Bernoulli **23**(4B), 2017, 3268–3310
DOI: 10.3150/16-BEJ846

Approximate local limit theorems with effective rate and application to random walks in random scenery

RITA GIULIANO[1] and MICHEL WEBER[2]

[1] *Dipartimento di Matematica, Via F. Buonarroti 2, 56127 Pisa, Italy. E-mail: rita.giuliano@unipi.it*
[2] *IRMA, UMR 7501, Université Louis-Pasteur, 7 rue René Descartes, 67084 Strasbourg Cedex, France. E-mail: michel.weber@math.unistra.fr*

We show that the Bernoulli part extraction method can be used to obtain approximate forms of the local limit theorem for sums of independent lattice valued random variables, with effective error term. That is with explicit parameters and universal constants. We also show that our estimates allow us to recover Gnedenko and provide a version with effective bounds of Gamkrelidze's local limit theorem. We further establish by this method a local limit theorem with effective remainder for random walks in random scenery.

Keywords: Bernoulli part; effective remainder; independent random variables; lattice distributed; local limit theorem; random walk in random scenery

1. Introduction and main result

The extraction method of the Bernoulli part of a lattice valued random variable was developed by McDonald in [10,23,26] for proving local limit theorems, under the assumption that the central limit theorem holds. Twenty years before McDonald's work, Kolmogorov [22] used a similar approach in the study of Lévy's concentration function, and is the first to have explored in this direction. For more details, see the recent paper by Aizenmann, Germinet, Klein and Warzel [1], where this idea is also developed for general random variables and applications are given.

We also mention related approaches developed by Röllin and Ross [32] based on Landau–Kolmogorov inequalities, as well as Johnson's recent survey [20] on entropy and thinning of random variables.

This method allows one to transfer results which are available for systems of Bernoulli random variables to systems of arbitrary random variables. It is based on a probabilistic device, and is proved to be an efficient alternative to the characteristic functions method. Kolmogorov remarked to this effect in his 1958s paper [22], p. 29: "…*Il semble cependant que nous restons toujours dans une période où la compétition de ces deux directions* [characteristic functions or direct methods from the calculus of probability] *conduit aux résultats les plus féconds*…." We believe that Kolmogorov's comment is still relevant today.

The main purpose of this article is to show that this approach can be used to obtain, in a rather simple way, approximate forms of the local limit theorem with effective error term for sums of independent lattice valued random variables. The method indeed shows that this can be done with an error term using explicit parameters and universal constants. The approximate form we obtain expresses quite simply, and is therefore easy to use. Further, it is precise enough to contain Gnedenko's theorem (Remark 1.10) and provide a version with effective bounds (Section 7) of Gamkrelidze's strong form of the local limit theorem ([15], see also [10]). Before stating the main results and for the purposes of comparing results, it is necessary to recall and discuss some classical facts and briefly describe the background of this problem. Let $\tilde{X} = \{X_n, n \geq 1\}$ be a sequence of independent, square integrable random variables taking values in a common lattice $\mathcal{L}(v_0, D)$ defined by the sequence $v_k = v_0 + Dk$, $k \in \mathbb{Z}$, where v_0 and $D > 0$ are real numbers. Let

$$S_n = \sum_{j=1}^{n} X_j, \qquad M_n = \sum_{j=1}^{n} \mathbb{E}X_j, \qquad \Sigma_n = \sum_{j=1}^{n} \mathrm{Var}(X_j). \tag{1.1}$$

Then S_n takes values in the lattice $\mathcal{L}(v_0 n, D)$. The sequence $\tilde{X}$ satisfies a local limit theorem if

$$\Delta_n := \sup_{N = v_0 n + Dk} \left| \sqrt{\Sigma_n}\mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-M_n)^2}{2\Sigma_n}} \right| = o(1). \tag{1.2}$$

This is a fine limit theorem in Probability Theory, which also has deep connections with Number Theory, see, for instance, Freiman [13] and Postnikov [31]. These two aspects of a same problem were much studied in the past decades by the Russian School of probability. It seems however that some of these results are now forgotten.

Assume that $\tilde{X}$ is an i.i.d. sequence and let $\mu = \mathbb{E}X_1$, $\sigma^2 = \mathrm{Var}(X_1)$. If for instance X_1 takes only even values, it is clear that (1.2) cannot be fulfilled with $D = 1$. In fact, (1.2) holds (with $M_n = n\mu$, $\Sigma_n = n\sigma^2$) if and only if the span D is maximal, that is, there are no other real numbers v_0' and $D' > D$ for which $\mathbb{P}\{X \in \mathcal{L}(v_0', D')\} = 1$. This is Gnedenko's well-known generalization of the De Moivre–Laplace theorem. See [16]. Notice that (1.2) is significant only for the bounded domains of values

$$|N - n\mu| \leq \sigma\sqrt{2n \log \frac{D}{\varepsilon_n}}, \tag{1.3}$$

where $\varepsilon_n \downarrow 0$ depends on the Landau symbol o. It is worth observing that (1.2) cannot be deduced from a central limit theorem with rate, even under stronger moment assumption. Suppose for instance $D = 1$, X is centered and $\mathbb{E}|X|^3 < \infty$. Using the Berry–Esseen estimate only implies that

$$\left| \sigma\sqrt{n}\mathbb{P}\{S_n = k\} - \sigma\sqrt{n} \int_{\frac{k}{\sigma\sqrt{n}}}^{\frac{k+1}{\sigma\sqrt{n}}} e^{-t^2/2} \frac{\mathrm{d}t}{\sqrt{2\pi}} \right| \leq C \frac{\mathbb{E}|X|^3}{\sigma^2}.$$

The comparison term has already the right order for all integers k such that $k + 1 \leq \sigma\sqrt{n}$ since,

$$\sup_{k+1 \leq \sigma\sqrt{n}} \left| \sigma\sqrt{n} \int_{\frac{k}{\sigma\sqrt{n}}}^{\frac{k+1}{\sigma\sqrt{n}}} e^{-t^2/2} \frac{\mathrm{d}t}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma^2 n}} \right| \leq \frac{C}{\sigma\sqrt{n}} \to 0.$$

By substituting, we only get

$$\left|\sigma\sqrt{n}\mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2\sigma^2 n}}\right| \le C\left(\frac{\mathbb{E}|X|^3}{\sigma^2} + \frac{1}{\sigma\sqrt{n}}\right).$$

Letting $k = k_n \to \infty$, $k_n + 1 \le \sigma\sqrt{n}$, the above right-hand side is only bounded with n, whereas by (1.2), this one tends to zero with n. Hence, (1.2) cannot follow from the Berry–Esseen estimate. Note however that if Cramer's condition is fulfilled, namely $\limsup_{|u|\to\infty} |\mathbb{E}e^{iuX_1}| < 1$, and higher moments exist, better rates of approximation in the Berry–Esseen theorem are available, see [5], p. 329 and [30], p. 130.

Gnedenko's theorem is optimal: Matskyavichyus [24] showed that for any nonnegative function $\varphi(n) \to 0$ as $n \to \infty$, there is an i.i.d. sequence $\tilde{X}$, with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 < \infty$ and the form of the characteristic function of X_1 is made explicit, such that for each $n \ge n_0$, $\sqrt{n}\Delta_n \ge \varphi(n)$. Stronger integrability properties yield better remainder terms.

Theorem 1.1. *Let F denote the distribution function of X_1.*

(i) ([19], Theorem 4.5.3) *In order that the property*

$$\sup_{N=an+Dk}\left|\sqrt{n}\mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}\sigma}e^{-\frac{(N-n\mu)^2}{2n\sigma^2}}\right| = \mathcal{O}(n^{-\alpha}), \qquad 0 < \alpha < 1/2, \tag{1.4}$$

holds, it is necessary and sufficient that the following conditions be satisfied:

$$\text{(1)} \quad D \text{ is maximal}, \qquad \text{(2)} \quad \int_{|x|\ge u} x^2 F(\mathrm{d}x) = \mathcal{O}(u^{-2\alpha}) \qquad \text{as } u \to \infty.$$

(ii) ([30], Theorem 6, p. 197) *If* $\mathbb{E}|X_1|^3 < \infty$, *then* (1.4) *holds with* $\alpha = 1/2$.

The local limit theorem in the independent case is often studied by using various structural characteristics, which are interrelated. There exists a subsequent literature. This unfortunately does not include a survey, and we can only report a little of the background here. Let X denote a random variable. The "smoothness" characteristic

$$\delta_X = \sum_{k\in\mathbb{Z}}\left|\mathbb{P}\{X = v_k\} - \mathbb{P}\{X = v_{k+1}\}\right|, \tag{1.5}$$

thoroughly investigated by Gamkrelidze is connected to the characteristic function $\varphi_X(t) = \mathbb{E}e^{itX}$ through the relation

$$\left(1 - e^{it}\right)\varphi_X(t) = \sum_{m\in\mathbb{Z}}\frac{(itm)}{m!}\left(\mathbb{P}\{X = m\} - \mathbb{P}\{X = m-1\}\right). \tag{1.6}$$

Hence,

$$\left|\varphi_X(t)\right| \le \frac{\delta_X}{2|\sin(t/2)|} \qquad (t \notin 2\pi\mathbb{Z}). \tag{1.7}$$

This is used in Gamkrelidze [15], to prove the following well-known result. Suppose a sequence $\tilde{X}$ of independent integer valued random variables satisfies:

(i) there exists an n_0 such that $\sup_k \delta_{X_k^1+\cdots+X_k^{n_0}} < \sqrt{2}$, where X_k^j, $1 \le j \le n_0$ are independent copies of X_k,
(ii) the central limit theorem is applicable,
(iii) $\mathrm{Var}(S_n) = \mathcal{O}(n)$.

Then the local limit theorem is applicable in the strong form. By this we mean that it remains true when changing or discarding a finite number of terms of $\tilde{X}$.

Later Davis and McDonald [10] proved a variant of Gamkrelidze's result using the Bernoulli part extraction method. Let X be a random variable such that $\mathbb{P}\{X \in \mathcal{L}(v_0, D)\} = 1$, and let

$$\vartheta_X = \sum_{k \in \mathbb{Z}} \mathbb{P}\{X = v_k\} \wedge \mathbb{P}\{X = v_{k+1}\}, \tag{1.8}$$

where $a \wedge b = \min(a, b)$. Note from Section 2.2 that necessarily $\vartheta_X < 1$. Moreover $\delta_X = 2 - 2\vartheta_X$. See Mukhin [29], p. 700. This simple characteristic is used in [10] and it is required that $\vartheta_X > 0$. More precisely, is the following theorem.

Theorem 1.2 ([10], Theorem 1.1). *Let $\{X_j, j \ge 1\}$ be independent, integer valued random variables with partial sums $S_n = X_1 + \cdots + X_n$ and let $f_j(k) = \mathbb{P}\{X_j = k\}$. Also for each j and n, let*

$$q(f_j) = \sum_k \bigl[f_j(k) \wedge f_j(k+1)\bigr], \qquad Q_n = \sum_{j=1}^n q(f_j).$$

Suppose that there are numbers $b_n > 0$, a_n such that $\lim_{n\to\infty} b_n = \infty$, $\limsup_{n\to\infty} b_n^2/Q_n < \infty$, and

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Then

$$\lim_{n\to\infty} \sup_k \left| b_n \mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(k-a_n)^2}{2b_n^2}} \right| = 0.$$

Remark 1.3.

– It may happen that $q(f_j) \equiv 0$ and so $Q_n \equiv 0$. In the above original statement, it is thus implicitly assumed that $Q_n > 0$, $Q_n \uparrow \infty$ and $q(f_j) > 0$, which is equivalent to $f_j(k) \wedge f_j(k+1) > 0$ for some $k \ge 0$.
– It was recently shown in Weber [33] that this method can also be used efficiently to prove the almost sure local limit theorem in the critical case, namely for sums of i.i.d. random variables with the minimal integrability assumption: square integrability.

As mentioned before, we are mainly interested in local limit theorems with explicit constants in the remainder term. There are generally speaking, much less related papers. Most of the local limit theorems with rate are usually stated with Landau symbols o, $\mathcal{O}$. The implicit constants may depend on the sequence itself. Consider the characteristic

$$H(X,d)=\mathbb{E}\langle X^*d\rangle^2,$$

where $\langle\alpha\rangle$ is the distance from α to the nearest integer and X^* denotes a symmetrization of X. In Mukhin [29] and [28], the two-sided inequality

$$1-2\pi^2H\left(X,\frac{t}{2\pi}\right)\le|\varphi_X(t)|\le 1-4H\left(X,\frac{t}{2\pi}\right), \tag{1.9}$$

is established. The following is the one-dimensional version of Theorem 5 in [29], which is stated without proof.

Theorem 1.4. *Let $X_1,\dots,X_n$ have zero mean and finite third moments. Let*

$$H_n=\inf_{1/4\le d\le 1/2}\sum_{j=1}^n H(X_j,d),\qquad L_n=\frac{\sum_{j=1}^n\mathbb{E}|X_j|^3}{(\sum_{j=1}^n\mathbb{E}|X_j|^2)^{3/2}}.$$

Then $\Delta_n\le CL_n(\Sigma_n/H_n)$.

The author also announced a manuscript devoted to the question of the estimates of the rate of convergence. We have however been unable to find any resulting publication. For the i.i.d. case with third moment condition, we also record Lemma 3 in Doney [12].

Before stating our main result, we say a few words concerning the method we will use, which is quite elementary.

Recall that $S_n=X_1+\cdots+X_n$, where X_j are independent random variables such that $\mathbb{P}\{X_j\in\mathcal{L}(v_0,D)\}=1$. For the moment, we do not assume any moment condition. We only suppose that

$$\vartheta_{X_j}>0,\qquad j=1,\dots,n. \tag{1.10}$$

Anticipating Lemma 2.8, we can write $S_n\overset{\mathcal{D}}{=}W_n+DM_n$ where

$$W_n=\sum_{j=1}^n V_j,\qquad M_n=\sum_{j=1}^n\varepsilon_jL_j. \tag{1.11}$$

The random variables $(V_j,\varepsilon_j), L_j$, $j=1,\dots,n$ are mutually independent. In addition ε_j, L_j are independent Bernoulli random variables with $\mathbb{P}\{L_j=0\}=\mathbb{P}\{L_j=1\}=1/2$.

Let also $B_n=\sum_{j=1}^n\varepsilon_j$, and note that $M_n\overset{\mathcal{D}}{=}\sum_{j=1}^{B_n}L_j$. The following result will be relevant.

Lemma 1.5 ([30], Chapter 7, Theorem 13). *Let $\mathcal{B}_n=\beta_1+\cdots+\beta_n$, $n=1,2,\dots$ where β_i are i.i.d. Bernoulli r.v.'s ($\mathbb{P}\{\beta_i=0\}=\mathbb{P}\{\beta_i=1\}=1/2$). There exists a numerical constant C_0 such* *that for all positive n*

$$\sup_z \left| \mathbb{P}\{\mathcal{B}_n = z\} - \sqrt{\frac{2}{\pi n}} e^{-\frac{(2z-n)^2}{2n}} \right| \le \frac{C_0}{n^{3/2}}.$$

Remark 1.6. In fact a little more is true, namely that we have $o(1/n^{3/2})$. It is also possible to give an estimate yielding a better error term using another comparison term. In particular, there exists an absolute constant C such that

$$\sup_k \left| \mathbb{P}\{\mathcal{B}_n = z\} - \sqrt{\frac{2}{\pi n}} \int_{\mathbb{R}} e^{-i\frac{2z-n}{\sqrt{n}}v - \frac{v^2}{2} - \frac{v^4}{12}} \, \mathrm{d}v \right| \le C \frac{\log^{7/2} n}{n^{5/2}}. \tag{1.12}$$

Let $\mathbb{E}_L$, $\mathbb{P}_L$ (resp. $\mathbb{E}_{(V,\varepsilon)}$, $\mathbb{P}_{(V,\varepsilon)}$) stand for the integration symbols and probability symbols relatively to the σ-algebra generated by the sequence $\{L_j, j = 1, \dots, n\}$ (resp. $\{(V_j, \varepsilon_j), j = 1, \dots, n\}$).

