{\mathrm{UM}}$) *the improper $b$-tilted density $f_b(t) = e^{bt} f(t)$ is ultimately monotone, that is, it is monotone for all $t > A$, for some $A$.*

For Example 1, note that Feller condition $\mathfrak{F}_{\rm UM}$ fails to hold since tilted density $e^t f(t) = 2/3(1+\sin t)$ is not ultimately monotone in $t$ as $t\to\infty$. Overall, condition $\mathfrak{F}_{\rm UM}$ can be weakened to the condition $\mathfrak{F}_{\rm UM2}$ that $f_b(t)\sim v(t)$ as $t\to\infty$ with $v(t)$ ultimately monotone as indicated in Section A.2.1.

In many simple practical applications, both the Feller and Ikehara conditions apply. For example, if $X = -\ln\{\text{Beta}\,(\alpha,\beta)\}$, then both Ikehara and Feller conditions hold for all values of $\alpha,\beta>0$; this gives asymptotic hazard rate $\alpha$ and tail behaviour $f(t)\sim\Gamma(\alpha+\beta)/\{\Gamma(\alpha)\Gamma(\beta)\}e^{-\alpha t}$.

In more complicated stochastic model settings, however, this is not the case and only the Ikehara conditions can be applied in this broader range of settings. Direct comparison of the two sets of conditions shows why the Ikehara conditions are more practically useful. Verifying the condition placed on density $f$ is the main difficulty. Feller condition $\mathfrak{F}_{\rm UM}$ supposes the $b$-tilted density is either ultimately non-decreasing or non-increasing with almost all applications being ultimately non-decreasing. Ikehara condition $\mathfrak{I}_{\rm UND}$ supposes there exists some $(b+\varepsilon)$-tilted density, with $\varepsilon>0$, that is ultimately non-decreasing. Feller condition $\mathfrak{F}_{\rm UM}$ (applied as ultimately non-decreasing) is much stronger and more restrictive and implies that Ikehara condition $\mathfrak{I}_{\rm UND}$ holds for all $\varepsilon>0$.

The main consequence of using the more restrictive Feller condition $\mathfrak{F}_{\rm UM}$ is that it is generally not possible to show that it holds in stochastic modelling settings whereas the more relaxed Ikehara condition $\mathfrak{I}_{\rm UND}$ is often easily shown to hold for a sufficiently large $\varepsilon>0$. The classic Cramér–Lundberg example of Section 6 provides an example. In this model, the ruin amount $R$ with density $f_R(t)$ has MGF of the form

$$\mathcal{M}_R(s) = \frac{1-\rho}{1-\rho\mathcal{M}_E(s)}, \qquad \text{Re}(s) < b. \tag{3.6}$$

Here, $\mathcal{M}_E$ is the MGF for the excess life distribution of the claim density $f_X(t)$ and convergent on $\{\text{Re}(s)<c\}$, while $b\in(0,c)$ is the smallest positive zero of the denominator in (3.6). We want to conclude that the associated ruin density $f_R(t)\sim c_1e^{-bt}$ as $t\to\infty$ for constant $c_1>0$ as stated in Theorem 3 of Section 6.1. Assuming the claim density $f_X$ satisfies Feller condition $\mathfrak{F}_{\rm UM}$ does not allow one to conclude that ruin density $f_R$ also satisfies $\mathfrak{F}_{\rm UM}$ since under the former assumption $e^{ct}f_X(t)$ is ultimately non-decreasing whilst under the latter assumption $e^{bt}f_R(t)$ must be ultimately non-decreasing. The problem is simply that $b<c$. Such problems are avoided by placing an Ikehara condition on $f_X$. As will be seen in Section 6, a uniform Ikehara assumption on $f_X$, in which an $\varepsilon>0$ exists for which $e^{(c+\varepsilon)t}f_X(t)$ is non-decreasing for all $t>0$, allows one to conclude the same uniform Ikehara property for $f_R$, that is, $e^{(c+\varepsilon)t}f_R(t)$ is also non-decreasing for all $t>0$. Thus, for this and other stochastic models, an Ikehara condition needed for Proposition 1 to apply to the intractable density $f_R$ can be deduced by assuming the same Ikehara condition on the more tractable input claim density $f_X$. This idea underlies all the asymptotic results developed in the major applications concerning compound and first-passage densities in Sections 6–7.

Further comparison of Ikehara and Feller conditions placed upon $\mathcal{M}$ reveals that the Ikehara condition $\mathfrak{I}_{\mathcal{M}}$ is stronger than the corresponding Feller condition $\mathfrak{F}_{\mathcal{M}}$ thus compensating for the weaker condition placed on $f$. However, in most all practical settings, both conditions $\mathfrak{I}_{\mathcal{M}}$ and $\mathfrak{F}_{\mathcal{M}}$ tend to hold together and showing either is typically quite straightforward when $\mathcal{M}$ is given.

***Example 2 (Excess life distribution).*** Suppose absolutely continuous $X$ satisfies all the conditions of Proposition 1. If $X$ is interpreted as an interarrival time, then the excess life $E$ associated with it has density $f_E(t) = S(t)/\mu$, with $\mu = E(X)$, and MGF $\mathcal{M}_E(s) = \{1 - \mathcal{M}(s)\}/(-\mu s)$ which is also convergent on $\{\mathrm{Re}(s) < b\}$. If $X$ satisfies the Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$, then so does $E$ if the singularity $b$ for $\mathcal{M}$ is restricted to being a $w$-pole (so $w > 0$ is an integer); see Section B.2.1 for a proof. A comparable result can be shown under Feller conditions; see Section A.2.2.

From a measure-theoretic point of view, Proposition 1 applies only to a Radon–Nykodym derivative $f$ that satisfies either $\mathfrak{I}_{\mathrm{UND}}$ or $\mathfrak{F}_{\mathrm{UM}}$. In most applications, there is no ambiguity since $f$ is ultimately continuous with at most a finite number of step discontinuities. The theorem can also allow $f$ to have an infinite number $\{t_n : n \geq 1\}$ of step discontinuities that extend into the tail. Under such conditions, both $\mathfrak{I}_{\mathrm{UND}}$ and $\mathfrak{F}_{\mathrm{UM}}$ may hold if all but a finite number are upward stepping so that ultimately $f(t_n^-) \leq f(t_n^+)$; if, however, $f(t_n^-) > f(t_n^+)$ i.o., then neither of the conditions can hold.

Proposition 1 may be extended to absolutely continuous distributions supported on the real line using slightly amended Ikehara conditions and Feller conditions; see Corollaries B1 and A1 respectively, in Sections B.2.2 and A.2.3. As an example, consider $X = -\ln\{\mathrm{Gamma}\,(\alpha, \beta)\}$ with MGF $\mathcal{M}(s) = \beta^s \Gamma(\alpha - s)/\Gamma(\alpha)$ which has a dominant singularity at $\alpha$. Both Corollaries B1 and A1 apply to give tail behaviour $f(t) \sim \beta^\alpha e^{-\alpha t}/\Gamma(\alpha)$ which is easily verified directly.

While the expansions in (3.4) of Proposition 1 apply to gamma-like distributions, they do not apply to heavy-tailed distributions on $(0, \infty)$, whose MGFs converge on the non-open region $(-\infty, 0]$. Existing methods for obtaining such expansions with subexponential distributions do not lead to tail approximations with the same accuracy and hence practical importance as the current light-tailed expansions in Proposition 1; see Tijms [27], pages 332–333, and Rolski *et al.* [25], Section 5.4.2, for discussion and numerical verification. Neither does Proposition 1 apply to very light-tailed distributions, such as a Normal $(\mu, \sigma^2)$, for which $b = \infty$ and whose MGF lacks finite singularities. Third, it does not deal with all distributions for which $b$ is a branch point of the MGF; for example, an inverse Gaussian MGF, which converges on non-open region $(-\infty, b]$, as well as other examples given in Section A.4. Expansions for such distributions are considered in related unpublished work by the author. Finally, it does not deal with branch points created from the logarithm multi-function; such examples are covered in Section 8. Even though the theorem does not apply to such distributions, the value for $\liminf_{t\to\infty} H(t)/t$ is still $b \in [0, \infty]$, as described in Theorem 1, and this conclusion does not depend upon the type of singularity at $b$ nor upon whether $b = 0$ or $\infty$.

## 3.1. Large deviation theory and numerical accuracy of the expansions

Large deviation theory is concerned with the decay of $S(t)$ as $t \to \infty$ and a typical theorem would show that $-b < 0$ is the exponential rate of decay for $S$ as expressed through the equality $\lim_{t\to\infty} t^{-1} \ln S(t) = -b$. The conclusions of Proposition 1, however, are stronger because they not only imply such results but also provide the rate of such convergence as expressed through the leading term in the expansion of $t^{-1} \ln S(t) + b = o(1)$. Consider, for example, the very common setting in which $b$ is a simple pole. Then, Proposition 1 gives

$$t^{-1}\ln S(t)+b=t^{-1}\ln\{g(b)/b\}+o(t^{-1}),\qquad t\to\infty,\tag{3.7}$$

so the leading term in $o(1)$ is $t^{-1}\ln\{g(b)/b\}$ to order $o(t^{-1})$.

The use of the approximation in (3.7), as opposed to $t^{-1}\ln S(t)\approx -b$, is particularly important when approximating tail probabilities in practical applications. Indeed, expression (3.7) can be quite accurate when $b$ is a dominant pole even for moderately large values of $t$. Alternatively, $t^{-1}\ln S(t)\approx -b$ is only accurate for extremely large values of $t$ ([5], Section 6.6). As an example, Tijms [27], Section 8.4, Table 8.4.1, shows the good numerical accuracy that can be obtained when using (3.7) in the context of the well-known Cramér–Lundberg approximation which is to be discussed in Section 6.1.

