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Pathwise stochastic integrals for model free finance

NICOLAS PERKOWSKI and DAVID J. PRÖMEL

[1]*CEREMADE & CNRS UMR 7534, Université Paris-Dauphine, France.*
E-mail: perkowski@ceremade.dauphine.fr
[2]*Humboldt-Universität zu Berlin, Institut für Mathematik, Germany. E-mail:* proemel@math.hu-berlin.de

We present two different approaches to stochastic integration in frictionless model free financial mathematics. The first one is in the spirit of Itô's integral and based on a certain topology which is induced by the outer measure corresponding to the minimal superhedging price. The second one is based on the controlled rough path integral. We prove that every "typical price path" has a naturally associated Itô rough path, and justify the application of the controlled rough path integral in finance by showing that it is the limit of non-anticipating Riemann sums, a new result in itself. Compared to the first approach, rough paths have the disadvantage of severely restricting the space of integrands, but the advantage of being a Banach space theory.

Both approaches are based entirely on financial arguments and do not require any probabilistic structure.

Keywords: Föllmer integration; model uncertainty; rough path; stochastic integration; Vovk's outer measure

1. Introduction

In this paper, we use Vovk's [40] game-theoretic approach to develop two different techniques of stochastic integration in frictionless model free financial mathematics. A priori the integration problem is highly non-trivial in the model free context since we do not want to assume any probabilistic, respectively, semimartingale structure. Therefore, we do not have access to Itô integration and most known techniques completely break down. There are only two general solutions to the integration problem in a non-probabilistic continuous time setting that we are aware of. One was proposed by [15], who simply restrict themselves to trading strategies (integrands) of bounded variation. While this already allows to solve many interesting problems, it is not a very natural assumption to make in a frictionless market model. Indeed, in [15] a general duality approach is developed for pricing path-dependent derivatives that are Lipschitz continuous in the supremum norm, but so far their approach does not allow to treat derivatives depending on the volatility.

Another interesting solution was proposed by [9] (using an idea which goes back to [31]). They restrict the set of "possible price paths" to those admitting a quadratic variation. This allows them to apply Föllmer's pathwise Itô calculus [17] to define pathwise stochastic integrals of the form $\int \nabla F(S)\,\mathrm{d}S$. In [31], that approach was used to derive prices for American and European options under volatility uncertainty. In [9], the given data is a finite number of European call and put prices and the derivative to be priced is a weighted variance swap. The restriction to the set of paths with quadratic variation is justified by referring to Vovk [40], who proved that "typical price paths" (to be defined below) admit a quadratic variation.

In our first approach, we do not restrict the set of paths and work on the space Ω of d-dimensional continuous paths (which represent all possible asset price trajectories). We follow Vovk in introducing an outer measure on Ω which is defined as the pathwise minimal superhedging price (in a suitable sense), and therefore has a purely financial interpretation and does not come from an artificially imposed probabilistic structure. Our first observation is that Vovk's outer measure allows us to define a topology on processes on Ω, and that the "natural Itô integral" on step functions is in a certain sense continuous in that topology. This allows us to extend the integral to càdlàg adapted integrands, and we call the resulting integral "model free Itô integral". We stress that the entire construction is based only on financial arguments.

Let us also stress that it is the *continuity* of our integral which is the most important aspect. Without reference to any topology, the construction would certainly not be very useful, since already in the classical probabilistic setting virtually all applications of the Itô integral (SDEs, stochastic optimization, duality theory, ...) are based on the fact that it is a continuous operator.

This also motivates our second approach, which is more in the spirit of [9,15,31]. While in the first approach we do have a continuous operator, it is only continuous with respect to a sequence of pseudometrics and it seems impossible to find a Banach space structure that is compatible with it. This is a pity since Banach space theory is one of the key tools in the classical theory of financial mathematics, as emphasized, for example, in [13]. However, using the model free Itô integral we are able to show that every "typical price path" has a natural Itô rough path associated to it. Since in financial applications we can always restrict ourselves to typical price paths, this observation opens the door for the application of the controlled rough path integral [21,32] in model free finance. Controlled rough path integration has the advantage of being an entirely linear Banach space theory which simultaneously extends:

- the Riemann–Stieltjes integral of S against functions of bounded variation which was used by [15];
- the Young integral [43]: typical price paths have finite p-variation for every $p > 2$, and therefore for every F of finite q-variation for $1 \leq q < 2$ (so that $1/p + 1/q > 1$), the integral $\int F \, \mathrm{d}S$ is defined as limit of non-anticipating Riemann sums;
- Föllmer's [17] pathwise Itô integral, which was used by [9,31]. That this last integral is a special case of the controlled rough path integral is, to the best of our knowledge, proved rigorously for the first time in this paper, although also [19] contains some observations in that direction.

In other words, our second approach covers all previously known techniques of integration in model free financial mathematics, while the first approach is much more general but at the price of leaving the Banach space world.

There is only one pitfall: the rough path integral is usually defined as a limit of compensated Riemann sums which have no obvious financial interpretation. This sabotages our entire philosophy of only using financial arguments. That is why we show that under some weak condition every rough path integral $\int F \, \mathrm{d}S$ is given as limit of non-anticipating Riemann sums that do not need to be compensated – the first time that such a statement is shown for general rough path integrals. Of course, this will not change anything in concrete applications, but it is of utmost importance from a philosophical point of view. Indeed, the justification for using the Itô integral in classical financial mathematics is crucially based on the fact that it is the limit of non-anticipating Riemann sums, even if in "every day applications" one never makes reference to that; see, for example, the discussion in [31].

Plan of the paper. Below we present a very incomplete list of solutions to the stochastic integration problem under model uncertainty and in a discrete time model free context (both a priori much simpler problems than the continuous time model free case), and we introduce some notations and conventions that will be used throughout the paper. In Section 2, we briefly recall Vovk's game-theoretic approach to mathematical finance and introduce our outer measure. We also construct a topology on processes which is induced by the outer measure. Section 3 is devoted to the construction of the model free Itô integral. Section 4 recalls some basic results from rough path theory, and continues by constructing rough paths associated to typical price paths. Here we also prove that the rough path integral is given as a limit of non-anticipating Riemann sums. Furthermore, we compare Föllmer's pathwise Itô integral with the rough path integral and prove that the latter is an extension of the former. Appendix A recalls Vovk's pathwise Hoeffding inequality. In Appendix B, we show that a result of Davie which also allows to calculate rough path integrals as limit of Riemann sums is a special case of our results in Section 4.

Stochastic integration under model uncertainty. The first works which studied the option pricing problem under model uncertainty were [3] and [31], both considering the case of volatility uncertainty. As described above, [31] is using Föllmer's pathwise Itô integral. In [3] the problem is reduced to the classical setting by deriving a "worst case" model for the volatility.

A powerful tool in financial mathematics under model uncertainty is Karandikar's pathwise construction of the Itô integral [5,24], which allows to construct the Itô integral of a càdlàg integrand simultaneously under all semimartingale measures. The crucial point that makes the construction useful is that the Itô integral is a continuous operator under every semimartingale measure. While its pathwise definition would allow us to use the same construction also in a model free setting, it is not even clear what the output should signify in that case (e.g., the construction depends on a certain sequence of partitions and changing the sequence will change the output). Certainly it is not obvious whether the Karandikar integral is continuous in any topology once we dispose of semimartingale measures. A more general pathwise construction of the Itô integral was given in [34], but it suffers from the same drawbacks with respect to applications in model free finance.

A general approach to stochastic analysis under model uncertainty was put forward in [14], and it is based on quasi sure analysis. This approach is extremely helpful when working under model uncertainty, but it also does not allow us to define stochastic integrals in a model free context.

In a related but slightly different direction, in [7] non-semimartingale models are studied (which do not violate arbitrage assumptions if the set of admissible strategies is restricted). While the authors work under one fixed probability measure, the fact that their price process is not a semimartingale prevents them from using Itô integrals, a difficulty which is overcome by working with the Russo–Vallois integral [37].

Of course all these technical problems disappear if we restrict ourselves to discrete time, and indeed in that case [4] develop an essentially fully satisfactory duality theory for the pricing of derivatives under model uncertainty.

Notation and conventions. Throughout the paper, we fix $T \in (0, \infty)$ and we write $\Omega := C([0, T], \mathbb{R}^d)$ for the space of d-dimensional continuous paths. The coordinate process on Ω is denoted by $S_t(\omega) = \omega(t)$, $t \in [0, T]$. For $i \in \{1, \dots, d\}$, we also write $S_t^i(\omega) := \omega^i(t)$, where $\omega = (\omega^1, \dots, \omega^d)$. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is defined as $\mathcal{F}_t := \sigma(S_s : s \leq t)$, and we set $\mathcal{F} := \mathcal{F}_T$. Stopping times τ and the associated σ-algebras $\mathcal{F}_\tau$ are defined as usual.

Unless explicitly stated otherwise, inequalities of the type $F_t \geq G_t$, where F and G are processes on Ω, are supposed to hold for all $\omega \in \Omega$, and not modulo null sets, as it is usually assumed in stochastic analysis.

The indicator function of a set A is denoted by $\mathbf{1}_A$.

A *partition* π of $[0, T]$ is a finite set of time points, $\pi = \{0 = t_0 < t_1 < \cdots < t_m = T\}$. Occasionally, we will identify π with the set of intervals $\{[t_0, t_1], [t_1, t_2], \dots, [t_{m-1}, t_m]\}$, and write expressions like $\sum_{[s,t]\in\pi}$.

For $f : [0, T] \to \mathbb{R}^n$ and $t_1, t_2 \in [0, T]$, denote $f_{t_1,t_2} := f(t_2) - f(t_1)$ and define the p-variation of f restricted to $[s, t] \subseteq [0, T]$ as

$$\|f\|_{p\text{-var},[s,t]} := \sup\left\{\left(\sum_{k=0}^{m-1} |f_{t_k,t_{k+1}}|^p\right)^{1/p} : s = t_0 < \cdots < t_m = t, m \in \mathbb{N}\right\}, \qquad p > 0, \quad (1)$$

(possibly taking the value $+\infty$). We set $\|f\|_{p\text{-var}} := \|f\|_{p\text{-var},[0,T]}$. We write $\Delta_T := \{(s, t) : 0 \leq s \leq t \leq T\}$ for the simplex and define the p-variation of a function $g : \Delta_T \to \mathbb{R}^n$ in the same manner, replacing $f_{t_k,t_{k+1}}$ in (1) by $g(t_k, t_{k+1})$.

For $\alpha > 0$ and $\lfloor\alpha\rfloor := \max\{z \in \mathbb{Z} : z \leq \alpha\}$, the space C^α consists of those functions that are $\lfloor\alpha\rfloor$ times continuously differentiable, with $(\alpha - \lfloor\alpha\rfloor)$-Hölder continuous partial derivatives of order $\lfloor\alpha\rfloor$ (and with continuous partial derivatives of order α in case $\alpha = \lfloor\alpha\rfloor$). The space C_b^α consists of those functions in C^α that are bounded, together with their partial derivatives, and we define the norm $\|\cdot\|_{C_b^\alpha}$ by setting

$$\|f\|_{C_b^\alpha} := \sum_{k=0}^{\lfloor\alpha\rfloor} \left\|D^k f\right\|_\infty + \mathbf{1}_{\alpha>\lfloor\alpha\rfloor} \left\|D^{\lfloor\alpha\rfloor} f\right\|_{\alpha-\lfloor\alpha\rfloor},$$

where $\|\cdot\|_\beta$ denotes the β-Hölder norm for $\beta \in (0, 1)$, and $\|\cdot\|_\infty$ denotes the supremum norm.

For $x, y \in \mathbb{R}^d$, we write $xy := \sum_{i=1}^d x_i y_i$ for the usual inner product. However, often we will encounter terms of the form $\int S \,\mathrm{d}S$ or $S_s S_{s,t}$ for $s, t \in [0, T]$, where we recall that S denotes the coordinate process on Ω. Those expressions are to be understood as the matrix $(\int S^i \,\mathrm{d}S^j)_{1\leq i,j\leq d}$, and similarly for $S_s S_{s,t}$. The interpretation will be usually clear from the context, otherwise we will make a remark to clarify things.

We use the notation $a \lesssim b$ if there exists a constant $c > 0$, independent of the variables under consideration, such that $a \leq c \cdot b$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. If we want to emphasize the dependence of c on the variable x, then we write $a(x) \lesssim_x b(x)$.

We make the convention that $0/0 := 0 \cdot \infty := 0$, $1 \cdot \infty := \infty$ and $\inf \varnothing := \infty$.

2. Superhedging and typical price paths

2.1. The outer measure and its basic properties

In a recent series of papers, Vovk [39–41] has introduced a model free, hedging based approach to mathematical finance that uses arbitrage considerations to examine which properties are satisfied by "typical price paths". This is achieved with the help of an outer measure given by the cheapest superhedging price.

Recall that $T \in (0, \infty)$ and $\Omega = C([0, T], \mathbb{R}^d)$ is the space of continuous paths, with coordinate process S, natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and $\mathcal{F} = \mathcal{F}_T$. A process $H : \Omega \times [0, T] \to \mathbb{R}^d$ is called a *simple strategy* if there exist stopping times $0 = \tau_0 < \tau_1 < \cdots$, and $\mathcal{F}_{\tau_n}$-measurable bounded functions $F_n : \Omega \to \mathbb{R}^d$, such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n, and such that

$$H_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t).$$

In that case, the integral

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} F_n(\omega)\big(S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n\wedge t}(\omega)\big) = \sum_{n=0}^{\infty} F_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega)$$

is well defined for all $\omega \in \Omega$, $t \in [0, T]$. Here $F_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega)$ denotes the usual inner product on $\mathbb{R}^d$. For $\lambda > 0$, a simple strategy H is called λ-*admissible* if $(H \cdot S)_t(\omega) \geq -\lambda$ for all $\omega \in \Omega$, $t \in [0, T]$. The set of λ-admissible simple strategies is denoted by $\mathcal{H}_\lambda$.

Definition 2.1. *The* outer measure *of $A \subseteq \Omega$ is defined as the cheapest superhedging price for $\mathbf{1}_A$, that is*

$$\overline{P}(A) := \inf\Big\{\lambda > 0 : \exists \big(H^n\big)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}\big(\lambda + \big(H^n \cdot S\big)_T(\omega)\big) \geq \mathbf{1}_A(\omega)\ \forall \omega \in \Omega\Big\}.$$

A set of paths $A \subseteq \Omega$ is called a null set *if it has outer measure zero.*

The term outer measure will be justified by Lemma 2.3 below. Our definition of $\overline{P}$ is very similar to the one used by Vovk [40], but not quite the same. For a discussion, see Section 2.3 below.

By definition, every Itô stochastic integral is the limit of stochastic integrals against simple strategies. Therefore, our definition of the cheapest superhedging price is essentially the same as in the classical setting, with one important difference: we require superhedging for *all* $\omega \in \Omega$, and not just almost surely.

