Approximation of a stochastic wave equation in dimension three, with application to a support theorem in Hölder norm: The non-stationary case

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This paper is a continuation of (*Bernoulli* **20** (2014) 2169–2216) where we prove a characterization of the support in Hölder norm of the law of the solution to a stochastic wave equation with three-dimensional space variable and null initial conditions. Here, we allow for non-null initial conditions and, therefore, the solution does not possess a stationary property in space. As in (*Bernoulli* **20** (2014) 2169–2216), the support theorem is a consequence of an approximation result, in the convergence of probability, of a sequence of evolution equations driven by a family of regularizations of the driving noise. However, the method of the proof differs from (*Bernoulli* **20** (2014) 2169–2216) since arguments based on the stationarity property of the solution cannot be used.

Keywords: approximating schemes; stochastic wave equation; support theorem

1. Introduction

This paper is a continuation of [7], where we prove a characterization of the topological support in Hölder norm for the law of the solution of a stochastic wave equation with vanishing initial conditions. Consider the stochastic partial differential equation (SPDE)

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t, x) = \sigma\big(u(t, x)\big)\dot{M}(t, x) + b\big(u(t, x)\big),$$

$$u(0, x) = v_0(x), \qquad \frac{\partial}{\partial t}u(0, x) = \tilde{v}_0(x), \tag{1.1}$$

where Δ denotes the Laplacian on $\mathbb{R}^3$, $T > 0$ is fixed, $t \in (0, T]$ and $x \in \mathbb{R}^3$. The nonlinear terms and the initial conditions are defined by functions $\sigma, b : \mathbb{R} \to \mathbb{R}$ and $v_0, \tilde{v}_0 : \mathbb{R}^3 \to \mathbb{R}$, respectively. The notation $\dot{M}(t, x)$ refers to the formal derivative of a Gaussian random field M white in the time variable and with a correlation in the space variable given by a Riesz kernel. More specifically,

$$\mathbb{E}\big(\dot{M}(t, x)\dot{M}(s, y)\big) = \delta_0(t - s)|x - y|^{-\beta}, \tag{1.2}$$

where δ_0 denotes the delta Dirac measure and $\beta \in (0, 2)$.

We consider a random field solution to the SPDE (1.1), which means a real-valued adapted (with respect to the natural filtration generated by the Gaussian process M) stochastic process $\{u(t,x), (t,x) \in (0,T] \times \mathbb{R}^3\}$ satisfying

$$u(t,x) = X^0(t,x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)\sigma\big(u(s,y)\big) M(\mathrm{d}s, \mathrm{d}y)$$
$$+ \int_0^t \big[G(t-s, \cdot) \star b\big(u(s,\cdot)\big)\big](x)\,\mathrm{d}s. \tag{1.3}$$

Here,

$$X^0(t,x) = \big[G(t) \star \tilde{v}_0 \big](x) + \Big[\frac{\mathrm{d}}{\mathrm{d}t} G(t) \star v_0 \Big](x), \tag{1.4}$$

$G(t)$ is the fundamental solution to the wave equation in dimension three, $G(t, \mathrm{d}x) = \frac{1}{4\pi t}\sigma_t(\mathrm{d}x)$, where $\sigma_t(x)$ denotes the uniform surface measure on the sphere of radius t with total mass $4\pi t^2$ (see, e.g., [8]), and the symbol "$\star$" denotes the convolution in the spatial argument.

The stochastic integral (also termed stochastic convolution) in (1.3) is defined as a stochastic integral with respect to a sequence of independent standard Brownian motions $\{W_j(s)\}_{j \in \mathbb{N}}$, as follows. Let $\mathcal{H}$ be the Hilbert space defined by the completion of $\mathcal{S}(\mathbb{R}^3)$, the space of rapidly decreasing functions on $\mathbb{R}^3$, endowed with the semi-inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi) \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)},$$

where $\mathcal{F}$ denotes the Fourier transform operator and $\mu(\mathrm{d}\xi) = \mathcal{F}^{-1}(|\xi|^{-\beta}\,\mathrm{d}\xi) = |\xi|^{\beta-3}\,\mathrm{d}\xi$. Then

$$\int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)\sigma\big(u(s,y)\big) M(\mathrm{d}s, \mathrm{d}y)$$
$$:= \sum_{j \in \mathbb{N}} \int_0^t \big\langle G(t-s, x-*)\sigma\big(u(s,*)\big), e_j \big\rangle_{\mathcal{H}} W_j(\mathrm{d}s), \tag{1.5}$$

where $(e_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3)$ is a complete orthonormal basis of $\mathcal{H}$.

Assume that $\varphi \in \mathcal{H}$ is a signed measure with finite total variation. Then, by applying [10], Theorem 5.2 (see also [11], Lemma 12.12, for the case of probability measures with compact support) and a polarization argument on the positive and negative parts of φ, we obtain

$$\|\varphi\|_{\mathcal{H}}^2 = C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(\mathrm{d}x)\varphi(\mathrm{d}y)|x-y|^{-\beta}. \tag{1.6}$$

For $t_0 \in [0,T]$, $K \subset \mathbb{R}^3$ compact and $\rho \in (0,1)$, we denote by $\mathcal{C}^\rho([t_0, T] \times K)$ the space of real functions g such that $\|g\|_{\rho, t_0, K} < \infty$, where

$$\|g\|_{\rho, t_0, K} := \sup_{(t,x) \in [t_0, T] \times K} \big|g(t,x)\big| + \sup_{\substack{(t,x),(\bar{t},\bar{x}) \in [t_0, T] \times K \\ (t,x) \neq (\bar{t},\bar{x})}} \frac{|g(t,x) - g(\bar{t}, \bar{x})|}{(|t - \bar{t}| + |x - \bar{x}|)^\rho}.$$

Let $0 < \rho' < \rho$ and $\mathcal{E}^{\rho'}([t_0, T] \times K)$ be the space of Hölder continuous functions g of degree ρ' such that

$$O_g(\delta) := \sup_{|t-s|+|x-y|<\delta} \frac{|g(t,x) - g(s,y)|}{(|t-s| + |x-y|)^{\rho'}} \to 0, \qquad \text{if } \delta \to 0. \tag{1.7}$$

The space $\mathcal{E}^{\rho'}([t_0, T] \times K)$ endowed with the norm $\| \cdot \|_{\rho', t_0, K}$ is a Polish space and the embedding $\mathcal{C}^{\rho}([t_0, T] \times K) \subset \mathcal{E}^{\rho'}([t_0, T] \times K)$ is compact.

Assume that the functions σ and b are Lipschitz continuous and the initial conditions v_0, $\tilde{v}_0$ satisfy the assumption (h2) of Theorem 1.1 below. Theorem 4.11 in [5] along with [4], Proposition 2.6, give the existence of a random field solution to (1.3) with sample paths in $\mathcal{C}^{\rho}([0, T] \times K)$, with $\rho \in (0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2})$.