Assume from now that the X_j's are square integrable. The study of the probability

$$\mathbb{P}\{S_n = \kappa\} = \mathbb{E}_{(V,\varepsilon)} \mathbb{P}_L\{D M_n = \kappa - W_n\}$$

relies upon the conditional sum $S_n' = \mathbb{E}_L S_n = W_n + \frac{D}{2} B_n$, which satisfies

$$\mathbb{E} S_n' = \mathbb{E} S_n, \qquad \mathbb{E}(S_n')^2 = \mathbb{E} S_n^2 - \frac{D^2 \Theta_n}{4}.$$

Set

$$H_n = \sup_{x\in\mathbb{R}} \left| \mathbb{P}_{(V,\varepsilon)} \left\{ \frac{S_n' - \mathbb{E}_{(V,\varepsilon)} S_n'}{\sqrt{\operatorname{Var}(S_n')}} < x \right\} - \mathbb{P}\{g < x\} \right|,$$

$$\rho_n(h) = \mathbb{P}\left\{ \left| \sum_{j=1}^n \varepsilon_j - \Theta_n \right| > h\Theta_n \right\},$$

where $\varepsilon_1, \dots, \varepsilon_n$ are independent random variables satisfying $\mathbb{P}\{\varepsilon_j = 1\} = 1 - \mathbb{P}\{\varepsilon_j = 0\} = \vartheta_j$, $0 < \vartheta_j \le \vartheta_{X_j}$, $j = 1, \dots, n$ and

$$\Theta_n = \sum_{j=1}^n \vartheta_j.$$

As $S_n' = \mathbb{E}_L S_n$, suitable moment conditions permit us to easily estimate H_n by using Berry–Esseen estimates. Further the concentration inequalities (Lemma 4.1) provide sharp estimates of $\rho_n(h)$.

We are now ready to state our main result. Let $C_1 = \max(8/\sqrt{2\pi}, C_0)$.

Theorem 1.7. *Let $X_1, \dots, X_n$ be independent square integrable random variables taking almost surely values in a common lattice $\mathcal{L}(v_0, D) = \{v_k, k \in \mathbb{Z}\}$, where $v_k = v_0 + Dk$, $k \in \mathbb{Z}$, v_0 and $D > 0$ are real numbers. Let $S_n = X_1 + \cdots + X_n$.*

For any $0 < h < 1$, $0 < \vartheta_j \le \vartheta_{X_j}$, *and all* $\kappa \in \mathcal{L}(v_0 n, D)$

$$\mathbb{P}\{S_n = \kappa\} \le \left(\frac{1+h}{1-h}\right) \frac{D}{\sqrt{2\pi \operatorname{Var}(S_n)}} e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2(1+h)\operatorname{Var}(S_n)}} + \frac{C_1}{\sqrt{(1-h)\Theta_n}} \left(H_n + \frac{1}{(1-h)\Theta_n}\right) + \rho_n(h).$$

Also

$$\mathbb{P}\{S_n = \kappa\} \ge \left(\frac{1-h}{1+h}\right) \frac{D}{\sqrt{2\pi \operatorname{Var}(S_n)}} e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2(1-h)\operatorname{Var}(S_n)}} - \frac{C_1}{\sqrt{(1-h)\Theta_n}} \left(H_n + \frac{1}{(1-h)\Theta_n} + 2\rho_n(h)\right) - \rho_n(h).$$

Corollary 1.8. *Assume that* $\frac{\log \Theta_n}{\Theta_n} \le 1/14$. *Then, for all* $\kappa \in \mathcal{L}(v_0 n, D)$ *such that*

$$\frac{(\kappa - \mathbb{E}S_n)^2}{\operatorname{Var}(S_n)} \le \left(\frac{\Theta_n}{14 \log \Theta_n}\right)^{1/2},$$

we have

$$\left|\mathbb{P}\{S_n = \kappa\} - \frac{D e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}}}{\sqrt{2\pi \operatorname{Var}(S_n)}}\right| \le C_2 \left\{ D \left(\frac{\log \Theta_n}{\operatorname{Var}(S_n)\Theta_n}\right)^{1/2} + \frac{H_n + \Theta_n^{-1}}{\sqrt{\Theta_n}} \right\}.$$

Here $C_2 = 2^{7/2} C_1$.

Corollary 1.9. *Assume that*

$$\lim_{n\to\infty} \left(\frac{\operatorname{Var}(S_n)}{\Theta_n}\right)^{1/2} \left(H_n + \frac{1}{\Theta_n}\right) = 0. \tag{1.13}$$

Then

$$\lim_{n\to\infty} \sup_{\kappa \in \mathcal{L}(v_0 n, D)} \left| \sqrt{\operatorname{Var}(S_n)}\, \mathbb{P}\{S_n = \kappa\} - \frac{D e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}}}{\sqrt{2\pi}} \right| = 0. \tag{1.14}$$

Remark 1.10. Condition (1.13) is for instance, satisfied if

(i) $\lim_{n\to\infty} \operatorname{Var}(S_n) = \infty$, (ii) $\lim_{n\to\infty} H_n = 0$, (iii) $\limsup_{n\to\infty} \frac{\operatorname{Var}(S_n)}{\Theta_n} < \infty$.

We note that if X_i are i.i.d., then by Theorem 8, p. 118 of [30], we have that $H_n \to 0$ as n tends to infinity. Hence condition (ii) is satisfied. As conditions (i) and (iii) trivially hold, Corollary 1.9 applies and Gnedenko's theorem follows from (1.14).

Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be even, convex and such that $\frac{\psi(x)}{x^2}$ and $\frac{x^3}{\psi(x)}$ are non-decreasing on $\mathbb{R}^+$. We further assume that

$$\mathbb{E}\psi(X_j) < \infty. \tag{1.15}$$

Put

$$L_n = \frac{\sum_{j=1}^n \mathbb{E}\psi(X_j)}{\psi(\sqrt{\mathrm{Var}(S_n)})}.$$

Then Corollary 1.8 can strengthened as follows.

Corollary 1.11. *Assume that* $\frac{\log \Theta_n}{\Theta_n} \le 1/14$. *Then, for all* $\kappa \in \mathcal{L}(v_0 n, D)$ *such that*

$$\frac{(\kappa - \mathbb{E}S_n)^2}{\mathrm{Var}(S_n)} \le \left(\frac{\Theta_n}{14 \log \Theta_n}\right)^{1/2},$$

we have

$$\left|\mathbb{P}\{S_n = \kappa\} - \frac{D e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}}}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}\right| \le C_3 \left\{ D\left(\frac{\log \Theta_n}{\mathrm{Var}(S_n)\Theta_n}\right)^{1/2} + \frac{L_n + \Theta_n^{-1}}{\sqrt{\Theta_n}} \right\}.$$

And $C_3 = \max(C_2, 2^{3/2} C_{\mathrm{E}})$, C_{E} *being an absolute constant arising from Berry–Esseen's inequality. See* [30], *p.* 111 *and* (6.1).

2. Preliminaries

2.1. Characteristics of a random variable

Let X be a random variable such that $\mathbb{P}(X \in \mathcal{L}(v_0, D)) = 1$ and recall according to (1.8) that

$$\vartheta_X = \sum_{k \in \mathbb{Z}} \mathbb{P}\{X = v_k\} \wedge \mathbb{P}\{X = v_{k+1}\}.$$

Then

$$0 \le \vartheta_X < 1. \tag{2.1}$$

Indeed, let k_0 be some integer such that $f(k_0) > 0$. Then

$$\sum_{k=k_0}^{\infty} f(k) \wedge f(k+1) \le \sum_{k=k_0}^{\infty} f(k+1) = \sum_{k=k_0+1}^{\infty} f(k),$$

and so $0 \le \vartheta_X \le \sum_{k<k_0} f(k) + \sum_{k=k_0+1}^{\infty} f(k) < 1$.

Now assume that X has finite mean μ and finite variance σ^2. The following inequality linking parameters σ, D, ϑ_X, is implicit in our proof (see (3.6)),

$$\sigma^2 \geq \frac{D^2}{4}\vartheta_X. \tag{2.2}$$

We begin with giving a proof valid for general lattice valued random variables. At first by Tchebycheff's inequality,

$$\frac{D^2}{4}\mathbb{P}\left\{|X-\mu| \geq \frac{D}{2}\right\} \leq \sigma^2.$$

Now

$$\begin{aligned}
\mathbb{P}\left\{|X-\mu| \geq \frac{D}{2}\right\} &= \sum_{v_k \geq \mu+\frac{D}{2}} \mathbb{P}\{X=v_k\} + \sum_{v_k \leq \mu-\frac{D}{2}} \mathbb{P}\{X=v_k\} \\
&\geq \sum_{v_{k+1} \geq \mu+\frac{D}{2}} \mathbb{P}\{X=v_k\} \wedge \mathbb{P}\{X=v_{k+1}\} + \sum_{v_k \leq \mu-\frac{D}{2}} \mathbb{P}\{X=v_k\} \wedge \mathbb{P}\{X=v_{k+1}\} \\
&= \sum_{v_k \geq \mu-\frac{D}{2}} \mathbb{P}\{X=v_k\} \wedge \mathbb{P}\{X=v_{k+1}\} + \sum_{v_k \leq \mu-\frac{D}{2}} \mathbb{P}\{X=v_k\} \wedge \mathbb{P}\{X=v_{k+1}\} \\
&\geq \vartheta_X.
\end{aligned}$$

Hence inequality (2.2).

Remark 2.1. In Lemma 2 of Mukhin [29], the following inequality is proved

$$\mathcal{D}(X,d) := \inf_{a\in\mathbb{R}} \mathbb{E}\langle (X-a)d\rangle^2 \geq \frac{|d|^2}{4}\vartheta_X.$$

Here d is a real number, $|d| \leq 1/2$ and $\langle\alpha\rangle$ is the distance from α to the nearest integer. Notice that $\mathcal{D}(X,d) = 0$ if and only if X is lattice valued with span $1/d$.

2.2. Bernoulli component of a random variable

Let X be a random variable such that $\mathbb{P}\{X \in \mathcal{L}(v_0, D)\} = 1$. It is not necessary to suppose here that the span D is maximal. Put

$$f(k) = \mathbb{P}\{X = v_k\}, \qquad k \in \mathbb{Z}.$$

We assume that

$$\vartheta_X > 0. \tag{2.3}$$

Recall that $\vartheta_X < 1$. Let $0 < \vartheta \le \vartheta_X$. One can associate to ϑ and X a sequence $\{\tau_k, k \in \mathbb{Z}\}$ of non-negative reals such that

$$\tau_{k-1} + \tau_k \le 2f(k), \qquad \sum_{k\in\mathbb{Z}} \tau_k = \vartheta. \tag{2.4}$$

Just take $\tau_k = \frac{\vartheta}{\nu_X}(f(k) \wedge f(k+1))$. Now define a pair of random variables (V, ε) as follows:

$$\begin{cases} \mathbb{P}\big\{(V,\varepsilon) = (v_k, 1)\big\} = \tau_k, \\ \mathbb{P}\big\{(V,\varepsilon) = (v_k, 0)\big\} = f(k) - \dfrac{\tau_{k-1}+\tau_k}{2} \end{cases} \qquad (\forall k \in \mathbb{Z}). \tag{2.5}$$

By assumption this is well-defined, and the margin laws verify

$$\begin{cases} \mathbb{P}\{V = v_k\} = f(k) + \dfrac{\tau_k - \tau_{k-1}}{2}, \\ \mathbb{P}\{\varepsilon = 1\} = \vartheta = 1 - \mathbb{P}\{\varepsilon = 0\}. \end{cases} \tag{2.6}$$

Indeed, $\mathbb{P}\{V = v_k\} = \mathbb{P}\{(V,\varepsilon) = (v_k,1)\} + \mathbb{P}\{(V,\varepsilon) = (v_k,0)\} = f(k) + \frac{\tau_k - \tau_{k-1}}{2}$. Further $\mathbb{P}\{\varepsilon = 1\} = \sum_{k\in\mathbb{Z}} \mathbb{P}\{(V,\varepsilon) = (v_k,1)\} = \sum_{k\in\mathbb{Z}} \tau_k = \vartheta$.

The whole approach is based on the lemma below, the proof of which is given for the sake of completeness.

Lemma 2.2. *Let L be a Bernoulli random variable which is independent of (V,ε), and put $Z = V + \varepsilon DL$. We have $Z \stackrel{\mathcal{D}}{=} X$.*

Proof. ([10,33]) Plainly,

$$\begin{aligned} \mathbb{P}\{Z = v_k\} &= \mathbb{P}\{V + \varepsilon DL = v_k, \varepsilon = 1\} + \mathbb{P}\{V + \varepsilon DL = v_k, \varepsilon = 0\} \\ &= \frac{\mathbb{P}\{V = v_{k-1}, \varepsilon = 1\} + \mathbb{P}\{V = v_k, \varepsilon = 1\}}{2} + \mathbb{P}\{V = v_k, \varepsilon = 0\} \\ &= \frac{\tau_{k-1}+\tau_k}{2} + f(k) - \frac{\tau_{k-1}+\tau_k}{2} = f(k). \end{aligned}$$

□

Now consider independent random variables X_j, $j = 1, \dots, n$, each satisfying assumption (2.3) and let $0 < \vartheta_i \le \vartheta_{X_i}$, $i = 1, \dots, n$. Iterated applications of Lemma 2.2 allow us to associate to them a sequence of independent vectors $(V_j, \varepsilon_j, L_j)$, $j = 1, \dots, n$ such that

$$\{V_j + \varepsilon_j DL_j, j = 1, \dots, n\} \stackrel{\mathcal{D}}{=} \{X_j, j = 1, \dots, n\}. \tag{2.7}$$

Further the sequences $\{(V_j, \varepsilon_j), j = 1, \dots, n\}$ and $\{L_j, j = 1, \dots, n\}$ are independent. For each $j = 1, \dots, n$, the law of (V_j, ε_j) is defined according to (2.5) with $\vartheta = \vartheta_j$. And $\{L_j, j = 1, \dots, n\}$ is a sequence of independent Bernoulli random variables. Set

$$S_n = \sum_{j=1}^n X_j, \qquad W_n = \sum_{j=1}^n V_j, \qquad M_n = \sum_{j=1}^n \varepsilon_j L_j, \qquad B_n = \sum_{j=1}^n \varepsilon_j. \tag{2.8}$$

Lemma 2.3. *We have the representation*

$$\{S_k, 1 \le k \le n\} \stackrel{\mathcal{D}}{=} \{W_k + DM_k, 1 \le k \le n\}.$$

And $M_n \stackrel{\mathcal{D}}{=} \sum_{j=1}^{B_n} L_j$.

3. Proof of Theorem 1.7

We denote again $X_j = V_j + D\varepsilon_j L_j$, $S_n = W_n + M_n$, $j, n \ge 1$, which is justified by the previous representation. Fix $0 < h < 1$ and let

$$A_n = \big\{|B_n - \Theta_n| \le h\Theta_n\big\}, \qquad \rho_n(h) = \mathbb{P}_{(V,\varepsilon)}\big(A_n^c\big). \tag{3.1}$$

For $\kappa \in \mathcal{L}(v_0, D)$,

$$\begin{aligned}\mathbb{P}\{S_n = \kappa\} &= \mathbb{E}_{(V,\varepsilon)}\mathbb{P}_L\left\{D\sum_{j=1}^n \varepsilon_j L_j = \kappa - W_n\right\} \\ &= \mathbb{E}_{(V,\varepsilon)}\big(\chi(A_n) + \chi\big(A_n^c\big)\big)\mathbb{P}_L\left\{D\sum_{j=1}^n \varepsilon_j L_j = \kappa - W_n\right\}.\end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned}\left|\mathbb{P}\{S_n = \kappa\} - \mathbb{E}_{(V,\varepsilon)}\chi(A_n)\mathbb{P}_L\left\{D\sum_{j=1}^n \varepsilon_j L_j = \kappa - W_n\right\}\right| &\le \mathbb{P}_{(V,\varepsilon)}\big(A_n^c\big) \\ &= \rho_n(h).\end{aligned} \tag{3.3}$$

We have $\sum_{j=1}^n \varepsilon_j L_j \stackrel{\mathcal{D}}{=} \sum_{j=1}^{B_n} L_j$. In view of Lemma 1.5,

$$\sup_z \left|\mathbb{P}_L\left\{\sum_{j=1}^N L_j = z\right\} - \frac{2}{\sqrt{2\pi N}} e^{-\frac{(z-(N/2))^2}{N/2}}\right| \le \frac{C_0}{N^{3/2}}.$$

On A_n, we have $(1-h)\Theta_n \le B_n \le (1+h)\Theta_n$. Therefore,

$$\begin{aligned}&\left|\mathbb{E}_{(V,\varepsilon)}\chi(A_n)\left\{\mathbb{P}_L\left\{D\sum_{j=1}^n \varepsilon_j L_j = \kappa - W_n\right\} - \frac{2e^{-\frac{(\kappa - W_n - D(B_n/2))^2}{D^2(B_n/2)}}}{\sqrt{2\pi B_n}}\right\}\right| \\ &\quad \le C_0\mathbb{E}_{(V,\varepsilon)}\chi(A_n)\cdot B_n^{-3/2} \le \frac{C_0}{(1-h)^{3/2}}\frac{1}{(\sum_{i=1}^n \vartheta_i)^{3/2}}.\end{aligned} \tag{3.4}$$

Inserting this in (3.3)

$$
\begin{aligned}
&\left|\mathbb{P}\{S_n = \kappa\} - \mathbb{E}_{(V,\varepsilon)}\chi(A_n)\frac{2e^{-\frac{(\kappa - W_n - D(B_n/2))^2}{D^2(B_n/2)}}}{\sqrt{2\pi B_n}}\right| \\
&\quad \le \frac{C_0}{(1-h)^{3/2}}\frac{1}{(\sum_{i=1}^n \vartheta_i)^{3/2}} + \rho_n(h).
\end{aligned}
\tag{3.5}
$$

Step 2. (*Second reduction*) Some elementary algebra is necessary in order to put the exponential term in a more appropriate form. Recall that $S_n = W_n + M_n$.