Remark 2.2 ([40], page 564). An equivalent definition of $\overline{P}$ would be

$$\widetilde{P}(A) := \inf\Big\{\lambda > 0 : \exists \big(H^n\big)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty} \sup_{t\in[0,T]} \big(\lambda + \big(H^n \cdot S\big)_t(\omega)\big) \geq \mathbf{1}_A(\omega)\ \forall \omega \in \Omega\Big\}.$$

Clearly, $\widetilde{P} \le \overline{P}$. To see the opposite inequality, let $\widetilde{P}(A) < \lambda$. Let $(H^n)_{n\in\mathbb{N}} \subset \mathcal{H}_\lambda$ be a sequence of simple strategies such that $\liminf_{n\to\infty} \sup_{t\in[0,T]}(\lambda + (H^n \cdot S)_t) \ge \mathbf{1}_A$, and let $\varepsilon > 0$. Define $\tau_n := \inf\{t \in [0,T] : \lambda + \varepsilon + (H^n \cdot S)_t \ge 1\}$. Then the stopped strategy $G^n_t(\omega) := H^n_t(\omega)\mathbf{1}_{[0,\tau_n(\omega))}(t)$ is in $\mathcal{H}_\lambda \subseteq \mathcal{H}_{\lambda+\varepsilon}$ and

$$\liminf_{n\to\infty}\big(\lambda + \varepsilon + \big(G^n \cdot S\big)_T(\omega)\big) \ge \liminf_{n\to\infty} \mathbf{1}_{\{\lambda+\varepsilon+\sup_{t\in[0,T]}(H^n\cdot S)_t \ge 1\}}(\omega) \ge \mathbf{1}_A(\omega).$$

Therefore $\overline{P}(A) \le \lambda + \varepsilon$, and since $\varepsilon > 0$ was arbitrary $\overline{P} \le \widetilde{P}$, and thus $\overline{P} = \widetilde{P}$.

Lemma 2.3 ([40], Lemma 4.1). *$\overline{P}$ is in fact an outer measure, that is, a non-negative function defined on the subsets of Ω such that*

- $\overline{P}(\varnothing) = 0$;
- $\overline{P}(A) \le \overline{P}(B)$ *if* $A \subseteq B$;
- *if* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of subsets of* Ω, *then* $\overline{P}(\bigcup_n A_n) \le \sum_n \overline{P}(A_n)$.

Proof. Monotonicity and $\overline{P}(\varnothing) = 0$ are obvious. So let (A_n) be a sequence of subsets of Ω. Let $\varepsilon > 0$, $n \in \mathbb{N}$, and let $(H^{n,m})_{m\in\mathbb{N}}$ be a sequence of $(\overline{P}(A_n) + \varepsilon 2^{-n-1})$-admissible simple strategies such that $\liminf_{m\to\infty}(\overline{P}(A_n) + \varepsilon 2^{-n-1} + (H^{n,m} \cdot S)_T) \ge \mathbf{1}_{A_n}$. Define for $m \in \mathbb{N}$ the $(\sum_n \overline{P}(A_n) + \varepsilon)$-admissible simple strategy $G^m := \sum_{n=0}^m H^{n,m}$. Then by Fatou's lemma

$$\begin{aligned}\liminf_{m\to\infty}\left(\sum_{n=0}^{\infty}\overline{P}(A_n) + \varepsilon + \big(G^m \cdot S\big)_T\right) &\ge \sum_{n=0}^{k}\Big(\overline{P}(A_n) + \varepsilon 2^{-n-1} + \liminf_{m\to\infty}\big(H^{n,m} \cdot S\big)_T\Big)\\ &\ge \mathbf{1}_{\bigcup_{n=0}^k A_n}\end{aligned}$$

for all $k \in \mathbb{N}$. Since the left-hand side does not depend on k, we can replace $\mathbf{1}_{\bigcup_{n=0}^k A_n}$ by $\mathbf{1}_{\bigcup_n A_n}$ and the proof is complete. □

Maybe the most important property of $\overline{P}$ is that there exists an arbitrage interpretation for sets with outer measure zero.

Lemma 2.4. *A set $A \subseteq \Omega$ is a null set if and only if there exists a sequence of 1-admissible simple strategies $(H^n)_n \subset \mathcal{H}_1$ such that*

$$\liminf_{n\to\infty}\big(1 + \big(H^n \cdot S\big)_T(\omega)\big) \ge \infty \cdot \mathbf{1}_A(\omega), \tag{2}$$

where we use the convention $0 \cdot \infty = 0$ and $1 \cdot \infty := \infty$.

Proof. If such a sequence exists, then we can scale it down by an arbitrary factor $\varepsilon > 0$ to obtain a sequence of strategies in $\mathcal{H}_\varepsilon$ that superhedge $\mathbf{1}_A$, and therefore $\overline{P}(A) = 0$.

If conversely $\overline{P}(A) = 0$, then for every $n \in \mathbb{N}$ there exists a sequence of simple strategies $(H^{n,m})_{m\in\mathbb{N}} \subset \mathcal{H}_{2^{-n-1}}$ such that $2^{-n-1} + \liminf_{m\to\infty}(H^{n,m} \cdot \omega)_T \ge \mathbf{1}_A(\omega)$ for all $\omega \in \Omega$. Define

$G^m := \sum_{n=0}^m H^{n,m}$, so that $G^m \in \mathcal{H}_1$. For every $k \in \mathbb{N}$, we obtain

$$\liminf_{m\to\infty}\big(1+\big(G^m\cdot S\big)_T\big) \geq \sum_{n=0}^{k}\Big(2^{-n-1}+\liminf_{m\to\infty}\big(H^{n,m}\cdot S\big)_T\Big) \geq (k+1)\mathbf{1}_A.$$

Since the left-hand side does not depend on k, the sequence (G^m) satisfies (2). □

In other words, if a set A has outer measure 0, then we can make infinite profit by investing in the paths from A, without ever risking to lose more than the initial capital 1.

This motivates the following definition.

Definition 2.5. *We say that a property* (P) *holds for* typical price paths *if the set* A *where* (P) *is violated is a null set.*

The basic idea of Vovk, which we shall adopt in the following, is that we only need to concentrate on typical price paths. Indeed, "non-typical price paths" can be excluded since they are in a certain sense "too good to be true": they would allow investors to realize infinite profit while at the same time taking essentially no risk.

2.2. Arbitrage notions and link to classical mathematical finance

Before we continue, let us discuss different notions of arbitrage and link our outer measure to classical mathematical finance. We start by observing that $\overline{P}$ is an outer measure which simultaneously dominates all local martingale measures on Ω.

Propostion 2.6 ([40], Lemma 6.3). *Let* $\mathbb{P}$ *be a probability measure on* $(\Omega,\mathcal{F})$, *such that the coordinate process* S *is a* $\mathbb{P}$*-local martingale, and let* $A\in\mathcal{F}$. *Then* $\mathbb{P}(A)\leq\overline{P}(A)$.

Proof. Let $\lambda>0$ and let $(H^n)_{n\in\mathbb{N}}\subseteq\mathcal{H}_\lambda$ be such that $\liminf_n(\lambda+(H^n\cdot S)_T)\geq\mathbf{1}_A$. Then

$$\mathbb{P}(A)\leq\mathbb{E}_{\mathbb{P}}\Big[\liminf_n\big(\lambda+\big(H^n\cdot S\big)_T\big)\Big]\leq\liminf_n\mathbb{E}_{\mathbb{P}}\big[\lambda+\big(H^n\cdot S\big)_T\big]\leq\lambda,$$

where in the last step we used that $\lambda+(H^n\cdot S)$ is a non-negative $\mathbb{P}$-local martingale and thus a $\mathbb{P}$-supermartingale. □

This already indicates that $\overline{P}$-null sets are quite degenerate, in the sense that they are null sets under all local martingale measures. However, if that was the only reason for our definition of typical price paths, then a definition based on model free arbitrage opportunities would be equally valid. A map $X\colon\Omega\to[0,\infty)$ is a *model free arbitrage opportunity* if X is not identically 0 and if there exists $c>0$ and a sequence $(H^n)\subseteq\mathcal{H}_c$ such that $\liminf_{n\to\infty}(H^n\cdot S)_T(\omega)=X(\omega)$ for all $\omega\in\Omega$. See [1,10] where (a similar) definition is used in the discrete time setting.

It might then appear more natural to say that a property holds for typical price paths if the indicator function of its complement is a model free arbitrage opportunity, rather than working with

Definition 2.5. This "arbitrage definition" would also imply that any property which holds for typical price paths is almost surely satisfied under every local martingale measure. Nonetheless, we decidedly claim that our definition is "the correct one". First of all, the arbitrage definition would make our life much more difficult since it seems not very easy to work with. But of course this is only a convenience and cannot serve as justification of our approach. Instead, we argue by relating the two notions to classical mathematical finance.

For that purpose, recall the fundamental theorem of asset pricing [11]: If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ under which S is a semimartingale, then there exists an equivalent measure $\mathbb{Q}$ such that S is a $\mathbb{Q}$-local martingale if and only if S admits *no free lunch with vanishing risk* (*NFLVR*). But (NFLVR) is equivalent to the two conditions *no arbitrage* (*NA*) (intuitively: no profit without risk) and *no arbitrage opportunities of the first kind* (*NA1*) (intuitively: no very large profit with a small risk). The (NA) property holds if for every $c > 0$ and every sequence $(H^n) \subseteq \mathcal{H}_c$ for which $\lim_{n\to\infty}(H^n \cdot S)_T(\omega)$ exists for all ω we have $\mathbb{P}(\lim_{n\to\infty}(H^n \cdot S)_T < 0) > 0$ or $\mathbb{P}(\lim_{n\to\infty}(H^n \cdot S)_T = 0) = 1$. The (NA1) property holds if $\{1 + (H \cdot S)_T : H \in \mathcal{H}_1\}$ is bounded in $\mathbb{P}$-probability, that is, if

$$\lim_{c\to\infty} \sup_{H\in\mathcal{H}_1} \mathbb{P}\big(1 + (H \cdot S)_T \geq c\big) = 0.$$

Strictly speaking this is (NA1) with simple strategies, but as observed by [26] (NA1) and (NA1) with simple strategies are equivalent; see also [2,23].

It turns out that the arbitrage definition of typical price paths corresponds to (NA), while our definition corresponds to (NA1).

Propostion 2.7. *Let $A \in \mathcal{F}$ be a null set, and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process satisfies* (*NA*1). *Then $\mathbb{P}(A) = 0$.*

Proof. Let $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_1$ be such that $1 + \liminf_n (H^n \cdot S)_T \geq \infty \cdot \mathbf{1}_A$. Then for every $c > 0$

$$\mathbb{P}(A) = \mathbb{P}\Big(A \cap \Big\{\liminf_{n\to\infty}\big(H^n \cdot S\big)_T > c\Big\}\Big) \leq \sup_{H\in\mathcal{H}_1} \mathbb{P}\big(\big\{(H \cdot S)_T > c\big\}\big).$$

By assumption, the right-hand side converges to 0 as $c \to \infty$ and thus $\mathbb{P}(A) = 0$. □

Remark 2.8. Proposition 2.7 is actually a consequence of Proposition 2.6, because if S satisfies (NA1) under $\mathbb{P}$, then there exists a dominating measure $\mathbb{Q} \gg \mathbb{P}$, such that S is a $\mathbb{Q}$-local martingale. See [36] for the case of continuous S, and [23] for the general case.

The crucial point is that (NA1) is *the* essential property which every sensible market model has to satisfy, whereas (NA) is nice to have but not strictly necessary. Indeed, (NA1) is equivalent to the existence of an unbounded utility function such that the maximum expected utility is finite [23,25]. (NA) is what is needed in addition to (NA1) in order to obtain equivalent local martingale measures [11]. But there are perfectly viable models which violate (NA), for example, the three dimensional Bessel process [12,25]. By working with the arbitrage definition of typical price paths, we would in a certain sense ignore these models.

2.3. Relation to Vovk's outer measure

Our definition of the outer measure $\overline{P}$ is not exactly the same as Vovk's [40]. We find our definition more intuitive and it also seems to be easier to work with. However, since we rely on some of the results established by Vovk, let us compare the two notions.

For $\lambda > 0$, Vovk defines the set of processes

$$\mathcal{S}_\lambda := \left\{ \sum_{k=0}^{\infty} H^k : H^k \in \mathcal{H}_{\lambda_k}, \lambda_k > 0, \sum_{k=0}^{\infty} \lambda_k = \lambda \right\}.$$

For every $G = \sum_{k\geq 0} H^k \in \mathcal{S}_\lambda$, every $\omega \in \Omega$ and every $t \in [0, T]$, the integral

$$(G \cdot S)_t(\omega) := \sum_{k\geq 0} \big(H^k \cdot S\big)_t(\omega) = \sum_{k\geq 0} \big(\lambda_k + \big(H^k \cdot S\big)_t(\omega)\big) - \lambda$$

is well defined and takes values in $[-\lambda, \infty]$. Vovk then defines for $A \subseteq \Omega$ the cheapest superhedging price as

$$\overline{Q}(A) := \inf\big\{\lambda > 0 : \exists G \in \mathcal{S}_\lambda \text{ s.t. } \lambda + (G \cdot S)_T \geq \mathbf{1}_A\big\}.$$

This definition corresponds to the usual construction of an outer measure from an outer content (i.e., an outer measure which is only finitely subadditive and not countably subadditive); see [16], Chapter 1.4, or [38], Chapter 1.7. Here, the outer content is given by the cheapest superhedging price using only simple strategies. It is easy to see that $\overline{P}$ is dominated by $\overline{Q}$.

Lemma 2.9. *Let $A \subseteq \Omega$. Then $\overline{P}(A) \leq \overline{Q}(A)$.*

Proof. Let $G = \sum_k H^k$, with $H^k \in \mathcal{H}_{\lambda_k}$ and $\sum_k \lambda_k = \lambda$, and assume that $\lambda + (G \cdot S)_T \geq \mathbf{1}_A$. Then $(\sum_{k=0}^n H^k)_{n\in\mathbb{N}}$ defines a sequence of simple strategies in $\mathcal{H}_\lambda$, such that

$$\liminf_{n\to\infty} \left(\lambda + \left(\left(\sum_{k=0}^{n} H^k\right) \cdot S\right)_T\right) = \lambda + (G \cdot S)_T \geq \mathbf{1}_A.$$

So if $\overline{Q}(A) < \lambda$, then also $\overline{P}(A) \leq \lambda$, and therefore $\overline{P}(A) \leq \overline{Q}(A)$. □

Corollary 2.10. *For every $p > 2$, the set $A_p := \{\omega \in \Omega : \|S(\omega)\|_{p\text{-var}} = \infty\}$ has outer measure zero, that is $\overline{P}(A_p) = 0$.*

Proof. Theorem 1 of Vovk [39] states that $\overline{Q}(A_p) = 0$, so $\overline{P}(A_p) = 0$ by Lemma 2.9. □

It is a remarkable result of [40] that if $\Omega = C([0, \infty), \mathbb{R})$ (i.e., if the asset price process is one-dimensional), and if $A \subseteq \Omega$ is "invariant under time changes" and such that $S_0(\omega) = 0$ for all $\omega \in A$, then $A \in \mathcal{F}$ and $\overline{Q}(A) = \mathbb{P}(A)$, where $\mathbb{P}$ denotes the Wiener measure. This can be interpreted as a pathwise Dambis Dubins–Schwarz theorem.