For any $t \in (0, T]$, let $\mathcal{H}_t = L^2([0, t]; \mathcal{H})$. Fix $h \in \mathcal{H}_t$ and consider the deterministic evolution equation

$$
\begin{aligned}
\Phi^h(t, x) = X^0(t, x) &+ \big\langle G(t - \cdot, x - *)\sigma\big(\Phi^h(\cdot, *)\big), h \big\rangle_{\mathcal{H}_t} \\
&+ \int_0^t \mathrm{d}s \big[G(t-s, \cdot) \star \big(\Phi^h(s, \cdot)\big) \big](x).
\end{aligned}
\tag{1.8}
$$

The main objective in [7] is to prove that, in the particular case $v_0 = \tilde{v}_0 = 0$, the topological support of the law of the solution to (1.3) in the space $\mathcal{E}^{\rho}([t_0, T] \times K)$ with $\rho \in (0, \frac{2-\beta}{2})$ is the closure in the Hölder norm of the set $\{\Phi^h, h \in \mathcal{H}_T\}$, for any $t_0 > 0$ (see [7], Theorem 3.1).

The aim of this paper is to prove a partial extension of this result allowing non-null initial conditions v_0, $\tilde{v}_0$, but restricting to affine coefficients σ. In particular, this will apply to the *hyperbolic Anderson model* ($\sigma(x) = \lambda x$, $\lambda \neq 0$). The theorem is as follows.

Theorem 1.1. *Assume that*

- (h1) *the function σ is affine and b is Lipschitz continuous;*
- (h2) *$v_0, \tilde{v}_0 : \mathbb{R}^3 \to \mathbb{R}$ are bounded, $v_0 \in \mathcal{C}^2(\mathbb{R}^3)$, ∇v_0 is bounded, $\tilde{v}_0 \in \mathcal{C}(\mathbb{R}^3)$, Δv_0 and $\tilde{v}_0$ are Hölder continuous functions of degree γ_1, γ_2, respectively.*

Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then the topological support of the law of the solution to (1.3) in the space $\mathcal{E}^{\rho}([t_0, T] \times K)$ with $\rho \in (0, \gamma_1 \wedge \gamma_2 \wedge (\frac{2-\beta}{2}))$ is the closure in the Hölder norm $\| \cdot \|_{\rho, t_0, K}$ of the set $\{\Phi^h, h \in \mathcal{H}_T\}$, where Φ^h is given in (1.8).

After the seminal paper [17], an extensive literature on support theorems for stochastic differential equations appeared (see, e.g., [1,9,13], and references herein). The analysis of the uniqueness of invariant measures is one of the motivations for the characterization of the support of stochastic evolution equations (see [7], Section 1, for more details).

As in [7], Theorem 1.1 will be a corollary of a general result on approximations of equation (1.3) by a sequence of SPDEs obtained by smoothing the noise M. The precise statement, given in Theorem 2.1, provides a Wong–Zakai-type theorem in Hölder norm. It is of interest by its own. The method relies on [1], further developed and used in [2,9,12–14]. We refer the reader

to [7], Section 1, for a detailed description of the method for the proof of support theorems based on approximations.

In contrast with the situation considered in [7], the solution to (1.3) with non-null initial conditions does not possess the spatial stationary property termed $\mathcal{S}$ *property* in [3]. This property is crucial in the proof of the analogue of Theorem 2.2 and more precisely, in establishing the upper bound of L^p norms of increments in space when the initial conditions are null. The new approach to the proof of a similar upper bound when the initial conditions do not vanish uses fractional Sobolev norms and the classical Sobolev's embeddings (see Proposition 2.5). To some extend, some of the results of this paper are a refinement and an extension of results of [5]. Compare, for example, Lemma 2.6 with [5], Proposition 3.5, and Proposition 2.5 with [5], Theorem 4.6. Others, like Proposition 2.7, are crucial results to establish the approximations. The proof of Proposition 2.7 requires the validity of the inequality $\|B(f) - B(g)\|_{\gamma,p,\mathcal{O}} \leq C\|f - g\|_{\gamma,p,\mathcal{O}}$ for $B : \mathbb{R} \to \mathbb{R}$ and functions f, g belonging to the fractional Sobolev space $W^{\gamma,p}(\mathcal{O})$ (see (2.8), (2.9) for the definition of these spaces). This holds when B is affine and not only Lipschitz, which explains the hypothesis on σ in Theorem 1.1. The use of fractional norms seems to be at the origin of this restriction, as was noticed for example in [15].

The paper is organized in the following way. In Section 2, we prove Theorem 2.1 – a general result on approximations of SPDEs in the convergence of probability and in the Hölder norm – which in turn follow from Theorems 2.2 and 2.3. As a particular case, the characterization of the support stated in Threorem 1.1 is established. Section 3 gathers some technical results used in the proofs.

2. Approximations of the wave equation

As in the companion paper [7], we consider smooth approximations of W defined as follows. For any $n \in \mathbb{N}$, we define the partition of $[0, T]$ consisting of the points $\frac{iT}{2^n}$, $i = 0, 1, \ldots, 2^n$. Denote by Δ_i the interval $[\frac{iT}{2^n}, \frac{(i+1)T}{2^n})$ and by $|\Delta_i|$ its length. We write $W_j(\Delta_i)$ for the increment $W_j(\frac{(i+1)T}{2^n}) - W_j(\frac{iT}{2^n})$, $i = 0, \ldots, 2^n - 1$, $j \in \mathbb{N}$. Then we define W^n as the sequence whose terms are

$$W_j^n = \int_0^{\cdot} \dot{W}_j^n(s)\,ds, \qquad j \in \mathbb{N},$$

where for $j > n$, $\dot{W}_j^n = 0$, and for $1 \leq j \leq n$,

$$\dot{W}_j^n(t) = \begin{cases} \sum_{i=0}^{2^n-2} 2^n T^{-1} W_j(\Delta_i) 1_{\Delta_{i+1}}(t), & \text{if } t \in \left[2^{-n}T, T\right], \\ 0, & \text{if } t \in \left[0, 2^{-n}T\right[. \end{cases}$$

Set

$$w^n(t, x) = \sum_{j \in \mathbb{N}} \dot{W}_j^n(t) e_j(x). \tag{2.1}$$

It can be easily checked that, for any $p \in [2, \infty)$,

$$\|w^n\|_{L^p(\Omega, \mathcal{H}_T)} \le Cn^{1/2}2^{n/2}. \tag{2.2}$$

Hence, w^n belongs to $\mathcal{H}_T$ a.s.

We consider the integral equations

$$
\begin{aligned}
X(t, x) &= X^0(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)\big(X(s, y)\big)M(ds, dy) \\
&\quad + \big\langle G(t - \cdot, x - *)D\big(X(\cdot, *)\big), h\big\rangle_{\mathcal{H}_t} + \int_0^t \big[G(t - s, \cdot) \star b\big(X(s, \cdot)\big)\big](x) ds,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
X_n(t, x) &= X^0(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)A\big(X_n(s, y)\big)M(ds, dy) \\
&\quad + \big\langle G(t - \cdot, x - *)B\big(X_n(\cdot, *)\big), w^n\big\rangle_{\mathcal{H}_t} + \big\langle G(t - \cdot, x - *)D\big(X_n(\cdot, *)\big), h\big\rangle_{\mathcal{H}_t} \\
&\quad + \int_0^t \big[G(t - s, \cdot) \star b\big(X_n(s, \cdot)\big)\big](x) ds,
\end{aligned}
\tag{2.4}
$$

where $h \in \mathcal{H}_T$, w^n defined as in (2.1), $A, B, D, b : \mathbb{R} \to \mathbb{R}$, and $X^0 : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is the deterministic function defined in (1.4).