Lemma 3.1. *Let $S'_n = W_n + D(B_n/2)$. Then*

$$
\mathbb{E}S'_n = \mathbb{E}S_n, \qquad \mathbb{E}\big(S'_n\big)^2 = \mathbb{E}S_n^2 - \frac{D^2\Theta_n}{4}.
$$

Proof. At first $\mathbb{E}S_n = \mathbb{E}_{(V,\varepsilon)}\mathbb{E}_L(W_n + D\sum_{j=1}^n \varepsilon_j L_j) = \mathbb{E}_{(V,\varepsilon)}(W_n + D\frac{B_n}{2}) = \mathbb{E}S'_n$. Further, by using independence,

$$
\begin{aligned}
\mathbb{E}_{(V,\varepsilon)}B_n^2 &= \sum_{\substack{1\le i,j\le n\\ i\ne j}} \mathbb{E}_{(V,\varepsilon)}\varepsilon_i \mathbb{E}_{(V,\varepsilon)}\varepsilon_j + \sum_{1\le i\le n}\mathbb{E}_{(V,\varepsilon)}\varepsilon_i^2 \\
&= \sum_{\substack{1\le i,j\le n\\ i\ne j}} \vartheta_i\vartheta_j + \sum_{i=1}^n \vartheta_i = \left(\sum_{i=1}^n \vartheta_i\right)^2 - \sum_{i=1}^n \vartheta_i^2 + \sum_{i=1}^n \vartheta_i,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^n \varepsilon_j L_j\right)^2 &= \sum_{\substack{1\le i,j\le n\\ i\ne j}} \mathbb{E}_{(V,\varepsilon)}\varepsilon_i\varepsilon_j \mathbb{E}_L L_i \mathbb{E}_L L_j + \sum_{i=1}^n \mathbb{E}_{(V,\varepsilon)}\varepsilon_i^2 \mathbb{E}_L L_i^2 \\
&= \frac{1}{4}\sum_{\substack{1\le i,j\le n\\ i\ne j}} \mathbb{E}_{(V,\varepsilon)}\varepsilon_i\varepsilon_j + \frac{1}{2}\sum_{i=1}^n \mathbb{E}_{(V,\varepsilon)}\varepsilon_i^2 = \frac{1}{4}\sum_{\substack{1\le i,j\le n\\ i\ne j}} \vartheta_i\vartheta_j + \frac{1}{2}\sum_{i=1}^n \vartheta_i \\
&= \frac{1}{4}\left\{\left(\sum_{i=1}^n \vartheta_i\right)^2 - \sum_{i=1}^n \vartheta_i^2\right\} + \frac{1}{2}\sum_{i=1}^n \vartheta_i.
\end{aligned}
$$

Now

$$
\mathbb{E}S_n^2 = \mathbb{E}\left(W_n + D\sum_{i=1}^n \varepsilon_i L_i\right)^2
$$

$$= \mathbb{E}_{(V,\varepsilon)} W_n^2 + 2D\mathbb{E}_{(V,\varepsilon)} W_n \left(\sum_{i=1}^n \varepsilon_i \mathbb{E}_L L_i\right) + D^2 \mathbb{E}\left(\sum_{i=1}^n \varepsilon_i L_i\right)^2$$

$$= \mathbb{E}_{(V,\varepsilon)}\left(W_n^2 + 2DW_n\left(\frac{B_n}{2}\right)\right) + \frac{D^2}{4}\left\{\left(\sum_{i=1}^n \vartheta_i\right)^2 - \sum_{i=1}^n \vartheta_i^2\right\} + \frac{D^2}{2}\sum_{i=1}^n \vartheta_i.$$

And

$$\mathbb{E}\big(S_n'\big)^2 = \mathbb{E}_{(V,\varepsilon)}\left(W_n^2 + 2DW_n\left(\frac{B_n}{2}\right)\right) + \frac{D^2}{4}\left\{\left(\sum_{i=1}^n \vartheta_i\right)^2 - \sum_{i=1}^n \vartheta_i^2 + \sum_{i=1}^n \vartheta_i\right\}$$

$$= \mathbb{E}S_n^2 - \frac{D^2}{4}\sum_{i=1}^n \vartheta_i.$$

Hence, Lemma 3.1 is established. □

We deduce

$$\operatorname{Var}\big(S_n'\big) = \operatorname{Var}(S_n) - \frac{D^2}{4}\sum_{i=1}^n \vartheta_i = \sum_{i=1}^n \left(\sigma_i^2 - \frac{D^2\vartheta_i}{4}\right). \tag{3.6}$$

Put

$$T_n = \frac{S_n' - \mathbb{E}_{(V,\varepsilon)} S_n'}{\sqrt{\operatorname{Var}(S_n')}}.$$

As $\mathbb{E}_{(V,\varepsilon)} S_n' = \mathbb{E}S_n$ we can write

$$\frac{(\kappa - W_n - D(B_n/2))^2}{D^2(B_n/2)} = \frac{(\kappa - \mathbb{E}S_n - \{S_n' - \mathbb{E}_{(V,\varepsilon)} S_n'\})^2}{D^2(B_n/2)}$$

$$= \frac{\operatorname{Var}(S_n')}{D^2(B_n/2)}\left(\frac{\kappa - \mathbb{E}S_n}{\sqrt{\operatorname{Var}(S_n')}} - T_n\right)^2.$$

And recalling that $\Theta_n = \sum_{i=1}^n \vartheta_i$, (3.5) is more conveniently rewritten as

$$\big|\mathbb{P}\{S_n = \kappa\} - \Upsilon_n\big| \le \frac{C_0}{(1-h)^{3/2}} \frac{1}{\Theta_n^{3/2}} + \rho_n(h), \tag{3.7}$$

where

$$\Upsilon_n = \mathbb{E}_{(V,\varepsilon)} \chi(A_n) \frac{2e^{-\frac{\operatorname{Var}(S_n')}{D^2(B_n/2)}\left(\frac{\kappa - \mathbb{E}S_n}{\sqrt{\operatorname{Var}(S_n')}} - T_n\right)^2}}{\sqrt{2\pi B_n}}. \tag{3.8}$$

Set for $-1 < u \le 1$,

$$Z_n(u) = \mathbb{E}_{(V,\varepsilon)} e^{-\frac{2\,\mathrm{Var}(S'_n)}{D^2(1+u)\Theta_n}\left(\frac{\kappa-\mathbb{E}S_n}{\sqrt{\mathrm{Var}(S'_n)}}-T_n\right)^2}.$$

Then

$$\frac{2Z_n(-h) - 2\rho_n(h)}{\sqrt{2\pi(1+h)\Theta_n}} \le \Upsilon_n \le \frac{2Z_n(h)}{\sqrt{2\pi(1-h)\Theta_n}}. \tag{3.9}$$

The second inequality is obvious, and the first follows from

$$\begin{aligned}
\Upsilon_n &\ge \frac{2}{\sqrt{2\pi(1+h)\Theta_n}} \mathbb{E}_{(V,\varepsilon)} \chi(A_n) e^{-\frac{2\,\mathrm{Var}(S'_n)}{D^2(1-h)\Theta_n}\left(\frac{\kappa-\mathbb{E}S_n}{\sqrt{\mathrm{Var}(S'_n)}}-T_n\right)^2} \\
&\ge \frac{2}{\sqrt{2\pi(1+h)\Theta_n}} \left\{\mathbb{E}_{(V,\varepsilon)} e^{-\frac{2\,\mathrm{Var}(S'_n)}{D^2(1-h)\Theta_n}\left(\frac{\kappa-\mathbb{E}S_n}{\sqrt{\mathrm{Var}(S'_n)}}-T_n\right)^2} - \mathbb{P}_{(V,\varepsilon)}\big(A_n^c\big)\right\} \\
&\ge \frac{2Z_n(-h) - 2\rho_n(h)}{\sqrt{2\pi(1+h)\Theta_n}}.
\end{aligned}$$

Step 3. (*Exponential moment*)

Lemma 3.2. *Let Y be a centered random variable. For any positive reals a and b*

$$\left|\mathbb{E}e^{-a(b-Y)^2} - \frac{e^{-\frac{b^2}{2+1/a}}}{\sqrt{1+2a}}\right| \le 4 \sup_{x\in\mathbb{R}} \big|\mathbb{P}\{Y < x\} - \mathbb{P}\{g < x\}\big|.$$

Proof. By the transfer formula,

$$\begin{aligned}
\big|\mathbb{E}e^{-a(b-Y)^2} - \mathbb{E}e^{-a(b-g)^2}\big| &= \left|\int_0^1 \big(\mathbb{P}\big\{e^{-a(b-Y)^2} > x\big\} - \mathbb{P}\big\{e^{-a(b-g)^2} > x\big\}\big)\,\mathrm{d}x\right| \\
\big(x = e^{-ay^2}\big) \qquad &= 2a\left|\int_0^\infty \big(\mathbb{P}\big\{|b-Y| < y\big\} - \mathbb{P}\big\{|b-g| < y\big\}\big) y e^{-ay^2}\,\mathrm{d}y\right| \\
&\le 4 \sup_{x\in\mathbb{R}} \big|\mathbb{P}\{Y < x\} - \mathbb{P}\{g < x\}\big|.
\end{aligned}$$

The claimed estimate follows from

$$\mathbb{E}e^{-a(b-g)^2} = \frac{e^{-\frac{b^2}{2+1/a}}}{\sqrt{1+2a}}. \tag{3.10}$$

□

We apply Lemma 3.2 to estimate $Z_n(u)$. Here $a = \frac{2\operatorname{Var}(S'_n)}{D^2(1+u)\Theta_n}$, $b = \frac{\kappa - \mathbb{E}S_n}{\sqrt{\operatorname{Var}(S'_n)}}$. Since by (3.6), $\operatorname{Var}(S'_n) = \operatorname{Var}(S_n) - \frac{D^2\Theta_n}{4}$, we have

$$\begin{aligned}\frac{b^2}{2+1/a} &= \frac{(\kappa - \mathbb{E}S_n)^2}{\operatorname{Var}(S'_n)(2 + \frac{D^2(1+u)\Theta_n}{2\operatorname{Var}(S'_n)})} = \frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S'_n) + \frac{D^2(1+u)\Theta_n}{2}} \\ &= \frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n) - \frac{D^2\Theta_n}{2} + \frac{D^2(1+u)\Theta_n}{2}} = \frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n) + \frac{D^2u\Theta_n}{2}} \\ &= \frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)(1+\delta(u))},\end{aligned}$$

where we have denoted

$$\delta(u) = \frac{D^2\Theta_n u}{4\operatorname{Var}(S_n)}.$$

Now

$$\begin{aligned}\frac{1}{\sqrt{1+2a}} &= \left(\frac{1}{1+\frac{4\operatorname{Var}(S'_n)}{D^2(1+u)\Theta_n}}\right)^{1/2} = \frac{D}{2}\left(\frac{(1+u)\Theta_n}{\operatorname{Var}(S'_n) + \frac{D^2(1+u)\Theta_n}{4}}\right)^{1/2} \\ &= \frac{D}{2}\left(\frac{(1+u)\Theta_n}{\operatorname{Var}(S_n) + \frac{D^2h\Theta_n}{4}}\right)^{1/2} = \frac{D}{2}\left(\frac{\Theta_n(1+u)}{\operatorname{Var}(S_n)(1+\delta(u))}\right)^{1/2}.\end{aligned}$$

This along with Lemma 3.2 provides the following bound,

$$\left| Z_n(u) - \frac{D}{2}\left(\frac{\Theta_n(1+u)}{\operatorname{Var}(S_n)(1+\delta(u))}\right)^{1/2} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)(1+\delta(u))}} \right| \le 4H_n, \tag{3.11}$$

with

$$H_n = \sup_{x\in\mathbb{R}} \left| \mathbb{P}_{(V,\varepsilon)}\{T_n < x\} - \mathbb{P}\{g < x\} \right|.$$

Besides, it follows from (2.2) that for $h \ge 0$,

$$0 \le \delta(h) \le h. \tag{3.12}$$

Step 4. (*Conclusion*) Consider the upper bound part. By reporting (3.11) into (3.9) and using (3.12), we get

$$\Upsilon_n \le \frac{8H_n}{\sqrt{2\pi(1-h)\Theta_n}} + \left(\frac{1+h}{1-h}\right)\frac{D}{\sqrt{2\pi\operatorname{Var}(S_n)}} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1+h)\operatorname{Var}(S_n)}}.$$

And by combining with (3.7),

$$\mathbb{P}\{S_n=\kappa\} \le \left(\frac{1+h}{1-h}\right)\frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1+h)\,\mathrm{Var}(S_n)}} + \frac{8H_n}{\sqrt{2\pi(1-h)\Theta_n}} + \frac{C_0}{(1-h)^{3/2}}\frac{1}{\Theta_n^{3/2}} + \rho_n(h). \tag{3.13}$$

Similarly, by using (3.9),

$$\Upsilon_n \ge \left(\frac{1-h}{1+h}\right)\frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h)\,\mathrm{Var}(S_n)}} - \frac{8H_n+2\rho_n(h)}{\sqrt{2\pi(1+h)\Theta_n}}. \tag{3.14}$$

By combining with (3.7), we obtain

$$\mathbb{P}\{S_n=\kappa\} \ge \left(\frac{1-h}{1+h}\right)\frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h)\,\mathrm{Var}(S_n)}} - \frac{8H_n+2\rho_n(h)}{\sqrt{2\pi(1+h)\Theta_n}} - \frac{C_0}{(1-h)^{3/2}}\frac{1}{\Theta_n^{3/2}} - \rho_n(h). \tag{3.15}$$

As $C_1=\max(8/\sqrt{2\pi},C_0)$, we deduce

$$\mathbb{P}\{S_n=\kappa\} \le \left(\frac{1+h}{1-h}\right)\frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1+h)\,\mathrm{Var}(S_n)}} + \frac{C_1}{\sqrt{(1-h)\Theta_n}}\left(H_n+\frac{1}{(1-h)\Theta_n}\right) + \rho_n(h). \tag{3.16}$$

And

$$\mathbb{P}\{S_n=\kappa\} \ge \left(\frac{1-h}{1+h}\right)\frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h)\,\mathrm{Var}(S_n)}} - \frac{C_1}{\sqrt{(1-h)\Theta_n}}\left(H_n+\frac{1}{(1-h)\Theta_n}+2\rho_n(h)\right) - \rho_n(h). \tag{3.17}$$

This achieves the proof.