2.4. A topology on path-dependent functionals

It will be very useful to introduce a topology on functionals on Ω. For that purpose let us identify $X, Y: \Omega \to \mathbb{R}$ if $X = Y$ for typical price paths. Clearly this defines an equivalence relation, and we write $\overline{L}_0$ for the space of equivalence classes. We then introduce the analog of convergence in probability in our context: (X_n) *converges in outer measure* to X if

$$\lim_{n\to\infty} \overline{P}\big(|X_n - X| > \varepsilon\big) = 0 \qquad \text{for all } \varepsilon > 0.$$

We follow [40] in defining an expectation operator. If $X: \Omega \to [0, \infty]$, then

$$\overline{E}[X] := \inf\Big\{\lambda > 0 : \exists \big(H^n\big)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}\big(\lambda + \big(H^n \cdot S\big)_T(\omega)\big) \geq X(\omega)\ \forall \omega \in \Omega\Big\}.$$

In particular, $\overline{P}(A) = \overline{E}[\mathbf{1}_A]$. The expectation $\overline{E}$ is countably subadditive, monotone, and positively homogeneous. It is an easy exercise to verify that

$$d(X, Y) := \overline{E}\big[|X - Y| \wedge 1\big]$$

defines a metric on $\overline{L}_0$.

Lemma 2.11. *The distance d metrizes the convergence in outer measure. More precisely, a sequence (X_n) converges to X in outer measure if and only if* $\lim_n d(X_n, X) = 0$. *Moreover, $(\overline{L}_0, d)$ is a complete metric space.*

Proof. The arguments are the same as in the classical setting. Using subadditivity and monotonicity of the expectation operator, we have

$$\varepsilon \overline{P}\big(|X_n - X| \geq \varepsilon\big) \leq \overline{E}\big[|X_n - X| \wedge 1\big] \leq \overline{P}\big(|X_n - X| > \varepsilon\big) + \varepsilon$$

for all $\varepsilon \in (0, 1]$, showing that convergence in outer measure is equivalent to convergence with respect to d.

As for completeness, let (X_n) be a Cauchy sequence with respect to d. Then there exists a subsequence (X_{n_k}) such that $d(X_{n_k}, X_{n_{k+1}}) \leq 2^{-k}$ for all k, so that

$$\overline{E}\Big[\sum_k \big(|X_{n_k} - X_{n_{k+1}}| \wedge 1\big)\Big] \leq \sum_k \overline{E}\big[|X_{n_k} - X_{n_{k+1}}| \wedge 1\big] = \sum_k d(X_{n_k}, X_{n_{k+1}}) < \infty,$$

which means that (X_{n_k}) converges for typical price paths. Define $X := \liminf_k X_{n_k}$. Then we have for all n and k

$$d(X_n, X) \leq d(X_n, X_{n_k}) + d(X_{n_k}, X) \leq d(X_n, X_{n_k}) + \sum_{\ell \geq k} d(X_{n_\ell}, X_{n_{\ell+1}}) \leq d(X_n, X_{n_k}) + 2^{-k}.$$

Choosing n and k large, we see that $d(X_n, X)$ tends to 0. $\square$

3. Model free Itô integration

The present section is devoted to the construction of a model free Itô integral. The main ingredient is a (weak) type of model free Itô isometry, which allows us to estimate the integral against a step function in terms of the amplitude of the step function and the quadratic variation of the price path. Using the topology introduced in Section 2.4, it is then easy to extend the integral to càdlàg integrands by a continuity argument.

Since we are in an unusual setting, let us spell out the following standard definitions.

Definition 3.1. *A process* $F\colon \Omega \times [0,T] \to \mathbb{R}^d$ *is called* adapted *if the random variable* $\omega \mapsto F_t(\omega)$ *is* $\mathcal{F}_t$*-measurable for all* $t \in [0,T]$.

The process F *is said to be* càdlàg *if the sample path* $t \mapsto F_t(\omega)$ *is càdlàg for all* $\omega \in \Omega$.

To prove our weak Itô isometry, we will need an appropriate sequence of stopping times: Let $n \in \mathbb{N}$. For each $i = 1, \dots, d$ define inductively

$$\sigma_0^{n,i} := 0, \qquad \sigma_{k+1}^{n,i} := \inf\{t \geq \sigma_k^{n,i} : |S_t^i - S^i_{\sigma_k^{n,i}}| \geq 2^{-n}\}, \qquad k \in \mathbb{N}.$$

Since we are working with continuous paths and we are considering entrance times into closed sets, the maps $(\sigma^{n,i})$ are indeed stopping times, despite the fact that $(\mathcal{F}_t)$ is neither complete nor right-continuous. Denote $\pi^{n,i} := \{\sigma_k^{n,i} : k \in \mathbb{N}\}$. To obtain an increasing sequence of partitions, we take the union of the $(\pi^{n,i})$, that is we define $\sigma_0^n := 0$ and then

$$\sigma_{k+1}^n(\omega) := \min\left\{t > \sigma_k^n(\omega) : t \in \bigcup_{i=1}^d \pi^{n,i}(\omega)\right\}, \qquad k \in \mathbb{N}, \tag{3}$$

and we write $\pi^n := \{\sigma_k^n : k \in \mathbb{N}\}$ for the corresponding partition.

Lemma 3.2 ([41], Theorem 4.1). *For typical price paths* $\omega \in \Omega$, *the quadratic variation along* $(\pi^{n,i}(\omega))_{n\in\mathbb{N}}$ *exists. That is,*

$$V_t^{n,i}(\omega) := \sum_{k=0}^{\infty} \bigl(S^i_{\sigma_{k+1}^{n,i}\wedge t}(\omega) - S^i_{\sigma_k^{n,i}\wedge t}(\omega)\bigr)^2, \qquad t \in [0,T], n \in \mathbb{N},$$

converges uniformly to a function $\langle S^i\rangle(\omega) \in C([0,T],\mathbb{R})$ *for all* $i \in \{1,\dots,d\}$.

For later reference, let us estimate $N_t^n := \max\{k \in \mathbb{N} : \sigma_k^n \leq t \text{ and } \sigma_k^n \neq 0\}$, the number of stopping times $\sigma_k^n \neq 0$ in π^n with values in $[0,t]$:

Lemma 3.3. *For all* $\omega \in \Omega$, $n \in \mathbb{N}$, *and* $t \in [0,T]$, *we have*

$$2^{-2n} N_t^n(\omega) \leq \sum_{i=1}^d V_t^{n,i}(\omega) =: V_t^n(\omega).$$

Proof. For $i \in \{1, \dots, d\}$ define $N_t^{n,i} := \max\{k \in \mathbb{N} : \sigma_k^{n,i} \le t \text{ and } \sigma_k^{n,i} \neq 0\}$. Since S^i is continuous, we have $|S^i_{\sigma^{n,i}_{k+1}} - S^i_{\sigma^{n,i}_k}| = 2^{-n}$ as long as $\sigma^{n,i}_{k+1} \le T$. Therefore, we obtain

$$N_t^n(\omega) \le \sum_{i=1}^d N_t^{n,i}(\omega) = \sum_{i=1}^d \sum_{k=0}^{N_t^{n,i}(\omega)-1} \frac{1}{2^{-2n}} \big(S_{\sigma^{n,i}_{k+1}}(\omega) - S_{\sigma^{n,i}_k}(\omega)\big)^2 \le 2^{2n} \sum_{i=1}^d V_t^{n,i}(\omega). \qquad \square$$

We will start by constructing the integral against step functions, which are defined similarly as simple strategies, except possibly unbounded: A process $F : \Omega \times [0, T] \to \mathbb{R}^d$ is called a *step function* if there exist stopping times $0 = \tau_0 < \tau_1 < \cdots$, and $\mathcal{F}_{\tau_n}$-measurable functions $F_n : \Omega \to \mathbb{R}^d$, such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n, and such that

$$F_t(\omega) = \sum_{n=0}^\infty F_n(\omega) \mathbf{1}_{[\tau_n(\omega), \tau_{n+1}(\omega))}(t).$$

For notational convenience, we are now considering the interval $[\tau_n(\omega), \tau_{n+1}(\omega))$ which is closed on the left-hand side. This allows us define the integral

$$(F \cdot S)_t := \sum_{n=0}^\infty F_n S_{\tau_n \wedge t, \tau_{n+1} \wedge t} = \sum_{n=0}^\infty F_{\tau_n} S_{\tau_n \wedge t, \tau_{n+1} \wedge t}, \qquad t \in [0, T].$$

The following lemma will be the main building block in the construction of our integral.

Lemma 3.4 (Model free version of Itô's isometry). *Let F be a step function. Then for all $a, b, c > 0$ we have*

$$\overline{P}\big(\big\{\|(F \cdot S)\|_\infty \ge ab\sqrt{c}\big\} \cap \big\{\|F\|_\infty \le a\big\} \cap \big\{\langle S\rangle_T \le c\big\}\big) \le 2\exp\big(-b^2/(2d)\big),$$

where the set $\{\langle S\rangle_T \le c\}$ should be read as $\{\langle S\rangle_T = \lim_n V_T^n$ exists and satisfies $\langle S\rangle_T \le c\}$.

Proof. Assume $F_t = \sum_{n=0}^\infty F_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t)$ and set $\tau_a := \inf\{t > 0 : |F_t| \ge a\}$. Let $n \in \mathbb{N}$ and define $\rho_0^n := 0$ and then for $k \in \mathbb{N}$

$$\rho_{k+1}^n := \min\big\{t > \rho_k^n : t \in \pi^n \cup \{\tau_m : m \in \mathbb{N}\}\big\},$$

where we recall that $\pi^n = \{\sigma_k^n : k \in \mathbb{N}\}$ is the nth generation of the dyadic partition generated by S. For $t \in [0, T]$, we have $(F \cdot S)_{\tau_a \wedge t} = \sum_k F_{\rho_k^n} S_{\tau_a \wedge \rho_k^n \wedge t, \tau_a \wedge \rho_{k+1}^n \wedge t}$, and by the definition of $\pi^n(\omega)$ and τ_a we get

$$\sup_{t \in [0,T]} \big|F_{\rho_k^n} S_{\tau_a \wedge \rho_k^n \wedge t, \tau_a \wedge \rho_{k+1}^n \wedge t}\big| \le a\sqrt{d} 2^{-n}.$$

Hence, the pathwise Hoeffding inequality, Lemma A.1 in Appendix A, yields for every $\lambda \in \mathbb{R}$ the existence of a 1-admissible simple strategy $H^{\lambda,n} \in \mathcal{H}_1$ such that

$$1 + \big(H^{\lambda,n} \cdot S\big)_t \ge \exp\bigg(\lambda(F \cdot S)_{\tau_a \wedge t} - \frac{\lambda^2}{2}\big(N_t^{(\rho^n)} + 1\big)2^{-2n} a^2 d\bigg) =: \mathcal{E}^{\lambda,n}_{\tau_a \wedge t}$$

for all $t \in [0, T]$, where

$$N_t^{(\rho^n)} := \max\{k : \rho_k^n \le t\} \le N_t^n + N_t^{(\tau)} := N_t^n + \max\{k : \tau_k \le t\}.$$

By Lemma 3.3, we have $N_t^n \le 2^{2n} V_t^n$, so that

$$\mathcal{E}_{\tau_a \wedge t}^{\lambda,n} \ge \exp\biggl(\lambda(F \cdot S)_t - \frac{\lambda^2}{2} V_T^n a^2 d - \frac{\lambda^2}{2}\bigl(N_T^{(\tau)} + 1\bigr) 2^{-2n} a^2 d\biggr).$$

If now $\|(F \cdot S)\|_\infty \ge ab\sqrt{c}$, $\|F(\omega)\|_\infty \le a$ and $\langle S\rangle_T \le c$, then

$$\liminf_{n\to\infty} \sup_{t\in[0,T]} \frac{\mathcal{E}_t^{\lambda,n} + \mathcal{E}_t^{-\lambda,n}}{2} \ge \frac{1}{2} \exp\biggl(\lambda ab\sqrt{c} - \frac{\lambda^2}{2} ca^2 d\biggr).$$

The argument inside the exponential is maximized for $\lambda = b/(a\sqrt{c}d)$, in which case we obtain $1/2 \exp(b^2/(2d))$. The statement now follows from Remark 2.2. □

Of course, we did not actually establish an isometry but only an upper bound for the integral. But this estimate is the key ingredient which allows us to extend the model free Itô integral to more general integrands, and it is this analogy to the classical setting that the terminology "model free version of Itô's isometry" alludes to.