# 4. Lattice distributions

It suffices to consider a non-negative integer-valued random variable with mass function $\{p(n):n\geq 0\}$ and hazard rate sequence

$$h_n=\frac{p(n)}{\sum_{j=n}^{\infty}p(j)},\qquad n\geq 0.$$

**Theorem 2.** *If $X$ has a mass function on the non-negative integer lattice and moment generating function $\mathcal{M}(s)$ which converges on $(-\infty,b)$ or $(-\infty,b]$, for $b\geq 0$, then*

$$\liminf_{n\to\infty}\left\{-\frac{1}{n}\sum_{k=0}^{n-1}\ln(1-h_k)\right\}=b.\tag{4.1}$$

**Proof.** The proof is either derived from first principles (see Section A.1), or directly from Lemma 1 by noting that

$$\ln S(n)=\ln P(X\geq n)=\ln\prod_{k=0}^{n-1}(1-h_k)=\sum_{k=0}^{n-1}\ln(1-h_k).$$ □

The result easily extends to arbitrary distributions on the integer lattice in which the MGF converges on $(a,b)$ or $(a,b]$ for $a\leq 0\leq b$. Following the proof in the continuous case, then $\liminf_{n\to\infty}\{-n^{-1}\sum_{k=-\infty}^{n-1}\ln(1-h_k)\}=b$.

Sufficient conditions for the existence of a limiting hazard rate as well as asymptotic expansions for the mass function $p(n)$ and survival function $S(n)$ are now given.

**Proposition 2.** *Suppose $X$ has non-negative integer support, and its moment generating function $\mathcal{M}(s)$, which converges on $\{s\in C:\mathrm{Re}(s)<b\}$ with $b>0$, satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$ below. Then, $\lim_{n\to\infty}h_n=1-e^{-b}$,*

$$p(n)\sim\frac{g(e^b)e^{-bw}}{\Gamma(w)}n^{w-1}e^{-bn}\quad\text{and}\quad S(n)\sim\frac{1}{1-e^{-b}}p(n),\tag{4.2}$$

*as* $n \to \infty$.

($\mathfrak{D}_{\mathcal{M}}$) $\mathcal{M}$ *has the form*

$$\mathcal{M}(s) = g(e^s)(e^b - e^s)^{-w} + h(e^s), \tag{4.3}$$

*where* $w > 0$, $g(e^s)$ *and* $h(e^s)$ *are analytic on* $\{s \in C : \mathrm{Re}(s) \leq b\}$ *with* $g(e^b) \neq 0$, *and* $(e^b - e^s)^{-w}$ *assumes principal branch values for non-integer* $w$. (*Condition* $\mathfrak{D}_{\mathcal{M}}$ *ensures that* $b$ *is a dominant singularity, that is, it is the only singularity on the boundary of the principal convergence region defined as* $\{s \in C : -\pi < \mathrm{Im}(s) \leq \pi; \mathrm{Re}(s) = b\}$.)

The novelty and importance of Proposition 2 are the Darboux condition $\mathfrak{D}_{\mathcal{M}}$ which ensures the well-known results in (4.2). Established conditions for deriving (4.2) from Tauberian Theorem 5 of Feller [20], XIII.5, require the additional assumption that $\{p(n)\}$ is ultimately non-increasing. Such an assumption is difficult to verify from $\mathcal{M}$ alone and this undermines all our applications to stochastic modelling. The Darboux condition $\mathfrak{D}_{\mathcal{M}}$, by comparison, does not limit such applications as it is quite weak and can be easily verified from $\mathcal{M}$.

A proof of Proposition 2 is given in Section A.5.2 and uses Darboux's theorem, which may be considered a lattice version of the Ikehara–Delange and Ikehara–Wiener theorems. Theorem A1 of Section A.5.1 is a modification of Darboux's theorem, as given in Theorem 5.11 of Wilf [29], page 194, and deals with generating functions (GFs) that converge on a disc of radius $c > 0$ rather than the usual radius of $c = 1$ in which the theorem is generally stated. The proof also uses the Stolz–Cesàro theorem in Lemma A1 of Section A.5.1 which is a discrete version of l'Hôpital's rule; see Huang [21], page 32.

For a simple distribution such as $X \sim$ Negative Binomial $(m, p)$, Proposition 2 applies with $w = m$ and $e^{-b} = p$ and the asymptotic order in (4.2) is easily verified. The expressions in (4.2) are exact for the Geometric mass function with $m = 1$. Our next class of examples is less trivial.

***Example 3 (Equilibrium distributions).*** Suppose $\{p(n) : n \geq 0\}$ has GF $\mathcal{P}(z)$ and is the equilibrium distribution for a positive recurrent queue with a countably infinite state space. Conditions on $\mathcal{P}$ that ensure $p(n) \sim \gamma e^{-bn}$ as $n \to \infty$ are given in Theorem C.1 of Tijms [27], pages 452–453, and they agree with the conditions in Proposition 2. The GF has form $\mathcal{P}(z) = \mathcal{N}(z)/\mathcal{D}(z)$ and is assumed to be analytic on $\{|z| \leq c\}$ apart from a simple pole at $c = e^b > 1$, which results from a simple zero of $\mathcal{D}(z)$. When written as in (4.3) of Proposition 2, the MGF $\mathcal{P}(e^s) = g(e^s)/(e^b - e^s)$ has factor $g(e^s) = \mathcal{N}(e^s)(e^b - e^s)/\mathcal{D}(e^s)$, which is analytic on $\{\mathrm{Re}(s) \leq b\}$, and $\mathcal{P}(e^s)$ admits a negative residue at $b$ given by

$$\gamma = g(e^b)e^{-b} = -\mathcal{N}(e^b)/\{e^b \mathcal{D}'(e^b)\}.$$

A wide range of examples in [27] result in equilibrium distributions with GFs of this form. Examples includes a discrete-time queue (Example 3.4.1), a continuous-time Markov process on a semi-infinite state space (Section 4.4), the M/G/1 queue (Sections 4.4 and 9.2), a bulk $\mathrm{M}^X$/G/1 queue (Section 9.3), and approximations to several GI/G/m queues (Sections 9.5–9.7).

Proposition 2 can be extended to lattice distributions over all integers. The results are summarised in Corollary A6 in Section A.5.3 under slightly modified conditions.

# 5. Finite convolution and mixture distributions

A considerable broadening of the theory in Sections 3 and 4 results when it is applied to finite convolutions and mixtures. Let $X$ be an absolutely continuous or integer-valued random variable that has an asymptotic hazard by virtue of satisfying the Ikehara or Darboux conditions of Proposition 1 or 2. Now, convolve $X$ with independent variable $Y$ and mix the resulting distribution with the distribution of independent random variable $Z$. This leads to a mixture-convolution random variable $W$ with density/mass function $pf_{X+Y}(t) + qf_Z(t)$ for some $p \in (0, 1]$ with $q = 1 - p$. Conditions are given below to ensure that the distribution of $W$ also satisfies Proposition 1 or 2 so that its asymptotic hazard rate exists and its tail behaviour can be characterised.

## 5.1. One strongest variable

Let $Y$ and $Z$ be strictly weaker components than $X$ in the sense that their MGFs have strictly larger non-negative convergence regions than that for $X$. Then, apart from some additional technical conditions, the mixture-convolution distribution for $W$ has the same asymptotic hazard rate as that for the strongest variable $X$. Also, the survival and density/mass functions for $W$ have the same gamma-like and negative binomial-like tails as for $X$ and differ only by having a different value for constant $g(b)$. The general interpretation that may be given is that the strongest component prevails asymptotically, and the two weaker components $Y$ and $Z$ only express themselves by changing the value of the constant $g(b)$. Block *et al.* [7] and Block and Joe [6], Theorem 4.1, pointed out the lack of influence of $Z$ on asymptotic hazard whilst the minimal influence of $Y$ is new. These results are formalized in Corollary 2 under Ikehara conditions and the proof is relegated to Section B.3.1. Comparable results under Feller conditions are given in Corollary A2 of Section A.3.1. We use subscripted notation so $\mathcal{M}_X(s)$ denotes the MGF of $X$, etc.

**Corollary 2.** *Let $X, Y$, and $Z$ be absolutely continuous and non-negative variables such that $X$ is stronger than $Y$ and $Z$; that is, let $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ converge on $\{\mathrm{Re}(s) < b\}$, $\{\mathrm{Re}(s) < b + \eta_Y\}$, and $\{\mathrm{Re}(s) < b + \eta_Z\}$ respectively for $b > 0$ and some values $\eta_Y > 0 < \eta_Z$.*

**Ikehara conditions**: *Suppose $X$ satisfies $\mathfrak{I}_{\mathcal{M}}$ and either $X$ or $Y$ (or both) satisfies the uniform Ikehara condition denoted by $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and given below. Let $Z$ satisfy condition $\mathfrak{I}_{\mathrm{UND}}$ if $p < 1$.*

($\mathfrak{I}_{\mathrm{ND}(0,\infty)}$) *$X$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ if an $\varepsilon > 0$ exists for which $e^{(b+\varepsilon)t} f_X(t)$ is non-decreasing for all $t > 0$.*

*Then, $X + Y$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and $W$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$. Thus, $W$ has asymptotic hazard rate $b > 0$, density*

$$f_W(t) \sim p\mathcal{M}_Y(b) f_X(t) \sim p\mathcal{M}_Y(b) g_X(b) \Gamma(w)^{-1} t^{w-1} e^{-bt}, \qquad t \to \infty, \tag{5.1}$$

*and survival $S_W(t) \sim f_W(t)/b$.*

***Example 4 (Sums and products of independent variables).*** Suppose $Z_1, \ldots, Z_k$ are independent with $Z_i \sim$ Gamma $(\alpha_i, \beta_i)$ and $\beta_1 < \min_{i\geq 2} \beta_i$. The sum $W = Z_1 + \sum_{i\geq 2} Z_i$ is the passage time though a series connection of states in a semi-Markov process with Gamma $(\alpha_i, \beta_i)$ holding times. The strongest addend of $X$ is $Z_1$. Ikehara conditions require $\min_i \alpha_i \geq 1$, since the conditions of Corollary 2 are that at least one $Z_i$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. The Feller conditions of Corollary A2 in Section A.3.1 are more restrictive and require $\alpha_1 \geq 1$ since they are more specific in requiring that the strongest variable satisfy the uniform Feller condition $\mathfrak{F}_{\mathrm{ND}(0,\infty)}$. Both sets of conditions lead to the conclusion that $W$ has a Gamma $(\alpha_1, \beta_1)$ tail.