4. Proof of Corollary 1.8

In order to estimate $\rho_n(h)$ we use the following Lemma ([25], Theorem 2.3)

Lemma 4.1. *Let $X_1,\dots,X_n$ be independent random variables, with $0\le X_k\le 1$ for each k. Let $S_n=\sum_{k=1}^n X_k$ and $\mu=\mathbb{E}S_n$. Then for any $\epsilon>0$,*

$$\text{(a)}\quad \mathbb{P}\big\{S_n\ge(1+\epsilon)\mu\big\}\le e^{-\frac{\epsilon^2\mu}{2(1+\epsilon/3)}},$$

$$\text{(b)} \quad \mathbb{P}\big\{S_n \le (1-\epsilon)\mu\big\} \le e^{-\frac{\epsilon^2\mu}{2}}.$$

By (a) and (b), and observing that $e^{-\frac{\epsilon^2\mu}{2}} \le e^{-\frac{\epsilon^2\mu}{2(1+\epsilon/3)}}$, we obtain

$$\rho_n(h) = \mathbb{P}\left\{\left|\sum_{k=1}^n \varepsilon_k - \Theta_n\right| > h\Theta_n\right\} = \mathbb{P}\left\{\sum_{k=1}^n \varepsilon_k > (1+h)\Theta_n\right\} + \mathbb{P}\left\{\sum_{k=1}^n \varepsilon_k < (1-h)\Theta_n\right\}$$

$$\le 2e^{-\frac{h^2\Theta_n}{2(1+h/3)}}.$$

Let $h_n = \sqrt{\frac{7\log\Theta_n}{2\Theta_n}}$. By assumption $\frac{\log\Theta_n}{\Theta_n} \le 1/14$. Thus, $h_n \le 1/2$ and so $\frac{h_n^2\Theta_n}{2(1+h_n/3)} \ge (3/2)\log\Theta_n$. It follows that

$$\rho_n(h_n) \le 2\Theta_n^{-3/2}. \tag{4.1}$$

Let $C_2 = 2^{7/2}\max(C_1, 1)$. Further

$$\begin{aligned}\frac{C_1}{\sqrt{(1-h_n)\Theta_n}}\left(H_n + \frac{1}{(1-h_n)\Theta_n}\right) + \rho_n(h_n) &\le 2^{1/2}C_1\frac{H_n}{\sqrt{\Theta_n}} + \frac{2^{3/2}C_1 + 2}{\Theta_n^{3/2}}\\ &\le \frac{2^{1/2}\max(C_1,1)}{\sqrt{\Theta_n}}\left(H_n + \frac{2+\sqrt{2}}{\Theta_n}\right)\\ &\le \frac{C_2}{\sqrt{\Theta_n}}\left(H_n + \frac{1}{\Theta_n}\right).\end{aligned}$$

Therefore

$$\mathbb{P}\{S_n = \kappa\} \le \frac{D(1+4h_n)}{\sqrt{2\pi\,\mathrm{Var}(S_n)}} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1+h_n)\mathrm{Var}(S_n)}} + \frac{C_2}{\sqrt{\Theta_n}}\left(H_n + \frac{1}{\Theta_n}\right).$$

Besides

$$\begin{aligned}&\frac{C_1}{\sqrt{(1-h_n)\Theta_n}}\left(H_n + \frac{1}{(1-h_n)\Theta_n} + 2\rho_n(h_n)\right) + \rho_n(h_n)\\ &\le \frac{2^{1/2}C_1}{\sqrt{\Theta_n}}\left(H_n + \frac{6}{\Theta_n}\right) + \frac{2}{\Theta_n^{3/2}} \le \frac{2^{1/2}\max(C_1,1)}{\sqrt{\Theta_n}}\left(H_n + \frac{6+\sqrt{2}}{\Theta_n}\right)\\ &\le \frac{C_2}{\sqrt{\Theta_n}}\left(H_n + \frac{1}{\Theta_n}\right).\end{aligned}$$

Consequently,

$$\mathbb{P}\{S_n = \kappa\} \ge \frac{D(1-2h_n)}{\sqrt{2\pi\,\mathrm{Var}(S_n)}} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h_n)\mathrm{Var}(S_n)}} - \frac{C_2}{\sqrt{\Theta_n}}\left(H_n + \frac{1}{\Theta_n}\right).$$

If

$$\frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)} \leq \frac{1+h_n}{h_n},$$

then by using the inequalities $e^u \leq 1+3u$ and $Xe^{-X} \leq e^{-1}$ valid for $0 \leq u \leq 1$, $X \geq 0$, we get

$$\begin{aligned} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h_n)\operatorname{Var}(S_n)}} &= e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} e^{\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}\frac{h_n}{1+h_n}} \\ &\leq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} \left\{1 + 3\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}\frac{h_n}{1+h_n}\right\} \\ &\leq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} + \frac{3h_n}{e(1+h_n)} \leq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} + 2h_n. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{D(1+4h_n)}{\sqrt{2\pi\operatorname{Var}(S_n)}} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h_n)\operatorname{Var}(S_n)}} &\leq \frac{D(1+4h_n)}{\sqrt{2\pi\operatorname{Var}(S_n)}} \left\{e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} + 2h_n\right\} \\ &\leq \frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}}}{\sqrt{2\pi\operatorname{Var}(S_n)}} + \frac{4h_n D}{\sqrt{2\pi\operatorname{Var}(S_n)}} + \frac{2h_n D(1+4h_n)}{\sqrt{2\pi\operatorname{Var}(S_n)}} \\ &\leq \frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}}}{\sqrt{2\pi\operatorname{Var}(S_n)}} + \frac{10h_n D}{\sqrt{2\pi\operatorname{Var}(S_n)}}. \end{aligned}$$

Therefore, recalling that $h_n = \sqrt{\frac{7\log\Theta_n}{2\Theta_n}}$,

$$\begin{aligned} \mathbb{P}\{S_n = \kappa\} - \frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}}}{\sqrt{2\pi\operatorname{Var}(S_n)}} &\leq \frac{10h_n D}{\sqrt{2\pi\operatorname{Var}(S_n)}} + C_2\frac{H_n + \Theta_n^{-1}}{\sqrt{\Theta_n}} \\ &\leq C_2\left\{D\left(\frac{\log\Theta_n}{\operatorname{Var}(S_n)\Theta_n}\right)^{1/2} + \frac{H_n + \Theta_n^{-1}}{\sqrt{\Theta_n}}\right\} \end{aligned}$$

since $5\sqrt{7/\pi} \leq C_2$. Similarly, if

$$\frac{(\kappa - \mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)} \leq \frac{1}{2h_n},$$

then as $e^{-u} \geq 1 - 3u$ if $0 \leq u \leq 1$, we get

$$\begin{aligned} e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h_n)\operatorname{Var}(S_n)}} &\geq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} e^{-h_n\frac{(\kappa-\mathbb{E}S_n)^2}{\operatorname{Var}(S_n)}} \geq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} \left\{1 - 3h_n\frac{(\kappa-\mathbb{E}S_n)^2}{\operatorname{Var}(S_n)}\right\} \\ &\geq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} - \frac{3h_n}{e} \geq e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\operatorname{Var}(S_n)}} - 2h_n. \end{aligned}$$

Hence,

$$\frac{D(1-2h_n)}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1-h_n)\,\mathrm{Var}(S_n)}} \geq \frac{D(1-2h_n)}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}\{e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}} - 2h_n\}$$
$$\geq \frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}} - \frac{3h_n D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}.$$

Consequently,

$$\mathbb{P}\{S_n=\kappa\} - \frac{D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}} \geq -\frac{3h_n D}{\sqrt{2\pi\,\mathrm{Var}(S_n)}} - \frac{C_2}{\sqrt{\Theta_n}}\left(H_n + \frac{1}{\Theta_n}\right)$$
$$\geq -C_2\left\{D\left(\frac{\log\Theta_n}{\mathrm{Var}(S_n)\Theta_n}\right)^{1/2} + \frac{H_n+\Theta_n^{-1}}{\sqrt{\Theta_n}}\right\}.$$

5. Proof of Corollary 1.9

It follows from Corollary 1.8 and assumption (1.13) that

$$\lim_{n\to\infty}\sup_{\frac{(\kappa-\mathbb{E}S_n)^2}{\mathrm{Var}(S_n)}\leq(\frac{\Theta_n}{14\log\Theta_n})^{1/2}}\left|\sqrt{\mathrm{Var}(S_n)}\mathbb{P}\{S_n=\kappa\} - \frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}}}{\sqrt{2\pi}}\right| = 0.$$

Now if $\frac{(\kappa-\mathbb{E}S_n)^2}{\mathrm{Var}(S_n)} > (\frac{\Theta_n}{14\log\Theta_n})^{1/2}$, then

$$e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}} \leq e^{-\frac{1}{2}(\frac{\Theta_n}{14\log\Theta_n})^{1/2}}.$$

By using the first part of Theorem 1.7, with $h=h_n$ (see previous proof) and (4.1),

$$\sqrt{\mathrm{Var}(S_n)}\mathbb{P}\{S_n=\kappa\} \leq \left(\frac{1+h_n}{1-h_n}\right)\frac{D}{\sqrt{2\pi}}e^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2(1+h_n)\,\mathrm{Var}(S_n)}}$$
$$+ \frac{C_1}{\sqrt{(1-h_n)}}\left(\frac{\mathrm{Var}(S_n)}{\Theta_n}\right)^{1/2}\left(H_n + \frac{1}{(1-h_n)\Theta_n}\right) + \sqrt{\mathrm{Var}(S_n)}\rho_n(h_n)$$
$$\leq \frac{3D}{\sqrt{2\pi}}e^{-\frac{1}{3}(\frac{\Theta_n}{14\log\Theta_n})^{1/2}}$$
$$+ C_1\sqrt{2}\left(\frac{\mathrm{Var}(S_n)}{\Theta_n}\right)^{1/2}\left(H_n + \frac{1}{(1-h_n)\Theta_n}\right) + 2\sqrt{\mathrm{Var}(S_n)}\Theta_n^{-3/2}.$$

Thus by assumption (1.13),

$$\lim_{n\to\infty}\sup_{\frac{(\kappa-\mathbb{E}S_n)^2}{\mathrm{Var}(S_n)}>(\frac{\Theta_n}{14\log\Theta_n})^{1/2}}\left|\sqrt{\mathrm{Var}(S_n)}\mathbb{P}\{S_n=\kappa\} - \frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}}}{\sqrt{2\pi}}\right| = 0.$$

Consequently,

$$\lim_{n\to\infty}\sup_{\kappa\in\mathcal{L}(v_0 n,D)}\left|\sqrt{\mathrm{Var}(S_n)}\mathbb{P}\{S_n=\kappa\}-\frac{De^{-\frac{(\kappa-\mathbb{E}S_n)^2}{2\,\mathrm{Var}(S_n)}}}{\sqrt{2\pi}}\right|=0.$$

6. Proof of Corollary 1.11

By using the generalization of Esseen's inequality given in [30], Theorem 5, p. 112, we have

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\{T_n<x\}-\mathbb{P}\{g<x\}\right|\le\frac{C_{\mathrm{E}}}{\psi(\sqrt{\mathrm{Var}(S_n')})}\sum_{j=1}^n\mathbb{E}\psi(\overline{\xi}_j). \tag{6.1}$$

And the constant C_{E} is numerical. Let $\xi_j=\mathbb{E}_L X_j=V_j+(D/2)\varepsilon_j$, $\overline{\xi}_j=\xi_j-\mathbb{E}_{(V,\varepsilon)}\xi_j$. By assumption $\psi(x)$ is convex and $\frac{x^3}{\psi(x)}$ is non-decreasing on $\mathbb{R}^+$. Thus $\psi(ax)\ge a^3\psi(x)$ for $0\le a\le 1$, $x\ge 0$. By Young's inequality,

$$\mathbb{E}\psi(2\xi_j)=\mathbb{E}_{(V,\varepsilon)}\psi(2\mathbb{E}_L X_j)\le\mathbb{E}\psi(2X_j).$$

Thus

$$\begin{aligned}\mathbb{E}\psi(\overline{\xi}_j)&\le\frac12\big(\mathbb{E}\psi(2\xi_j)+\mathbb{E}\psi(2\mathbb{E}_{(V,\varepsilon)}\xi_j)\big)\le\frac12\big(\mathbb{E}\psi(2X_j)+\mathbb{E}\psi(2X_j)\big)\\&\le\frac12\big(8\mathbb{E}\psi(X_j)+8\mathbb{E}\psi(X_j)\big)=8\mathbb{E}\psi(X_j).\end{aligned}$$

By reporting into (6.1), we get

$$H_n\le 2^{3/2}C_{\mathrm{E}}L_n \tag{6.2}$$

recalling that

$$L_n=\frac{\sum_{j=1}^n\mathbb{E}\psi(X_j)}{\psi(\sqrt{\mathrm{Var}(S_n)})}.$$

The conclusion then follows directly from Corollary 1.8.

7. Gamkrelidze's local limit theorem

We indicate in this section how to obtain a strong form of the local limit theorem (as in the case of Gamkrelidze's theorem [15]) and provide an effective bound. To this extent we first restate Lemma 1 of [10] for the particular case of a sequence $\tilde{X}=\{X_n,n\geqslant 1\}$ such that the X_n are independent, integer-valued random variables. We prove it in greater detail than in the original paper. Let (a_n) and (b_n) be two sequences of real numbers, and assume that $b_n > 0$ for every n. As usual we denote $S_n = \sum_{k=1}^n X_k$. Put

$$\rho_n := \sup_{p,q:p<q}\left|\mathbb{P}\{p \le S_n \le q\} - \frac{1}{\sqrt{2\pi}}\int_{\frac{p-1-a_n}{\sqrt{b_n}}}^{\frac{q-a_n}{\sqrt{b_n}}} e^{\frac{-t^2}{2}}\,\mathrm{d}t\right|.$$

First, observe that

$$\begin{aligned}&\mathbb{P}\{p \le S_n \le q\} - \frac{1}{\sqrt{2\pi}}\int_{\frac{p-1-a_n}{\sqrt{b_n}}}^{\frac{q-a_n}{\sqrt{b_n}}} e^{\frac{-t^2}{2}}\,\mathrm{d}t\\ &\quad= \sum_{h=p}^{q}\left\{\mathbb{P}(S_n = h) - \frac{1}{\sqrt{2\pi}}\int_{\frac{h-1-a_n}{\sqrt{b_n}}}^{\frac{h-a_n}{\sqrt{b_n}}} e^{\frac{-t^2}{2}}\,\mathrm{d}t\right\} = \sum_{h=p}^{q} d_{h,n},\end{aligned}$$

where we set

$$d_{h,n} := \mathbb{P}\{S_n = h\} - \frac{1}{\sqrt{2\pi}}\int_{\frac{h-1-a_n}{\sqrt{b_n}}}^{\frac{h-a_n}{\sqrt{b_n}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t.$$

Proposition 7.1. *Suppose that*

$$\sup_{n\in\mathbb{N}} b_n\Big(\sup_{k\in\mathbb{Z}}\big|\mathbb{P}\{S_n = k+1\} - \mathbb{P}\{S_n = k\}\big|\Big) = M < \infty. \tag{7.1}$$

Then there exists a constant C depending on M only such that

$$\sup_{k\in\mathbb{Z}}\sqrt{b_n}|d_{k,n}| \le C\sqrt{\rho_n}. \tag{7.2}$$

Further,

$$\sup_{k\in\mathbb{Z}}\left|\sqrt{b_n}\mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(k-a_n)^2}{2b_n}}\right| \le C\sqrt{\rho_n} + \frac{1}{\sqrt{2\pi e}\sqrt{b_n}}. \tag{7.3}$$

The value of C will be explicit in the course of the proof.

Proof of Proposition 7.1. Put

$$\ell_{k,n} := \frac{1}{\sqrt{2\pi}}\int_{\frac{k-1-a_n}{\sqrt{b_n}}}^{\frac{k-a_n}{\sqrt{b_n}}} e^{\frac{-t^2}{2}}\,\mathrm{d}t,$$

and observe that

$$|\ell_{k+1,n} - \ell_{k,n}| = \frac{1}{\sqrt{2\pi}\sqrt{b_n}}\left|e^{-\frac{\xi_k^2}{2}} - e^{-\frac{\eta_k^2}{2}}\right|,$$

with $\frac{k-1-a_n}{\sqrt{b_n}} \leq \xi_k \leq \frac{k-a_n}{\sqrt{b_n}} \leq \eta_k \leq \frac{k+1-a_n}{\sqrt{b_n}}$. By Lagrange's theorem

$$\left|e^{-\frac{\xi_k^2}{2}} - e^{-\frac{\eta_k^2}{2}}\right| = |\xi_k - \eta_k| \cdot \left|\theta_k e^{-\frac{\theta_k^2}{2}}\right| \leq \frac{2}{\sqrt{eb_n}},$$

with $\xi_k \leq \theta_k \leq \eta_k$ and $\sup_{z\in\mathbb{R}} |ze^{-\frac{z^2}{2}}| = e^{-1/2}$. Hence,

$$|\ell_{k+1,n} - \ell_{k,n}| \leq \left(\sqrt{\frac{2}{e\pi}}\right)\frac{1}{b_n}. \tag{7.4}$$

Now we write

$$\begin{aligned} d_{k,n} &= \mathbb{P}\{S_n = k\} - \ell_{k,n} \\ &= \big\{\mathbb{P}\{S_n = k\} - \mathbb{P}\{S_n = k+1\}\big\} + \{\ell_{k+1,n} - \ell_{k,n}\} + d_{k+1,n} \\ &\leq \sup_{k\in\mathbb{Z}}\big|\mathbb{P}\{S_n = k+1\} - \mathbb{P}\{S_n = k\}\big| + \sup_{k\in\mathbb{Z}} |\ell_{k+1,n} - \ell_{k,n}| + d_{k+1,n} \leq \frac{R}{b_n} + d_{k+1,n}, \end{aligned}$$

where we set

$$R := M + \sqrt{\frac{2}{e\pi}}.$$

Similarly we also have

$$d_{k,n} \leq \frac{R}{b_n} + d_{k-1,n}. \tag{7.5}$$

Proceeding by induction, we find that for $h < k$,

$$d_{h,n} \leq \frac{R(k-h)}{b_n} + d_{k,n}, \qquad d_{k,n} \leq \frac{R(k-h)}{b_n} + d_{h,n}.$$

By combining we get

$$|d_{h,n} - d_{k,n}| \leq \frac{R|k-h|}{b_n} \qquad \forall h, k. \tag{7.6}$$

We now show that for all $\delta > 0$, all n and all k,

$$4R\rho_n < \delta^2 \quad \Longrightarrow \quad \sqrt{b_n}|d_{k,n}| < \delta,$$

which will imply (7.2) with $C = 2\sqrt{R}$.