For each $t \in [0, T]$, let $t_n = \max\{\underline{t}_n - 2^{-n}T, 0\}$, with

$$\underline{t}_n = \max\{k2^{-n}T, k = 0, \ldots, 2^n - 1 : k2^{-n}T \le t\}. \tag{2.5}$$

By means of the following expressions, we define stochastic processes close to $X(t_n, x)$ and $X_n(t_n, x)$, $(t, x) \in [0, T] \times \mathbb{R}^3$, respectively:

$$
\begin{aligned}
X(t, t_n, x) &= X^0(t, x) + \int_0^{t_n} \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)\big(X(s, y)\big)M(ds, dy) \\
&\quad + \big\langle G(t - \cdot, x - *)D\big(X(\cdot, *)\big)1_{[0, t_n]}(\cdot), h\big\rangle_{\mathcal{H}_t} \\
&\quad + \int_0^{t_n} \big[G(t - s, \cdot) \star b\big(X(s, \cdot)\big)\big](x) ds,
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
X_n^-(t, x) &= X^0(t, x) + \int_0^{t_n} \int_{\mathbb{R}^3} G(t - s, x - y)A\big(X_n(s, y)\big)M(ds, dy) \\
&\quad + \big\langle G(t - \cdot, x - *)B\big(X_n(\cdot, *)\big)1_{[0, t_n]}(\cdot), w^n\big\rangle_{\mathcal{H}_t} \\
&\quad + \big\langle G(t - \cdot, x - *)D\big(X_n(\cdot, *)\big)1_{[0, t_n]}(\cdot), h\big\rangle_{\mathcal{H}_t} \\
&\quad + \int_0^{t_n} \big[G(t - s, \cdot) \star b\big(X_n(s, \cdot)\big)\big](x) ds.
\end{aligned}
\tag{2.7}
$$

We will consider the following set of assumptions.

Hypothesis (H).

(H1) *The functions A, B, D, $b : \mathbb{R} \mapsto \mathbb{R}$ are globally Lipschitz continuous.*

(H2) *$v_0, \tilde{v}_0 : \mathbb{R}^3 \to \mathbb{R}$ are bounded, $v_0 \in \mathcal{C}^2(\mathbb{R}^3)$, ∇v_0 is bounded, $\tilde{v}_0 \in \mathcal{C}(\mathbb{R}^3)$, Δv_0 and $\tilde{v}_0$ are Hölder continuous functions of degree γ_1, γ_2, respectively.*

Let $\mathcal{O}$ be a bounded or unbounded open subset of $\mathbb{R}^3$, $p \in [1, \infty)$, $\gamma \in (0, 1)$. We define

$$\|g\|_{\gamma, p, \mathcal{O}} = \left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|g(x) - g(y)|^p}{|x - y|^{3 + \gamma p}} \right)^{1/p}. \tag{2.8}$$

Then we denote by $W^{\gamma, p}(\mathcal{O})$ the Banach space consisting of functions $g : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|g\|_{W^{\gamma, p}(\mathcal{O})} := \left(\|g\|_{L^p(\mathcal{O})}^p + \|g\|_{\gamma, p, \mathcal{O}}^p \right)^{1/p} < \infty. \tag{2.9}$$

From [5], *Lemmas* 4.2 *and* 4.4 *and* [4], *Lemma* 4.2, *it follows that* (H2) *implies the following.*

(H2.1) *For any $t \in [0, T]$ and any bounded domain $\mathcal{O} \subset \mathbb{R}^3$, for any $p \in [2, \infty)$ such that $\frac{2-\beta}{2} > \frac{3}{p}$, and for any $\gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge (\frac{2-\beta}{2} - \frac{3}{p}))$,*

$$\left\| X^0(t) \right\|_{W^{\gamma, p}(\mathcal{O})} < \infty.$$

(H2.2) *$(t, x) \mapsto X^0(t, x)$ is continuous and $\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} |X^0(t, x)| < \infty$.*

The existence and uniqueness of a random field solution to the equations (2.3), (2.4) is established as in [7], Theorem 5.1. It is proved using the convergence of a Picard iteration scheme. For (2.3), the Picard approximations converge in $L^p(\Omega)$, uniformly in $(t, x) \in [0, T] \times \mathbb{R}^3$. For (2.4), the convergence of the Picard approximations holds in probability. It is obtained using a localization in Ω. Notice that equation (2.4) is more general than (2.3).

The aim of this section is to prove the following theorem, which is the analogue of [7], Theorem 2.2, in the context of this article.

Theorem 2.1. *We assume Hypothesis* (H) *and in addition that the function B is affine. Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then for any $\rho \in (0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2})$ and $\lambda > 0$,*

$$\lim_{n \to \infty} \mathbb{P}\left(\|X_n - X\|_{\rho, t_0, K} > \lambda \right) = 0. \tag{2.10}$$

With a particular choice of the functions A, B and D in equations (2.3), (2.4), this theorem yields the characterization of the support stated in Theorem 1.1 (see the proof of Theorem 3.1 in [7]).

The proof of Theorem 2.1 entails several steps. As in the stationary case considered in [7], the main ingredients are *local L^p estimates* of increments of X_n and X, in time and in space, and a *local L^p convergence* of the sequence $X_n(t, x)$ to $X(t, x)$. Here, in contrast with [7], *local L^p* estimates of increments of X_n and X in space are obtained via Sobolev's embeddings.

We remind the *localization* procedure introduced in [14] and also used in [7]. For any integer $n \geq 1$ and $t \in [0, T]$, define

$$L_n(t) = \left\{ \sup_{1 \leq j \leq n} \sup_{0 \leq i \leq [2^n t T^{-1} - 1]^+} \left| W_j(\Delta_i) \right| \leq \alpha n^{1/2} 2^{-n/2} \right\}, \tag{2.11}$$

with $\alpha > (2 \ln 2)^{1/2}$. The mapping $t \mapsto L_n(t)$ is decreasing and $\lim_{n \to \infty} \mathbb{P}(L_n(t)^c) = 0$ (see [14], Lemma 2.1). It is easy to check that

$$\left\| w^n(t, *) 1_{L_n(t)} \right\|_{\mathcal{H}} \leq C n^{3/2} 2^{n/2}, \tag{2.12}$$

and also

$$\left\| w^n 1_{L_n(t')} 1_{[t, t']} \right\|_{\mathcal{H}_T} \leq C n^{3/2} 2^{n/2} \left| t' - t \right|^{1/2}, \qquad 0 \leq t \leq t' \leq T.$$

In particular, if $[t, t'] \subset \Delta_i$ for some $i = 0, \ldots, 2^n - 1$, then

$$\left\| w^n 1_{L_n(t')} 1_{[t, t']} \right\|_{\mathcal{H}_T} \leq C n^{3/2}. \tag{2.13}$$

As has been said in the Introduction, the proof of Theorem 2.1 will follow from Theorems 2.2 and 2.3 below. These are the analogues of [7], Theorems 2.3 and 2.4, in the context of this article. We denote by $\| \cdot \|_p$ the $L^p(\Omega)$ norm, and for any compact set $K \subset \mathbb{R}^3$, we define

$$K(t) = \left\{ x \in \mathbb{R}^3 \colon d(x, K) \leq T - t \right\}, \qquad t \in [0, T], \tag{2.14}$$

where d denotes the Euclidean distance. Notice that $t \mapsto K(t)$ is a decreasing mapping.