If $Z_1, \ldots, Z_k$ are independent with $Z_i \sim$ Beta $(\alpha_i, \beta_i)$ and $\alpha_1 < \min_{i\geq 2} \alpha_i$, then the product $\prod_{i\geq 1} Z_i$ arises as the posterior distribution for the probability that a series connection of $k$ independent components is working. It is also the null distribution for Wilks' likelihood ratio test and most of the other likelihood ratio test statistics in multivariate analysis of variance; see Anderson [3], Chapters 9–10. Variable $W = -\ln Z_1 - \sum_{i\geq 2} \ln Z_i$ has strongest addend $-\ln Z_1$ and Corollaries 2 and A2 imply that $W$ has an exponential-like tail of order $O(e^{-\alpha_1 t})$ under Ikehara condition $\max_i \beta_i \geq 1$ or Feller condition $\beta_1 \geq 1$.

For absolutely continuous densities, the additional uniformity restriction of $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ in Corollary 2 over $\mathfrak{I}_{\mathrm{UND}}$ does not adversely restrict the range of applicability of the corollary. Indeed, the only practical densities in $\mathfrak{I}_{\mathrm{UND}} \setminus \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ seem to be those that are either unbounded at $t = 0$ or have a downward stepping discontinuity, that is, $f(t^-) > f(t^+)$ for some $t > 0$. Also, only one of the two variables $X$ and $Y$ needs to satisfy $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ while the other may have an unbounded density or may have downward stepping discontinuities.

When the random variables in Corollary 2 have support on $(-\infty, \infty)$, then comparable results can be derived and are given in Corollary B2 of Section B.3.2. The same results using Feller conditions are given in Corollary A3 of Section A.3.2. As an example, consider the gamma variables in Example 4 and suppose $\alpha_1 < \min_{i\geq 2} \alpha_i$. The product $\prod_{i\geq 1} Z_i$ describes the distribution for the determinant of a $k \times k$ Wishart matrix based on independent components. Taking logarithms, then $W = -\ln Z_1 - \sum_{i\geq 2} \ln Z_i$ has strongest addend $-\ln Z_1$ with an exponential-like tail of order $O(e^{-\alpha_1 t})$ under both Ikehara and Feller conditions.

**Corollary 3.** *Suppose integer-valued $X \geq 0$ satisfies the conditions of Proposition 2 with $\mathcal{M}_X(s) = g_X(e^s)(e^b - e^s)^{-w} + h_X(e^s)$. Let independent integer-valued variables $Y \geq 0$ and $Z \geq 0$ be such that $\mathcal{M}_Y(s)$ and $\mathcal{M}_Z(s)$ are analytic on $\{s \in C : \mathrm{Re}(s) < b + \eta_Y\}$ and $\{s \in C : \mathrm{Re}(s) \leq b + \eta_Z\}$ respectively for $\eta_Y > 0 < \eta_Z$. Then, the distribution of $W$ has asymptotic hazard rate $1 - e^{-b}$ and mass and survival functions of asymptotic order given in (4.2) with $g(e^b)$replaced by $p\mathcal{M}_Y(b)g_X(e^b)$.*

## 5.2. Equally strong variables

Suppose random variables $X, Y$, and $Z$ are ostensibly of equal strength with MGFs that share the common convergence region $\{s \in C : \mathrm{Re}(s) < b\}$. Subject to some Ikehara conditions, convolution/mixture variable $W$ has asymptotic hazard rate $b$ with asymptotic tail behaviour as given below. See Section B.3.3 for a proof. Comparable results under Feller conditions are given in Corollary A4 in Section A.3.3.

**Corollary 4.** *Suppose absolutely continuous, non-negative, and independent variables $X$, $Y$, and $Z$ have moment generating functions $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ which share the common convergence region $\{s \in C : \mathrm{Re}(s) < b\}$.*

**Ikehara conditions**: *Let $X, Y$, and $Z$ all satisfy $\mathfrak{I}_{\mathcal{M}}$ of Proposition 1, and suppose the singularities for $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ at $b > 0$ are poles with positive integer orders $w_X, w_Y$, and $w_Z$, respectively. Furthermore, suppose either $X$ or $Y$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and $Z$ satisfies $\mathfrak{I}_{\mathrm{UND}}$ if $p < 1$.*

*Then, $X + Y$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and $W$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{UND}}$. Thus, $W$ has asymptotic hazard rate $b > 0$, density*

$$f_W(t) \sim g_W(b)\Gamma(w_*)^{-1} t^{w_*-1} e^{-bt}, \qquad t \to \infty, \tag{5.2}$$

*and survival $S_W(t) \sim f_W(t)/b$, where $w_* = \max\{w_X + w_Y, w_Z\}$ and*

$$g_W(b) = \begin{cases} p g_X(b) g_Y(b), & \text{if } w_X + w_Y > w_Z, \\ p g_X(b) g_Y(b) + q g_Z(b), & \text{if } w_X + w_Y = w_Z, \\ q g_Z(b), & \text{if } w_X + w_Y < w_Z. \end{cases} \tag{5.3}$$

When variables in a finite convolution/mixture share a common convergence region, then, according to Corollary 4, the resulting distribution still reflects the strongest component but that component is now the one with the highest order singularity at $b$. If multiple components share the highest common order, as occurs when $w_X + w_Y = w_Z$ in (5.3), then all such components contribute to the asymptotic order through the value of coefficient $g_W(b)$. The following result for i.i.d. variables follows directly from Corollary 4. The same result under Feller conditions is given in Corollary A5 of Section A.3.4.

**Corollary 5 (Convolution of i.i.d. variables).** *Let $W = X_1 + \cdots + X_m$ where $X_1, \ldots, X_m$ are non-negatively-valued i.i.d. variables from an absolutely continuous distribution.*

**Ikehara conditions:** *Suppose $X_1$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and singularity $b > 0$ for $\mathcal{M}_{X_1}$ is a $w$-pole, for integer $w$.*

*Then, $W$ also satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Proposition 1 applies to give asymptotic hazard rate $b > 0$, density*

$$f_W(t) \sim g_{X_1}^m(b)\Gamma(mw)^{-1} t^{mw-1} e^{-bt}, \qquad t \to \infty, \tag{5.4}$$

*and survival function $S_W(t) \sim f_W(t)/b$. If $\mathcal{M}_{X_1}(s)$ takes the form $g_{X_1}(s)(b - s)^{-w}$ in (3.5) with addend $h_{X_1}(s) \equiv 0$, then the same conclusions hold for arbitrary $w > 0$ (it need not be an integer).*

**Proof.** We only comment on showing that $W$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Apply Lemma B1 of Section B.3.1 successively to the sequence $W_2 = X_1 + X_2$, $W_3 = W_2 + X_3, \ldots, W = W_{m-1} + X_m$ to show that $\{W_2, \ldots, W_{m-1}, W\}$ all satisfy condition $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Thus, $W$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$. □

***Example 5 (Excess life distribution).*** Consider an $m$-fold convolution $W = E_1 + \cdots + E_m$ of i.i.d. excess life variables $E$ as in Example 2. Two results are shown in Section B.3.4. First, if interarrival time $X$ satisfies $\mathfrak{I}_{\mathcal{M}}$ with $\mathcal{M}_X$ convergent on $\{\mathrm{Re}(s) < b\}$ and $b > 0$ is an $w$-pole, then $E$ satisfies $\mathfrak{I}_{\mathcal{M}}$ and $b > 0$ is a $w$-pole for $\mathcal{M}_E$.

Second, if $X$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$, then $E$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Putting the two results together, if $X$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ with $b$ as an $w$-pole of $\mathcal{M}_X$, then $E$ satisfies the requirements for Corollary 5, i.e. $E$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and singularity $b$ is a $w$-pole for $\mathcal{M}_E$. Thus, $W$ also satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and has density and survival function with asymptotic orders as in (5.4).

# 6. Compound distributions

A rich Tauberian theory is derived below for such infinite mixture distributions under Ikehara conditions. Corresponding results cannot be derived under Feller conditions since it is generally not possible to show that such distributions satisfy condition $\mathfrak{F}_{\mathrm{UM}}$.

## 6.1. Cramér–Lundberg approximation

Suppose the arrival of claims filed against an insurance company follows a Poisson process $\{N(\tau) : \tau > 0\}$ with rate $\lambda > 0$. Let successive positive claim amounts be the i.i.d. absolutely continuous values $\{X_i\}$ with mean $\mu$ so the compound Poisson process $L(\tau) = \sum_{i=1}^{N(\tau)} X_i$ describes the company payout after time $\tau$. Also, suppose the company's premiums increase revenues at constant rate $\sigma > 0$ with $\sigma > \lambda\mu$ so the premium rate exceeds the claim rate and $\rho = \lambda\mu/\sigma < 1$. If the company starts with initial reserve $t$, then the probability of eventual ruin for the company is $S(t) = P\{L(\tau) - \sigma\tau > t\ \exists_{\tau>0}\}$. With claims filed after interarrival times $\{T_i\}$, which are i.i.d. Exponential ($\lambda$), then the ruin must occur at an arrival epoch of a claim so that

$$S(t) = P\left\{ S_n = \sum_{i=1}^{n} (X_i - \sigma T_i) > t\ \exists_{n\geq 1} \right\} = P(R > t) = S_R(t),$$

where $R = \sup_{n\geq 1} S_n$ is the maximum loss that occurs with claim $N = \arg\sup_{n\geq 1} S_n$. Denote the density of $R$ as $f_R(t)$ so $-S_R'(t) = f_R(t)$ for a.e. $t$.

Suppose the claim amount $X$ has MGF $\mathcal{M}_X$ which converges on $\{\mathrm{Re}(s) < c\}$ with $c > 0$. Let $E$ have the excess life distribution for $X$ with MGF $\mathcal{M}_E$. Then, $R$ has MGF

$$\mathcal{M}_R(s) = \int_{0^-}^{\infty} e^{st}\, dF_R(t) = \frac{1-\rho}{1-\rho\mathcal{M}_E(s)} = (1-\rho) + \rho \sum_{k=0}^{\infty} (1-\rho)\rho^k \mathcal{M}_E^{k+1}(s) \qquad (6.1)$$

which converges on $\{\mathrm{Re}(s) < b\}$ where $b \in (0, c)$; see Tijms [27], Section 8.4. Convergence bound $b$ is a simple pole and results as a simple zero of $1 - \rho\mathcal{M}_E(s)$ since $\mathcal{M}_E(s)$ is strictly increasing. The rightmost expression in (6.1) reveals $R$ as a mixture distribution with a point mass of $1 - \rho$ at $t = 0$ and an absolutely continuous component with mass $\rho$ on $(0, \infty)$. The following result, based on Ikehara conditions, is shown in Section B.4.1.