Let us extend the topology of Section 2.4 to processes: we identify $X, Y : \Omega \times [0, T] \to \mathbb{R}^m$ if for typical price paths we have $X_t = Y_t$ for all $t \in [0, T]$, and we write $\overline{L}_0([0, T], \mathbb{R}^m)$ for the resulting space of equivalence classes which we equip with the distance

$$d_\infty(X, Y) := \overline{E}\bigl[\|X - Y\|_\infty \wedge 1\bigr].$$

Ideally, we would like the stochastic integral on step functions to be continuous with respect to d_∞. However, using Proposition 2.6 it is easy to see that $\overline{P}(\|((1/n) \cdot S)\|_\infty > \varepsilon) = 1$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$. This is why we also introduce for $c > 0$ the pseudometric

$$d_c(X, Y) := \overline{E}\bigl[\bigl(\|X - Y\|_\infty \wedge 1\bigr)\mathbf{1}_{\langle S\rangle_T \le c}\bigr] \le d_\infty(X, Y),$$

and then

$$d_{\mathrm{loc}}(X, Y) := \sum_{n=1}^{\infty} 2^{-n} d_{2^n}(X, Y) \le d_\infty(X, Y).$$

The distance d_{loc} is somewhat analogous to the distance used to metrize the topology of uniform convergence on compacts, except that we do not localize in time but instead we control the size of the quadratic variation. For step functions F and G, we get from Lemma 3.4

$$\begin{aligned} d_c\bigl((F \cdot S), (G \cdot S)\bigr) &\le \overline{P}\bigl(\bigl\{\bigl\|\bigl((F - G) \cdot S\bigr)\bigr\|_\infty \ge ab\sqrt{c}\bigr\} \cap \bigl\{\|F - G\|_\infty \le a\bigr\} \cap \bigl\{\langle S\rangle_T \le c\bigr\}\bigr) \\ &\quad + \frac{d_c(F, G)}{a} + ab\sqrt{c} \\ &\le 2\exp\biggl(-\frac{b^2}{2d}\biggr) + \frac{d_c(F, G)}{a} + ab\sqrt{c} \end{aligned}$$

whenever $a, b > 0$. Setting $a := \sqrt{d_c(F, G)}$ and $b := \sqrt{d|\log a|}$, we deduce that

$$d_c\big((F \cdot S), (G \cdot S)\big) \lesssim (1 + \sqrt{c}) d_c(F, G)^{1/2-\varepsilon} \tag{4}$$

for all $\varepsilon > 0$, and in particular

$$d_{\mathrm{loc}}\big((F \cdot S), (G \cdot S)\big) \lesssim \sum_{n=1}^{\infty} 2^{-n/2} d_{2^n}(F, G)^{1/2-\varepsilon} \lesssim d_\infty(F, G)^{1/2-\varepsilon}.$$

Theorem 3.5. *Let F be an adapted, càdlàg process with values in $\mathbb{R}^d$. Then there exists $\int F\,\mathrm{d}S \in \overline{L}_0([0, T], \mathbb{R})$ such that for every sequence of step functions (F^n) satisfying $\lim_n d_\infty(F^n, F) = 0$ we have $\lim_n d_{\mathrm{loc}}((F^n \cdot S), \int F\,\mathrm{d}S) = 0$. The integral process $\int F\,\mathrm{d}S$ is continuous for typical price paths, and there exists a representative $\int F\,\mathrm{d}S$ which is adapted, although it may take the values $\pm\infty$. We usually write $\int_0^t F_s\,\mathrm{d}S_s := \int F\,\mathrm{d}S(t)$, and we call $\int F\,\mathrm{d}S$ the* model free Itô integral of F with respect to S.

The map $F \mapsto \int F\,\mathrm{d}S$ is linear, satisfies

$$d_{\mathrm{loc}}\bigg(\int F\,\mathrm{d}S, \int G\,\mathrm{d}S\bigg) \lesssim d_\infty(F, G)^{1/2-\varepsilon}$$

for all $\varepsilon > 0$, and the model free version of Itô's isometry extends to this setting:

$$\overline{P}\bigg(\bigg\{\bigg\|\int F\,\mathrm{d}S\bigg\|_\infty \geq ab\sqrt{c}\bigg\} \cap \big\{\|F\|_\infty \leq a\big\} \cap \big\{\langle S\rangle_T \leq c\big\}\bigg) \leq 2\exp\big(-b^2/(2d)\big)$$

for all $a, b, c > 0$.

Proof. Everything follows in a straightforward way from (4) in combination with Lemma 2.11. We have to use the fact that F is adapted and càdlàg in order to approximate it uniformly by step functions. □

Another simple consequence of our model free version of Itô's isometry is a strengthened version of Karandikar's [24] pathwise Itô integral which works for all typical price paths and not just quasi surely under the local martingale measures.

Corollary 3.6. *In the setting of Theorem 3.5, let $(F^m)_{m\in\mathbb{N}}$ be a sequence of step functions with $\|F^m(\omega) - F(\omega)\|_\infty \leq c_m$ for all $\omega \in \Omega$ and all $m \in \mathbb{N}$. Then for typical price paths ω there exists a constant $C(\omega) > 0$ such that*

$$\bigg\|\big(F^m \cdot S\big)(\omega) - \int F\,\mathrm{d}S(\omega)\bigg\|_\infty \leq C(\omega) c_m \sqrt{\log m} \tag{5}$$

for all $m \in \mathbb{N}$. So, if $c_m = o((\log m)^{-1/2})$, then for typical price paths $(F^m \cdot S)$ converges to $\int F\,\mathrm{d}S$.

Proof. For $c > 0$ the model free Itô isometry gives

$$\overline{P}\left(\left\{\left\|(F^m \cdot S) - \int F \,\mathrm{d}S\right\|_\infty \geq c_m\sqrt{4d \log m}\sqrt{c}\right\} \cap \left\{\langle S\rangle_T \leq c\right\}\right) \leq \frac{1}{m^2}.$$

Since this is summable in m, the claim follows from Borel Cantelli (which only requires countable subadditivity and can thus be applied for the outer measure $\overline{P}$). □

Remark 3.7. The speed of convergence (5) is better than the one that can be obtained using the arguments in [24], where the summability of (c_m) is needed.

Remark 3.8. It would be desirable to extend the robust Itô integral obtained in Theorem 3.5 to general locally square integrable integrands, that is adapted processes H with measurable trajectories and such that $\int_0^t H_s^2(\omega)\,\mathrm{d}\langle S\rangle_s(\omega) < \infty$ for all t and for all ω which have a continuous quadratic variation $\langle S\rangle(\omega)$ up to time t. The reason why our methods break down in this setting is that our "model free version of Itô's isometry" requires as input a uniform bound on the integrand. However, even with the restriction to càdlàg integrands our robust Itô integral is suitable for all (financial) applications which use Karandikar's pathwise stochastic integral [24], with the great advantage of being a "model free" and not just a "quasi sure" object.

Similarly, it would be nice to have an extension of Theorem 3.5 to càdlàg integrators. Unfortunately, neither the outer measure $\overline{P}$ nor Vovk's outer measure $\overline{Q}$ have an obvious reasonable extension to the space $D([0, T], \mathbb{R}^d)$ of all càdlàg functions. The problem is that on this space there are no non-zero admissible strategies. As initiated in [41], it is possible to consider $\overline{P}$ or $\overline{Q}$ on the subspace of all paths in $D([0, T], \mathbb{R}^d)$ whose jump size at time $t > 0$ is bounded by a function of their supremum up to time t. However, it would be necessary to develop new techniques to obtain Theorem 3.5 in this setting since, for instance, the pathwise Hoeffding inequality (Lemma A.1) would not be applicable anymore.

4. Rough path integration for typical price paths

Our second approach to model free stochastic integration is based on the rough path integral, which has the advantage of being a continuous linear operator between Banach spaces. The disadvantage is that we have to restrict the set of integrands to those "locally looking like S", modulo a smoother remainder. Our two main results in this section are that every typical price path has a naturally associated Itô rough path, and that the rough path integral can be constructed as limit of Riemann sums.

Let us start by recalling the basic definitions and results of rough path theory.

4.1. The Lyons–Gubinelli rough path integral

Here we follow more or less the lecture notes [19], to which we refer for a gentle introduction to rough paths. More advanced monographs are [20,29,33]. The main difference to [19] in the derivation below is that we use p-variation to describe the regularity, and not Hölder continuity, because it is not true that all typical price paths are Hölder continuous. Also, we make an effort to give reasonably sharp results, whereas in [19] the focus lies more on the pedagogical presentation of the material. We stress that in this subsection we are merely collecting classical results.

Definition 4.1. *A* control function *is a continuous map* $c\colon \Delta_T \to [0,\infty)$ *with* $c(t,t)=0$ *for all* $t\in[0,T]$ *and such that* $c(s,u)+c(u,t)\le c(s,t)$ *for all* $0\le s\le u\le t\le T$.

Observe that if $f\colon[0,T]\to\mathbb{R}^d$ satisfies $|f_{s,t}|^p\le c(s,t)$ for all $(s,t)\in\Delta_T$, then the p-variation of f is bounded from above by $c(0,T)^{1/p}$.

Definition 4.2. *Let* $p\in(2,3)$. *A* p*-rough path is a map* $\mathbb{S}=(S,A)\colon\Delta_T\to\mathbb{R}^d\times\mathbb{R}^{d\times d}$ *such that* Chen's relation

$$S^i(s,t)=S^i(s,u)+S^i(u,t)\quad\textit{and}\quad A^{i,j}(s,t)=A^{i,j}(s,u)+A^{i,j}(u,t)+S^i(s,u)S^j(u,t)$$

holds for all $1\le i,j\le d$ *and* $0\le s\le u\le t\le T$ *and such that there exists a control function* c *with*

$$\big|S(s,t)\big|^p+\big|A(s,t)\big|^{p/2}\le c(s,t)$$

(*in other words* S *has finite* p*-variation and* A *has finite* $p/2$*-variation*). *In that case*, *we call* A *the* area *of* S.

Remark 4.3. Chen's relation simply states that S is the increment of a function, that is $S(s,t)=S(0,t)-S(0,s)=S_{s,t}$ for $S_t:=S(0,t)$, and that for all i,j there exists a function $f^{i,j}\colon[0,T]\to\mathbb{R}$ such that $A^{i,j}(s,t)=f^{i,j}(t)-f^{i,j}(s)-S^i_sS^j_{s,t}$. Indeed, it suffices to set $f^{i,j}(t):=A^{i,j}(0,t)+S^i_0S^j_{0,t}$.

Remark 4.4. The (strictly speaking incorrect) name "area" stems from the fact that if $S\colon[0,T]\to\mathbb{R}^2$ is a two-dimensional smooth function and if

$$A^{i,j}(s,t)=\int_s^t\int_s^{r_2}\mathrm{d}S^i_{r_1}\,\mathrm{d}S^j_{r_2}=\int_s^t S^i_{s,r_2}\,\mathrm{d}S^j_{r_2},$$

then the antisymmetric part of $A(s,t)$ corresponds to the algebraic area enclosed by the curve $(S_r)_{r\in[s,t]}$. It is a deep insight of Lyons [32], proving a conjecture of Föllmer, that the area is exactly the additional information which is needed to solve differential equations driven by S in a pathwise continuous manner, and to construct stochastic integrals as continuous maps. Actually, [32] solves a much more general problem and proves that if the driving signal is of finite p-variation for some $p>1$, then it has to be equipped with the iterated integrals up to order $\lfloor p\rfloor-1$ to obtain a continuous integral map. The for us relevant case $p\in(2,3)$ was already treated in [30].

Example 4.5. If S is a continuous semimartingale and if we set $S(s,t):=S_{s,t}$ as well as

$$A^{i,j}(s,t):=\int_s^t\int_s^{r_2}\mathrm{d}S^i_{r_1}\,\mathrm{d}S^j_{r_2}=\int_s^t S^i_{s,r_2}\,\mathrm{d}S^j_{r_2},$$

where the integral can be understood either in the Itô or in the Stratonovich sense, then almost surely $\mathbb{S} = (S, A)$ is a p-rough path for all $p \in (2, 3)$. This is shown in [6], and we will give a simplified model free proof below (indeed we will show that every typical price path together with its model free Itô integral is a p-rough path for all $p \in (2, 3)$, from where the statement about continuous semimartingales easily follows).

From now on, we fix $p \in (2, 3)$ and we assume that $\mathbb{S}$ is a p-rough path. Gubinelli [21] observed that for every rough path there is a naturally associated Banach space of integrands, the space of *controlled paths*. Heuristically, a path F is controlled by S, if it locally "looks like S", modulo a smooth remainder. The precise definition is the following.

Definition 4.6. *Let $p \in (2, 3)$ and $q > 0$ be such that $2/p + 1/q > 1$. Let $\mathbb{S} = (S, A)$ be a p-rough path and let $F: [0, T] \to \mathbb{R}^n$ and $F': [0, T] \to \mathbb{R}^{n\times d}$. We say that the pair (F, F') is* controlled *by S if the* derivative *F' has finite q-variation, and the* remainder *$R_F: \Delta_T \to \mathbb{R}^n$, defined by*

$$R_F(s, t) := F_{s,t} - F'_s S_{s,t},$$

has finite r-variation for $1/r = 1/p + 1/q$. In this case, we write $(F, F') \in \mathscr{C}^q_{\mathbb{S}} = \mathscr{C}^q_{\mathbb{S}}(\mathbb{R}^n)$, and define

$$\|(F, F')\|_{\mathscr{C}^q_{\mathbb{S}}} := \|F'\|_{q\text{-var}} + \|R_F\|_{r\text{-var}}.$$

Equipped with the norm $|F_0| + |F'_0| + \|(F, F')\|_{\mathscr{C}^q_{\mathbb{S}}}$, the space $\mathscr{C}^q_{\mathbb{S}}$ is a Banach space.

Naturally, the function F' should be interpreted as the derivative of F with respect to S. The reason for considering couples (F, F') and not just functions F is that the regularity requirement on the remainder R_F usually does not determine F' uniquely for a given path F. For example, if F and S both have finite r-variation rather than just finite p-variation, then for every F' of finite q-variation we have $(F, F') \in \mathscr{C}^q_{\mathbb{S}}$.

Note that we do not require F or F' to be continuous. We will point out in Remark 4.10 below why this does not pose any problem.

To gain a more "quantitative" feeling for the condition on q, let us assume for the moment that we can choose $p > 2$ arbitrarily close to 2 (which is the case in the example of a continuous semimartingale rough path). Then $2/p + 1/q > 1$ as long as $q > 0$, so that the derivative F' may essentially be as irregular as we want. The remainder R_F has to be of finite r-variation for $1/r = 1/p + 1/q$, so in other words it should be of finite r-variation for some $r < 2$ and thus slightly more regular than the sample path of a continuous local martingale.

Example 4.7. Let $\varepsilon \in (0, 1]$ be such that $(2 + \varepsilon)/p > 1$. Let $\varphi \in C^{1+\varepsilon}_b$ and define $F_s := \varphi(S_s)$ and $F'_s := \varphi'(S_s)$. Then $(F, F') \in \mathscr{C}^{p/\varepsilon}_{\mathbb{S}}$: Clearly F' has finite p/ε-variation. For the remainder, we have

$$\big|R_F(s, t)\big|^{p/(1+\varepsilon)} = \big|\varphi(S_t) - \varphi(S_s) - \varphi'(S_s)S_{s,t}\big|^{p/(1+\varepsilon)} \le \|\varphi\|_{C^{1+\varepsilon}_b} c(s, t),$$

where c is a control function for S. As the image of the continuous path S is compact, it is not actually necessary to assume that φ is bounded. We may always consider a $C^{1+\varepsilon}$ function ψ of compact support, such that ψ agrees with φ on the image of S.

This example shows that in general $R_F(s,t)$ is not a path increment of the form $R_F(s,t) = G(t) - G(s)$ for some function G defined on $[0,T]$, but really a function of two variables.

Example 4.8. Let G be a path of finite r-variation for some r with $1/p + 1/r > 1$. Setting $(F, F') = (G, 0)$, we obtain a controlled path in $\mathscr{C}^q_{\mathbb{S}}$, where $1/q = 1/r - 1/p$. In combination with Theorem 4.9 below, this example shows in particular that the controlled rough path integral extends the Young integral and the Riemann–Stieltjes integral.

The basic idea of rough path integration is that if we already know how to define $\int S \,\mathrm{d}S$, and if F looks like S on small scales, then we should be able to define $\int F \,\mathrm{d}S$ as well. The precise result is given by the following theorem.