Theorem 2.2. *We assume Hypothesis* (H) *and also that the function B is affine. Fix $t_0 > 0$ and $t_0 \leq t \leq \bar{t} \leq T$, $x, \bar{x} \in \mathbb{R}^3$. Then, for any $p \in [1, \infty)$ and $\rho \in (0, \gamma_1 \wedge \gamma_2 \wedge \frac{2 - \beta}{2})$, there exists a positive constant C such that*

$$\sup_{n \geq 1} \left\| \left[X_n(t, x) - X_n(\bar{t}, \bar{x}) \right] 1_{L_n(\bar{t})} \right\|_p \leq C \left(|\bar{t} - t| + |\bar{x} - x| \right)^{\rho}. \tag{2.15}$$

Theorem 2.3. *Assume Hypothesis* (H)*. Fix a compact set $K \subset \mathbb{R}^3$. Then, for any $p \in [1, \infty)$,*

$$\lim_{n \to \infty} \sup_{\substack{t \in [0, T] \\ x \in K(t)}} \left\| \left(X_n(t, x) - X(t, x) \right) 1_{L_n(t)} \right\|_p = 0. \tag{2.16}$$

The proof of Theorem 2.2 consists of two parts. First, we shall consider $t = \bar{t}$ and obtain (2.15), uniformly in $t \in [0, T]$. This is the difficult and novel part, and where the additional assumption on B being affine is needed. Then, using this result, we consider $x = \bar{x}$ and following the proof of [7], Proposition 2.9, we can establish (2.15), uniformly in x over compact sets. The details of the proof of the estimates of L^p increments in time are omitted, since they can be reconstructed from [7], Proposition 2.9, with minor changes.

Remark 2.4. Assume that Hypothesis (H) holds and, moreover,

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\big\|\big[X_n(t,x)-X_n(t,\bar{x})\big]1_{L_n(t)}\big\|_p \leq C|\bar{x}-x|^\rho,$$

with ρ as in Theorem 2.2. Then, with the same proof of [7], Proposition 2.9, one has

$$\sup_{n\geq 1}\big\|\big[X_n(t,x)-X_n(\bar{t},x)\big]1_{L_n(\bar{t})}\big\|_p \leq C|\bar{t}-t|^\rho,$$

for any $t_0 \leq t \leq \bar{t} \leq T$, $t_0 > 0$, uniformly over x on a compact set of $\mathbb{R}^3$.

The proof of Theorem 2.3 is very similar to [7], Theorem 2.4, and will also be omitted. Notice that the initial condition $X^0(t,x)$ cancels in the difference $X_n(t,x) - X(t,x)$, and also that in the proof of [7], Theorem 2.4, the stationarity property is never used.

The rest of the section is devoted to establish L^p estimates of increments in space. They will be derived from Proposition 2.5 below.

Proposition 2.5. *We assume Hypothesis* (H) *and that the function B is affine. Fix a compact set* $K \subset \mathbb{R}^3$ *and* $p \in (\frac{6\vee[2(4-\beta)]}{2-\beta}, \infty)$. *Then, for any* $t \in [0,T]$ *and* $\gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge (\frac{2-\beta}{2} - \frac{3}{p}))$,

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}\big(\big[\big\|X_n(t)\big\|^p_{W^{\gamma,p}(K(t))} + \big\|X_n^-(t)\big\|^p_{W^{\gamma,p}(K(t))}\big]1_{L_n(t)}\big) < +\infty. \tag{2.17}$$

Assume this has been proved. By the Sobolev embedding theorem (see, for instance, [16], Theorem E.12, page 257), for any bounded or unbounded domain $\mathcal{O} \subset \mathbb{R}^d$, $W^{\rho,p}(\mathcal{O}) \subset \mathcal{C}^{\bar{\rho}}(\mathcal{O})$, for each $\bar{\rho} < \rho - \frac{3}{p}$. Since Proposition 2.5 holds for any p large enough, (2.17) yields

$$\sup_{n\geq 0}\sup_{t\in[0,T]}\big\|\big(X_n(t,x)-X_n(t,\bar{x})\big)1_{L_n(t)}\big\|_p \leq C|x-\bar{x}|^\rho, \tag{2.18}$$

for any $\rho \in (0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2})$.

The next Lemma 2.6 and Proposition 2.7 are important ingredients in the proof of Proposition 2.5. Roughly speaking, Lemma 2.6 is an abstract result about upper bounds of L^p moments of fractional norms of indefinite stochastic integral, taking into account the size of the domain of integration in time. In Proposition 2.7, it is used to establish the discrepancy in the fractional norm, and in terms of n, between the Picard's iterations of $X_n(t,x)$ and $X_n^-(t,x)$ (see (2.4), (2.7), resp.).

For a function $f:\mathbb{R}^3 \mapsto \mathbb{R}$, we set

$$Df(u,x) = f(u+x) - f(u),$$
$$D^2f(u,x) = f(u-x) - 2f(u) + f(u+x).$$

Given a bounded set $\mathcal{O} \in \mathbb{R}^3$ and $\varepsilon > 0$, we denote by $\mathcal{O}^\varepsilon$ the open set

$$\mathcal{O}^\varepsilon = \big\{x \in \mathbb{R}^3 \colon \exists z \in \mathcal{O} \text{ such that } |x-z| < \varepsilon\big\}. \tag{2.19}$$

Lemma 2.6. *Fix* $p \in (\frac{6}{2-\beta}, \infty)$, $\gamma \in (0, \frac{2-\beta}{2} - \frac{3}{p})$, $t \in (0, T]$ *and a bounded domain* $\mathcal{O} \subset R^3$. *Let* $Z = \{Z(s, x), (s, x) \in [0, T] \times \mathbb{R}^3\}$ *be a stochastic process such that*

$$\int_0^t ds \mathbb{E}\big(\|Z(s)\|_{W^{\gamma,p}(\mathcal{O}^s)}^p\big) < \infty. \tag{2.20}$$