**Theorem 3.** *Suppose absolutely continuous claim amount $X$ is as described above and satisfies uniform Ikehara condition $\Im_{\mathrm{ND}(0,\infty)}$. Then, $R^+ = R|R > 0$, the positive and absolutely continuous portion of maximum loss $R$, satisfies $\Im_{\mathcal{M}} \cap \Im_{\mathrm{ND}(0,\infty)}$. Thus,*

$$f_R(t) = \rho f_{R^+}(t) \sim \frac{b(1-\rho)}{\lambda \mathcal{M}'_X(b)/\sigma - 1} e^{-bt}, \qquad t \to \infty, \tag{6.2}$$

*so that $S_R(t) = \rho S_{R^+}(t) \sim f_R(t)/b$.*

Theorem 3 is a new stronger form of the Cramér–Lundberg approximation since it provides an expansion for the density of $R$ in (6.2) and not just $S_R(t)$ as traditionally given in, for example, Asmussen [4], III.5 Theorem 5.3. The conclusions of Theorem 3 are stronger but they also require the stronger assumption that $X$ satisfies $\Im_{\mathrm{ND}(0,\infty)}$. Theorem 3 is proved by using Tauberian theory whereas the traditional expansion for $S_R(t)$ is derived by using renewal theory as in Feller [20], Sections XI.6 and XII.5.

## 6.2. Sparre Andersen risk model

This model generalises the Cramér–Lundberg model so claims can arrive according to a general renewal process rather than as a Poisson process. Assume absolutely continuous interarrival times $\{T_i\}$ are i.i.d. with a MGF $\mathcal{M}_T(s)$ which converges on $\{\mathrm{Re}(s) < a\}$ for $a > 0$. For all other quantities, we use the notation from the previous subsection. Assume i.i.d. absolutely continuous claims $X_1, X_2, \ldots$ have a MGF $\mathcal{M}_X(s)$ convergent on $\{\mathrm{Re}(s) < c\}$ for $c > 0$, and that $E(X - \sigma T) < 0$ so the drift of random walk $\{S_n = \sum_{i=1}^n (X_i - \sigma T_i)\}$ is negative. Let $b \in (0, c)$ be the unique (real) positive root of $\mathcal{M}_{X-\sigma T}(s) = \mathcal{M}_X(s)\mathcal{M}_T(-\sigma s) = 1$.

**Theorem 4.** *Suppose the conditions on $\mathcal{M}_T(s)$ and $\mathcal{M}_X(s)$ above. If $X$ satisfies uniform Ikehara condition $\Im_{\mathrm{ND}(0,\infty)}$, then $R^+ = R|R > 0$ satisfies $\Im_{\mathcal{M}} \cap \Im_{\mathrm{ND}(0,\infty)}$ and*

$$f_R(t) \sim \alpha e^{-bt}, \qquad t \to \infty, \tag{6.3}$$

*so that $S_R(t) \sim \alpha e^{-bt}/b$. Here, $\alpha > 0$ is given in* (9.37) *of Section* B.4.2 *as the negative residue of $\mathcal{M}_R(s)$ at $b$.*

The expansion for density $f_R(t)$ in (6.3) is new and requires the additional assumption that $X$ satisfies $\Im_{\mathrm{ND}(0,\infty)}$. Expansion (6.3) ensures that the established expansion for survival function $S_R(t)$ also holds as previously given, for example, in Theorem 3.1, case (i), of [19]. Note that the conditions on $X$ in Theorem 4 are exactly the same as those used in Theorem 3 for the Cramér–Lundberg setting.

The rather long proof of Theorem 4 is given in Section B.4.2 and demonstrates that $R^+$ satisfies the Ikehara conditions of Proposition 1. That $\mathcal{M}_{R^+}$ satisfies condition $\Im_{\mathcal{M}}$ follows from the Wiener–Hopf factorization that determines $\mathcal{M}_R$. To show condition $\Im_{\mathrm{UND}}$, we use the compound geometric sum characterization of $R$ in which $R = \sum_{i=0}^{N_+} L_i^+$ where $\{L_i^+ : i \geq 1\}$ are i.i.d. with the ascending ladder distribution of the random walk $\{S_n\}$ and $L_0^+$ denotes a point mass at 0.

From this compound geometric sum, we determine that $R^+$ satisfies $\mathfrak{I}_{\text{ND}(0,\infty)}$ if $L_1^+$ satisfies $\mathfrak{I}_{\text{ND}(0,\infty)}$ and this holds if $X$ satisfies $\mathfrak{I}_{\text{ND}(0,\infty)}$. The latter result follows by noting that the last convolved step amount (or $Y = X - \sigma T$ addend) in the ladder height $L_1^+$ is necessarily a positive step which depends on the $X$ contribution in the step amount $X - \sigma T$.

Feller conditions do not suffice in this line of proof. Tilting parameter $b$ must be used to show $R^+$ satisfies $\mathfrak{F}_{\text{UM}}$ whereas the assumption that $X$ satisfies $\mathfrak{F}_{\text{UM}}$ uses the larger tilting parameter $c > b$. Thus, the fact that $X$ satisfies $\mathfrak{F}_{\text{UM}}$ has no bearing on whether $R^+$ can satisfy $\mathfrak{F}_{\text{UM}}$.

## 6.3. More general compound distributions

In the two previous subsections, the ruin amount $R = \sum_{k=0}^{N} X_k$ assumes that $N$ has a geometric mass function, and this results in an infinite mixture distribution with geometric weights. We now give an expansion for the density of $R$ when $N$ assumes a more general mass function with probability generating function (PGF) $\mathcal{P}(z) = \sum_{k=0}^{\infty} p(k) z^k$ that converges on $\{|z| < r\}$ for $r > 1$. Suppose $X_0$ has a point mass at zero and let $\{X_k : k \geq 1\}$ be i.i.d. absolutely continuous positive claim amounts with MGF $\mathcal{M}(s)$ that converges on $\{\text{Re}(s) < c\}$ or $\{\text{Re}(s) \leq c\}$ for $c > 0$. Now, $R$ has a compound distribution with MGF $\mathcal{P}\{\mathcal{M}(s)\}$.

The next result shows that the density of $R$ has a gamma-like tail under two conditions: $\mathcal{M}$ satisfies Ikehara condition $\mathfrak{I}_{\text{ND}(0,\infty)}$, and $\mathcal{P}$ satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$. Condition $\mathfrak{D}_{\mathcal{M}}$ holds if convergence bound $r > 1$ for $\mathcal{P}$ is the only singularity on the circle $\{z \in C : |z| = r\}$ and $\mathcal{P}(z) = g(z)(r - z)^{-w} + h(z)$, where $w > 0$, $g(z)$ and $h(z)$ are analytic on the closed disc $\{|z| \leq r\}$, and $g(r) \neq 0$. See Section B.4.3 for a proof.

**Theorem 5.** *Let $\mathcal{P}(z)$ and $\mathcal{M}(s)$ be as described above and suppose $r < \mathcal{M}(c) \leq \infty$. Also suppose $\mathcal{M}$ satisfies uniform Ikehara condition $\mathfrak{I}_{\text{ND}(0,\infty)}$ and $\mathcal{P}$ satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$. Then, $R^+ = R|R > 0$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{\text{ND}(0,\infty)}$, $R$ has limiting hazard rate $b \in (0, c)$, where $b$ is the unique root of $\mathcal{M}(s) = r$ in $(0, c)$, and*

$$f_R(t) \sim \frac{g(r)}{\Gamma(w)\{\mathcal{M}'(b)\}^w} t^{w-1} e^{-bt}, \qquad t \to \infty. \tag{6.4}$$

*Consequently, $S_R(t) \sim f_R(t)/b$ and, for the stop-loss premium, $\int_t^\infty S_R(u)\,du \sim f_R(t)/b^2$ as $t \to \infty$.*

The density expansion is new while the expansion $S_R(t) \sim f_R(t)/b$ replicates the result of Embrechts *et al.* [18] who assume $p(k) \sim C_1 k^{w-1} r^{-k}$ as $k \to \infty$ so that $N$ has a Negative Binomial $(w, 1/r)$ tail. Note that the Darboux condition on $\mathcal{P}$ is slightly stronger, and, by Proposition 2, implies that the expansion for $p(k)$ holds with $C_1 = g(r) r^{-w} / \Gamma(w)$. Thus, Theorem 5 uses a slightly stronger condition but also returns a stronger result by giving the density expansion in (6.4). In addition, the conditions of Theorem 5 are more useful since the easiest way to justify the expansion for $p(k)$ is to verify the Darboux assumption on $\mathcal{P}$.

***Example 6 (Negative Binomial** $(w, p)$ **weights).*** This example is of practical importance since such weights represent over-dispersed Poisson weights that result from a gamma mixture of

Poisson weights. Let $N$ have mass function $p(k)=\binom{w+k-1}{k}p^w q^k$ on $k\geq 0$, with $q=1-p$, so that $\mathcal{P}(z)=p^w/(1-qz)^w$ for $z<1/q$. Then, $R$ has the density expansion in (6.4) where $g(r)=(p/q)^w$ and $b$ solves $\mathcal{M}(b)=1/q$. The expansion for $S_R(t)$ was originally derived in [26] under the assumption that $e^{bt}S_R(t)$ is ultimately monotone.

A version of Theorem 5 holds with integer-valued non-negative claim amounts $\{X_k : k\geq 1\}$ which are i.i.d. with MGF $\mathcal{M}(s)=\mathcal{P}_X(e^s)$. The proof is given in Section B.4.3.