Assume that the contrary is true, namely that there exist a real $\delta > 0$ and integers k_0, $n_0 > 0$ such that

$$4R\rho_{n_0} < \delta^2 \quad \text{and} \quad \sqrt{b_{n_0}} \cdot |d_{k_0,n_0}| \geq \delta.$$

Suppose that $\sqrt{b_{n_0}} \cdot d_{k_0,n_0} \geq \delta$, and consider the set of integers

$$A = \left\{ h \in \mathbb{Z} : \frac{R|k_0 - h|}{b_{n_0}} \leq \frac{\delta}{2\sqrt{b_{n_0}}} \right\} = \left\{ h \in \mathbb{Z} : k_0 - \frac{\delta\sqrt{b_{n_0}}}{2R} \leq h \leq k_0 + \frac{\delta\sqrt{b_{n_0}}}{2R} \right\}.$$

From

$$\#\big([r-\alpha, r+\alpha] \cap \mathbb{Z}\big) = 2\alpha + 1 - 2\{\alpha\} \geq \alpha \qquad \big(\alpha \in \mathbb{R}^+ \text{ and } r \in \mathbb{Z}\big)$$

we get

$$\#(A) \geq \frac{\delta\sqrt{b_{n_0}}}{2R}, \tag{7.7}$$

and by (7.6), for every $h \in A$

$$\frac{\delta}{\sqrt{b_{n_0}}} \leq d_{k_0,n_0} \leq |d_{k_0,n_0} - d_{h,n_0}| + d_{h,n_0} \leq \frac{R|k_0 - h|}{b_{n_0}} + d_{h,n_0} \leq \frac{\delta}{2\sqrt{b_{n_0}}} + d_{h,n_0},$$

which implies

$$d_{h,n_0} \geq \frac{\delta}{2\sqrt{b_{n_0}}}. \tag{7.8}$$

This along with (7.7) and (7.8) implies,

$$\begin{aligned} 4R\rho_{n_0} &= 4R \cdot \sup_{p,q:p<q} \left| \sum_{h=p}^{q} d_{h,n_0} \right| \geq 4R \left| \sum_{h=p_0}^{q_0} d_{h,n_0} \right| = 4R \left(\sum_{h=p_0}^{q_0} d_{h,n_0} \right) \\ &= 4R \left(\sum_{h\in A} d_{h,n_0} \right) \geq 4R \frac{\delta}{2\sqrt{b_{n_0}}} \#(A) \geq 4R \frac{\delta}{2\sqrt{b_{n_0}}} \frac{\delta\sqrt{b_{n_0}}}{2R} = \delta^2, \end{aligned}$$

thus providing a contradiction. This proves (7.2). In order to get (7.3), we note that for some suitable $\xi_k \in (k-1, k)$,

$$\begin{aligned} &\left| \sqrt{b_n}\mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(k-a_n)^2}{2b_n}} \right| \\ &\leq \sqrt{b_n}|d_{k,n}| + \left| \frac{\sqrt{b_n}}{\sqrt{2\pi}} \int_{\frac{k-1-a_n}{\sqrt{b_n}}}^{\frac{k-a_n}{\sqrt{b_n}}} e^{\frac{-t^2}{2}}\, \mathrm{d}t - \frac{1}{\sqrt{2\pi}} e^{-\frac{(k-a_n)^2}{2b_n}} \right| \\ &= \sqrt{b_n}|d_{k,n}| + \frac{1}{\sqrt{2\pi}} \left| e^{-\frac{(\xi_k-a_n)^2}{2b_n}} - e^{-\frac{(k-a_n)^2}{2b_n}} \right| \leq \sqrt{b_n}|d_{k,n}| + \frac{1}{\sqrt{2\pi}} \cdot \frac{|\xi_k - k|}{\sqrt{b_n}} \sup_{z\in\mathbb{R}} \left| z e^{-\frac{z^2}{2}} \right| \\ &\leq \sqrt{b_n}|d_{k,n}| + \frac{1}{\sqrt{2\pi e}\sqrt{b_n}}. \end{aligned}$$

□

Now we estimate M in (7.1) by using (3.5) which we recall,

$$\left|\mathbb{P}\{S_n = k\} - \mathbb{E}_{(V,\epsilon)}\left\{\mathbf{1}_{A_n} \cdot \frac{2}{\sqrt{2\pi B_n}} e^{-\frac{(k-W_n-D\frac{B_n}{2})^2}{D^2\frac{B_n}{2}}}\right\}\right| \leq 2e^{-\frac{h^2\Theta_n}{2(1+h/3))}} + \frac{C}{(1-h)^{\frac{3}{2}}\Theta_n^{3/2}}.$$

Thus

$$\begin{aligned}
&\left|\mathbb{P}\{S_n = k\} - \mathbb{P}\{S_n = k+1\}\right| \\
&\leq 2e^{-\frac{h^2\Theta_n}{2(1+h/3))}} + \frac{2C}{(1-h)^{\frac{3}{2}}\Theta_n^{3/2}} \\
&\quad + \left|\mathbb{E}_{(V,\epsilon)}\left\{\mathbf{1}_{A_n} \cdot \frac{2}{\sqrt{2\pi B_n}}\left(e^{-\frac{(k+1-W_n-D\frac{B_n}{2})^2}{D^2\frac{B_n}{2}}} - e^{-\frac{(k-W_n-D\frac{B_n}{2})^2}{D^2\frac{B_n}{2}}}\right)\right\}\right|.
\end{aligned}$$

As on A_n we have $(1-h)\Theta_n \leq B_n \leq (1+h)\Theta_n$, it follows that

$$\begin{aligned}
&\left|\mathbb{E}_{(V,\epsilon)}\left\{\mathbf{1}_{A_n} \cdot \frac{2}{\sqrt{2\pi B_n}}\left(e^{-\frac{(k+1-W_n-D\frac{B_n}{2})^2}{D^2\frac{B_n}{2}}} - e^{-\frac{(k-W_n-D\frac{B_n}{2})^2}{D^2\frac{B_n}{2}}}\right)\right\}\right| \\
&\quad \leq \left|\mathbb{E}_{(V,\epsilon)}\left\{\mathbf{1}_{A_n} \cdot \frac{2}{\sqrt{2\pi B_n}} \cdot \frac{\sqrt{2}}{D\sqrt{eB_n}}\right\}\right| \\
&\quad \leq \frac{2}{\sqrt{\pi e}(1-h)\Theta_n}.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&b_n \sup_{k\in\mathbb{Z}}\left|\mathbb{P}\{S_n = k\} - \mathbb{P}\{S_n = k+1\}\right| \\
&\quad \leq 2b_n e^{-\frac{h^2\Theta_n}{2(1+h/3))}} + \frac{2Cb_n}{(1-h)^{\frac{3}{2}}\Theta_n^{3/2}} + \frac{2b_n}{\sqrt{\pi e}(1-h)\Theta_n},
\end{aligned} \tag{7.9}$$

which is bounded if

$$\limsup_{n\in\mathbb{N}} \frac{b_n}{\Theta_n} < \infty. \tag{7.10}$$

In particular, in the case $b_n = \mathrm{Var}(S_n)$, assumption (7.10) is exactly assumption (iii) in Remark 1.10. Note that (7.10) is satisfied if

$$\liminf_j \vartheta_{X_j} > 0, \qquad \limsup_j \mathrm{Var}(X_j) < \infty.$$

Remark 7.2. Assume that we have an effective bound for ρ_n, as it happens with the Berry–Esseen theorem. In such a situation from (7.3), we automatically get an effective bound for

$$\sup_{k\in\mathbb{Z}}\left|\sqrt{b_n}\mathbb{P}\{S_n=k\}-\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(k-a_n)^2}{2b_n}}\right|.$$

We shall now prove the following strong local limit theorem.

Theorem 7.3. *Assume that*

$$\mathcal{L}:=\limsup_{n\to\infty}\frac{b_n}{\Theta_n}<\infty;\qquad \ell:=\liminf_{n\to\infty}\vartheta_{X_n}>0. \tag{7.11}$$

Let $\{Y_n, n\geqslant 1\}$ be a sequence obtained from the sequence $\{X_n, n\geqslant 1\}$ by changing (or discarding) a finite number of terms and denote

$$\widetilde{\Theta}_n=\sum_{k=1}^n\vartheta_{Y_k};\qquad \widetilde{S}_n=\sum_{k=1}^n Y_n,\qquad \widetilde{\rho}_n:=\sup_{p,q:p<q}\left|\mathbb{P}\{p\le\widetilde{S}_n\le q\}-\frac{1}{\sqrt{2\pi}}\int_{\frac{p-1-a_n}{\sqrt{b_n}}}^{\frac{q-a_n}{\sqrt{b_n}}}e^{\frac{-t^2}{2}}\,\mathrm{d}t\right|.$$

Then

$$\sup_{k\in\mathbb{Z}}\left|\sqrt{b_n}\mathbb{P}\{\widetilde{S}_n=k\}-\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(k-a_n)^2}{2b_n}}\right|\le\widetilde{C}\sqrt{\widetilde{\rho}_n}+\frac{1}{\sqrt{2\pi e}\sqrt{b_n}}, \tag{7.12}$$

where

$$\widetilde{C}=2\sqrt{\widetilde{M}+2/e\pi}$$

and

$$\widetilde{M}\le 2b_ne^{-\frac{h^2\widetilde{\Theta}_n}{2(1+h/3)}}+\frac{2Cb_n}{(1-h)^{\frac{3}{2}}\widetilde{\Theta}_n^{3/2}}+\frac{2b_n}{\sqrt{\pi e}(1-h)\widetilde{\Theta}_n}<\infty. \tag{7.13}$$

Proof. In view of the proof of Proposition 7.1 and relation (7.9), it is sufficient to check the boundedness of

$$2b_ne^{-\frac{h^2\widetilde{\Theta}_n}{2(1+h/3)}}+\frac{2Cb_n}{(1-h)^{\frac{3}{2}}\widetilde{\Theta}_n^{3/2}}+\frac{2b_n}{\sqrt{\pi e}(1-h)\widetilde{\Theta}_n}.$$

See (7.13). This easily follows from the fact that

$$\widetilde{\mathcal{L}}:=\limsup_{n\to\infty}\frac{b_n}{\widetilde{\Theta}_n}\le\mathcal{L},$$

which holds by the following argument: let n_0 be an integer such that, for every $n>n_0$, we have

$$Y_n=X_n,\qquad \vartheta_{X_n}>\frac{\ell}{2}.$$

Then for $n > n_0$, recalling that $\vartheta_{X_k} > 0$ for every k, we can write

$$\frac{b_n}{\widetilde{\Theta}_n} = \frac{b_n}{\sum_{k=1}^{n_0} \vartheta_{Y_k} + \sum_{k=n_0+1}^{n} \vartheta_{X_k}} \le \frac{b_n}{\sum_{k=n_0+1}^{n} \vartheta_{Y_k}} = \frac{\frac{b_n}{\Theta_n}}{1 - \frac{\Theta_{n_0}}{\Theta_n}}.$$

Now as $\vartheta_{X_k} \le 1$ for every k, we obtain that

$$0 \le \frac{\Theta_{n_0}}{\Theta_n} = \frac{\sum_{k=1}^{n_0} \vartheta_{X_k}}{\sum_{k=1}^{n_0} \vartheta_{X_k} + \sum_{k=n_0+1}^{n} \vartheta_{X_k}} \le \frac{\sum_{k=1}^{n_0} \vartheta_{X_k}}{\sum_{k=n_0+1}^{n} \vartheta_{X_k}} \le \frac{n_0}{(n-n_0)(\ell/2)} \to 0,$$

as $n \to \infty$. □

8. Application to random walks in random scenery

Let $X = \{X_j, j \ge 1\}$ be a sequence of i.i.d. square integrable random variables taking values in a lattice $\mathcal{L}(v_0, D)$. Suppose we are given another sequence $U = \{U_j, j \ge 1\}$ of integer-valued random variables, independent of X. We form the sequence of composed sums

$$S = \{S_n, n \ge 1\}, \qquad \text{where } S_n = \sum_{k=1}^{n} X_{U_k}.$$

This defines a random walk in a random scenery (RWRS), described by the sequence U. The notion of RWRS goes back to the work of Kesten and Spitzer [21], see also Borodin [3] and [4]. The central limit theorem for such processes was proved by Bolthausen [2]. A large literature about the study of RWRS has been produced since [2] was written. We refer for instance, to Guillotin-Plantard and Prieur [18] or Guillotin-Plantard and Pène [6,17] and the references they contain. For a survey of recent results, we refer to den Hollander and Steif [11]. Concerning local limit theorems, we can mention Castell, Guillotin-Plantard, Pène and Schapira [7] as notable.

We establish an effective local limit theorem for the sequence S. In a first step, we prove the analog of Theorem 1.7 for the sequence S. Next, we find a reasonable condition under which Berry–Esseen's estimate is applicable, see (8.9). This is due to the surprising fact that under this condition, the intermediate conditioned sums in the Bernoulli part construction, are sums of *i.i.d.* random variables.

We remark that although the results in [7] are more general than ours, since variables lying in the domain of attraction of a stable law of index ≤ 2 are considered in [7], Theorems 1 and 2 of [7] do not however provide effective error terms, as it happens in our case.

It is also worth noticing that our approach, based on the Bernoulli part extraction, has never been used before in the framework of local limit theorems for a RWRS. Furthermore, the usual notion of local time is not used in our proofs.

8.1. Preliminary calculations

By Lemma 2.3, $\{X_j, 1 \le j \le n\} \overset{\mathcal{D}}{=} \{V_j + \varepsilon_j L_j, 1 \le j \le n\}$ where the random variables $(V_j, \varepsilon_j), L_j, j = 1, \dots, n$ are mutually independent. Also ε_j, L_j are independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_j = 1\} = 1 - \mathbb{P}\{\varepsilon_j = 0\} = \vartheta_j$ and $\mathbb{P}\{L_j = 0\} = \mathbb{P}\{L_j = 1\} = 1/2$. We again set $X_j = V_j + D\varepsilon_j L_j$ $1 \le j \le n$. The corollary below is straightforward. Put

$$W_n = \sum_{k=1}^{n} V_{U_k}, \qquad M_n = \sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k}, \qquad B_n = \sum_{k=1}^{n} \varepsilon_{U_k}.$$

Corollary 8.1. *For every $n \geqslant 1$ we have the representation*

$$\{S_k, 1 \leqslant k \leqslant n\} \overset{\mathcal{D}}{=} \{W_k + DM_k, 1 \leqslant k \leqslant n\}.$$

Remark 8.2 (Local time). We also have that $S_n = \sum_{j=1}^{\infty} X_j \nu_n(j)$, where $\nu_n(j)$ is the local time of the sequence (U_j), that is,

$$\nu_n(j) = \begin{cases} 0 & \text{if } U_k \ne j, 1 \le k \le n, \\ \#\{k; 1 \le k \le n : U_k = j\} & \text{otherwise.} \end{cases}$$

Hence, $S_n = \sum_{j=1}^{\infty} (V_j + \varepsilon_j D L_j) \nu_n(j) \overset{\mathcal{D}}{=} \sum_{k=1}^{n} V_{U_k} + D \sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k}$. However, we will not use properties of local time as is standard for proving strong laws or local limit theorem.

In this regard, our approach is new in the context of random scenery. We will still use the Bernoulli part extraction approach, in developing further the algebra inherent to this construction, which in the setting of random scenery, is revealed to be richer than expected.

In what follows, we set $V = \{V_j, j \ge 1\}$, $\varepsilon = \{\varepsilon_j, j \ge 1\}$, $L = \{L_j, j \ge 1\}$.