Then

$$\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{\mathbb{E}(\|[G(\cdot, x - *) - G(\cdot, y - *)]Z(\cdot, *)\|_{\mathcal{H}_t}^p)}{|x - y|^{3+\gamma p}}$$
$$\leq C t^{\eta(p/2-1)} \int_0^t ds \mathbb{E}\big(\|Z(s)\|_{W^{\gamma,p}(\mathcal{O}^s)}^p\big), \tag{2.21}$$

with $\eta := \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta) \in (1, 2)$. *Consequently,*

$$\mathbb{E}\big(\big\|\|G(\cdot, \bullet - *)Z(\cdot, *)\|_{\mathcal{H}_t}\big\|_{\gamma, p, \mathcal{O}}^p\big) \leq C t^{\eta(p/2-1)} \int_0^t ds \mathbb{E}\big(\|Z(s)\|_{W^{\gamma,p}(\mathcal{O}^s)}^p\big). \tag{2.22}$$

Proof. Throughout the proof, $\beta \in (0, 2)$ is fixed and we denote by $f(x)$ the Riesz kernel $|x|^{-\beta}$. Remember that the symbols "$\cdot$", "$*$" denote the relevant variables for the $\mathcal{H}_t$ norm, and "$\bullet$" the argument for the fractional norm $\|\cdot\|_{\gamma, p, \mathcal{O}}$.

Fix $x, y \in \mathbb{R}^3$. By applying the triangular inequality, we have

$$\big|\|G(\cdot, x - *)Z(\cdot, *)\|_{\mathcal{H}_t} - \|G(\cdot, y - *)Z(\cdot, *)\|_{\mathcal{H}_t}\big|$$
$$\leq \|[G(\cdot, x - *) - G(\cdot, y - *)]Z(\cdot, *)\|_{\mathcal{H}_t}. \tag{2.23}$$

Hence, (2.22) is a trivial consequence of (2.21).

Set

$$T_t := \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{\mathbb{E}(\|[G(\cdot, x - *) - G(\cdot, y - *)]Z(\cdot, *)\|_{\mathcal{H}_t}^p)}{|x - y|^{3+\gamma p}}.$$

By (1.6), we write

$$\|[G(\cdot, x - *) - G(\cdot, y - *)]Z(\cdot, *)\|_{\mathcal{H}_t}$$
$$= C\bigg(\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Z(s, u) f(u - v) Z(s, v) \tag{2.24}$$
$$\times \big[G(s, x - du) - G(s, y - du)\big]\big[G(s, x - dv) - G(s, y - dv)\big]\bigg)^{1/2}.$$

Fix $p \in (\frac{6}{2-\beta}, \infty)$, $\gamma \in (0, \frac{2-\beta}{2} - \frac{3}{p})$ and $\mathcal{O} \subset \mathbb{R}^3$, and let $\bar{\rho} = \gamma + \frac{3}{p}$. By (2.24) and using the method of the proof of [5], Proposition 3.5, increments of G are transferred to increments of the

factors f and Z. We obtain (see [5], pages 19–20),

$$T_t \leq C \sum_{i=1}^{4} \mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_i^t(x, y)|^{p/2}}{|x - y|^{\bar{\rho}p}} \right), \tag{2.25}$$

where

$$J_1^t(x, y) = \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) f(y - x + v - u)$$
$$\times \big[Z(s, x - u) - Z(s, y - u) \big] \big[Z(s, x - v) - Z(s, y - v) \big],$$

$$J_2^t(x, y) = \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) Df(v - u, x - y) Z(s, x - u)$$
$$\times \big[Z(s, x - v) - Z(s, y - v) \big],$$

$$J_3^t(x, y) = \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) Df(u - v, x - y)$$
$$\times Z(s, x - v) \big[Z(s, x - u) - Z(s, y - u) \big],$$

$$J_4^t(x, y) = \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) D^2 f(v - u, x - y) Z(s, x - u) Z(s, x - v).$$

Let

$$\mu_1(t, x, y) := \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) f(y - x + v - u). \tag{2.26}$$

The following properties hold:

$$\mu_1(t, x, y) \leq C t^{3-\beta}, \tag{2.27}$$

$$\sup_{s \in [0,T]} \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) f(y - x + v - u) \leq C \tag{2.28}$$

(see, e.g., [6], Lemma 5.1). Thus, by Hölder's inequality and (2.27) we obtain

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_1^t(x, y)|^{p/2}}{|x - y|^{\bar{\rho}p}} \right)$$
$$\leq C t^{(3-\beta)(p/2-1)} \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) f(y - x + v - u) \mathbb{E}\big(\|Z(s)\|_{\gamma, p, \mathcal{O}^s}^p \big).$$

By (2.28), this yields

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_1^t(x, y)|^{p/2}}{|x - y|^{\bar{\rho}p}} \right) \leq C t^{(3-\beta)(p/2-1)} \int_0^t ds \big(\|Z(s)\|_{\gamma, p, \mathcal{O}^s}^p \big). \tag{2.29}$$

By symmetry, the contributions of the terms $J_2^t(x, y)$ and $J_3^t(x, y)$ are the same. Hence, we will focus on $J_2^t(x, y)$. Set

$$\mu_2(t, x, y) = \int_0^t ds \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) |Df(v - u, x - y)|. \qquad (2.30)$$

The following properties hold:

$$\mu_2(t, x, y) \leq C|x - y|^\alpha t^{3-(\alpha+\beta)}, \qquad (2.31)$$

$$\sup_{s \in [0,T]} \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) |Df(v - u, x - y)| \leq C|x - y|^\alpha, \qquad (2.32)$$

for any $\alpha \in (0, (2 - \beta) \wedge 1)$ (see [6], Lemma 5.4, and a slight modification of [5], Lemma 6.1, resp.). By applying Hölder's and Schwarz' inequalities and (2.31), we can write

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_2^t(x, y)|^{p/2}}{|x - y|^{\bar{\rho}p}} \right)$$

$$= \mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \left(\frac{|J_2^t(x, y)|}{|x - y|^\alpha |x - y|^{2\bar{\rho} - \alpha}} \right)^{p/2} \right)$$

$$\leq \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \left(\frac{\mu_2(t, x, y)}{|x - y|^\alpha} \right)^{p/2 - 1} \int_0^t ds \int_{\mathbb{R}^3} G(s, du)$$

$$\times \int_{\mathbb{R}^3} G(s, dv) \frac{|Df(v - u, x - y)|}{|x - y|^\alpha} \mathbb{E}\left(|Z(s, x - u)|^{p/2} \left(\frac{|Z(s, x - v) - Z(s, y - v)|}{|x - y|^{2\bar{\rho} - \alpha}} \right)^{p/2} \right)$$

$$\leq Ct^{[3-(\alpha+\beta)][p/2-1]}$$

$$\times \int_0^t ds \left[\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \int_{\mathbb{R}^3} G(s, du) \int_{\mathbb{R}^3} G(s, dv) \frac{|Df(v - u, x - y)|}{|x - y|^\alpha} E(|Z(s, x - u)|^p) \right]^{1/2}$$

$$\times \left[\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \int_{\mathbb{R}^3} G(s, du) \right.$$

$$\times \left. \int_{\mathbb{R}^3} G(s, dv) \frac{|Df(v - u, x - y)|}{|x - y|^\alpha} E\left(\frac{|Z(s, x - v) - Z(s, y - v)|}{|x - y|^{2\bar{\rho} - \alpha}} \right)^p \right]^{1/2}.$$