**Corollary 6.** *Suppose claim amount $X$ is non-negative and integer-valued (but not degenerate at 0) and let $\mathcal{P}(z)$ and $\mathcal{M}(s)=\mathcal{P}_X(e^s)$ have the same properties as in Theorem 5 (except now $X$ does not satisfy $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$). Then compound sum $R=\sum_{k=0}^N X_k$ has limiting hazard rate $1-e^{-b}$, where $b\in(0,c)$ is the unique root of $\mathcal{P}_X(e^s)=r$ in $(0,c)$. Furthermore,*

$$p_R(n)\sim\frac{g(r)e^{-bw}}{\Gamma(w)\{\mathcal{P}_X'(e^b)\}^w}n^{w-1}e^{-bn},\qquad n\to\infty,\tag{6.5}$$

*$S_R(n)\sim(1-e^{-b})^{-1}p_R(n)$, and stop-loss premium, $\sum_{k=n+1}^\infty S_R(k)\sim(1-e^{-b})^{-2}p_R(n+1)$ as $n\to\infty$.*

These expansions agree with those in [30] who assumes that $N$ has mass function $p(k)\sim C_1k^{w-1}r^{-k}$ as $k\to\infty$; note this is implied by the Darboux condition on $\mathcal{P}$.

### 6.3.1. *Compound distributions with multiple claim distributions and multivariate weights*

Compound distributions are generalised to allow $M$ distinct categories of positive claim amounts $X_1,\dots,X_M$. Suppose $\{X_i\}$ are absolutely continuous and independent, and $X_i$ has MGF $\mathcal{M}_i(s)$ which converges on $\{\mathrm{Re}(s)<c_i\}$ or $\{\mathrm{Re}(s)\leq c_i\}$ for $c_i>0$. If $\{X_{ij}: j\geq 1\}$ are i.i.d. absolutely continuous claims from category $i$, then $R=\sum_{i=1}^M 1\{N_i\geq 1\}\sum_{j=1}^{N_i}X_{ij}$ denotes a compound variable over the $M$ categories of claims, where $\mathbf{N}=(N_1,\dots,N_M)^T$ tallies counts for the various categories. Let the components of $\mathbf{N}$ have a general distribution, non-degenerate in each component, with multivariate PGF $\mathcal{P}(\mathbf{z})$, where $\mathbf{z}=(z_1,\dots,z_M)^T\in\Re^M$. Suppose $\mathcal{P}$ converges on maximal set $\mathcal{O}\supset\{\mathbf{z}\in\Re^M : |z_i|\leq 1 \text{ for } i=1,\dots,M\}$ so as to avoid heavy-tailed counting components. Also, let $\mathcal{M}(s)=\{\mathcal{M}_1(s),\dots,\mathcal{M}_M(s)\}^T$, with scalar $s\in\Re$, be the vector of MGFs for $\mathbf{X}=(X_1,\dots,X_M)^T$. Then, $R$ has MGF $\mathcal{P}\{\mathcal{M}(s)\}$ which converges in a neighbourhood of 0. The assumption that Ikehara condition $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ holds for the components of $\mathcal{M}(s)$ can be used to show that $f_R(t)$ and $S_R(t)$ have Gamma tails with a limiting hazard rate. A proof is given in Section B.4.3.

**Theorem 6.** *Let $\mathcal{P}(\mathbf{z})$ and $\mathcal{M}(s)$ be as described above and suppose each component of $\mathcal{M}(s)$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Let $\mathcal{P}(\mathbf{z})=\mathcal{N}(\mathbf{z})/\mathcal{D}(\mathbf{z})$ have maximal convergence region $\mathcal{O}$, where $\{\mathbf{z}:|z_i|\leq 1 \text{ for } i=1,\dots,M\}\subset\mathcal{O}\subseteq\{\mathbf{z}\in\Re^M:\mathcal{D}(\mathbf{z})>0\}$. Take $c_*=\min_i c_i$ and suppose $\mathcal{D}\{\mathcal{M}(s)\}=0$ admits a smallest positive root $b\in(0,c_*)$ which is an $m$-zero. Let $\mathcal{N}(\mathbf{z})$ and $\mathcal{D}(\mathbf{z})$ be analytic at $\mathcal{M}(b)\in\Re^M$ with $\mathcal{N}\{\mathcal{M}(b)\}\neq 0$. Then, $R^+=R|R>0$ satisfies $\mathfrak{I}_\mathcal{M}\cap\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ and*

$$f_R(t)\sim\beta t^{m-1}e^{-bt},\qquad S_R(t)\sim b^{-1}f_R(t),\qquad t\to\infty,\tag{6.6}$$

*with*

$$\beta = \frac{1}{(m-1)!} \lim_{s\to b} (b-s)^m \mathcal{P}\{\mathcal{M}(s)\} = \frac{(-1)^m m \mathcal{N}\{\mathcal{M}(b)\}}{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}. \tag{6.7}$$

An important feature of Theorem 6 is that it lacks the assumption that $b$ is a dominant pole for $\mathcal{P}\{\mathcal{M}(s)\}$; this emerges subject to the quite minimal conditions placed upon $\mathcal{P}$.

The important applications for Theorem 6 are deferred to the next section since they are rather involved and concern the derivation of exponential tail behaviour for the densities and survival functions of first-passage distributions in finite-state continuous-time semi-Markov processes. For now, we consider two simpler examples.

***Example 7 (Independent counts).*** Let $\mathbf{N}$ have independent components, so $\mathcal{P}(\mathbf{z}) = \prod_{i=1}^M \mathcal{P}_i(z_i)$ and $R = \sum_{i=1}^M R_i$ is a sum of independent compound sum variables with $R_i = 1\{N_i \geq 1\} \sum_{j=1}^{N_i} X_{ij}$ having MGF $\mathcal{P}_i\{\mathcal{M}_i(s)\}$ that converges on $\mathrm{Re}(s) < b_i > 0$. For simplicity, suppose $b_1 < \min_{i\geq 2} b_i$ so $R_1$ is the strongest term of $R$. For the setting in which $\mathcal{P}_1\{\mathcal{M}_1(s)\}$ admits a pole of order $w_1$ at $b_1 > 0$, then Theorem 6 applies by taking $\mathcal{D}(\mathbf{z}) = 1/\mathcal{P}_1(z_1)$ and $\mathcal{N}(\mathbf{z}) = \prod_{i=2}^M \mathcal{P}_i(z_i)$. In this case, $\mathcal{M}_1(b_1) = r_1 > 1$, the radius of convergence for $\mathcal{P}_1$, and

$$f_R(t) \sim f_{R_1}(t) \prod_{i=2}^M \mathcal{P}_i\{\mathcal{M}_i(b_1)\} \sim \beta t^{w_1-1} e^{-b_1 t}, \qquad t \to \infty, \tag{6.8}$$

where

$$\beta = \frac{g_1(r_1)}{\Gamma(w_1)\{\mathcal{M}_1'(b_1)\}^{w_1}} \prod_{i=2}^M \mathcal{P}_i\{\mathcal{M}_i(b_1)\}$$

and $g_1(r_1) = \lim_{z_1\to r_1}(r_1 - z_1)^{w_1}\mathcal{P}_1(z_1)$.

The expansion in (6.8) also holds when $b_1$ is a singularity of order $w_1 > 0$ and not a pole. The argument for this requires expressing $R$ as a finite mixture of $2^M$ terms, where terms are determined by which components in $\{R_i\}$ are positive and which ones are point masses at 0. There are $2^{M-1}$ dominant terms each convolving the dominant component $R_1^+ = R_1|R_1 > 0$. Expansion (6.8) results when Corollary 2 of Section 5.1 is applied to these mixture terms and Theorem 5 is applied to $R_1$. See Section B.4.3 for details.

***Example 8 (Multivariate Negative Binomial*** $(m, \mathbf{p})$ ***weights).*** Consider $M + 1$ claim categories with claims sampled as independent multivariate Bernoulli trials in which category $i$ has probability $p_i$, and the components of $\mathbf{p} = (p_1, \ldots, p_M, p_{M+1})$ sum to 1. If sampling stops on the $m$th occurrence of category $M + 1$, then $\mathbf{N}$, the count totals for the first $M$ categories, has PGF

$$\mathcal{P}(\mathbf{z}) = p_{M+1}^m \left(1 - \sum_{i=1}^M p_i z_i\right)^{-m}.$$

Taking $\mathcal{M}(s) = \{\mathcal{M}_1(s), \ldots, \mathcal{M}_M(s)\}$, then the claim over the first $M$ categories is $R$ with MGF $\mathcal{P}\{\mathcal{M}(s)\}$ and the total claim is $R_T = R + \sum_{j=1}^m X_{M+1,j}$ with MGF $\mathcal{P}\{\mathcal{M}(s)\}\mathcal{M}_{M+1}^m(s)$. For simplicity, we assume that $R$ is stronger than $\{X_{M+1,j}\}$. Subject to the components of $\mathcal{M}(s)$ satisfying $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$, then direct application of Theorem 6 gives

$$f_{R_T}(t) \sim \frac{\gamma^m}{(m-1)!} t^{m-1} e^{-bt} \mathcal{M}_{M+1}^m(b), \qquad t \to \infty, \tag{6.9}$$

where $b$ is the smallest positive root of $1 - \sum_{i=1}^M p_i \mathcal{M}_i(s) = 0$ and $\gamma = p_{M+1}/\{\sum_{i=1}^M p_i \mathcal{M}_i'(b)\}$.

Now consider the setting in which claim amounts $X_1, \dots, X_M$ are non-negative and integer-valued. Then, subject to most of the same conditions as in Theorem 6 along with $\mathfrak{A}$ below, the compound distribution $R$ has a negative binomial tail; see Section B.4.3 for a proof.

**Theorem 7.** *Suppose all the assumptions in Theorem 6 but now let $X_1, \dots, X_M$ be non-negative, integer-valued (so components of $\mathcal{M}(s)$ need not satisfy $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$) and non-degenerate with mass functions that satisfy the aperiodic condition $\mathfrak{A}$ below. Then,*

$$p_R(n) \sim \beta n^{m-1} e^{-bn}, \qquad S_R(t) \sim \frac{1}{1-e^{-b}} p_R(n), \qquad n \to \infty, \tag{6.10}$$

*where $\beta$ is given by the rightmost expression in (6.7).*

($\mathfrak{A}$) *Mass function $p$ is aperiodic in the sense that $1 = \gcd\{n_2 - n_1 : n_1 < n_2$ and $p(n_1) > 0 < p(n_2)\}$, where* gcd *indicates the greatest common denominator.*

All claim amount mass functions are required to be non-degenerate and aperiodic to ensure that the mass function of $R$ has a dominate pole at $b$ on its boundary. Condition $\mathfrak{A}$ also ensures that each component of $\mathcal{M}(s)$ has a unique singularity on its convergence boundary.