Theorem 4.9 (Theorem 4.9 in [19], see also [21], Theorem 1). *Let $p \in (2,3)$ and $q > 0$ be such that $2/p + 1/q > 1$. Let $\mathbb{S} = (S, A)$ be a p-rough path and let $(F, F') \in \mathscr{C}^q_{\mathbb{S}}$. Then there exists a unique function $\int F \,\mathrm{d}S \in C([0,T], \mathbb{R}^n)$ which satisfies*

$$\left| \int_s^t F_u \,\mathrm{d}S_u - F_s S_{s,t} - F'_s A(s,t) \right| \lesssim \|S\|_{p\text{-var},[s,t]} \|R_F\|_{r\text{-var},[s,t]} + \|A\|_{p/2\text{-var},[s,t]} \left\|F'\right\|_{q\text{-var},[s,t]}$$

for all $(s,t) \in \Delta_T$. The integral is given as limit of the compensated Riemann sums

$$\int_0^t F_u \,\mathrm{d}S_u = \lim_{m\to\infty} \sum_{[s_1,s_2]\in\pi^m} \left[F_{s_1} S_{s_1,s_2} + F'_{s_1} A(s_1,s_2)\right], \tag{6}$$

where (π^m) is any sequence of partitions of $[0,t]$ with mesh size going to 0.

The map $(F, F') \mapsto (G, G') := (\int F_u \,\mathrm{d}S_u, F)$ is continuous from $\mathscr{C}^q_{\mathbb{S}}$ to $\mathscr{C}^p_{\mathbb{S}}$ and satisfies

$$\left\|(G, G')\right\|_{\mathscr{C}^p_{\mathbb{S}}} \lesssim \|F\|_{p\text{-var}} + \left(\left\|F'\right\|_\infty + \left\|F'\right\|_{q\text{-var}}\right) \|A\|_{p/2\text{-var}} + \|S\|_{p\text{-var}} \|R_F\|_{r\text{-var}}.$$

Remark 4.10. To the best of our knowledge, there is no publication in which the controlled path approach to rough paths is formulated using p-variation regularity. The references on the subject all work with Hölder continuity. But in the p-variation setting, all the proofs work exactly as in the Hölder setting, and it is a simple exercise to translate the proof of Theorem 4.9 in [19] (which is based on Young's maximal inequality which we will encounter below) to obtain Theorem 4.9.

There is only one small pitfall: We did not require F or F' to be continuous. The rough path integral for discontinuous functions is somewhat tricky, see [18,42]. But here we do not run into any problems, because the integrand $\mathbb{S} = (S, A)$ is continuous. The construction based on Young's maximal inequality works as long as integrand and integrator have no common discontinuities, see the theorem on page 264 of [43].

If now $\varphi \in C_b^{1+\varepsilon}$ for some $\varepsilon > 0$, then using a Taylor expansion one can show that there exist $p > 2$ and $q > 0$ with $2/p + 1/q > 0$, such that $(F, F') \mapsto (\varphi(F), \varphi'(F)F')$ is a locally bounded map from $\mathscr{C}_{\mathbb{S}}^p$ to $\mathscr{C}_{\mathbb{S}}^q$. Combining this with the fact that the rough path integral is a bounded map from $\mathscr{C}_{\mathbb{S}}^q$ to $\mathscr{C}_{\mathbb{S}}^p$, it is not hard to prove the *existence* of solutions to the rough differential equation

$$X_t = x_0 + \int_0^t \varphi(X_s)\,\mathrm{d}S_s, \tag{7}$$

$t \in [0, T]$, where $X \in \mathscr{C}_{\mathbb{S}}^p$, $\int \varphi(X_s)\,\mathrm{d}S_s$ denotes the rough path integral, and S is a typical price path. Similarly, if $\varphi \in C_b^{2+\varepsilon}$, then the map $(F, F') \mapsto (\varphi(F), \varphi'(F)F')$ is locally Lipschitz continuous from $\mathscr{C}_{\mathbb{S}}^p$ to $\mathscr{C}_{\mathbb{S}}^q$, and this yields the *uniqueness* of the solution to (7) – at least among the functions in the Banach space $\mathscr{C}_{\mathbb{S}}^p$. See Section 5.3 of [21] for details.

A remark is in order about the stringent regularity requirements on φ. In the classical Itô theory of SDEs, the function φ is only required to be Lipschitz continuous. But to solve a Stratonovich SDE, we need better regularity of φ. This is natural, because the Stratonovich SDE can be rewritten as an Itô SDE with a Stratonovich correction term: the equations

$$\mathrm{d}X_t = \varphi(X_t) \circ \mathrm{d}W_t \quad \text{and}$$
$$\mathrm{d}X_t = \varphi(X_t)\,\mathrm{d}W_t + \tfrac{1}{2}\varphi'(X_t)\varphi(X_t)\,\mathrm{d}t$$

are equivalent (where W is a standard Brownian motion, $\mathrm{d}W_t$ denotes Itô integration, and $\circ\,\mathrm{d}W_t$ denotes Stratonovich integration). To solve the second equation, we need $\varphi'\varphi$ to be Lipschitz continuous, which is always satisfied if $\varphi \in C_b^2$. But rough path theory cannot distinguish between Itô and Stratonovich integrals: If we define the area of W using Itô (resp., Stratonovich) integration, then the rough path solution of the equation will coincide with the Itô (resp., Stratonovich) solution. So in the rough path setting, the function φ should satisfy at least the same conditions as in the Stratonovich setting. The regularity requirements on φ are essentially sharp, see [8], but the boundedness assumption can be relaxed, see [28]. See also Section 10.5 of [20] for a slight relaxation of the regularity requirements in the Brownian case.

Of course, the most interesting result of rough path theory is that the solution to a rough differential equation depends continuously on the driving signal. This is a consequence of the following observation.

Propostion 4.11 (Proposition 9.1 of [19]). *Let $p \in (2, 3)$ and $q > 0$ with $2/p + 1/q > 0$. Let $\mathbb{S} = (S, A)$ and $\tilde{\mathbb{S}} = (\tilde{S}, \tilde{A})$ be two p-rough paths, let $(F, F') \in \mathscr{C}_{\mathbb{S}}^q$ and $(\tilde{F}, \tilde{F}') \in \mathscr{C}_{\tilde{\mathbb{S}}}^q$. Then for every $M > 0$ there exists $C_M > 0$ such that*

$$\begin{aligned}
&\left\| \int_0^\cdot F_s\,\mathrm{d}S_s - \int_0^\cdot \tilde{F}_s\,\mathrm{d}\tilde{S}_s \right\|_{p\text{-var}} \\
&\quad \le C_M\big(|F_0 - \tilde{F}_0| + \big|F'_0 - \tilde{F}'_0\big| + \|F' - \tilde{F}'\|_{q\text{-var}} \\
&\qquad + \|R_F - R_{\tilde{F}}\|_{r\text{-var}} + \|S - \tilde{S}\|_{p\text{-var}} + \|A - \tilde{A}\|_{p/2\text{-var}}\big),
\end{aligned}$$

as long as

$$\max\Big\{\big|F_0'\big| + \big\|\big(F, F'\big)\big\|_{\mathscr{C}^q_{\mathbb{S}}}, \big|\tilde{F}_0'\big| + \big\|\big(\tilde{F}, \tilde{F}'\big)\big\|_{\mathscr{C}^q_{\tilde{\mathbb{S}}}}, \|S\|_{p\text{-var}}, \|A\|_{p/2\text{-var}}, \|\tilde{S}\|_{p\text{-var}}, \|\tilde{A}\|_{p/2\text{-var}}\Big\} \le M.$$

In other words, the rough path integral depends on integrand and integrator in a locally Lipschitz continuous way, and therefore it is no surprise that the solutions to differential equations driven by rough paths depend continuously on the signal.

4.2. Typical price paths as rough paths

Our second approach to stochastic integration in model free financial mathematics is based on the rough path integral. Here we show that for every typical price path, the pair (S, A) is a p-rough path for all $p \in (2, 3)$, where A corresponds to the model free Itô integral $\int S\,\mathrm{d}S$ which we constructed in Section 3. We also show that many Riemann sum approximations to $\int S\,\mathrm{d}S$ uniformly satisfy a certain coarse grained regularity condition, which we will use in the following section to prove that in our setting rough path integrals can be calculated as limit of Riemann sums (and not compensated Riemann sums as in Theorem 4.9). The main ingredient in the proofs will be our speed of convergence (5).

Theorem 4.12. *For $(s, t) \in \Delta_T$, $\omega \in \Omega$, and $i, j \in \{1, \dots, d\}$ define*

$$A^{i,j}_{s,t}(\omega) := \int_s^t S^i_r\,\mathrm{d}S^j_r(\omega) - S^i_s(\omega)S^j_{s,t}(\omega) := \int_0^t S^i_r\,\mathrm{d}S^j_r(\omega) - \int_0^s S^i_r\,\mathrm{d}S^j_r(\omega) - S^i_s(\omega)S^j_{s,t}(\omega),$$

where $\int S^i\,\mathrm{d}S^j$ is the integral constructed in Theorem 3.5. If $p > 2$, then for typical price paths $A = (A^{i,j})_{1\le i,j\le d}$ has finite $p/2$-variation, and in particular $\mathbb{S} = (S, A)$ is a p-rough path.

Proof. Define the dyadic stopping times $(\tau^n_k)_{n,k\in\mathbb{N}}$ by $\tau^n_0 := 0$ and

$$\tau^n_{k+1} := \inf\big\{t \ge \tau^n_k : |S_t - S_{\tau^n_k}| = 2^{-n}\big\},$$

and set $S^n_t := \sum_k S_{\tau^n_k}\mathbf{1}_{[\tau^n_k, \tau^n_{k+1})}(t)$, so that $\|S^n - S\|_\infty \le 2^{-n}$. Accorcing to (5), for typical price paths ω there exists $C(\omega) > 0$ such that

$$\bigg\|\big(S^n \cdot S\big)(\omega) - \int S\,\mathrm{d}S(\omega)\bigg\|_\infty \le C(\omega)2^{-n}\sqrt{\log n}.$$

Fix such a typical price path ω, which is also of finite q-variation for all $q > 2$ (recall from Corollary 2.10 that this is satisfied by typical price paths). Let us show that for such ω, the process A is of finite $p/2$-variation for all $p > 2$.

We have for $(s,t) \in \Delta_T$, omitting the argument ω of the processes under consideration,

$$|A_{s,t}| \le \left| \int_s^t S_r \, \mathrm{d}S_r - \left(S^n \cdot S\right)_{s,t} \right| + \left|\left(S^n \cdot S\right)_{s,t} - S_s S_{s,t}\right|$$
$$\le C 2^{-n} \sqrt{\log n} + \left|\left(S^n \cdot S\right)_{s,t} - S_s S_{s,t}\right| \lesssim_\varepsilon C 2^{-n(1-\varepsilon)} + \left|\left(S^n \cdot S\right)_{s,t} - S_s S_{s,t}\right|$$

for every $n \in \mathbb{N}$, $\varepsilon > 0$. The second term on the right-hand side can be estimated, using an argument based on Young's maximal inequality (see [33], Theorem 1.16), by

$$\left|\left(S^n \cdot S\right)_{s,t} - S_s S_{s,t}\right| \lesssim \max\left\{2^{-n} c(s,t)^{1/q}, \left(\#\left\{k : \tau_k^n \in [s,t]\right\}\right)^{1-2/q} c(s,t)^{2/q} + c(s,t)^{2/q}\right\}, \quad (8)$$

where $c(s,t)$ is a control function with $|S_{s,t}|^q \le c(s,t)$ for all $(s,t) \in \Delta_T$. Indeed, if there exists no k with $\tau_k^n \in [s,t]$, then $|(S^n \cdot S)_{s,t} - S_s S_{s,t}| \le 2^{-n} c(s,t)^{1/q}$, using that $|S_{s,t}| \le c(s,t)^{1/q}$. This corresponds to the first term in the maximum in (8).

Otherwise, note that at the price of adding $c(s,t)^{2/q}$ to the right-hand side, we may suppose that $s = \tau_{k_0}^n$ for some k_0. Let now $\tau_{k_0}^n, \dots, \tau_{k_0+N-1}^n$ be those $(\tau_k^n)_k$ which are in $[s,t)$. Without loss of generality we may suppose $N \ge 2$, because otherwise $(S^n \cdot S)_{s,t} = S_s S_{s,t}$. Abusing notation, we write $\tau_{k_0+N}^n = t$. The idea is now to successively delete points $(\tau_{k_0+\ell}^n)$ from the partition, in order to pass from $(S^n \cdot S)$ to $S_s S_{s,t}$. By super-additivity of c, there must exist $\ell \in \{1, \dots, N-1\}$, for which

$$c\left(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n\right) \le \frac{2}{N-1} c(s,t).$$

Deleting $\tau_{k_0+\ell}^n$ from the partition and subtracting the resulting integral from $(S^n \cdot S)_{s,t}$, we get

$$|S_{\tau_{k_0+\ell-1}^n} S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n} + S_{\tau_{k_0+\ell}^n} S_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n} - S_{\tau_{k_0+\ell-1}^n} S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n}|$$
$$= |S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n} S_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n}| \le c\left(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n\right)^{2/q} \le \left(\frac{2}{N-1} c(s,t)\right)^{2/q}.$$

Successively deleting all the points except $\tau_{k_0}^n = s$ and $\tau_{k_0+N}^n = t$ from the partition gives

$$\left|\left(S^n \cdot S\right)_{s,t} - S_s S_{s,t}\right| \le \sum_{k=2}^N \left(\frac{2}{k-1} c(s,t)\right)^{2/q} \lesssim N^{1-2/q} c(s,t)^{2/q},$$

and therefore (8). Now it is easy to see that $\#\{k : \tau_k^n \in [s,t]\} \le 2^{nq} c(s,t)$ (compare also the proof of Lemma 3.3), and thus

$$\begin{aligned} |A_{s,t}| &\lesssim_\varepsilon C 2^{-n(1-\varepsilon)} + \max\left\{2^{-n} c(s,t)^{1/q}, \left(2^{nq} c(s,t)\right)^{1-2/q} c(s,t)^{2/q} + c(s,t)^{2/q}\right\} \\ &= C 2^{-n(1-\varepsilon)} + \max\left\{2^{-n} c(s,t)^{1/q}, 2^{-n(2-q)} c(s,t) + c(s,t)^{2/q}\right\}. \end{aligned} \quad (9)$$

This holds for all $n \in \mathbb{N}$, $\varepsilon > 0$, $q > 2$. Let us suppose for the moment that $c(s,t) \le 1$ and let $\alpha > 0$ to be determined later. Then there exists $n \in \mathbb{N}$ for which $2^{-n-1} < c(s,t)^{1/\alpha(1-\varepsilon)} \le 2^{-n}$.