By applying (2.32), we conclude

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_2^t(x, y)|^{p/2}}{|x - y|^{\bar{\rho}p}} \right)$$

$$\leq Ct^{[3-(\alpha+\beta)][p/2-1]} \int_0^t ds \left[\mathbb{E}\left(\|Z(s)\|_{L^p(\mathcal{O}^s)}^p \right) \mathbb{E}\left(\|Z(s)\|_{2\bar{\rho}-\alpha-3/p, p, \mathcal{O}^s}^p \right) \right]^{1/2}. \qquad (2.33)$$

Choose $\alpha = \gamma + \frac{3}{p}$. Remember that $\gamma < \frac{2-\beta}{2} - \frac{3}{p}$. Hence $\alpha \in (0, \frac{2-\beta}{2})$. This implies $\alpha \in (0, (2 - \beta) \wedge 1)$, as required. Notice also that $2\bar{\rho} - \alpha - \frac{3}{p} = \gamma$, and $3 - (\alpha + \beta) > \frac{4-\beta}{2} > 1$. Therefore,

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_2^t(x,y)|^{p/2}}{|x-y|^{\bar{\rho}p}}\right) \le Ct^{((4-\beta)/2)(p/2-1)} \int_0^t \mathbb{E}\left(\|Z(s)\|_{W^{\gamma,p}(\mathcal{O}^s)}^p\right). \quad (2.34)$$

Set

$$\mu_4(t,x,y) = \int_0^t ds \int_{\mathbb{R}^3} G(s,du) \int_{\mathbb{R}^3} G(s,dv) |D^2 f(v-u, x-y)|.$$

The following properties hold:

$$\mu_4(t,x,y) \le C|x-y|^\alpha t^{3-(\alpha+\beta)}, \quad (2.35)$$

$$\sup_{s \in [0,T]} \int_{\mathbb{R}^3} G(s,du) \int_{\mathbb{R}^3} G(s,dv) D^2 f(v-u, x-y) \le C|x-y|^\alpha, \quad (2.36)$$

with $\alpha \in (0, 2-\beta)$. The former is proved in [6], Lemma 5.5, and the latter is a slight modification of [5], Lemma 6.2.

Choose $\alpha = 2\bar{\rho}$. By applying Hölder's and Schwarz's inequalities, we obtain

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_4^t(x,y)|^{p/2}}{|x-y|^{\bar{\rho}p}}\right)$$

$$= \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \, \mathbb{E}\left(\frac{|J_4^t(x,y)|}{|x-y|^\alpha}\right)^{p/2}$$

$$\le Ct^{[3-(\alpha+\beta)][p/2-1]}$$

$$\times \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \int_0^t ds \int_{\mathbb{R}^3} G(s,du) \int_{\mathbb{R}^3} G(s,dv) \frac{|D^2 f(v-u, x-y)|}{|x-y|^\alpha}$$

$$\times E\left(|Z(s,x-u)|^{p/2} |Z(s,x-v)|^{p/2}\right)$$

$$\le Ct^{[3-(\alpha+\beta)][p/2-1]}$$

$$\times \left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \int_0^t ds \int_{\mathbb{R}^3} G(s,du) \int_{\mathbb{R}^3} G(s,dv) \frac{|D^2 f(v-u, x-y)|}{|x-y|^\alpha} \mathbb{E}\left(|Z(s,x-u)|^p\right)\right).$$

The estimate (2.36) yields

$$\mathbb{E}\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|J_4^t(x,y)|^{p/2}}{|x-y|^{\bar{\rho}p}}\right) \le Ct^{(3-(2\bar{\rho}+\beta))(p/2-1)} \int_0^t ds \mathbb{E}\left(\|Z(s)\|_{L^p(\mathcal{O}^s)}^p\right)$$

$$= Ct^{(3-2\gamma-6/p-\beta)(p/2-1)} \int_0^t ds \mathbb{E}\left(\|Z(s)\|_{L^p(\mathcal{O}^s)}^p\right). \quad (2.37)$$

For $\gamma < \frac{2-\beta}{2} - \frac{3}{p}$, we have $\bar{\eta} := 3 - 2\gamma - \frac{6}{p} - \beta > 1$, and for $\beta \in (0, 2)$, $3 - \beta > \frac{4-\beta}{2}$. Hence, from the results proved so far, we see that (2.21) follows from (2.29), (2.34) and (2.37). $\qquad\square$

For the proof of Proposition 2.5, it is convenient to consider localizations of the processes X_n, X_n^- in the space variable defined by $\{X_n(t,x)1_{K(t)}(x), (t,x) \in [0,T] \times \mathbb{R}^3\}$, $\{X_n^-(t,x)1_{K(t)}(x), (t,x) \in [0,T] \times \mathbb{R}^3\}$, respectively, with $K(t)$ given in (2.14).

Let $x, y \in \mathbb{R}^3$ be such that $x \in K(t)$ and $|x - y| = t - s$. This choice is motivated by the fact that the Green function $G(t - s, x - *)$ has support on the sphere centered at x and with radius $t - s$. By the triangular inequality, $d(y, K) \le d(y,x) + d(x,K) \le T - s$. Thus, $y \in K(s)$. Consequently, $\{X_n(t,x)1_{K(t)}(x), (t,x) \in [0,T] \times \mathbb{R}^3\}$ satisfies the following localized evolution equation:

$$X_n(t,x)1_{K(t)}(x) = X^0(t,x)1_{K(t)}(x)$$

$$+ 1_{K(t)}(x) \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) A\big(X_n(s,y)\big) 1_{K(s)}(y) M(\mathrm{d}s, \mathrm{d}y)$$

$$+ 1_{K(t)}(x) \big\langle G(t-\cdot, x-*) B\big(X_n(\cdot,*)\big) 1_{K(\cdot)}(*), w^n \big\rangle_{\mathcal{H}_t} \qquad (2.38)$$

$$+ 1_{K(t)}(x) \big\langle G(t-\cdot, x-*) D\big(X_n(\cdot,*)\big) 1_{K(\cdot)}(*), h \big\rangle_{\mathcal{H}_t}$$

$$+ 1_{K(t)}(x) \int_0^t \big[G(t-s,*) \star b\big(X_n(s,*)\big) 1_{K(s)}(*) \big](x) \, \mathrm{d}s.$$

A similar equation also holds for $\{X_n^-(t,x)1_{K(t)}(x), (t,x) \in [0,T] \times \mathbb{R}^3\}$, with the obvious changes.