# 7. First-passage distributions for semi-Markov processes

Consider a semi-Markov process (SMP) with $M < \infty$ states. Under relatively mild conditions, we show that first-passage distributions from one state to another state have limiting hazard rates and exponential-like or geometric-like tails. Such results are derived by characterizing first-passage distributions as compound sums as described in Proposition 3. Then, for continuous-time processes, exponential-like tails follow from Theorem 6 under Ikehara $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ assumptions, and, for integer-time processes, geometric-like tails follow from Theorem 7 under aperiodic assumptions and the minimal conditions afforded by Darboux's theorem.

Let $\mathcal{S} = \{1, \dots, M\}$ be the states of a continuous- or integer-time SMP and consider a sojourn from state 1 to $M$. The SMP is characterised by its $M \times M$ kernel matrix $\mathbf{K}(t) = \{p_{ij} G_{ij}(t) : i, j \in \mathcal{S}\}$, where $\mathbf{P} = \{p_{ij}\}$ is the transition probability matrix of the associated jump chain for state transitions, and $G_{ij}$ is the holding time distribution in state $i$ given state $j$ is certain to be the next destination. The $M \times M$ Laplace–Stieltjes transform of $\mathbf{K}(t)$ also characterises the SMP and is given by

$$\mathbf{T}(s) = \int_0^\infty e^{st} \, d\mathbf{K}(t) = \{p_{ij} \mathcal{M}_{ij}(s)\} = \mathbf{P} \odot \mathcal{M}(s),$$

where $\mathcal{M}(s) = \{\mathcal{M}_{ij}(s)\}$ is $M \times M$, $\mathcal{M}_{ij}(s)$ is the MGF of $G_{ij}(t)$ and convergent on $\{\mathrm{Re}(s) < c_{ij}\}$ or $\{\mathrm{Re}(s) \le c_{ij}\}$, and $\odot$ denotes a Hadamard product. Matrix $\mathbf{T}(s)$ is called the transmittance matrix since its entries consist of transmittances defined as a probability $\times$ a MGF.

If $X$ is the first-passage time from $1 \to M$, then it has a potentially defective distribution in which $f_{1M} = P(X < \infty) \in (0, 1]$ and $\mathcal{F}_{1M}(s) = E\{e^{sX} | X < \infty\}$ is the MGF given a finite sojourn. The product of these two quantities determines the first-passage transmittance associated with the sojourn or $f_{1M}\mathcal{F}_{1M}(s) = E\{e^{sX} 1_{(X<\infty)}\}$. Butler [9] has shown that this first-passage transmittance takes the following simple form in terms of matrix $\mathbf{T}(s)$:

$$f_{1M}\mathcal{F}_{1M}(s) = \frac{(M,1)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{T}(s)\}}{(M,M)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{T}(s)\}} = \frac{(-1)^{M+1}|\mathbf{\Psi}_{M1}(s)|}{|\mathbf{\Psi}_{MM}(s)|}, \tag{7.1}$$

where $\mathbf{\Psi}_{ij}(s)$ is the $(i, j)$th minor of $\mathbf{I}_M - \mathbf{T}(s)$, or the submatrix of $\mathbf{I}_M - \mathbf{T}(s)$ with the $i$th row and $j$th column removed. In either continuous or integer time, the ratio (7.1) has a maximal convergence region that contains 0 of the form $\{\mathrm{Re}(s) < b\}$ or $\{\mathrm{Re}(s) \le b\}$ for some $b > 0$ under these conditions:

**(i)** The system states $\mathcal{S}$ consist of exactly those states that are relevant to passage from $1 \to M$ and contain no non-relevant states, that is, states that are not accessible while completing a sojourn from $1 \to M$. (Thus, all row sums for $\mathbf{P}$ may not be 1; see below.)

Non-relevant states include absorbing (classes of) states other than $M$ and perhaps some transient states that are not accessible during the sojourn. To determine $\mathcal{S}$ and hence $\mathbf{P}$, start with all states, both relevant and non-relevant, so transition probability rows all sum to 1. Now, delete the rows and columns associated with all non-relevant states; if absorbing (classes of) states have been removed then some row $i \in \mathcal{S} \setminus \{M\}$ of $\mathbf{P}$ will not sum to 1. If $p_{i\cdot} = \sum_{j\in\mathcal{S}} p_{ij} < 1$, then the jump chain $\mathbf{P}$ may pass from $i$ into an absorbing (class of) state(s) other than $M$ w.p. $1 - p_{i\cdot}$. When such occurs, the sojourn time is $\infty$, $f_{1M} < 1$, and the first-passage distribution is defective.

**(ii)** The convergence regions for the MGFs in the first $M - 1$ rows of $\mathbf{T}(s)$, which are those used in the ratio (7.1), include an open neighbourhood of 0, that is, $0 < c_* = \min\{c_{ij} : i \in \mathcal{S} \setminus \{M\}; j \in \mathcal{S}\}$.

We now show that MGF $\mathcal{F}_{1M}$ represents a compound distribution. During a first-passage sojourn from $1 \to M$, let $N_{ij}$ count the number of $i \to j$ transitions of the jump chain and denote $\mathbf{N}_{\setminus M} = \{N_{ij} : i \le M - 1, j \le M\}$ as the $(M - 1) \times M$ matrix of transition counts. Let $\mathbf{Z} = \{z_{ij} : i, j \le M\}$ be $M \times M$ and use the subscripted $\setminus M$ notation $\mathbf{Z}_{\setminus M} = \{z_{ij} : i \le M - 1, j \le M\}$ to denote its $(M - 1) \times M$ sub-block for $\mathbf{Z}$ and all other matrices used below. Let $\mathcal{P}(\mathbf{Z}_{\setminus M}|Y < \infty)$ denote the conditional PGF of $\mathbf{N}_{\setminus M}$ given $Y = \sum_{i=1}^{M-1}\sum_{j=1}^{M} N_{ij} < \infty$, where $Y$ counts the total number of steps required for first passage. Then, the first-passage time MGF is $\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s)_{\setminus M}|Y < \infty\}$ as stated below and shown in Section B.5.1.

**Proposition 3.** *Assuming* (i), *the conditional probability generating function for* $\mathbf{N}_{\setminus M}$ *given* $Y < \infty$ *is*

$$\mathcal{P}(\mathbf{Z}_{\setminus M}|Y < \infty) = \frac{1}{f_{1M}} \frac{(M,1)\text{-}cofactor\ of \{\mathbf{I}_M - \mathbf{P} \odot \mathbf{Z}\}}{(M,M)\text{-}cofactor\ of \{\mathbf{I}_M - \mathbf{P} \odot \mathbf{Z}\}}. \tag{7.2}$$

*The convergence in* (7.2) *is on*

$$\left\{\mathbf{Z}_{\setminus M} : \left|\lambda_1(\tilde{\boldsymbol{P}} \odot \mathbf{Z})\right| < 1\right\} \supset \left\{\mathbf{Z}_{\setminus M} : |z_{ij}| \le 1 \text{ for } i \le M-1,\ j \le M\right\},$$

*where* $\lambda_1\{\cdot\}$ *denotes the eigenvalue of largest modulus for the matrix argument, and* $\tilde{\mathbf{P}}$ *is* $\mathbf{P}$ *with its* $M$*th row replaced by zeros. First-passage sojourn time* $X$, *when* $X < \infty$, *is a compound distribution of the form*

$$X = \sum_{i=1}^{M-1} \sum_{j=1}^{M} 1\{N_{ij} \ge 1\} \sum_{k=1}^{N_{ij}} X_{ijk}, \tag{7.3}$$

*where* $\{X_{ijk} : k \ge 1\}$ *are i.i.d.* $G_{ij}(t)$. *Based upon this, the conditional MGF of* $X|X < \infty$ *is* $\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s)_{\setminus M}|Y < \infty\}$ *as given in* (7.1).

In this first-passage characterisation, exit from state $M$ is not possible. This is reflected in (7.3) which depends only on $\mathbf{N}_{\setminus M}$ as well as the fact that $\mathcal{M}(s)_{\setminus M}$ is the argument used in $\mathcal{P}(\mathbf{Z}_{\setminus M}|Y < \infty)$ for the compound-distribution characterisation. Consistent with this, both cofactors in (7.1) do not depend on the $M$th row of $\mathbf{T}(s) = \mathbf{P} \odot \mathbf{M}(s)$. If some components of $\mathbf{P}_{\setminus M}$ are 0, so certain branches cannot be traversed, then Proposition 3 still holds but with the corresponding components in $\mathbf{Z}_{\setminus M}$ removed and with $\mathcal{P}(\mathbf{Z}_{\setminus M}|Y < \infty)$ as a PGF in a lower dimensional subspace of $R^{(M-1)\times M}$.

Proposition 3 makes two important points. First, it provides another formal derivation of the identity in (7.1) thus confirming the initial proof in [9]. Second, and most crucially for our purposes, it characterises the first-passage distribution as a compound distribution to which we can apply Theorems 6 and 7. We now make some additional assumptions that are needed for using Theorem 6 in the continuous-time SMP setting. These additional assumptions can be verified as holding in all the various practical examples that have been considered in Butler [9,10], Chapter 13, and in the additional references therein.

**(iii)** Assume convergence bound $b \in (0, c_*)$ for $\mathcal{F}_{1M}(s)$ is a simple pole that results as the smallest positive zero of $|\boldsymbol{\Psi}_{MM}(s)|$ with $|\boldsymbol{\Psi}_{M1}(b)| \ne 0$.

**(iv)** Suppose the first $M-1$ rows of $\mathbf{K}(t)$ consist of absolutely continuous component distributions. Define $\mathcal{B} \subset \mathcal{S} \times \mathcal{S}$ as a *blockade* of state transitions for the sojourn $1 \to M$ if all paths from $1 \to M$ must incur at least one state transition in $\mathcal{B}$. Assume there exists a blockade $\mathcal{B}$ such that each blockade member $(i, j) \in \mathcal{B}$ has a density $g_{ij}(t)$ which satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$.