Lemma 8.3. *For every k, ε_{U_k} is a Bernoulli random variable such that*

$$\mathbb{P}\{\varepsilon_{U_k} = 1\} = \mathbb{E}\vartheta_{U_k}.$$

Moreover for $h \ne k$ we have,

$$\mathbb{P}\{\varepsilon_{U_h} = 1, \varepsilon_{U_k} = 1\} = \mathbb{E}\vartheta_{U_h}\vartheta_{U_k} + \sum_{r=1}^{\infty} \big(\vartheta_r - \vartheta_r^2\big) \mathbb{P}\{U_h = r, U_k = r\}.$$

Proof. Using independence of U and ε, we can write

$$\begin{aligned} \mathbb{P}\{\varepsilon_{U_k} = 1\} &= \sum_{r=1}^{\infty} \mathbb{P}\{\varepsilon_{U_k} = 1, U_k = r\} = \sum_{r=1}^{\infty} \mathbb{P}\{\varepsilon_r = 1, U_k = r\} \\ &= \sum_{r=1}^{\infty} \mathbb{P}\{\varepsilon_r = 1\} \mathbb{P}\{U_k = r\} = \sum_{r=1}^{\infty} \vartheta_r \mathbb{P}\{U_k = r\} = \mathbb{E}\vartheta_{U_k}. \end{aligned}$$

Similarly, using also the independence of the variables $\{\varepsilon_j, j \geq 1\}$,

$$\begin{aligned}
&\mathbb{P}\{\varepsilon_{U_h} = 1, \varepsilon_{U_k} = 1\} \\
&\quad = \sum_{r,s=1}^{\infty} \mathbb{P}\{\varepsilon_{U_h} = 1, \varepsilon_{U_k} = 1, U_h = r, U_k = s\} = \sum_{r,s=1}^{\infty} \mathbb{P}\{\varepsilon_r = 1, \varepsilon_s = 1, U_h = r, U_k = s\} \\
&\quad = \sum_{r=1}^{\infty} \mathbb{P}\{\varepsilon_r = 1\}\mathbb{P}\{U_h = r, U_k = r\} + \sum_{\substack{r,s=1 \\ r\neq s}}^{\infty} \mathbb{P}\{\varepsilon_r = 1\}\mathbb{P}\{\varepsilon_s = 1\}\mathbb{P}\{U_h = r, U_k = s\} \\
&\quad = \sum_{r} \vartheta_r \mathbb{P}\{U_h = r, U_k = r\} + \sum_{\substack{r,s=1 \\ r\neq s}}^{\infty} \vartheta_r\vartheta_s \mathbb{P}\{U_h = r, U_k = s\} \\
&\quad = \sum_{r=1}^{\infty} \big(\vartheta_r - \vartheta_r^2\big)\mathbb{P}\{U_h = r, U_k = r\} + \sum_{r,s=1}^{\infty} \vartheta_r\vartheta_s \mathbb{P}\{U_h = r, U_k = s\} \\
&\quad = \sum_{r=1}^{\infty} \big(\vartheta_r - \vartheta_r^2\big)\mathbb{P}\{U_h = r, U_k = r\} + \mathbb{E}\vartheta_{U_h}\vartheta_{U_k}.
\end{aligned}$$

□

Let

$$S_n' = W_n + D\frac{B_n}{2}, \qquad n = 1, 2, \ldots.$$

The two following lemmas generalize Lemma 3.1.

Lemma 8.4. *We have*

$$\mathbb{E}S_n = \mathbb{E}S_n'.$$

Proof. Just observe that

$$\begin{aligned}
\mathbb{E}M_n &= \sum_{k=1}^{n} \mathbb{E}\varepsilon_{U_k} L_{U_k} = \sum_{k=1}^{n}\sum_{r=1}^{\infty} \mathbb{E}\varepsilon_r L_r \mathbf{1}_{\{U_k = r\}} \\
&= \sum_{k=1}^{n}\sum_{r=1}^{\infty} \mathbb{E}\varepsilon_r \mathbb{E}L_r \mathbb{P}\{U_k = r\} = \sum_{k=1}^{n}\sum_{r=1}^{\infty} \frac{\vartheta_r}{2}\mathbb{P}\{U_k = r\} = \frac{1}{2}\sum_{k=1}^{n} \mathbb{E}\vartheta_{U_k} = \mathbb{E}\frac{B_n}{2}.
\end{aligned}$$

□

Lemma 8.5. *Let* $\Theta_n = \sum_{j=1}^{n} \mathbb{E}\vartheta_{U_j}$. *We have*

$$\mathbb{E}{S_n}^2 = \mathbb{E}\big(S_n'\big)^2 + \frac{D^2\Theta_n}{4} + \frac{D^2}{4}\sum_{\substack{1\leqslant h,k\leqslant n \\ h\neq k}} c_{h,k},$$

where

$$c_{h,k} = \sum_{r=1}^{\infty} \left(\frac{3\vartheta_r^2}{4} - \frac{\vartheta_r}{2} \right) \mathbb{P}\{U_h = r, U_k = r\}.$$

Proof. First, note that

$$\begin{aligned} \mathbb{E}{S_n}^2 &= \mathbb{E}\left(W_n + D \sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k} \right)^2 \\ &= \mathbb{E}W_n^2 + 2D\mathbb{E}\left[W_n \left(\sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k} \right) \right] + D^2 \mathbb{E}\left(\sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k} \right)^2. \end{aligned} \tag{8.1}$$

Now

$$\begin{aligned} \mathbb{E}W_n \left(\sum_{k=1}^{n} \varepsilon_{U_k} L_{U_k} \right) &= \sum_{k=1}^{n} \mathbb{E}W_n \varepsilon_{U_k} L_{U_k} = \sum_{k=1}^{n} \mathbb{E}\left\{ \left(\sum_{h=1}^{n} V_{U_h} \right) \varepsilon_{U_k} L_{U_k} \right\} \\ &= \sum_{h,k=1}^{n} \mathbb{E}[V_{U_h} \varepsilon_{U_k} L_{U_k}] \\ &= \sum_{k=1}^{n} \mathbb{E}(V_{U_k} \varepsilon_{U_k} L_{U_k}) + \sum_{h \neq k=1}^{n} \mathbb{E}(V_{U_h} \varepsilon_{U_k} L_{U_k}). \end{aligned} \tag{8.2}$$

Using the fact that U and (V, ε, L) are independent, further L and (V, ε) are independent, we get

$$\begin{aligned} \mathbb{E}(V_{U_k} \varepsilon_{U_k} L_{U_k}) &= \sum_{r=1}^{\infty} \mathbb{E}(V_{U_k} \varepsilon_{U_k} L_{U_k} \mathbf{1}_{\{U_k = r\}}) = \sum_{r=1}^{\infty} \mathbb{E}(V_r \varepsilon_r L_r) \mathbb{P}\{U_k = r\} \\ &= \sum_{r=1}^{\infty} \mathbb{E}(V_r \varepsilon_r) \mathbb{E}(L_r) \mathbb{P}\{U_k = r\} = \frac{1}{2} \sum_{r=1}^{\infty} \mathbb{E}(V_r \varepsilon_r) \mathbb{P}\{U_k = r\} \\ &= \frac{1}{2} \sum_{r=1}^{\infty} \mathbb{E}(V_r \varepsilon_r \mathbf{1}_{\{U_k = r\}}) = \frac{1}{2} \mathbb{E}(V_{U_k} \varepsilon_{U_k}). \end{aligned}$$

Similarly,

$$\mathbb{E}(V_{U_h} \varepsilon_{U_k} L_{U_k}) = \frac{1}{2} \mathbb{E}(V_{U_h} \varepsilon_{U_k}), \qquad h \neq k.$$

This along with (8.2) implies,

$$
\begin{aligned}
&\mathbb{E}\left(W_n\left(\sum_{k=1}^{n}\varepsilon_{U_k}L_{U_k}\right)\right)\\
&\quad=\frac{1}{2}\left(\sum_{k=1}^{n}\mathbb{E}(V_{U_k}\varepsilon_{U_k})+\sum_{1\leqslant h\neq k\leqslant n}\mathbb{E}(V_{U_h}\varepsilon_{U_k})\right)\\
&\quad=\frac{1}{2}\left(\sum_{k=1}^{n}\mathbb{E}(V_{U_k}\varepsilon_{U_k})+\sum_{k=1}^{n}\sum_{h\neq k}\mathbb{E}(V_{U_h}\varepsilon_{U_k})\right)=\frac{1}{2}\mathbb{E}\left(\sum_{k=1}^{n}V_{U_k}\varepsilon_{U_k}+\sum_{h\neq k}V_{U_h}\varepsilon_{U_k}\right)\\
&\quad=\frac{1}{2}\mathbb{E}\left(\sum_{k=1}^{n}\varepsilon_{U_k}\left(V_{U_k}+\sum_{h\neq k}V_{U_h}\right)\right)=\frac{1}{2}\mathbb{E}\left(\left(\sum_{k=1}^{n}\varepsilon_{U_k}\right)W_n\right)=\mathbb{E}\left(\frac{B_n}{2}W_n\right).
\end{aligned}
\tag{8.3}
$$

Further,

$$
\begin{aligned}
\mathbb{E}\left(\sum_{k=1}^{n}\varepsilon_{U_k}L_{U_k}\right)^2&=\sum_{k=1}^{n}\mathbb{E}\big(\varepsilon_{U_k}^2L_{U_k}^2\big)+\sum_{1\leqslant h\neq k\leqslant n}\mathbb{E}(\varepsilon_{U_h}L_{U_h}\varepsilon_{U_k}L_{U_k})\\
&=\sum_{k=1}^{n}\mathbb{E}(\varepsilon_{U_k}L_{U_k})+\sum_{1\leqslant h\neq k\leqslant n}\mathbb{E}(\varepsilon_{U_h}\varepsilon_{U_k}L_{U_h}L_{U_k}).
\end{aligned}
\tag{8.4}
$$

Also

$$
\begin{aligned}
\mathbb{E}(\varepsilon_{U_k}L_{U_k})&=\sum_{r=1}^{\infty}\mathbb{E}(\varepsilon_{U_k}L_{U_k}\mathbf{1}_{\{U_k=r\}})=\sum_{r=1}^{\infty}\mathbb{E}(\varepsilon_rL_r\mathbf{1}_{\{U_k=r\}})\\
&=\sum_{r=1}^{\infty}\mathbb{E}(\varepsilon_r)\mathbb{E}(L_r)\mathbb{P}\{U_k=r\}=\frac{1}{2}\sum_{r=1}^{\infty}\mathbb{E}(\vartheta_r)\mathbb{P}\{U_k=r\}=\frac{1}{2}\mathbb{E}\vartheta_{U_k}.
\end{aligned}
\tag{8.5}
$$

Now,

$$
\begin{aligned}
&\mathbb{E}\varepsilon_{U_h}\varepsilon_{U_k}L_{U_h}L_{U_k}\\
&\quad=\sum_{r,s=1}^{\infty}\mathbb{E}(\varepsilon_{U_h}\varepsilon_{U_k}L_{U_h}L_{U_k}\mathbf{1}_{\{U_h=r,U_k=s\}})=\sum_{r,s=1}^{\infty}\mathbb{E}(\varepsilon_r\varepsilon_sL_rL_s\mathbf{1}_{\{U_h=r,U_k=s\}})\\
&\quad=\sum_{r=1}^{\infty}\mathbb{E}\varepsilon_r\mathbb{E}L_r\mathbb{P}\{U_h=r,U_k=r\}+\sum_{\substack{r,s=1\\ r\neq s}}^{\infty}\mathbb{E}\varepsilon_r\mathbb{E}\varepsilon_s\mathbb{E}L_r\mathbb{E}L_s\mathbb{P}\{U_h=r,U_k=s\}
\end{aligned}
\tag{8.6}
$$

$$= \frac{1}{2}\sum_{r=1}^{\infty}\vartheta_r \mathbb{P}\{U_h = r, U_k = r\} + \frac{1}{4}\sum_{\substack{r,s=1\\ r\neq s}}^{\infty}\vartheta_r\vartheta_s\mathbb{P}\{U_h = r, U_k = s\}$$

$$= \sum_{r=1}^{\infty}\left(\frac{\vartheta_r}{2} - \frac{\vartheta_r^2}{4}\right)\mathbb{P}\{U_h = r, U_k = r\} + \frac{1}{4}\sum_{r,s=1}^{\infty}\vartheta_r\vartheta_s\mathbb{P}\{U_h = r, U_k = s\}$$

$$= \frac{1}{4}\mathbb{E}\vartheta_{U_h}\vartheta_{U_k} + \sum_{r=1}^{\infty}\left(\frac{\vartheta_r}{2} - \frac{\vartheta_r^2}{4}\right)\mathbb{P}\{U_h = r, U_k = r\} = \frac{1}{4}\mathbb{E}\vartheta_{U_h}\vartheta_{U_k} + a_{h,k},$$

where we set

$$a_{h,k} = \sum_{r=1}^{\infty}\left(\frac{\vartheta_r}{2} - \frac{\vartheta_r^2}{4}\right)\mathbb{P}\{U_h = r, U_k = r\}.$$

Thus by inserting (8.5) and (8.6) into (8.4) we get,

$$\mathbb{E}\left(\sum_{k=1}^{n}\varepsilon_{U_k}L_{U_k}\right)^2 = \frac{1}{2}\sum_{k=1}^{n}\mathbb{E}\vartheta_{U_k} + \sum_{1\leqslant h\neq k\leqslant n}\left(\frac{1}{4}\mathbb{E}\vartheta_{U_h}\vartheta_{U_k} + a_{h,k}\right)$$

$$= \frac{\Theta_n}{2} + \frac{1}{4}\sum_{1\leqslant h,k\leqslant n}\mathbb{E}\vartheta_{U_h}\vartheta_{U_k} - \frac{1}{4}\sum_{k=1}^{n}\mathbb{E}\vartheta_{U_k}^2 + \sum_{1\leqslant h\neq k\leqslant n}a_{h,k} \tag{8.7}$$

$$= \frac{\Theta_n}{2} + \frac{1}{4}\mathbb{E}\left\{\left(\sum_{k=1}^{n}\vartheta_{U_k}\right)^2 - \sum_{k=1}^{n}\vartheta_{U_k}^2\right\} + \sum_{1\leqslant h\neq k\leqslant n}a_{h,k}.$$

Inserting next (8.3), (8.7) in (8.1) we obtain

$$\mathbb{E}{S_n}^2 = \mathbb{E}W_n^2 + 2D\mathbb{E}\left(W_n\sum_{k=1}^{n}\varepsilon_{U_k}L_{U_k}\right) + D^2\mathbb{E}\left(\sum_{k=1}^{n}\varepsilon_{U_k}L_{U_k}\right)^2$$

$$= \mathbb{E}W_n^2 + 2D\mathbb{E}\left(\frac{B_n}{2}W_n\right) + \frac{D^2}{2}\Theta_n + \frac{D^2}{4}\mathbb{E}\left\{\left(\sum_{k=1}^{n}\vartheta_{U_k}\right)^2 - \sum_{k=1}^{n}\vartheta_{U_k}^2\right\}$$

$$+ \frac{D^2}{4}\sum_{1\leq h\neq k\leqslant n}a_{h,k}.$$

Besides,

$$\mathbb{E}W_n^2 + 2D\mathbb{E}\left(\frac{B_n}{2}W_n\right) = \mathbb{E}\left(W_n + D\frac{B_n}{2}\right)^2 - \frac{D^2}{4}\mathbb{E}B_n^2$$

$$= \mathbb{E}\big(S_n'\big)^2 - \frac{D^2}{4}\mathbb{E}B_n^2.$$

Hence,

$$\mathbb{E}{S_n}^2 = \mathbb{E}\big(S'_n\big)^2 - \frac{D^2}{4}\mathbb{E}B_n^2 + \frac{D^2}{2}\Theta_n + \frac{D^2}{4}\mathbb{E}\left\{\left(\sum_{k=1}^n \vartheta_{U_k}\right)^2 - \sum_{k=1}^n \vartheta_{U_k}^2\right\} + \frac{D^2}{4}\sum_{1\leqslant h\neq k\leqslant n} a_{h,k}. \tag{8.8}$$

Now, in a similar way to what we used in getting (8.7), we observe that

$$\begin{aligned}\mathbb{E}B_n^2 &= \mathbb{E}\left(\sum_{k=1}^n \varepsilon_{U_k}\right)^2 \\ &= \Theta_n + \mathbb{E}\left\{\left(\sum_{k=1}^n \vartheta_{U_k}\right)^2 - \sum_{k=1}^n \vartheta_{U_k}^2\right\} + \sum_{1\leqslant h\neq k\leqslant n} b_{h,k},\end{aligned}$$

where $b_{h,k} = \sum_{r=1}^\infty (\vartheta_r - \vartheta_r^2)\mathbb{P}\{U_h = r, U_k = r\}$. By inserting it in (8.8), we obtain

$$\begin{aligned}\mathbb{E}{S_n}^2 &= \mathbb{E}\big(S'_n\big)^2 - \frac{D^2}{4}\left\{\Theta_n + \mathbb{E}\left\{\left(\sum_{k=1}^n \vartheta_{U_k}\right)^2 - \sum_{k=1}^n \vartheta_{U_k}^2\right\} + \sum_{1\leqslant h\neq k\leqslant n} b_{h,k}\right\} \\ &\quad + \frac{D^2}{2}\Theta_n + \frac{D^2}{4}\mathbb{E}\left\{\left(\sum_{k=1}^n \vartheta_{U_k}\right)^2 - \sum_{k=1}^n \vartheta_{U_k}^2\right\} + \frac{D^2}{4}\sum_{1\leqslant h\neq k\leqslant n} a_{h,k} \\ &= \mathbb{E}\big(S'_n\big)^2 + \frac{D^2\Theta_n}{4} + \frac{D^2}{4}\sum_{1\leqslant h\neq k\leqslant n}(a_{h,k} - b_{h,k}) \\ &= \mathbb{E}\big(S'_n\big)^2 + \frac{D^2\Theta_n}{4} + \frac{D^2}{4}\sum_{1\leqslant h\neq k\leqslant n} c_{h,k},\end{aligned}$$

where

$$c_{h,k} = a_{h,k} - b_{h,k} = \sum_{r=1}^\infty\left(\frac{3\vartheta_r^2}{4} - \frac{\vartheta_r}{2}\right)\mathbb{P}\{U_h = r, U_k = r\}.$$

□

Remark 8.6. (i) Assume that the variables (U_j) verify

$$\mathbb{P}\{U_h = r, U_k = r\} = 0, \qquad \forall h \neq k \text{ and } \forall r. \tag{8.9}$$

Then from Lemma 8.5, we get

$$\mathbb{E}{S_n}^2 = \mathbb{E}\big(S'_n\big)^2 + \frac{D^2\Theta_n}{4}.$$

Condition (8.9) holds for instance in the following important case: let the U_j be the partial sums of a sequence of random variables (Y_i) taking positive integer values

$$U_j = \sum_{i=1}^{j} Y_i.$$

This is the case if $Y_i \equiv 1$ for every i, so that $U_j = j$ for every j. Hence, our present discussion is a generalization of the previous one.