Along with (2.38), we will also consider the Picard's iterations defined by

$$X_n^0(t,x)1_{K(t)}(x) = X^0(t,x)1_{K(t)}(x),$$

$$X_n^m(t,x)1_{K(t)}(x) = X^0(t,x)1_{K(t)}(x)$$

$$+ 1_{K(t)}(x) \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) A\big(X_n^{m-1}(s,y)\big) 1_{K(s)}(y) M(\mathrm{d}s, \mathrm{d}y)$$

$$+ 1_{K(t)}(x) \big\langle G(t-\cdot, x-*) B\big(X_n^{m-1}(\cdot,*)\big) 1_{K(\cdot)}(*), w^n \big\rangle_{\mathcal{H}_t} \qquad (2.39)$$

$$+ 1_{K(t)}(x) \big\langle G(t-\cdot, x-*) D\big(X_n^{m-1}(\cdot,*)\big) 1_{K(\cdot)}(*), h \big\rangle_{\mathcal{H}_t}$$

$$+ 1_{K(t)}(x) \int_0^t \big[G(t-s,*) \star b\big(X_n^{m-1}(s,*)\big) 1_{K(s)}(*) \big](x) \, \mathrm{d}s, \qquad m \ge 1.$$

For these Picard's iterations, and similarly as in (2.7), we define

$$X_n^{-,0}(t,x)1_{K(t)}(x) = X^0(t,x)1_{K(t)}(x),$$

$$X_n^{-,m}(t,x)1_{K(t)}(x)$$

$$= X^0(t,x)1_{K(t)}(x)$$

$$+ 1_{K(t)}(x) \int_0^{t_n} \int_{\mathbb{R}^3} G(t-s, x-y) A\big(X_n^{m-1}(s,y)\big) 1_{K(s)}(y) M(\mathrm{d}s, \mathrm{d}y) \qquad (2.40)$$

$$+ 1_{K(t)}(x)\langle G(t - \cdot, x - *)B(X_n^{m-1}(\cdot, *))1_{K(\cdot)}(*)1_{[0,t_n]}(\cdot), w^n\rangle_{\mathcal{H}_t}$$

$$+ 1_{K(t)}(x)\langle G(t - \cdot, x - *)D(X_n^{m-1}(\cdot, *))1_{K(\cdot)}(*)1_{[0,t_n]}(\cdot), h\rangle_{\mathcal{H}_t}$$

$$+ 1_{K(t)}(x)\int_0^{t_n}\left[G(t - s, *) \star b(X_n^{m-1}(s, *))1_{K(s)}(*)\right](x)\,\mathrm{d}s, \qquad m \geq 1.$$

In the next proposition, we consider the stochastic processes $\{X_n^m(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$, $\{X_n^{-,m}(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ given in (2.39), (2.40), respectively.

Proposition 2.7. *Let $p > \frac{6 \vee [2(4-\beta)]}{2-\beta}$ and γ be as in Lemma 2.6. We also assume that the function B is affine. Fix $m \geq 1$ and assume that*

$$\sup_{t \in [0,T]} \mathbb{E}\left(\left[\left\|X_n^{m-1}(t)\right\|_{W^{\gamma,p}(K(t))}^p + \left\|X_n^{-,m-1}(t)\right\|_{W^{\gamma,p}(K(t))}^p\right]1_{L_n(t)}\right) \leq C, \tag{2.41}$$

for some constant C independent of n, m.

Then there exists $\bar{\eta} \in (1, \infty)$ independent of n, m but depending on p, and $C > 0$, such that

$$\sup_{t \in [0,T]} \mathbb{E}\left(\left\|X_n^m(t) - X_n^{-,m}(t)\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right) \leq C2^{-n\bar{\eta}p/2}. \tag{2.42}$$

Proof. Fix $p > \frac{6}{2-\beta}$, $\gamma \in (0, \frac{2-\beta}{2} - \frac{3}{p})$, $m \geq 1$, $t \in (0, T]$. Remember that if $x \in K(t)$ and $|x - y| = t - s$, then $y \in K(s)$. Hence, from (2.39), (2.40), we have

$$\mathbb{E}\left(\left\|X_n^m(t) - X_n^{-,m}(t)\right\|_{\gamma, p, K(t)}^p 1_{L_n(s)}\right) \leq C\sum_{i=1}^5 V_{n,m}^i(t), \tag{2.43}$$

where

$$V_{n,m}^1(t) = \mathbb{E}\left(\left\|\int_{t_n}^t \int_{\mathbb{R}^3} G(t - s, \bullet - y)A(X_n^{m-1}(s, y))M(\mathrm{d}s, \mathrm{d}y)\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right),$$

$$V_{n,m}^2(t) = \mathbb{E}\left(\left\|\langle G(t - \cdot, \bullet - *)B(X_n^{-,m-1}(\cdot, *))1_{[t_n,t]}(\cdot), w^n\rangle_{\mathcal{H}_t}\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right),$$

$$V_{n,m}^3(t) = \mathbb{E}\left(\left\|\langle G(t - \cdot, \bullet - *)\left[B(X_n^{-,m-1}(\cdot, *)) - B(X_n^{m-1}(\cdot, *))\right]\right.\right.$$
$$\left.\left.\times 1_{[t_n,t]}(\cdot), w^n\rangle_{\mathcal{H}_t}\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right),$$

$$V_{n,m}^4(t) = \mathbb{E}\left(\left\|\langle G(t - \cdot, \bullet - *)D(X_n^{m-1}(\cdot, *))1_{[t_n,t]}(\cdot), h\rangle_{\mathcal{H}_t}\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right),$$

$$V_{n,m}^5(t) = \mathbb{E}\left(\left\|\int_{t_n}^t\left[G(t - s, *) \star b(X_n^{m-1}(s, *))\right](\bullet)\,\mathrm{d}s\right\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\right).$$

Set $\bar{\rho} = \gamma + \frac{3}{p}$. By writing explicitly the norm $\|\cdot\|_{\gamma, p, K(t)}$, and then applying Fubini's theorem and Burkholder's inequality, we obtain

$$V_{n,m}^1(t) \le C \int_{K(t)} dx$$
$$\times \int_{K(t)} dz \frac{\mathbb{E}(\|[G(t-\cdot, x-*) - G(t-\cdot, z-*)]A(X_n^{m-1}(\cdot,*))1_{[t_n,t]}(\cdot)1_{L_n(t)}\|_{\mathcal{H}_t}^p)}{|x-z|^{\bar{\rho}p}}.$$

Set $Z(s,y) := A(X_n^{m-1}(t-s,y))1_{L_n(t-s)}$. With the change of variable $s \mapsto t-s$, the preceding inequality implies

$$V_{n,m}^1(t) \le C \int_{K(t)} dx \int_{K(t)} dz \frac{\mathbb{E}(\int_0^{(2T2^{-n})\wedge t} ds(\|[G(s, x-*) - G(s, z-*)]Z(\cdot,*)\|_{\mathcal{H}}^2)^{p/2})}{|x-z|^{\bar{\rho}p}}.$$

The right-hand side of the preceding expression coincides up to a constant with the left-hand side of (2.21) with $t := (2T2^{-n}) \wedge t$ and $\mathcal{O} := K(t)$.