**Theorem 8.** *Suppose a continuous-time semi-Markov process satisfies conditions* (i)–(iv) *above. Then the first-passage time distribution of* $X|X < \infty$ *from* $1 \to M$ *has asymptotic hazard rate* $b > 0$, *as given in* (iii), *with density and survival functions*

$$f(t) \sim \beta e^{-bt} \quad \text{and} \quad S(t) \sim \frac{1}{b} f(t), \qquad t \to \infty,$$

*where*

$$\beta = -\operatorname{Res}\left\{\mathcal{F}_{1M}(s); b\right\} = -\frac{|\boldsymbol{\Psi}_{M1}(b)|}{|\boldsymbol{\Psi}_{M1}(0)|} \frac{|\boldsymbol{\Psi}_{MM}(0)|}{\operatorname{tr}[\operatorname{adj}\{\boldsymbol{\Psi}_{MM}(b)\}\dot{\boldsymbol{\Psi}}_{MM}(b)]}, \tag{7.4}$$

adj$\{\cdot\}$ *denotes the* $(M-1)\times(M-1)$ *adjoint of the matrix argument, and* $\dot{\mathbf{\Psi}}_{MM}(b) = d\mathbf{\Psi}_{MM}(s)/ds|_{s=b}$ *is* $(M-1)\times(M-1)$.

**Proof.** Under the conditions of Theorem 8, the compound distribution for $X|X<\infty$, as characterised in Proposition 3, satisfies the conditions of Theorem 6; see Section B.5.2 for details. □

It is noteworthy that Theorem 8 lacks the full Ikehara assumption $\mathfrak{I}_{\mathcal{M}}$. Assumption (iii) stipulates that $b$ is a simple pole for $\mathcal{F}_{1M}$ but it does not require that $b$ be a dominant pole as also stipulated in $\mathfrak{I}_{\mathcal{M}}$. This latter fact emerges as a consequence of the method of proof in which Theorem 6 is applied to the compound distribution of $\mathcal{F}_{1M}$ as described in Proposition 3.

***Example 9 (Cramér–Lundberg and Sparre Andersen).*** In both of these models, the conditional distribution for positive ruin $R^+ = R|R>0$ can be considered as an example of a first-passage distribution for a certain SMP. For the Cramér–Lundberg model, the two state SMP with transmittance matrix

$$\mathbf{T}(s) = \begin{pmatrix} \rho\mathcal{M}_E(s) & (1-\rho)\mathcal{M}_E(s) \\ 0 & 0 \end{pmatrix} \tag{7.5}$$

has first-passage transmittance from $1\to 2$ computed from (7.1) as $f_{12}\mathcal{F}_{12}(s) = (1-\rho)\mathcal{M}_E(s)/\{1-\rho\mathcal{M}_E(s)\}$, which is $\mathcal{M}_{R^+}(s)$ computed from the rightmost summation component in (6.1). For the Sparre Andersen model, if the first row entries in (7.5) are $e^{-B}\mathcal{M}_{L^+}(s)$ and $(1-e^{-B})\mathcal{M}_{L^+}(s)$, then $f_{12}\mathcal{F}_{12}(s)$ yields $\mathcal{M}_{R^+}$ as given in the middle expression for $\mathcal{M}_{R^+}$ in (9.39) of Section B.4.2. The conditions and proofs used in Theorems 3 and 4 are needed to ensure that the associated SMPs satisfy conditions (i)–(iv) of Theorem 8. In particular, the densities for excess life $E$ and the ascending ladder variable $L^+$ must satisfy $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ so condition (iv) is satisfied. Thus, the conclusions of Theorems 3 and 4 follow as special cases of Theorem 8 applied to simple SMPs as in (7.5).

***Example 10 (GI/M/1 and M/G/1 queues).*** The first passage time from an empty queue (state 0) to queue length $M$ for either of these queues is a passage time for a SMP; see Butler [9], Section 6 and [10], Section 13.2.5, and Butler and Huzurbazar [14], Section 7, respectively. In either setting, it can be shown that all entries of the $(M+1)\times(M+1)$ kernel matrix $\mathbf{K}(t)$ satisfy $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ (so condition (iv) is satisfied) when the interarrival distribution of the renewal process satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$. Three such interarrival distributions are the particular Gamma, compound Poisson, and inverse Gaussian distributions used as numerical examples in the references above. Indeed, these examples lead to transmittance matrices with entries that also satisfy the remaining conditions (i)–(iii) so that Theorem 8 applies. Thus, the true hazard functions approach an asymptote with value $b$ as suggested in the plots of saddlepoint approximations for such hazard functions in Butler [9], Section 6 and [10], Section 13.2.5. Theorem 8 also proves that the survival functions for these examples have exponential orders $\beta e^{-bt}/b$. This is also suggested in the additional plots of saddlepoint approximations for these survival functions.

The analogous integer-time results follow directly from Theorem 7 and are based upon the minimal conditions of Proposition 2. Having proven Proposition 2 by using Darboux's theorem rather than Theorem 5 in Feller [20], XIII.5, we avoid the need to assume that the sojourn mass function is ultimately non-decreasing as required in Feller's theorem. Making such an assumption would be rather pointless as it requires additional and mostly unverifiable knowledge about the very function for which we are providing an asymptotic expansion.

**Theorem 9.** *Suppose an integer-time semi-Markov process satisfies conditions* (i)–(iii) *and assume the* $1$*-step mass functions for transitions from* $\mathcal{S}\setminus\{M\}\to\mathcal{S}$ *are non-degenerate and aperiodic as in condition* $\mathfrak{A}$. *Then the first-passage time distribution of* $X|X<\infty$ *from* $1\to M$ *has asymptotic hazard rate* $1-e^{-b}$, *with* $b$ *given in* (iii), *and mass and survival functions*

$$p(n)\sim\beta e^{-bn}\quad\text{and}\quad S(n)\sim\frac{1}{1-e^{-b}}p(n),\qquad n\to\infty,\tag{7.6}$$

*where* $\beta$ *is given in* (7.4).

Theorem 9 lacks the full Darboux $\mathfrak{D}_{\mathcal{M}}$ assumption since condition (iii) only assumes that $b$ is a simple pole for $\mathcal{F}_{1M}$. The requirement that it also be a dominant pole emerges as a consequence of Theorem 7 with the additional assumptions on 1-step mass functions for transitions from $\mathcal{S}\setminus\{M\}\to\mathcal{S}$.

In the special case of integer-time Markov chains with $\mathcal{M}_{ij}(s)=e^s$ the obvious limitation to applying Theorem 9 is degenerate 1-step mass functions. As it concerns first passage $1\to M$, however, this is not a limitation if $p_{ii}>0$ for $i=1,\dots,M-1$ but is otherwise. If $p_{ii}>0$ for all $i$, the first-passage can be considered using an equivalent SMP $\mathbf{P}^*\odot\mathcal{M}^*(s)$ in which 1-step returns to states have been removed, so $p_{ii}^*=0$ and $p_{ij}^*=p_{ij}/(1-p_{ii})$ for $j\neq i$, and holding times in state $i$ are Geometric $(p_{ii})$ with MGF $\mathcal{M}_{ij}^*(s)=(1-p_{ii})e^s/(1-p_{ii}e^s)$. Note when $p_{ii}=0$ for all $i$ and the chain is periodic, then $\mathcal{F}_{1M}$ cannot have a dominant pole $b$ on its boundary of convergence. For a periodic chain, additional terms resulting from the other dominant poles are needed for $p(n)$ and $S(n)$ to capture the $O(e^{-bn})$ order of these expansions.

In the continuous-time Markov setting, the results of Theorem 8 are, of course, well-known as they simply state the known asymptotic behaviour of the phase distributions which represent the sojourn times.

For SMPs in which $\mathbf{T}(s)$ is composed of rational MGFs, the results follow directly from partial fraction expansion of $\mathcal{F}_{1M}(s)$ based upon (7.1). The importance of these two theorems, however, is not that they apply to such rational settings, but rather that they apply to non-rational settings in which the entries of $\mathbf{T}(s)$ may be non-rational MGFs as occurs in the broader class of SMPs. In such non-rational settings, the theorems show that exponential-like/geometric-like tails result quite generally for first-passage distributions of SMPs as they do for the more restrictive class of Markov processes. Such conclusions reinforce the *insensitivity* properties of SMPs discussed by Tijms [27], Section 5.4, in which, for large $t$ or $n$, SMPs behave much like Markov processes and exhibit insensitivity to the actual non-exponential/non-geometric holding time distributions used in the kernel $\mathbf{K}(t)$.

Exponential-like/geometric-like tails can also be shown to hold for other types of sojourns in finite state SMPs, such as first return to a single state or first-passage from one state to a subset of states. These results make use of other first-passage transmittances as given in Theorems 2 and 3 of Butler [9] or [10], Sections 13.2.6 and 13.3. Details of this will be presented elsewhere.

Assumption (iii), forming part of the Ikehara $\mathfrak{I}_{\mathcal{M}}$ condition, is most likely unnecessary for the conclusions of Theorems 8 and 9 to hold and can be replaced by weaker assumptions concerning the composition of states in $\mathcal{S}$. For example, if $\mathcal{S} \setminus \{M\}$ consists of states that all communicate, then in Markov processes $b \in (0, c_*)$ is a simple pole, which is a result that follows from the associated Perron–Frobenius theory. Similar results should apply to SMPs and will be addressed in future work. Furthermore, assumption (iv), used for continuous-time processes, can likely be replaced with alternative integrability assumptions on the components of $\mathbf{T}(s)$. However, even with the potential for relaxing some of these assumptions, Theorems 8 and 9 are the first formal results of their kind for general SMPs and apply to a very broad class of SMPs used in many classical applications of stochastic modelling and multi-state survival analysis.

## 8. Logarithmic singularities

Asymptotic expressions are given for distributions whose boundary singularity $b$ is logarithmic. Proofs are given in Section A.6.