Using this n in (9), we get

$$
\begin{aligned}
&|A_{s,t}|^\alpha \\
&\quad \lesssim_{\varepsilon,\omega,\alpha} c(s,t) + \max\big\{c(s,t)^{1/(1-\varepsilon)} c(s,t)^{\alpha/q}, c(s,t)^{(2-q)/(1-\varepsilon)+\alpha} + c(s,t)^{2\alpha/q}\big\} \\
&\quad = c(s,t) + \max\big\{c(s,t)^{\frac{q+\alpha(1-\varepsilon)}{q(1-\varepsilon)}}, c(s,t)^{\frac{2-q+\alpha(1-\varepsilon)}{1-\varepsilon}} + c(s,t)^{2\alpha/q}\big\}.
\end{aligned}
$$

We would like all the exponents in the maximum on the right-hand side to be larger or equal to 1. For the first term, this is satisfied as long as $\varepsilon < 1$. For the third term, we need $\alpha \geq q/2$. For the second term, we need $\alpha \geq (q-1-\varepsilon)/(1-\varepsilon)$. Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, it suffices if $\alpha > q-1$. Now, since $q > 2$ can be chosen arbitrarily close to 2, we see that α can be chosen arbitrarily close to 1. In particular, we may take $\alpha = p/2$ for any $p > 2$, and we obtain $|A_{s,t}|^{p/2} \lesssim_{\omega,\delta} c(s,t)$.

It remains to treat the case $c(s,t) > 1$, for which we simply estimate

$$
|A_{s,t}|^{p/2} \lesssim_p \left\| \int_0^\cdot S_r \,\mathrm{d}S_r \right\|_\infty^{p/2} + \|S\|_\infty^p \leq \left(\left\| \int_0^\cdot S_r \,\mathrm{d}S_r \right\|_\infty^{p/2} + \|S\|_\infty^p \right) c(s,t).
$$

So for every interval $[s,t]$ we can estimate $|A_{s,t}|^{p/2} \lesssim_{\omega,p} c(s,t)$, and the proof is complete. □

Remark 4.13. To the best of our knowledge, this is one of the first times that a non-geometric rough path is constructed in a non-probabilistic setting, and certainly we are not aware of any works where rough paths are constructed using financial arguments.

We also point out that, thanks to Proposition 2.6, we gave a simple, model free, and pathwise proof for the fact that a local martingale together with its Itô integral defines a rough path. While this seems intuitively clear, the only other proof that we know of is somewhat involved: it relies on a strong version of the Burkholder–Davis–Gundy inequality, a time change, and Kolmogorov's continuity criterion; see [6] or Chapter 14 of [20].

The following auxiliary result will allow us to obtain the rough path integral as a limit of Riemann sums, rather than compensated Riemann sums, which are usually used to define it.

Lemma 4.14. *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $c_n = o((\log n)^{-c})$ for all $c > 0$. For $n \in \mathbb{N}$ define $\tau_0^n := 0$ and $\tau_{k+1}^n := \inf\{t \geq \tau_k^n : |S_t - S_{\tau_k^n}| = c_n\}$, $k \in \mathbb{N}$, and set $S_t^n := \sum_k S_{\tau_k^n} \mathbf{1}_{[\tau_k^n, \tau_{k+1}^n)}(t)$. Then for typical price paths, $((S^n \cdot S))$ converges uniformly to $\int S \,\mathrm{d}S$ defined in Theorem 3.5. Moreover, for $p > 2$ and for typical price paths there exists a control function $c = c(p,\omega)$ such that*

$$
\sup_n \sup_{k<\ell} \frac{|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n}(\omega) - S_{\tau_k^n}(\omega) S_{\tau_k^n, \tau_\ell^n}(\omega)|^{p/2}}{c(\tau_k^n, \tau_\ell^n)} \leq 1.
$$

Proof. The uniform convergence of $((S^n \cdot S))$ to $\int S \, \mathrm{d}S$ follows from Corollary 3.6. For the second claim, fix $n \in \mathbb{N}$ and $k < \ell$ such that $\tau^n_\ell \le T$. Then

$$\begin{aligned} \big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big| &\lesssim \Big\|(S^n \cdot S) - \int_0^\cdot S_s \, \mathrm{d}S_s\Big\|_\infty + |A_{\tau^n_k,\tau^n_\ell}| \\ &\lesssim_\omega c_n \sqrt{\log n} + v_{p/2}\big(\tau^n_k, \tau^n_\ell\big)^{2/p} \lesssim_\varepsilon c_n^{1-\varepsilon} + v_{p/2}\big(\tau^n_k, \tau^n_\ell\big)^{2/p}, \end{aligned} \tag{10}$$

where $\varepsilon > 0$ and the last estimate holds by our assumption on the sequence (c_n), and where $v_{p/2}(s,t) := \|A\|^{p/2}_{p/2\text{-var},[s,t]}$ for $(s,t) \in \Delta_T$. Of course, this inequality only holds for typical price paths and not for all $\omega \in \Omega$.

On the other side, the same argument as in the proof of Theorem 4.12 (using Young's maximal inequality and successively deleting points from the partition) shows that

$$\big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big| \lesssim c_n^{2-q} v_q\big(\tau^n_k, \tau^n_\ell\big), \tag{11}$$

where $v_q(s,t) := \|S\|^q_{q\text{-var},[s,t]}$ for $(s,t) \in \Delta_T$.

Let us define the control function $\tilde{c} := v_q + v_{p/2}$. Take $\alpha > 0$ to be determined below. If $c_n > \tilde{c}(s,t)^{1/\alpha(1-\varepsilon)}$, then we use (11) and the fact that $2 - q < 0$, to obtain

$$\big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big|^\alpha \lesssim \big(\tilde{c}\big(\tau^n_k, \tau^n_\ell\big)\big)^{\frac{2-q}{(1-\varepsilon)}} v_q\big(\tau^n_k, \tau^n_\ell\big)^\alpha \le \tilde{c}\big(\tau^n_k, \tau^n_\ell\big)^{\frac{2-q+\alpha(1-\varepsilon)}{(1-\varepsilon)}}.$$

The exponent is larger or equal to 1 as long as $\alpha \ge (q - 1 - \varepsilon)/(1 - \varepsilon)$. Since q and ε can be chosen arbitrarily close to 2 and 0, respectively, we can take $\alpha = p/2$, and get

$$\big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big|^{p/2} \lesssim \tilde{c}\big(\tau^n_k, \tau^n_\ell\big)\big(1 + \tilde{c}(0,T)^\delta\big)$$

for a suitable $\delta > 0$.

On the other side, if $c_n \le \tilde{c}(s,t)^{1/\alpha(1-\varepsilon)}$, then we use (10) to obtain

$$\big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big|^\alpha \lesssim \tilde{c}\big(\tau^n_k, \tau^n_\ell\big) + \tilde{c}\big(\tau^n_k, \tau^n_\ell\big)^{2\alpha/p},$$

so that also in this case we may take $\alpha = p/2$, and thus we have in both cases

$$\big|(S^n \cdot S)_{\tau^n_k,\tau^n_\ell} - S_{\tau^n_k} S_{\tau^n_k,\tau^n_\ell}\big|^{p/2} \le c\big(\tau^n_k, \tau^n_\ell\big),$$

where c is a suitable (ω-dependent) multiple of $\tilde{c}$. □

4.3. The rough path integral as limit of Riemann sums

Theorem 4.12 shows that we can apply the controlled rough path integral in model free financial mathematics since every typical price path is a rough path. But there remains a philosophical problem: As we have seen in Theorem 4.9, the rough path integral $\int F \,\mathrm{d}S$ is given as limit of the compensated Riemann sums

$$\int_0^t F_s \,\mathrm{d}S_s = \lim_{m\to\infty} \sum_{[r_1,r_2]\in\pi^m} \big[F_{r_1} S_{r_1,r_2} + F'_{r_1} A(r_1,r_2)\big],$$

where (π^m) is an arbitrary sequence of partitions of $[0,t]$ with mesh size going to 0. The term $F_{r_1} S_{r_1,r_2}$ has an obvious financial interpretation as profit made by buying F_{r_1} units of the traded asset at time r_1 and by selling them at time r_2. However, for the "compensator" $F'_{r_1} A(r_1,r_2)$ there seems to be no financial interpretation, and therefore it is not clear whether the rough path integral can be understood as profit obtained by investing in S.

However, we observed in Section 3 that along suitable stopping times $(\tau_k^n)_{n,k}$, we have

$$\int_0^t S_s \,\mathrm{d}S_s = \lim_{n\to\infty} \sum_k S_{\tau_k^n} S_{\tau_k^n\wedge t, \tau_{k+1}^n\wedge t}.$$

By the philosophy of controlled paths, we expect that also for F which looks like S on small scales we should obtain

$$\int_0^t F_s \,\mathrm{d}S_s = \lim_{n\to\infty} \sum_k F_{\tau_k^n} S_{\tau_k^n\wedge t, \tau_{k+1}^n\wedge t},$$

without having to introduce the compensator $F'_{\tau_k^n} A(\tau_k^n \wedge t, \tau_{k+1}^n \wedge t)$ in the Riemann sum. With the results we have at hand, this statement is actually relatively easy to prove. Nonetheless, it seems not to have been observed before.

For the remainder of this section, we fix $S \in C([0,T],\mathbb{R}^d)$, and we work under the following assumption:

Assumption **(RIE).** *Let* $\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, *be a given sequence of partitions such that* $\sup\{|S_{t_k^n,t_{k+1}^n}| : k = 0,\dots,N_n-1\}$ *converges to* 0, *and let* $p \in (2,3)$. *Set*

$$S_t^n := \sum_{k=0}^{N_n-1} S_{t_k^n} \mathbf{1}_{[t_k^n,t_{k+1}^n)}(t).$$

We assume that the Riemann sums $(S^n \cdot S)$ *converge uniformly to a limit that we denote by* $\int S\,\mathrm{d}S$, *and that there exists a control function c for which*

$$\sup_{(s,t)\in\Delta_T} \frac{|S_{s,t}|^p}{c(s,t)} + \sup_n \sup_{0\le k<\ell\le N_n} \frac{|(S^n\cdot S)_{t_k^n,t_\ell^n} - S_{t_k^n} S_{t_k^n,t_\ell^n}|^{p/2}}{c(t_k^n,t_\ell^n)} \le 1. \tag{12}$$

Remark 4.15. We expect that "coarse-grained" regularity conditions as in (12) have been used for a long time, but were only able to find quite recent references: condition (12) was previously used in [35], see also [22], and has also appeared independently in [27]. In our setting this is quite a natural relaxation of a uniform p-variation bound since say for $s, t \in [t^n_k, t^n_{k+1}]$ with $|t-s| \ll |t^n_{k+1} - t^n_k|$ the increment of the discrete integral $(S^n \cdot S)_{s,t}$ is not a good approximation of $\int_s^t S_r \,\mathrm{d}S_r$, and therefore we cannot expect it to be close to $S_s S_{s,t}$.

Remark 4.16. Every typical price path satisfies (RIE) if we choose (t^n_k) to be a partition of stopping times such as the (τ^n_k) in Lemma 4.14.

It is not hard to see that if S satisfies (RIE) and if we define $A(s,t) := \int_s^t S_r \,\mathrm{d}S_r - S_s S_{s,t}$, then (S, A) is a p-rough path. This means that we can calculate the rough path integral $\int F \,\mathrm{d}S$ whenever (F, F') is controlled by S, and the aim of the remainder of this section is to show that this integral is given as limit of (uncompensated) Riemann sums. Our proof is somewhat indirect. We translate everything from Itô type integrals to related Stratonovich type integrals, for which the convergence follows from the continuity of the rough path integral, Proposition 4.11. Then we translate everything back to our Itô type integrals. To go from Itô to Stratonovich, we need the quadratic variation.

Lemma 4.17. *Under Assumption* (RIE), *let* $1 \le i, j \le d$, *and define*

$$\langle S^i, S^j \rangle_t := S^i_t S^j_t - S^i_0 S^j_0 - \int_0^t S^i_r \,\mathrm{d}S^j_r - \int_0^t S^j_r \,\mathrm{d}S^i_r.$$

Then $\langle S^i, S^j \rangle$ *is a continuous function and*

$$\langle S^i, S^j \rangle_t = \lim_{n\to\infty} \langle S^i, S^j \rangle^n_t = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \big(S^i_{t^n_{k+1}\wedge t} - S^i_{t^n_k \wedge t}\big)\big(S^j_{t^n_{k+1}\wedge t} - S^j_{t^n_k\wedge t}\big). \tag{13}$$

The sequence $(\langle S^i, S^j\rangle^n)_n$ *is of uniformly bounded total variation, and in particular* $\langle S^i, S^j\rangle$ *is of bounded variation. We write* $\langle S \rangle = \langle S, S\rangle = (\langle S^i, S^j\rangle)_{1\le i,j\le d}$, *and call* $\langle S\rangle$ *the* quadratic variation *of* S.

Proof. The function $\langle S^i, S^j\rangle$ is continuous by definition. The specific form (13) of $\langle S^i, S^j\rangle$ follows from two simple observations:

$$S^i_t S^j_t - S^i_0 S^j_0 = \sum_{k=0}^{N_n-1} \big(S^i_{t^n_{k+1}\wedge t} S^j_{t^n_{k+1}\wedge t} - S^i_{t^n_k\wedge t} S^j_{t^n_k\wedge t}\big)$$

for every $n \in \mathbb{N}$, and

$$S^i_{t^n_{k+1}\wedge t} S^j_{t^n_{k+1}\wedge t} - S^i_{t^n_k\wedge t} S^j_{t^n_k\wedge t} = S^i_{t^n_k\wedge t} S^j_{t^n_k\wedge t, t^n_{k+1}\wedge t} + S^j_{t^n_k\wedge t} S^i_{t^n_k\wedge t, t^n_{k+1}\wedge t} + S^i_{t^n_k\wedge t, t^n_{k+1}\wedge t} S^j_{t^n_k\wedge t, t^n_{k+1}\wedge t},$$

so that the convergence in (13) is a consequence of the convergence of $(S^n \cdot S)$ to $\int S \,\mathrm{d}S$.