(ii) Let the U_j be the partial sums of a sequence of independent random variables (Y_i). Then for $h < k$,

$$\mathbb{P}\{U_h = r, U_k = r\} = \mathbb{P}\{U_h = r\}\mathbb{P}\left\{\sum_{i=h+1}^{k} Y_i = 0\right\}.$$

Hence,

$$c_{h,k} = \rho_{h,k}\sum_{r=1}^{\infty}\left(\frac{3\vartheta_r^2}{4} - \frac{\vartheta_r}{2}\right)\mathbb{P}\{U_h = r\} = \mathbb{P}\left\{\sum_{i=h+1}^{k} Y_i = 0\right\}\mathbb{E}\left(\frac{3\vartheta_{U_h}^2}{4} - \frac{\vartheta_{U_h}}{2}\right).$$

Notice that if the random variables (Y_i) are i.i.d., then

$$\mathbb{P}\left\{\sum_{i=h+1}^{k} Y_i = 0\right\} = \mathbb{P}\left\{\sum_{i=1}^{k-h} Y_i = 0\right\} = \sigma_{k-h},$$

where $\sigma_n = \mathbb{P}\{U_n = 0\}$.

8.2. The local limit theorem with effective rate

In this section, we still use the previous notation. We set

$$H_n = \sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S'_n - \mathbb{E}S'_n}{\sqrt{\mathrm{Var}(S'_n)}} < x\right\} - \Phi(x)\right|,$$

$$\rho_n = \mathbb{P}\left\{\left|\sum_{k=1}^{n}\varepsilon_{U_k} - \Theta_n\right| > h\Theta_n\right\},$$

where Φ denotes the distribution function of the standard Gaussian law.

The following theorem now generalizes Theorem 1.7 in the case of random scenery. The proof is identical to that of Theorem 1.7, just replace ϑ_k with ϑ_{U_k} in each formula of Theorem 1.7, so we omit it.

Theorem 8.7. *For any* $0 < h < 1$, $0 < \vartheta_j \leqslant \vartheta_{X_j}$ *and all* $\kappa \in \mathcal{L}(v_0 n, D)$

$$\mathbb{P}\{S_n = \kappa\} \le \left(\frac{1+h}{1-h}\right) \frac{D}{\sqrt{2\pi \operatorname{Var}(S_n)}} e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2(1+h)\operatorname{Var}(S_n)}} + \frac{C_1}{\sqrt{(1-h)\Theta_n}} \left(H_n + \frac{1}{(1-h)\Theta_n}\right) + \rho_n(h),$$

$$\mathbb{P}\{S_n = \kappa\} \ge \left(\frac{1-h}{1+h}\right) \frac{D}{\sqrt{2\pi \operatorname{Var}(S_n)}} e^{-\frac{(\kappa - \mathbb{E}S_n)^2}{2(1-h)\operatorname{Var}(S_n)}} - \frac{C_1}{\sqrt{(1-h)\Theta_n}} \left(H_n + \frac{1}{(1-h)\Theta_n} + 2\rho_n(h)\right) - \rho_n(h).$$

8.3. A correlation property of the sequence $\{V_{U_k} + \frac{D}{2}\varepsilon_{U_k}, k \ge 1\}$

Put

$$Y_k = V_{U_k} + \frac{D}{2}\varepsilon_{U_k}.$$

We observe that

$$S_n' = W_n + \frac{D}{2}B_n = \sum_{k=1}^n \left(V_{U_k} + \frac{D}{2}\varepsilon_{U_k}\right) = \sum_{k=1}^n Y_k$$

and that the quantity H_n appearing in Theorem 8.7 is expressed in terms of the partial sums S_n'. The aim of the present section is to discuss suitable assumptions assuring the independence of the variables $\{Y_k, k \ge 1\}$, thus enabling us to give an estimate of "Berry–Esseen type" for H_n.

Throughout this section, we assume that the variables $\{U_j, j \ge 1\}$ verify condition (8.9) appeared in Remark 8.6(i). That is

$$\mathbb{P}\{U_h = r, U_k = r\} = 0, \qquad \forall h \ne k \text{ and } \forall r \ge 1.$$

Theorem 8.8. *Let the* $\{X_n, n \ge 1\}$ *be i.i.d. Assume moreover that, for every pair* (h, k) *with* $h \ne k$, *the random variables* ϑ_{U_h} *and* ϑ_{U_k} *are uncorrelated. Then the sequence* $\{Y_k, k \ge 1\}$ *is i.i.d.*

Remark 8.9. The assumption of the above theorem is valid if either

(i) $r \mapsto \vartheta_r$ is constant (for instance, $\vartheta_r = \vartheta_{X_r} = \vartheta_X$ for every r),
(ii) U_h and U_k are independent (and trivially if $U_h = h$, for every h).

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and denote

$$\Delta\phi(t) = \phi\left(t + \frac{D}{2}\right) - \phi(t).$$

The above theorem is a straightforward consequence of the following properties.

Proposition 8.10. *Let the sequence* $\{X_n, n \ge 1\}$ *be i.i.d. Then, for every pair* ϕ, ψ *of measurable functions* $\mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\big[\phi(Y_h)\psi(Y_k)\big] = \mathbb{E}\big[(\alpha_\phi + \beta_\phi \vartheta_{U_h})(\alpha_\psi + \beta_\psi \vartheta_{U_k})\big], \qquad h \ne k,$$

where

$$\alpha_\phi = \mathbb{E}\phi(X_1) = \sum_{k=1}^{\infty} f(k)\phi(v_k), \qquad \beta_\phi = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{f(k) \wedge f(k+1)}{\vartheta_X} \Delta^2 \phi(v_k),$$

$$\alpha_\psi = \mathbb{E}\psi(X_1) = \sum_{k=1}^{\infty} f(k)\psi(v_k), \qquad \beta_\psi = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{f(k) \wedge f(k+1)}{\vartheta_X} \Delta^2 \psi(v_k).$$

In particular, for every pair A *and* B *of Borel subsets of* $\mathbb{R}$,

$$\mathbb{P}\{Y_h \in A, Y_k \in B\} - \mathbb{P}\{Y_h \in A\}\mathbb{P}\{Y_k \in B\} = \operatorname{Cov}\big(\mathbf{1}_A(Y_h), \mathbf{1}_B(Y_k)\big) = \beta_A \beta_B \operatorname{Cov}(\vartheta_{U_h}, \vartheta_{U_k}),$$

where

$$\beta_A = \beta_{\mathbf{1}_A}, \qquad \beta_B = \beta_{\mathbf{1}_B}.$$

Proof. Since the X_r are identically distributed, we shall omit the symbol r in the definition of f_r. Moreover,

$$\tau_k^{(r)} = \vartheta_r \frac{f(k) \wedge f(k+1)}{\vartheta_X}.$$

See Section 1, before (2.4). First, for every r,

$$\begin{aligned}
&\mathbb{E}\phi\left(V_r + \frac{D}{2}\varepsilon_r\right) \\
&= \sum_{k=1}^{\infty} \phi\left(v_k + \frac{D}{2}\right)\mathbb{P}\{V_r = v_k, \varepsilon_r = 1\} + \sum_{k=1}^{\infty} \phi(v_k)\mathbb{P}\{V_r = v_k, \varepsilon_r = 0\} \\
&= \sum_{k=1}^{\infty} \phi\left(v_k + \frac{D}{2}\right)\tau_k^{(r)} + \sum_{k=1}^{\infty} \phi(v_k)\left(f(k) - \frac{\tau_{k-1}^{(r)} + \tau_k^{(r)}}{2}\right) \\
&= \sum_{k=1}^{\infty} \phi\left(v_k + \frac{D}{2}\right)\tau_k^{(r)} + \sum_{k=1}^{\infty} \phi(v_k) f(k) - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_k)\tau_{k-1}^{(r)} - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_k)\tau_k^{(r)} \\
&= \sum_{k=1}^{\infty} \phi\left(v_k + \frac{D}{2}\right)\tau_k^{(r)} + \sum_{k=1}^{\infty} \phi(v_k) f(k) - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_{k-1} + D)\tau_{k-1}^{(r)} - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_k)\tau_k^{(r)} \\
&= \sum_{k=1}^{\infty} \phi\left(v_k + \frac{D}{2}\right)\tau_k^{(r)} + \sum_{k=1}^{\infty} \phi(v_k) f(k) - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_k + D)\tau_k^{(r)} - \frac{1}{2}\sum_{k=1}^{\infty} \phi(v_k)\tau_k^{(r)}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \tau_k^{(r)} \left\{ \phi\left(v_k + \frac{D}{2} \right) - \frac{\phi(v_k + D) + \phi(v_k)}{2} \right\} + \sum_{k=1}^{\infty} \phi(v_k) f(k)$$

$$= \sum_{k=1}^{\infty} \phi(v_k) f(k) - \frac{1}{2} \sum_{k=1}^{\infty} \tau_k^{(r)} \Delta^2 \phi(v_k)$$

$$= \sum_{k=1}^{\infty} \phi(v_k) f(k) - \frac{\vartheta_r}{2} \sum_{k=1}^{\infty} \frac{f(k) \wedge f(k+1)}{\vartheta_X} \Delta^2 \phi(v_k) = \alpha_\phi + \beta_\phi \vartheta_r.$$

Similarly,

$$\mathbb{E}\psi\left(V_s + \frac{D}{2} \varepsilon_s \right) = \alpha_\psi + \beta_\psi \vartheta_s.$$

Hence, observing that for $r \neq s$ the random variables $V_r + \frac{D}{2}\varepsilon_r$ and $V_s + \frac{D}{2}\varepsilon_s$ are independent, we have

$$\begin{aligned}
\mathbb{E}\big[\phi(Y_h)\psi(Y_k)\big] &= \sum_{r,s=1}^{\infty} \mathbb{E}\left[\phi\left(V_r + \frac{D}{2}\varepsilon_r \right) \psi\left(V_s + \frac{D}{2}\varepsilon_s \right) \right] \mathbb{P}\{U_h = r, U_k = s\} \\
&= \sum_{\substack{r,s=1 \\ r \neq s}}^{\infty} \mathbb{E}\left[\phi\left(V_r + \frac{D}{2}\varepsilon_r \right) \psi\left(V_s + \frac{D}{2}\varepsilon_s \right) \right] \mathbb{P}\{U_h = r, U_k = s\} \\
&= \sum_{\substack{r,s=1 \\ r \neq s}}^{\infty} \mathbb{E}\left[\phi\left(V_r + \frac{D}{2}\varepsilon_r \right) \right] \mathbb{E}\left[\psi\left(V_s + \frac{D}{2}\varepsilon_s \right) \right] \mathbb{P}\{U_h = r, U_k = s\} \\
&= \sum_{\substack{r,s=1 \\ r \neq s}}^{\infty} (\alpha_\phi + \beta_\phi \vartheta_r)(\alpha_\psi + \beta_\psi \vartheta_r) \mathbb{P}\{U_h = r, U_k = s\} \\
&= \sum_{r,s=1}^{\infty} (\alpha_\phi + \beta_\phi \vartheta_r)(\alpha_\psi + \beta_\psi \vartheta_r) \mathbb{P}\{U_h = r, U_k = s\} \\
&= \mathbb{E}(\alpha_\phi + \beta_\phi \vartheta_{U_h})(\alpha_\psi + \beta_\psi \vartheta_{U_k}).
\end{aligned}$$

□

Remark 8.11. Let $A = [a, b]$ be a closed interval in $\mathbb{R}$. Let

$$p = \max\{k : v_k < a\}, \qquad q = \max\{k : v_k \leqslant b\}.$$

It is easy to see that

$$-\frac{1}{2}\Delta^2 \phi(v_p) = \begin{cases} +\frac{1}{2} & \text{if } v_p + \frac{D}{2} \in A, \\ -\frac{1}{2} & \text{if } v_p + \frac{D}{2} \notin A. \end{cases}$$

Similarly,

$$-\frac{1}{2}\Delta^2\phi(v_q) = \begin{cases} +\frac{1}{2} & \text{if } v_q + \frac{D}{2} \in A, \\ -\frac{1}{2} & \text{if } v_q + \frac{D}{2} \notin A. \end{cases}$$

It follows that

$$|\beta_A| = \left| -\frac{1}{2}\frac{f(p) \wedge f(p+1)}{\vartheta_X}\Delta^2\phi(v_p) - \frac{1}{2}\frac{f(q) \wedge f(q+1)}{\vartheta_X}\Delta^2\phi(v_q) \right| \leqslant 1,$$

since

$$\frac{f(k) \wedge f(k+1)}{\vartheta_X} \leqslant 1, \qquad \forall k.$$

As a consequence, we get

$$\left| \mathbb{P}\{Y_h \in A, Y_k \in B\} - \mathbb{P}\{Y_h \in A\}\mathbb{P}\{Y_k \in B\} \right| \leqslant \left| \mathrm{Cov}(\vartheta_{U_h}, \vartheta_{U_k}) \right|.$$

A similar argument yields the above inequality for any interval in $\mathbb{R}$ (open, or half-closed, or unbounded).

9. Concluding remarks and open problems

We conclude with discussing two important questions concerning the approach used. The first concerns moderate deviations, and the second is related to weighted sums.

9.1. Moderate deviation local limit theorems

In the i.i.d. case, the general form of the local limit theorem ([19], Th. 4.2.1) states the following theorem.

Theorem 9.1. *In order that for some choice of constants a_n and b_n*

$$\lim_{n\to\infty} \sup_{N \in \mathcal{L}(v_0 n, D)} \left| \frac{b_n}{\lambda}\mathbb{P}\{S_n = N\} - g\left(\frac{N - a_n}{b_n}\right) \right| = 0,$$

where g is the density of some stable distribution G with exponent $0 < \alpha \leq 2$, it is necessary and sufficient that

$$\text{(i)} \quad \frac{S_n - a_n}{b_n} \overset{\mathcal{D}}{\Rightarrow} G \qquad \textit{as } n \to \infty, \qquad \text{(ii)} \quad D \textit{ is maximal.}$$

This provides a useful estimate of $\mathbb{P}\{S_n = N\}$ for the values of N such that $|N|/b_n$ is bounded, as already mentioned when $\alpha = 2$ (with $b_n = \sqrt{\Sigma_n}$ using notation (1.1)). When $|N|/b_n \to \infty$, it is known, at least when $0 < \alpha < 1$, that another estimate exists. More precisely,

$$\mathbb{P}\{S_n = N\} \sim n\mathbb{P}\{X = N\} \qquad \text{as } n \to \infty,$$

uniformly in n such that $|N|/b_n \to \infty$. We refer to Doney [12] for large deviation local limit theorems. In the intermediate range of values where $|N|/b_n$ can be large but not too large with respect to n, it was known already three centuries ago that in the binomial case finer estimates are available for this range of values.