**Proposition 4.** *Let $X$ have an absolutely continuous distribution on $(0, \infty)$ and moment generating function $\mathcal{M}(s)$ which converges on $\{s \in C : \mathrm{Re}(s) < b\}$ for $b > 0$. If $X$ satisfies condition $\mathfrak{I}_{\mathrm{UND}}$ of Proposition 1 and condition $\mathfrak{L}_{\mathcal{M}}$ below, then $X$ has limiting hazard $b$ with*

$$f(t) \sim g_m(b) m (\ln t)^{m-1} t^{-1} e^{-bt} \quad \textit{and} \quad S(t) \sim f(t)/b, \qquad t \to \infty. \tag{8.1}$$

($\mathfrak{L}_{\mathcal{M}}$) *$b$ is a logarithmic singularity for $\mathcal{M}(s)$ of the form*

$$\mathcal{M}(s) = \sum_{j=1}^{m} g_j(s) \{-\ln(b-s)\}^j + h(s), \tag{8.2}$$

*where $\{g_j(s)\}$ and $h(s)$ are analytic on $\{s \in C : \mathrm{Re}(s) \le b\}$ and $g_m(b) \neq 0$.*

***Example 11.*** The exponential integral function $E_1(z) = \int_z^\infty t^{-1} e^{-t}\, dt$ defines the density

$$f(t) = E_1(1)^{-1} t^{-1} e^{-t}, \qquad t > 1 \tag{8.3}$$

with MGF $\mathcal{M}(s) = E_1(1-s)/E_1(1)$ which converges on $\{s \in C : \mathrm{Re}(s) < 1\}$. Simple computations show that $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ holds for any tilting parameter exceeding 1. Condition $\mathfrak{L}_{\mathcal{M}}$ holds due to the relationship of $E_1(1-s)$ to $-\ln(1-s)$, given in Abramowitz and Stegun [2], equation 5.1.11, page 229, in which $\mathcal{M}(s) = -E_1(1)^{-1}\ln(1-s) + h(s)$, for $h$ analytic on $\{s \in C\}$. From Proposition 4, the asymptotic hazard is 1 and the order of $f(t)$ in (8.1) agrees with the exact density in (8.3) with $m = 1$ and $g_1(1) = E_1(1)^{-1}$.

If $X_m$ is a convolution of $m$ such i.i.d. variables, then MGF $\mathcal{M}(s)^m$ can be expanded using the binomial theorem to show it has form (8.2) with $g_m(s) = E_1(1)^{-m}$. The density for $X_m$ satisfies $\mathfrak{I}_{\mathrm{ND}(0,\infty)}$ as shown by repeatedly using the convolution argument of Lemma B1 in Section B.3.1

starting with $X_1$. Thus, the asymptotic hazard is 1 and

$$f_{X_m}(t) \sim E_1(1)^{-m} m (\ln t)^{m-1} t^{-1} e^{-t}, \qquad t \to \infty. \tag{8.4}$$

For the case $m = 2$, the expansion in (8.4) is confirmed by direct computation in Section A.6.

Similar expansions hold for lattice distributions with a log-singularity as in (8.6). A proof is given in Section A.6.

**Proposition 5.** *Suppose $X$ has non-negative integer support and moment generating function $\mathcal{M}(s)$ that converges on $\{s \in C : \mathrm{Re}(s) < b\}$, for $b > 0$. If $X$ satisfies condition $\mathfrak{DL}_{\mathcal{M}}$ below, then $X$ has limiting hazard $1 - e^{-b}$, with*

$$p(n) \sim g_m\big(e^b\big) m (\ln n)^{m-1} n^{-1} e^{-bn}, \qquad n \to \infty, \tag{8.5}$$

*and $S(n) \sim (1 - e^{-b})^{-1} p(n)$.*

($\mathfrak{DL}_{\mathcal{M}}$) $b$ is a singularity for $\mathcal{M}(s)$ which has the form

$$\mathcal{M}(s) = \sum_{j=1}^{m} g_j\big(e^s\big)\big\{-\ln\big(e^b - e^s\big)\big\}^j + h\big(e^s\big), \tag{8.6}$$

*where $\{g_j(e^s)\}$ and $h(e^s)$ are analytic on $\{s \in C : \mathrm{Re}(s) \le b\}$, and $g_m(e^b) \neq 0$.*

***Example 12.*** The Logarithmic Series $(p)$ distribution, with $p = 1 - q \in (0, 1)$, has mass function

$$p(n) = n^{-1} p^n / (-\ln q), \qquad n \ge 1 \tag{8.7}$$

and MGF $\mathcal{M}(s) = \ln(1 - pe^s)/\ln q$, which converges for $\mathrm{Re}(s) < b = -\ln p$. Proposition 5 determines the asymptotic hazard as $q$, which can be verified directly by using the Stolz–Cesàro theorem (Lemma A2, Section A.5.1). The asymptotic order for $p(n)$ in (8.5) is exact.

The sum of $m$ independent Logarithmic Series $(p)$ variables has a MGF with the form (8.6) and Proposition 5 yields

$$p_m(n) \sim \frac{m}{(-\ln q)^m} (\ln n)^{m-1} n^{-1} p^n, \qquad n \to \infty. \tag{8.8}$$

This mass function is highly intractable except when $m = 2$ as considered in Section A.6.

# Acknowledgments

The author is grateful to Michael Woodroofe and Andrew Wood for helpful comments on this research at various stages of its development. Valuable comments by a referee and an associate editor have greatly improved the paper. Research was supported in part by the National Science Foundation under grant DMS-11-04474.

## Supplementary Material

**Supplement to "Asymptotic expansions and hazard rates for compound and first-passage distributions"** (DOI: 10.3150/16-BEJ854SUPP; .pdf). The Appendices can be found in the supplementary material referenced as [11] below. This material is comprised of two Appendices.

APPENDIX A: Contains proofs for asymptotic hazards, proofs using Feller conditions, examples with branch-point singularities, proofs for integer-valued distributions, and proofs with logarithmic singularities.

APPENDIX B: Contains proofs using Ikehara conditions and all derivations for compound distributions and first-passage distributions in SMPs.

## References


[1] Aalen, O.O. and Gjessing, H.K. (2001). Understanding the shape of the hazard rate: A process point of view. *Statist. Sci.* **16** 1–22. MR1838599

[2] Abramowitz, M. and Stegun, I.A. (1972). *Handbook of Mathematical Functions*, 9th ed. New York: Dover.

[3] Anderson, T.W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR1990662

[4] Asmussen, S. (2000). *Ruin Probabilities*. *Advanced Series on Statistical Science & Applied Probability* **2**. River Edge, NJ: World Scientific Co. MR1794582

[5] Barndorff-Nielsen, O.E. and Cox, D.R. (1989). *Asymptotic Techniques for Use in Statistics*. *Monographs on Statistics and Applied Probability*. London: Chapman & Hall. MR1010226

[6] Block, H. and Joe, H. (1997). Tail behavior of the failure rate functions of mixtures. *Lifetime Data Anal.* **3** 269–288.

[7] Block, H.W., Mi, J. and Savits, T.H. (1993). Burn-in and mixed populations. *J. Appl. Probab.* **30** 692–702. MR1232744

[8] Block, H.W. and Savits, T.H. (2001). Comment on Aalen and Gjessing.

[9] Butler, R.W. (2000). Reliabilities for feedback systems and their saddlepoint approximation. *Statist. Sci.* **15** 279–298. MR1821330

[10] Butler, R.W. (2007). *Saddlepoint Approximations with Applications*. Cambridge: Cambridge Univ. Press. MR2357347

[11] Butler, R.W. (2016). Supplement to "Asymptotic expansions and hazard rates for compound and first-passage distributions." DOI:10.3150/16-BEJ854SUPP.

[12] Butler, R.W. and Bronson, D.A. (2002). Bootstrapping survival times in stochastic systems by using saddlepoint approximations. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **64** 31–49. MR1881843

[13] Butler, R.W. and Bronson, D.A. (2012). Bootstrap confidence bands for sojourn distributions in multistate semi-Markov models with right censoring. *Biometrika* **99** 959–972. MR2999171

[14] Butler, R.W. and Huzurbazar, A.V. (2000). Bayesian prediction of waiting times in stochastic models. *Canad. J. Statist.* **28** 311–325. MR1792052

[15] Chandrasekharan, K. (1968). *Introduction to Analytic Number Theory*. *Die Grundlehren der Mathematischen Wissenschaften*, *Band* 148. New York: Springer. MR0249348

[16] Chung, S. (2010). Saddlepoint Approximation to Functional Equations in Queueing Theory and Insurance Mathematics, Ph.D. Dissertation, Colorado State University, Fort Collins, CO.

[17] Doetsch, G. (1972). *Handbuch der Laplace-Transformation*. Basel-Stuttgart: Birkhäuser. MR0344808

[18] Embrechts, P., Maejima, M. and Teugels, J. (1985). Asymptotic behaviour of compound distributions. *Astin Bull*. **15** 45–48.

[19] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math*. *Econom*. **1** 55–72. MR0652832

[20] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* **II**, 2nd ed. New York: Wiley. MR0270403

[21] Huang, X.C. (1988). A discrete L'Hôpital's rule. *College Math*. *J*. **19** 321–329. MR0962484

[22] Jensen, J.L. (1995). *Saddlepoint Approximations*. New York: Oxford Univ. Press. MR1354837

[23] Korevaar, J. (2004). *Tauberian Theory*: *A Century of Developments*. Berlin: Springer. MR2073637

[24] Narkiewicz, W. (1983). *Number Theory*. Singapore: World Scientific Co. MR0748136

[25] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*. Chichester: Wiley. MR1680267

[26] Sundt, B. (1982). Asymptotic behaviour of compound distributions and stop-loss premiums. *Astin Bull*. **13** 89–98. MR0712915

[27] Tijms, H.C. (2003). *A First Course in Stochastic Models*. Chichester: Wiley. MR2190630

[28] Widder, D.V. (1941). *The Laplace Transform*. Princeton, NJ: Princeton Univ. Press. MR0005923

[29] Wilf, H.S. (2006). *Generatingfunctionology*, 3rd ed. Wellesley, MA: A. K. Peters. MR2172781

[30] Willmot, G.E. (1989). Limiting tail behaviour of some discrete compound distributions. *Insurance Math*. *Econom*. **8** 175–185. MR1031371

*Received February 2015 and revised December 2015*