To see that $\langle S^i, S^j\rangle$ is of bounded variation, note that

$$S^i_{t^n_k\wedge t, t^n_{k+1}\wedge t} S^j_{t^n_k\wedge t, t^n_{k+1}\wedge t} = \tfrac{1}{4}\Big(\big((S^i + S^j)_{t^n_k\wedge t, t^n_{k+1}\wedge t}\big)^2 - \big((S^i - S^j)_{t^n_k\wedge t, t^n_{k+1}\wedge t}\big)^2\Big)$$

(read $\langle S^i, S^j\rangle = 1/4(\langle S^i + S^j\rangle - \langle S^i - S^j\rangle)$). In other words, the nth approximation of $\langle S^i, S^j\rangle$ is the difference of two increasing functions, and its total variation is bounded from above by

$$\sum_{k=0}^{N_n-1}\big(\big((S^i+S^j)_{t_k^n,t_{k+1}^n}\big)^2+\big((S^i-S^j)_{t_k^n,t_{k+1}^n}\big)^2\big)\lesssim \sup_m \sum_{k=0}^{N_m-1}\big(\big(S^i_{t_k^m,t_{k+1}^m}\big)^2+\big(S^j_{t_k^m,t_{k+1}^m}\big)^2\big).$$

Since the right-hand side is finite, also the limit $\langle S^i, S^j\rangle$ is of bounded variation. □

Given the quadratic variation, the existence of the Stratonovich integral is straightforward:

Lemma 4.18. *Under Assumption* (RIE), *define* $\tilde S^n|_{[t_k^n,t_{k+1}^n]}$ *as the linear interpolation of* $S_{t_k^n}$ *and* $S_{t_{k+1}^n}$ *for* $k = 0,\dots,N_n-1$. *Then* $(\int \tilde S^n \,\mathrm{d}\tilde S^n)$ *converges uniformly to*

$$\int_s^t S_r\circ \mathrm{d}S_r := \int_s^t S_r\,\mathrm{d}S_r + \frac{1}{2}\langle S\rangle_{s,t}. \tag{14}$$

Moreover, setting $\tilde A^n(s,t) := \int_s^t \tilde S^n_{s,r}\,\mathrm{d}\tilde S^n_r$ *for* $(s,t)\in\Delta_T$, *we have* $\sup_n \|\tilde A^n\|_{p/2\text{-var}} < \infty$.

Proof. Let $n\in\mathbb{N}$ and $k\in\{0,\dots,N_n-1\}$. Then for $t\in[t_k^n,t_{k+1}^n]$ we have

$$\tilde S^n_t = S_{t_k^n} + \frac{t-t_k^n}{t_{k+1}^n-t_k^n} S_{t_k^n,t_{k+1}^n},$$

so that

$$\int_{t_k^n}^{t_{k+1}^n} \tilde S^n_r\,\mathrm{d}\tilde S^n_r = S_{t_k^n} S_{t_k^n,t_{k+1}^n} + \frac{1}{2} S_{t_k^n,t_{k+1}^n} S_{t_k^n,t_{k+1}^n}, \tag{15}$$

from where the uniform convergence and the representation (14) follow by Lemma 4.17.

To prove that $\tilde A^n$ has uniformly bounded $\frac{p}{2}$-variation, consider $(s,t)\in\Delta_T$. If there exists k such that $t_k^n \le s < t \le t_{k+1}^n$, then we estimate

$$\begin{aligned}\big|\tilde A^n(s,t)\big|^{p/2} &= \left|\int_s^t \tilde S^n_{s,r}\,\mathrm{d}\tilde S^n_r\right|^{p/2} \le \left|\int_s^t (r-s)\frac{|S_{t_k^n,t_{k+1}^n}|^2}{|t_{k+1}^n-t_k^n|^2}\,\mathrm{d}r\right|^{p/2}\\ &= \frac{1}{2^{p/2}}|t-s|^p \frac{|S_{t_k^n,t_{k+1}^n}|^p}{|t_{k+1}^n-t_k^n|^p} \le \frac{|t-s|}{|t_{k+1}^n-t_k^n|}\|S\|^p_{p\text{-var},[t_k^n,t_{k+1}^n]}.\end{aligned} \tag{16}$$

Otherwise, let k_0 be the smallest k such that $t_k^n\in(s,t)$, and let k_1 be the largest such k. We decompose

$$\tilde A^n(s,t) = \tilde A^n\big(s,t_{k_0}^n\big) + \tilde A^n\big(t_{k_0}^n,t_{k_1}^n\big) + \tilde A^n\big(t_{k_1}^n,t\big) + \tilde S^n_{s,t_{k_0}^n}\tilde S^n_{t_{k_0}^n,t_{k_1}^n} + \tilde S^n_{s,t_{k_1}^n}\tilde S^n_{t_{k_1}^n,t}.$$

We get from (15) that

$$\big|\tilde{A}^n\big(t^n_{k_0}, t^n_{k_1}\big)\big|^{p/2} \lesssim \big|\big(S^n \cdot S\big)_{t^n_{k_0}, t^n_{k_1}} - S_{t^n_{k_0}} S_{t^n_{k_0}, t^n_{k_1}}\big|^{p/2} + \big(\langle S\rangle^n_{t^n_{k_0}, t^n_{k_1}}\big)^{p/2},$$

where $\langle S\rangle^n$ denotes the nth approximation of the quadratic variation. By the assumption (RIE) and Lemma 4.17, there exists a control function $\tilde{c}$ so that the right-hand side is bounded from above by $\tilde{c}(t^n_{k_0}, t^n_{k_1})$. Combining this with (16) and a simple estimate for the terms $\tilde{S}^n_{s,t^n_{k_0}} \tilde{S}^n_{t^n_{k_0}, t^n_{k_1}}$ and $\tilde{S}^n_{s,t^n_{k_1}} \tilde{S}^n_{t^n_{k_1}, t}$, we deduce that $\|\tilde{A}^n\|_{p/2\text{-var}} \lesssim \tilde{c}(0,T) + \|S\|^2_{p\text{-var}}$, and the proof is complete. □

We are now ready to prove the main result of this section.

Theorem 4.19. *Under Assumption (RIE), let $q > 0$ be such that $2/p + 1/q > 1$. Let $(F, F') \in \mathscr{C}^q_{\mathbb{S}}$ be a controlled path such that F is continuous. Then the rough path integral $\int F \,\mathrm{d}S$ which was defined in Theorem 4.9 is given by*

$$\int_0^t F_s \,\mathrm{d}S_s = \lim_{n\to\infty} \sum_{k=0}^{N_n - 1} F_{t^n_k} S_{t^n_k \wedge t, t^n_{k+1} \wedge t},$$

where the convergence is uniform in t.

Proof. For $n \in \mathbb{N}$ define $\tilde{F}^n$ as the linear interpolation of F between the points in π^n. Then $(\tilde{F}^n, F')$ is controlled by $\tilde{S}^n$: Clearly, $\|\tilde{F}^n\|_{q\text{-var}} \le \|F\|_{q\text{-var}}$. The remainder $\tilde{R}^n_{\tilde{F}^n}$ of $\tilde{F}^n$ with respect to $\tilde{S}^n$ is given by $\tilde{R}^n_{\tilde{F}^n}(s,t) = \tilde{F}^n_{s,t} - F'_s \tilde{S}^n_{s,t}$ for $(s,t) \in \Delta_T$. We need to show that $\tilde{R}^n_{\tilde{F}^n}$ has finite r-variation for $1/r = 1/p + 1/q$.

If $t^n_k \le s \le t \le t^n_{k+1}$, we have

$$\begin{aligned} \big|\tilde{R}^n_{\tilde{F}^n}(s,t)\big|^r &= \bigg|\frac{t-s}{t^n_{k+1} - t^n_k} F_{t^n_k, t^n_{k+1}} - F'_s \frac{t-s}{t^n_{k+1} - t^n_k} S_{t^n_k, t^n_{k+1}}\bigg|^r \\ &\le \bigg|\frac{t-s}{t^n_{k+1} - t^n_k}\bigg|^r \big(\|R_F\|_{r\text{-var},[t^n_k, t^n_{k+1}]} + \|F'\|^{r/q}_{q\text{-var},[t^n_k, s]} \|S\|^{r/p}_{p\text{-var},[t^n_k, t^n_{k+1}]}\big) \qquad (17) \\ &\le \frac{|t-s|}{|t^n_{k+1} - t^n_k|} \big(\|R_F\|_{r\text{-var},[t^n_k, t^n_{k+1}]} + \|F'\|_{q\text{-var},[t^n_k, t^n_{k+1}]} + \|S\|_{p\text{-var},[t^n_k, t^n_{k+1}]}\big), \end{aligned}$$

where in the last step we used that $1/r = 1/p + 1/q$, and thus $r/q + r/p = 1$.

Otherwise, there exists $k \in \{1, \dots, N_n - 1\}$ with $t^n_k \in (s,t)$. Let k_0 and k_1 the smallest and largest such k, respectively. Then

$$\begin{aligned} \big|\tilde{R}^n_{\tilde{F}^n}(s,t)\big|^r \lesssim_r{}& \big|\tilde{R}^n_{\tilde{F}^n}\big(s, t^n_{k_0}\big)\big|^r + \big|\tilde{R}^n_{\tilde{F}^n}\big(t^n_{k_0}, t^n_{k_1}\big)\big|^r + \big|\tilde{R}^n_{\tilde{F}^n}\big(t^n_{k_1}, t\big)\big|^r \\ &+ \big|F'_{s,t^n_{k_0}} S_{t^n_{k_0}, t^n_{k_1}}\big|^r + \big|F'_{s,t^n_{k_1}} S_{t^n_{k_1}, t}\big|^r. \end{aligned}$$

Now $\tilde{R}^n_{\tilde{F}^n}(t^n_{k_0}, t^n_{k_1}) = R_F(t^n_{k_0}, t^n_{k_1})$, and therefore we can use (17), the assumption on R_F, and the fact that $1/r = 1/p + 1/q$ (which is needed to treat the last two terms on the right-hand side), to obtain

$$\left\| \tilde{R}^n_{\tilde{F}^n} \right\|_{r\text{- var}} \lesssim_r \|R_F\|_{r\text{- var}} + \left\|F'\right\|_{q\text{- var}} + \|S\|_{p\text{- var}}.$$

On the other side, since F and R_F are continuous, $(\tilde{F}^n, \tilde{R}^n_{\tilde{F}^n})$ converges uniformly to (F, R_F). Now for continuous functions, uniform convergence with uniformly bounded p-variation implies convergence in p'-variation for every $p' > p$. See Exercise 2.8 in [19] for the case of Hölder continuous functions.

Thus, using Lemma 4.18, we see that if $p' > p$ and $q' > q$ are such that $2/p' + 1/q' > 0$, then $((\tilde{S}^n, \tilde{A}^n)_n)$ converges in $(p', p'/2)$-variation to (S, A°), where $A^\circ(s,t) = A(s,t) + 1/2\langle S\rangle_{s,t}$. Similarly, $((\tilde{F}^n, F', \tilde{R}^n_{\tilde{F}^n}))$ converges in (q', p', r')-variation to (F, F', R_F), where $1/r' = 1/p' + 1/q'$.

Proposition 4.11 now yields the uniform convergence of $\int \tilde{F}^n \,\mathrm{d}\tilde{S}^n$ to $\int F \circ \mathrm{d}S$, by which we denote the rough path integral of the controlled path (F, F') against the rough path (S, A°). But for every $t \in [0, T]$, we have

$$\begin{aligned}\lim_{n\to\infty} \int_0^t \tilde{F}^n_s \,\mathrm{d}\tilde{S}^n_s &= \lim_{n\to\infty} \sum_{k: t^n_{k+1} \le t} \frac{1}{2}(F_{t^n_k} + F_{t^n_{k+1}}) S_{t^n_k, t^n_{k+1}} \\ &= \lim_{n\to\infty} \Bigg(\sum_{k: t^n_{k+1} \le t} F_{t^n_k} S_{t^n_k, t^n_{k+1}} + \frac{1}{2} \sum_{k: t^n_{k+1} \le t} F_{t^n_k, t^n_{k+1}} S_{t^n_k, t^n_{k+1}} \Bigg).\end{aligned}$$

Using that F is controlled by S, it is easy to see that the second term on the right-hand side converges uniformly to $1/2 \int_0^t F'_s \,\mathrm{d}\langle S\rangle_s$, $t \in [0, T]$. Thus, the Riemann sums $\sum_k F_{t^n_k} S_{t^n_k \wedge \cdot, t^n_{k+1} \wedge \cdot}$ converge uniformly to $\int F \circ \mathrm{d}S - 1/2 \int F' \,\mathrm{d}\langle S\rangle$, and from the representation of the rough path integral as limit of compensated Riemann sums (6), it is easy to see that $\int F \circ \mathrm{d}S = \int F \,\mathrm{d}S + 1/2 \int F' \,\mathrm{d}\langle S\rangle$, which completes the proof. □

Remark 4.20. Given Theorem 4.19 it is natural to conjecture that if (S, A) is the rough path which we constructed in Theorem 4.12 and Lemma 4.14, then for typical price paths and for adapted, controlled, and continuous integrands F the rough path integral agrees with the model free integral of Section 3. This seems not very easy to show, but what can be verified is that if $F \in C^{1+\varepsilon}$, then for the integrand $F(S)$ both integrals coincide – simply take Riemann sums along the dyadic stopping times defined in (3).

Theorem 4.19 is reminiscent of Föllmer's pathwise Itô integral [17]. Föllmer assumes that the quadratic variation $\langle S\rangle$ of S exists along a given sequence of partitions and is continuous, and uses this to prove an Itô formula for S: if $F \in C^2$, then

$$F(S_t) = F(S_0) + \int_0^t \nabla F(S_s) \,\mathrm{d}S_s + \frac{1}{2} \int_0^t \mathrm{D}^2 F(S_s) \,\mathrm{d}\langle S\rangle_s, \tag{18}$$

where the integral $\int_0^\cdot \nabla F(S_s)\,\mathrm{d}S_s$ is given as limit of Riemann sums along that same sequence of partitions. Friz and Hairer [19] observe that if for $p \in (2,3)$ the function S is of finite p-variation and $\langle S\rangle$ is an arbitrary continuous function of finite $p/2$-variation, then setting

$$\mathrm{Sym}(A)(s,t) := \tfrac{1}{2}\big(S^i_{s,t}S^j_{s,t} + \langle S\rangle_{s,t}\big)$$

one obtains a "reduced rough path" $(S, \mathrm{Sym}(A))$. They continue to show that if F is controlled by S with *symmetric* derivative F', then it is possible to define the rough path integral $\int F\,\mathrm{d}S$. This is not surprising since then we have $F'_s A_{s,t} = F'_s \mathrm{Sym}(A)_{s,t}$ for the compensator term in the definition of the rough path integral. They also derive an Itô formula for reduced rough paths, which takes the same form as (18), except that now $\int \nabla F(S)\,\mathrm{d}S$ is a rough path integral (and therefore defined as limit of compensated Riemann sums).

So both the assumption and the result of [19] are slightly different from the ones in [17], and while it seems intuitively clear, it is still not shown rigorously that Föllmer's pathwise Itô integral is a special case of the rough path integral. We will now show that Föllmer's result is a special case of Theorem 4.19. For that purpose, we only need to prove that Föllmer's condition on the convergence of the quadratic variation is a special case of the assumption in Theorem 4.19, at least as long as we only need the symmetric part of the area.