Lemma 9.2 (De Moivre–Laplace, 1730). *Let $0 < p < 1$, $q = 1 - p$. Let X be such that $\mathbb{P}\{X = 1\} = p = 1 - \mathbb{P}\{X = 0\}$. Let $X_1, X_2, \dots$ be independent copies of X and let $S_n = X_1 + \cdots + X_n$. Let $0 < \gamma < 1$ and let $\beta \le \gamma\sqrt{pq}n^{1/3}$. Then for all k such that letting $x = \frac{k-np}{\sqrt{npq}}$, $|x| \le \beta n^{1/6}$, we have*

$$\mathbb{P}\{S_n = k\} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi npq}} e^E,$$

with $|E| \le \frac{|x|^3}{\sqrt{npq}} + \frac{|x|^4}{npq} + \frac{|x|^3}{2(npq)^{\frac{3}{2}}} + \frac{1}{4n\min(p,q)(1-\gamma)}$.

See Chow and Teicher [9]. Although the uniform estimate given in Lemma 1.5 is optimal (it is derived from a fine local limit theorem with asymptotic expansion), it is for a moderate deviation like $x \sim n^{1/7}$, considerably less precise than the one due to De Moivre which is the case $p = q$.

Problem I. *Under which moment assumptions, does the De Moivre–Laplace estimate extend to sums of independent random variables?*

A partial answer can be given by means of the following result proved by Chen, Fang and Shao [8].

Theorem 9.3. *Let X_i, $1 \leqslant i \leqslant n$ be a sequence of independent random variables with $\mathbb{E}X_i = 0$. Put $S_n = \sum_{i=1}^n X_i$ and $B_n^2 = \sum_{i=1}^n \mathbb{E}X_i^2$. Assume that there exist positive constants c_1, c_2 and t_0 such that*

$$B_n^2 \geqslant c_1^2 n, \qquad \mathbb{E}e^{t_0\sqrt{|X_i|}} \leqslant c_2 \qquad \text{for } 1 \leqslant i \leqslant n.$$

Then

$$\left|\frac{\mathbb{P}\{S_n/B_n \geqslant x\}}{1 - \Phi(x)} - 1\right| \le c_3 \frac{(1 + x^3)}{\sqrt{n}},$$

for $0 \leqslant x \leqslant (c_1 t_0^2)^{1/3} n^{1/6}$, where c_3 depends on c_2 and $c_1 t_0^2$.

Assume that the random variables X_i are integer–valued and let k be an integer. From Theorem 9.3, follows that

$$\begin{aligned} &\Big|\mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{k^2}{2B_n^2}}\Big| \\ &\quad \le \frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{k^2}{2B_n^2}} \bigg(1 - e^{-\frac{2k+1}{2B_n^2}} + \frac{cc_3 B_n^2}{k\sqrt{n}}\Big(1 + \Big(\frac{k}{B_n}\Big)^3\Big)\bigg), \end{aligned} \tag{9.1}$$

for $\frac{k}{B_n} \leqslant (c_1 t_0^2)^{1/3} n^{1/6}$. Here and in what follows, c denotes some positive numerical constant.

Indeed, as

$$\frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{(k+1)^2}{2B_n^2}} \leqslant \Phi\big((k+1)/B_n\big) - \Phi(k/B_n) \leqslant \frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{k^2}{2B_n^2}}$$

by using Boyd's estimate ([27], Section 2.26) of Mills' ratio $R(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,\mathrm{d}t$

$$\frac{2}{\sqrt{x^2+4}+x} \le R(x) \le \frac{2}{\sqrt{x^2+8/\pi}+x}, \qquad \forall x \ge 0,$$

we get in the one hand,

$$\begin{aligned} \mathbb{P}\{S_n = k\} &= \mathbb{P}\Big\{\frac{S_n}{B_n} \geqslant \frac{k}{B_n}\Big\} - \mathbb{P}\Big\{\frac{S_n}{B_n} \geqslant \frac{k+1}{B_n}\Big\} \\ &\leqslant \Big(1 + c_3\frac{1+(\frac{k}{B_n})^3}{\sqrt{n}}\Big)\Big(1 - \Phi\Big(\frac{k}{B_n}\Big)\Big) - \Big(1 - c_3\frac{1+(\frac{k+1}{B_n})^3}{\sqrt{n}}\Big)\Big(1 - \Phi\Big(\frac{k+1}{B_n}\Big)\Big) \\ &\leqslant \Big(\Phi\Big(\frac{k+1}{B_n}\Big) - \Phi\Big(\frac{k}{B_n}\Big)\Big) + \frac{2c_3}{\sqrt{n}}\Big(1 + \Big(\frac{k+1}{B_n}\Big)^3\Big)\Big(1 - \Phi\Big(\frac{k}{B_n}\Big)\Big) \\ &\leqslant \frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{k^2}{2B_n^2}} + \frac{2c_3}{\sqrt{n}}\Big(1 + \Big(\frac{k+1}{B_n}\Big)^3\Big)\Big(1 - \Phi\Big(\frac{k}{B_n}\Big)\Big) \\ &\le \frac{1}{\sqrt{2\pi}\,B_n} e^{-\frac{k^2}{2B_n^2}} \Big\{1 + \frac{cc_3 B_n^2}{k\sqrt{n}}\Big(1 + \Big(\frac{k+1}{B_n}\Big)^3\Big)\Big\}. \end{aligned}$$

And in the other

$$\begin{aligned} \mathbb{P}\{S_n = k\} &= \mathbb{P}\Big\{\frac{S_n}{B_n} \geqslant \frac{k}{B_n}\Big\} - \mathbb{P}\Big\{\frac{S_n}{B_n} \geqslant \frac{k+1}{B_n}\Big\} \\ &\geqslant \Big(1 - c_3\frac{1+(\frac{k}{B_n})^3}{\sqrt{n}}\Big)\Big(1 - \Phi\Big(\frac{k}{B_n}\Big)\Big) - \Big(1 + c_3\frac{1+(\frac{k+1}{B_n})^3}{\sqrt{n}}\Big)\Big(1 - \Phi\Big(\frac{k+1}{B_n}\Big)\Big) \\ &\geqslant \Big(\Phi\Big(\frac{k+1}{B_n}\Big) - \Phi\Big(\frac{k}{B_n}\Big)\Big) - \frac{2c_3}{\sqrt{n}}\Big(1 + \Big(\frac{k}{B_n}\Big)^3\Big)\Big(1 - \Phi\Big(\frac{k+1}{B_n}\Big)\Big) \end{aligned}$$

$$\geqslant \frac{1}{\sqrt{2\pi}\, B_n} e^{-\frac{(k+1)^2}{2B_n^2}} - \frac{2c_3}{\sqrt{n}}\left(1 + \left(\frac{k}{B_n}\right)^3\right)\left(1 - \Phi\left(\frac{k+1}{B_n}\right)\right)$$

$$\geq \frac{1}{\sqrt{2\pi}\, B_n} e^{-\frac{k^2}{2B_n^2}} \cdot e^{\frac{2k+1}{2B_n^2}} \left\{1 - \frac{c c_3 B_n^2}{(k+1)\sqrt{n}}\left(1 + \left(\frac{k}{B_n}\right)^3\right)\right\}.$$

Estimate (9.1) easily follows.

However assumption $\mathbb{E}e^{t_0\sqrt{|X_i|}} \leq c_2$ is restrictive, since the constant c_2 can be quite large and c_3 in turn depends increasingly from c_2. Consider for instance, the following remarkable example.

Probabilistic model of the partition function: We refer to Freiman–Pitman [14]. Let σ be a real. Fix some positive integer n, and let $1 \leq m \leq n$. Let $X_m, \dots, X_n$ be independent random variables defined by

$$\mathbb{P}\{X_j = 0\} = \frac{1}{1 + e^{-\sigma j}}, \qquad \mathbb{P}\{X_j = j\} = \frac{e^{-\sigma j}}{1 + e^{-\sigma j}}.$$

The random variable $Y = X_m + \cdots + X_n$ can serve to model the partition function $q_m(n)$ counting the number of partitions of n into distinct parts, each of which is at least m, namely the number of ways to express n as

$$n = i_1 + \cdots + i_r, \qquad m \leq i_1 < \cdots < i_r \leq n.$$

(By Euler's penthagonal theorem, $q_0(n)$ for instance appears as a coefficient in the expansion of $\prod_{k\leq n}(1 + e^{ik\theta})$.) Notice that we have the following formula (in which σ only appears in the right-hand side)

$$q_m(n) = e^{\sigma n} \int_0^1 \prod_{j=m}^{n} \left(1 + e^{-\sigma j} e^{2i\pi\alpha j}\right) e^{-2i\pi\alpha n}\, \mathrm{d}\alpha. \tag{9.2}$$

By using characteristic functions and Fourier inversion formula, we deduce from (9.2),

$$q_m(n) = e^{\sigma n}\left(\prod_{j=m}^{n}\left(1 + e^{-\sigma j}\right)\right)\mathbb{P}\{Y = n\}. \tag{9.3}$$

Choosing σ as the unique solution of the equation $\sum_{j=m}^{n} \frac{j}{1+e^{\sigma j}} = n$ gives $\mathbb{P}\{Y = n\} = \mathbb{P}\{\overline{Y} = 0\}$ where $\overline{Y} = Y - \mathbb{E}Y$. But here we have $\mathbb{E}e^{t_0\sqrt{|X_i - \mathbb{E}X_i|}} \approx e^{t_0'\sqrt{j}}$. $c_2 \approx e^{t_0'\sqrt{n}}$, and so $c_3 \gg \sqrt{n}$. Freiman and Pitman lacked a result of this kind, and in place, directly estimated the integral in (9.2) in a painstriking work.

9.2. Weighted i.i.d. sums

The requirement on the random variables to take values in a common lattice is generally no longer satisfied when replacing X_j by $w_j X_j$, where w_j, $j = 1, \dots, n$ are real numbers. This occurs if

$X_j = w_j \beta_j$, where β_j is a Bernoulli random variable and w_j are distinct integers having greatest common divisor d. In this case, $\mathbb{P}\{X_j \in \mathcal{L}(0, w_j)\} = 1$ for each j, but one cannot select a smaller *common* span (e.g., $D = d$) since condition (1.10) would be violated. See also (2.3). This example in turn covers important classes of independent random variables used as probabilistic models in arithmetic. See [13,14,31]. However, the representation given in Lemma 2.3 extends to weighted sums. Set for $m = 1, \dots, n$,

$$S_m = \sum_{j=1}^{m} w_j X_j, \qquad W_m = \sum_{j=1}^{m} w_j V_j, \qquad M_m = \sum_{j=1}^{m} w_j \varepsilon_j L_j, \qquad B_m = \sum_{j=1}^{m} \varepsilon_j.$$

A direct consequence of (2.7) is

Lemma 9.4. *We have the representation*

$$\{S_m, 1 \le m \le n\} \overset{\mathcal{D}}{=} \{W_m + D M_m, 1 \le m \le n\}.$$

And, conditionally to the σ-algebra generated by the sequence $\{(V_j, \varepsilon_j), j = 1, \dots, n\}$, M_n is a weighted Bernoulli random walk.

Problem II. *Show an approximate form of the local limit theorem for weighted i.i.d. sums.*

Acknowledgements

The authors thank the referees for helpful remarks. We also thank Kit Nair for useful comments.

References

[1] Aizenman, M., Germinet, F., Klein, A. and Warzel, S. (2009). On Bernoulli decompositions for random variables, concentration bounds, and spectral localization. *Probab. Theory Related Fields* **143** 219–238. MR2449128
[2] Bolthausen, E. (1989). A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.* **17** 108–115. MR0972774
[3] Borodin, A.N. (1979). A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR* **246** 786–787. MR0543530
[4] Borodin, A.N. (1979). Limit theorems for sums of independent random variables defined on a transient random walk. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **85** 17–29, 237, 244. MR0535455
[5] Butzer, P.L., Hahn, L. and Westphal, U. (1975). On the rate of approximation in the central limit theorem. *J. Approx. Theory* **13** 327–340. Collection of articles dedicated to G.G. Lorentz on the occasion of his sixty-fifth birthday, III. MR0394809
[6] Castell, F., Guillotin-Plantard, N. and Pène, F. (2013). Limit theorems for one and two-dimensional random walks in random scenery. *Ann. Inst. Henri Poincaré B*, *Probab. Stat.* **49** 506–528. MR3088379
[7] Castell, F., Guillotin-Plantard, N., Pène, F. and Schapira, B. (2011). A local limit theorem for random walks in random scenery and on randomly oriented lattices. *Ann. Probab.* **39** 2079–2118. MR2932665

[8] Chen, L.H.Y., Fang, X. and Shao, Q.-M. (2013). From Stein identities to moderate deviations. *Ann. Probab.* **41** 262–293. MR3059199

[9] Chow, Y.S. and Teicher, H. (2003). *Probability Theory*: *Independence*, *Interchangability*, *Martingales*, 3rd ed. *Springer Texts in Statistics*. New York: Springer.

[10] Davis, B. and McDonald, D. (1995). An elementary proof of the local central limit theorem. *J. Theoret. Probab.* **8** 693–701. MR1340834

[11] den Hollander, F. and Steif, J.E. (2006). Random walk in random scenery: A survey of some recent results. In *Dynamics & Stochastics*. *Institute of Mathematical Statistics Lecture Notes – Monograph Series* **48** 53–65. Beachwood, OH: IMS. MR2306188

[12] Doney, R.A. (1997). One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107** 451–465. MR1440141

[13] Freiman, G.A. (1999). Structure theory of set addition. *Astérisque* **258** 1–33. MR1701187

[14] Freiman, G.A. and Pitman, J. (1994). Partitions into distinct large parts. *J. Aust. Math. Soc. A* **57** 386–416. MR1297011

[15] Gamkrelidze, N.G. (1988). Application of a smoothness function in the proof of a local limit theorem. *Teor. Veroyatn. Primen.* **33** 373–376. MR0954586

[16] Gnedenko, B.V. (1948). On a local limit theorem of the theory of probability. *Uspekhi Mat. Nauk* **3** 187–194. MR0026275

[17] Guillotin-Plantard, N. and Pène, F. (2012). Renewal theorems for random walks in random scenery. *Electron. J. Probab.* **17** 78. MR2981903

[18] Guillotin-Plantard, N. and Prieur, C. (2010). Limit theorem for random walk in weakly dependent random scenery. *Ann. Inst. Henri Poincaré B*, *Probab. Stat.* **46** 1178–1194. MR2744890

[19] Ibragimov, I.A. and Linnik, Y.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. With a supplementary chapter by I.A. Ibragimov and V.V. Petrov, translation from the Russian edited by J.F.C. Kingman. MR0322926

[20] Johnson, O. (2015). Entropy and thinning of discrete random variables. Available at arXiv:1510.05390v2.

[21] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50** 5–25. MR0550121

[22] Kolmogorov, A. (1958). Sur les propriétés des fonctions de concentrations de M. P. Lévy. *Ann. Inst. Henri Poincaré* **16** 27–34. MR0101545

[23] MacDonald, D. (1979). On local limit theorems for integer valued random variables. *Theory Probab. Appl.* **33** 352–355.

[24] Matskyavichyus, V.K. On a lower bound for the convergence rate in a local limit theorem. *Theory Probab. Appl.* 810–814.

[25] McDiarmid, C. (1998). Concentration. In *Probabilistic Methods for Algorithmic Discrete Mathematics*. *Algorithms Combin.* **16** 195–248. Berlin: Springer. MR1678578

[26] McDonald, D. (1979). A local limit theorem for large deviations of sums of independent, nonidentically distributed random variables. *Ann. Probab.* **7** 526–531. MR0528328

[27] Mitrinović, D.S. (1970). *Analytic Inequalities*. New York: Springer. In cooperation with P.M. Vasić. *Die Grundlehren der mathematischen Wissenschaften*, Band 165. MR0274686

[28] Mukhin, A.B. (1984). Local limit theorems for distributions of sums of independent random vectors. *Teor. Veroyatn. Primen.* **29** 360–366. MR0749924

[29] Mukhin, A.B. (1991). Local limit theorems for lattice random variables. *Teor. Veroyatn. Primen.* **36** 660–674. MR1147168

[30] Petrov, V.V. (1975). *Sums of Independent Random Variables*. New York: Springer. Translated from the Russian by A.A. Brown, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 82. MR0388499

[31] Postnikov, A.G. (1988). *Introduction to Analytic Number Theory*. *Translations of Mathematical Monographs* **68**. Providence, RI: Amer. Math. Soc.. Translated from the Russian by G.A. Kandall, translation edited by Ben Silver, with an appendix by P.D.T.A. Elliott. MR0932727

[32] Röllin, A. and Ross, N. (2015). Local limit theorems via Landau–Kolmogorov inequalities. *Bernoulli* **21** 851–880. MR3338649

[33] Weber, M. (2011). A sharp correlation inequality with application to almost sure local limit theorem. *Probab*. *Math*. *Statist*. **31** 79–98. MR2804977

Received January 2015 and revised February 2016