Definition 4.21. *Let $f \in C([0,T],\mathbb{R})$ and let $\pi^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$, $n \in \mathbb{N}$ be such that* $\sup\{|f_{t^n_k,t^n_{k+1}}| : k = 0,\dots,N_n-1\}$ *converges to* 0. *We say that f has* quadratic variation along (π^n) in the sense of Föllmer *if the sequence of discrete measures (μ^n) on $([0,T],\mathcal{B}[0,T])$, defined by*

$$\mu_n := \sum_{k=0}^{N_n-1} |f_{t^n_k,t^n_{k+1}}|^2 \delta_{t^n_k}, \tag{19}$$

converges weakly to a non-atomic measure μ. We write $[f]_t$ for the "distribution function" of μ (in general μ will not be a probability measure). The function $f = (f^1,\dots,f^d) \in C([0,T],\mathbb{R}^d)$ has quadratic variation along (π^n) in the sense of Föllmer *if this holds for all f^i and $f^i + f^j$, $1 \le i,j \le d$. In this case, we set*

$$[f^i,f^j]_t := \tfrac{1}{2}\big([f^i+f^j]_t - [f^i]_t - [f^j]_t\big), \qquad t \in [0,T].$$

Lemma 4.22 (see also [41], Proposition 6.1). *Let $p \in (2,3)$, and let $S = (S^1,\dots,S^d) \in C([0,T],\mathbb{R}^d)$ have finite p-variation. Let $\pi^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$, $n \in \mathbb{N}$, be a sequence of partitions such that* $\sup\{|S_{t^n_k,t^n_{k+1}}| : k = 0,\dots,N_n-1\}$ *converges to* 0. *Then the following conditions are equivalent*:

(1) *The function S has quadratic variation along (π^n) in the sense of Föllmer.*
(2) *For all $1 \le i,j \le d$, the discrete quadratic variation*

$$\langle S^i,S^j\rangle^n_t := \sum_{k=0}^{N_n-1} S^i_{t^n_k\wedge t,t^n_{k+1}\wedge t} S^j_{t^n_k\wedge t,t^n_{k+1}\wedge t}$$

converges uniformly in $C([0,T],\mathbb{R})$ *to a limit* $\langle S^i, S^j\rangle$.

(3) *For* $S^{n,i} := \sum_{k=0}^{N_n-1} S^i_{t^n_k}\mathbf{1}_{[t^n_k,t^n_{k+1})}$, $i\in\{1,\dots,d\}$, $n\in\mathbb{N}$, *the Riemann sums* $(S^{n,i}\cdot S^j)+(S^{n,j}\cdot S^i)$ *converge uniformly to a limit* $\int S^i\,\mathrm{d}S^j + \int S^j\,\mathrm{d}S^i$. *Moreover, the symmetric part of the approximate area,*

$$\mathrm{Sym}\big(A^n\big)^{i,j}(s,t) = \tfrac{1}{2}\big(\big(S^{n,i}\cdot S^j\big)_{s,t} + \big(S^{n,j}\cdot S^i\big)_{s,t} - S^i_s S^j_{s,t} - S^j_s S^i_{s,t}\big),\quad 1\le i,j\le d,\,(s,t)\in\Delta_T,$$

has uniformly bounded $p/2$*-variation along* (π^n), *in the sense of* (12).

If these conditions hold, then $[S^i,S^j]=\langle S^i,S^j\rangle$ *for all* $1\le i,j\le d$.

Proof. Assume (1) and note that

$$S^i_{t^n_k\wedge t,t^n_{k+1}\wedge t} S^j_{t^n_k\wedge t,t^n_{k+1}\wedge t} = \tfrac{1}{2}\big(\big((S^i+S^j)_{t^n_k\wedge t,t^n_{k+1}\wedge t}\big)^2 - \big(S^i_{t^n_k\wedge t,t^n_{k+1}\wedge t}\big)^2 - \big(S^j_{t^n_k\wedge t,t^n_{k+1}\wedge t}\big)^2\big).$$

Thus, the uniform convergence of $\langle S^i,S^j\rangle^n$ and the fact that $\langle S^i,S^j\rangle=[S^i,S^j]$ follow once we show that Föllmer's weak convergence of the measures (19) implies the uniform convergence of their distribution functions. But since the limiting distribution is continuous by assumption, this is a standard result.

Next, assume (2). The uniform convergence of the Riemann sums $(S^{n,i}\cdot S^j)+(S^{n,j}\cdot S^i)$ is shown as in Lemma 4.17. To see that $\mathrm{Sym}(A^n)$ has uniformly bounded $p/2$-variation along (π^n), note that for $0\le k\le \ell\le N_n$ and $1\le i,j\le d$ we have

$$\begin{aligned}&\big|\big(S^{n,i}\cdot S^j\big)_{t^n_k,t^n_\ell} + \big(S^{n,j}\cdot S^i\big)_{t^n_k,t^n_\ell} - S^i_s S^j_{t^n_k,t^n_\ell} - S^j_s S^i_{t^n_k,t^n_\ell}\big|^{p/2}\\ &\quad= \big|S^i_{t^n_k,t^n_\ell} S^j_{t^n_k,t^n_\ell} - \big\langle S^i,S^j\big\rangle^n_{t^n_k,t^n_\ell}\big|^{p/2}\\ &\quad\le \|S\|_{p\text{-var},[t^n_k,t^n_\ell]} + \big\|\big\langle S^i,S^j\big\rangle^n\big\|_{1\text{-var},[t^n_k,t^n_\ell]}.\end{aligned}$$

That $\|\langle S^i,S^j\rangle^n\|_{1\text{-var}}$ is uniformly bounded in n is shown in Lemma 4.17.

That (3) implies (1) is also shown in Lemma 4.17. □

Remark 4.23. With Theorem 4.19 we can only derive an Itô formula for $F\in C^{2+\varepsilon}$, since we are only able to integrate $\nabla F(S)$ if $\nabla F\in C^{1+\varepsilon}$. But this only seems to be due to the fact that our analysis is not sharp. We expect that typical price paths have an associated rough path of finite 2-variation, up to logarithmic corrections. For such rough paths, the integral extends to integrands $F\in C^1$, see Chapter 10.5 of [20]. For typical price paths (but not for the area), it is shown in [40], Section 4.3, that they are of finite 2-variation up to logarithmic corrections.

Appendix A: Pathwise Hoeffding inequality

In the construction of the pathwise Itô integral for typical price processes, we needed the following result, a pathwise formulation of the Hoeffding inequality which is due to Vovk. Here we present a slightly adapted version.

Lemma A.1 ([40], Theorem A.1). *Let* $(\tau_n)_{n\in\mathbb{N}}$ *be a strictly increasing sequence of stopping times with* $\tau_0 = 0$, *such that for every* $\omega \in \Omega$ *we have* $\tau_n(\omega) = \infty$ *for all but finitely many* $n \in \mathbb{N}$. *Let for* $n \in \mathbb{N}$ *the function* $h_n : \Omega \to \mathbb{R}^d$ *be* $\mathcal{F}_{\tau_n}$*-measurable, and suppose that there exists a* $\mathcal{F}_{\tau_n}$*-measurable bounded function* $b_n : \Omega \to \mathbb{R}$, *such that*

$$\sup_{t\in[0,T]} \left| h_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega) \right| \le b_n(\omega) \tag{20}$$

for all $\omega \in \Omega$. *Then for every* $\lambda \in \mathbb{R}$ *there exists a simple strategy* $H^\lambda \in \mathcal{H}_1$ *such that*

$$1 + \left(H^\lambda \cdot S\right)_t \ge \exp\left(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n\wedge t, \tau_{n+1}\wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2\right)$$

for all $t \in [0, T]$, *where* $N_t := \max\{n \in \mathbb{N} : \tau_n \le t\}$.

Proof. Let $\lambda \in \mathbb{R}$. The proof is based on the following deterministic inequality: if (20) is satisfied, then for all $\omega \in \Omega$ and all $t \in [0, T]$ we have that

$$\begin{aligned}
&\exp\left(\lambda h_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega) - \frac{\lambda^2}{2} b_n^2(\omega)\right) - 1 \\
&\quad\le \exp\left(-\frac{\lambda^2}{2} b_n^2(\omega)\right) \frac{e^{\lambda b_n(\omega)} - e^{-\lambda b_n(\omega)}}{2 b_n(\omega)} h_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega) \\
&\quad=: f_n(\omega) S_{\tau_n\wedge t, \tau_{n+1}\wedge t}(\omega).
\end{aligned} \tag{21}$$

This inequality is shown in (A.1) of [40]. We define $H_t^\lambda := \sum_n F_n 1_{(\tau_n, \tau_{n+1}]}(t)$, with F_n that have to be specified. We choose $F_0 := f_0$, which is bounded and $\mathcal{F}_{\tau_0}$-measurable, and on $[0, \tau_1]$ we obtain

$$1 + \left(H^\lambda \cdot S\right)_t \ge \exp\left(\lambda h_0 S_{\tau_n\wedge t, \tau_{n+1}\wedge t} - \frac{\lambda^2}{2} b_0^2\right).$$

Observe also that $1 + (H^\lambda \cdot S)_{\tau_1} = 1 + f_0 S_{\tau_0, \tau_1}$ is bounded, because by assumption $h_0 S_{\tau_0, \tau_1}$ is bounded by the bounded random variable b_0.

Assume now that F_k has been defined for $k = 0, \dots, m-1$, that

$$1 + \left(H^\lambda \cdot S\right)_t \ge \exp\left(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n\wedge t, \tau_{n+1}\wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2\right)$$

for all $t \in [0, \tau_m]$, and that $1 + (H^\lambda \cdot S)_{\tau_m}$ is bounded. We define $F_m := (1 + (H^\lambda \cdot S)_{\tau_m}) f_m$, which is $\mathcal{F}_{\tau_m}$-measurable and bounded. From (21), we obtain for $t \in [\tau_m, \tau_{m+1}]$

$$\begin{aligned}
&1 + \left(H^\lambda \cdot S\right)_t \\
&\quad= 1 + \left(H^\lambda \cdot S\right)_{\tau_m} + \left(1 + \left(H^\lambda \cdot S\right)_{\tau_m}\right) f_m S_{\tau_m\wedge t, \tau_{m+1}\wedge t}
\end{aligned}$$

$$\geq \big(1 + (H^\lambda \cdot S)_{\tau_m}\big) \exp\bigg(\lambda h_m S_{\tau_m \wedge t, \tau_{m+1}\wedge t} - \frac{\lambda^2}{2} b_m^2\bigg)$$

$$\geq \exp\Bigg(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n \wedge t, \tau_{n+1}\wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2\Bigg),$$

where in the last step we used the induction hypothesis. From the first line of the previous equation, we also obtain that $1 + (H^\lambda \cdot S)_{\tau_{m+1}}$ is bounded because $f_m S_{\tau_m, \tau_{m+1}}$ is bounded for the same reason that $f_0 S_{\tau_0, \tau_1}$ is bounded. □

Appendix B: Davie's criterion

It was already observed by Davie [8] that in certain situations the rough path integral can be constructed as limit of Riemann sums and not just compensated Riemann sums. Davie shows that under suitable conditions, the usual Euler scheme (without "area compensation") converges to the solution of a given rough differential equation. But from there it is easily deduced that then also the rough path integral is given as limit of Riemann sums. Here we show that Davie's criterion implies our assumption (RIE).

Let $p \in (2,3)$ and let $\mathbb{S} = (S, A)$ be a $1/p$-Hölder continuous rough path, that is $|S_{s,t}| \lesssim |t-s|^{1/p}$ and $|A(s,t)| \lesssim |t-s|^{2/p}$. Write $\alpha := 1/p$ and let $\beta \in (1-\alpha, 2\alpha)$. Davie assumes that there exists $C > 0$ such that the area process A satisfies

$$\Bigg|\sum_{j=k}^{\ell-1} A\big(jh, (j+1)h\big)\Bigg| \leq C(\ell - k)^\beta h^{2\alpha}, \tag{B.1}$$

whenever $0 < k < \ell$ are integers and $h > 0$ such that $\ell h \leq T$. Under these conditions, Theorem 7.1 of [8] implies that for $F \in C^\gamma$ with $\gamma > p$ and for $t_k^n = kT/n$, $n, k \in \mathbb{N}$, the Riemann sums

$$\sum_{k=0}^{n-1} F(S_{t_k^n}) S_{t_k^n \wedge t, t_{k+1}^n \wedge t}, \quad t \in [0, T],$$

converge uniformly to the rough path integral. But it can be easily deduced from (B.1) that the area process A is given as limit of non-anticipating Riemann sums along $(t^n)_n$. Indeed, letting $h = T/n$,

$$\Bigg|\int_0^t S_s \,\mathrm{d}S_s - \sum_{k=0}^{n-1} S_{t_k^n} S_{t_k^n \wedge t, t_{k+1}^n \wedge t}\Bigg|$$

$$= \Bigg|\sum_{k=0}^{n-1}\bigg(\int_{t_k^n \wedge t}^{t_{k+1}^n \wedge t} S_s \,\mathrm{d}S_s - S_{t_k^n \wedge t} S_{t_k^n \wedge t, t_{k+1}^n \wedge t}\bigg)\Bigg|$$

$$= \left|\sum_{k=0}^{n-1} A\big(t_k^n \wedge t, t_{k+1}^n \wedge t\big)\right| \le \left|\sum_{k=0}^{\lfloor t/h\rfloor-1} A_{kh,(k+1)h}\right| + \big|A\big(\lfloor t/h\rfloor, t\big)\big|$$
$$\lesssim C\lfloor t/h\rfloor^{\beta} h^{2\alpha} + h^{2\alpha}\|A\|_{2\alpha} \lesssim Cth^{2\alpha-\beta} + h^{2\alpha}\|A\|_{2\alpha}.$$

Since $\beta < 2\alpha$, the right-hand side converges to 0 as n goes to ∞ (and thus h goes to 0). Furthermore, (B.1) implies the "uniformly bounded $p/2$-variation" condition (12):

$$\big|(S^n \cdot S)_{t_k^n, t_\ell^n} - S_{t_k^n} S_{t_k^n, t_\ell^n}\big| \le \left|\int_{t_k^n}^{t_\ell^n} S_s \,\mathrm{d}S_s - S_{t_k^n} S_{t_k^n, t_\ell^n}\right| + \left|\sum_{j=k}^{\ell-1}\left(\int_{t_j^n}^{t_{j+1}^n} S_s \,\mathrm{d}S_s - S_{t_j^n} S_{t_j^n, t_{j+1}^n}\right)\right|$$
$$\le \|A\|_{2\alpha}\big|t_\ell^n - t_k^n\big|^{2\alpha} + \left|\sum_{j=k}^{\ell-1} A_{t_k^n, t_{k+1}^n}\right|$$
$$\le \|A\|_{2\alpha}\big|t_\ell^n - t_k^n\big|^{2\alpha} + C(\ell - k)^{\beta} h^{2\alpha}$$
$$\le \|A\|_{2\alpha}\big|t_\ell^n - t_k^n\big|^{2\alpha} + C\big|t_\ell^n - t_k^n\big|^{2\alpha}.$$

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