We are assuming $\sup_{t\in[0,T]} \mathbb{E}(\|X_n^{m-1}(t)\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}) < \infty$. Hence, using the linear growth of A and Lemma 3.3 [see (3.9)], we have

$$\sup_{s\in[0,t]} \mathbb{E}(\|A(X_n^{m-1}(t-s))1_{L_n(t-s)}\|_{W^{\gamma,p}((K(t))^s)}^p)$$
$$\le \sup_{s\in[0,t]} C(\mathbb{E}(\|A(X_n^{m-1}(t-s))1_{L_n(t-s)}\|_{L^p((K(t))^s)}^p)$$
$$+ \mathbb{E}(\|A(X_n^{m-1}(t-s))1_{L_n(t-s)}\|_{p,\gamma,(K(t))^s}^p))$$
$$\le C_1 + C_2 \sup_{s\in[0,t]} \mathbb{E}(\|A(X_n^{m-1}(s))1_{L_n(s)}\|_{p,\gamma,(K(t))^{t-s}}^p)$$
$$\le C_1 + C_2 \sup_{s\in[0,t]} \mathbb{E}(\|X_n^{m-1}(s)1_{L_n(s)}\|_{\gamma,p,K(s)}^p)$$
$$\le C.$$

By Lemma 2.6, we conclude

$$V_{n,m}^1(t) \le C 2^{-(np/2)\eta}, \tag{2.44}$$

with $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta)$.

The term $V_{n,m}^2(t)$ is also a stochastic integral with respect to M. Indeed, for a given function $f: [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $t \in [0,T]$, let τ_n be the operator defined by

$$\tau_n(f)(s,x) = f((s+2^{-n}) \wedge t, x). \tag{2.45}$$

Let $\mathcal{E}_n$ be the closed subspace of $\mathcal{H}_T$ generated by the orthonormal system

$$2^n T^{-1} 1_{\Delta_i}(\cdot) \otimes e_j(*), \qquad i = 0, \ldots, 2^n - 1, j = 1, \ldots, n,$$

and denote by π_n the orthogonal projection operator on $\mathcal{E}_n$. Notice that $\pi_n \circ \tau_n$ is a bounded operator on $\mathcal{H}_T$, uniformly in n.

The random vector $X_n^{-,m-1}(s, *)$ is $\mathcal{F}_{s_n}$-measurable. Then, using the definition of w^n, it is easy to check that

$$\left\langle G(t - \cdot, \bullet - *) B\left(X_n^{-,m-1}(\cdot, *)\right) 1_{[t_n, t]}(\cdot), w^n \right\rangle_{\mathcal{H}_t}$$
$$= \int_{t_n}^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n) \left(G(t - \cdot, \bullet - *) B\left(X_n^{-,m-1}(\cdot, *)\right)\right)(s, y) M(\mathrm{d}s, \mathrm{d}y).$$

Therefore, $V_{n,m}^2(t)$ can be studied in a similar way than $V_{n,m}^1(t)$, with $Z(s, y) := B(X_n^{-,m-1}(t - s, y)) 1_{L_n(t-s)}$. We obtain

$$V_{n,m}^2(t) \le C 2^{-(np/2)\eta}, \tag{2.46}$$

with $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta)$.

Using Schwarz's inequality and (2.12), we have

$$V_{n,m}^3(t) \le C n^{3p/2} 2^{np/2} \mathbb{E}\left(\left\| \left\| G(t - \cdot, \bullet - *) \left[B\left(X_n^{-,m-1}(\cdot, *)\right) - B\left(X_n^{m-1}(\cdot, *)\right)\right]\right. \right.\right.$$
$$\left.\left.\left. \times 1_{[t_n, t]}(\cdot) \right\|_{\mathcal{H}_t} 1_{L_n(t)} \right\|_{\gamma, p, K(t)}^p\right).$$

Let

$$Z(s, y) := \left[B\left(X_n^{-,m-1}(t - s, y)\right) - B\left(X_n^{m-1}(t - s, y)\right)\right] 1_{L_n(t-s)}. \tag{2.47}$$

With the change of variable $s \mapsto t - s$, we see that

$$V_{n,m}^3(t) \le C n^{3p/2} 2^{np/2} \mathbb{E}\left(\left\| \left\| G(\cdot, \bullet - *) Z(\cdot, *) \right\|_{\mathcal{H}_{(2T2^{-n}) \wedge t}} \right\|_{\gamma, p, K(t)}^p\right).$$

We are assuming (2.41). Hence, as in the analysis of $V_{n,m}^1(t)$, using Lemma 3.3 we can prove that the assumptions of Lemma 2.6 are satisfied for $\mathcal{O} := K(t)$. Consequently,

$$V_{n,m}^3(t) \le C n^{3p/2} 2^{np/2} 2^{-n\eta(p/2-1)} \int_0^{(2T2^{-n}) \wedge t} E\left(\left\| Z(s) \right\|_{W^{\gamma, p}(K(t)^s)}^p\right), \tag{2.48}$$

with Z given in (2.47) and $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta)$.

From (2.19), it follows that for $0 \le s \le t \le T$, $K(t)^s = K(t - s)$. Therefore,

$$E\left(\left\| \left[B\left(X_n^{-,m-1}(t - s, y)\right) - B\left(X_n^{m-1}(t - s, y)\right)\right] 1_{L_n(t-s)} \right\|_{L^p((K(t)^s))}^p\right)$$
$$\le C \int_{K(t-s)} \mathrm{d}y\, E\left(\left| X_n^{-,m-1}(t - s, y) - X_n^{m-1}(t - s, y) \right|^p 1_{L_n(t-s)}\right)$$
$$\le C \sup_{(s,y) \in [0,T] \times \mathbb{R}^3} E\left(\left| X_n^{-,m-1}(s, y) - X_n^{m-1}(s, y) \right|^p 1_{L_n(s)}\right)$$
$$\le C n^{3p/2} 2^{-np(3-\beta)/2},$$

where we have applied Lemma 3.2.

Since B is affine, we have

$$\big\| \big[B\big(X_n^{-,m-1}(s)\big) - B\big(X_n^{m-1}(s)\big)\big]1_{L_n(s)}\big\|_{\gamma,p,K(s)}^p$$
$$\leq C\big\| \big[X_n^{-,m-1}(s) - X_n^{m-1}(s)\big]1_{L_n(s)}\big\|_{\gamma,p,K(s)}^p, \qquad s \in [0,T].$$

Applying these estimates to (2.48) yields

$$V_{n,m}^3(t) \leq Cn^{3p/2}2^{np/2}2^{-n\eta(p/2-1)}$$
$$\times \left[n^{3p/2}2^{-np(3-\beta)/2} + \int_0^{(2T2^{-n})\wedge t} \mathrm{d}s \mathbb{E}\big(\big\|X_n^{-,m-1}(s) - X_n^{m-1}(s)\big\|_{\gamma,p,K(s)}^p\big)\right]$$
$$\leq C_1 n^{3p}2^{-(np/2)(\eta+2-\beta-2\eta/p)}$$
$$+ C_2 n^{3p/2}2^{-(np/2)(\eta-1-2\eta/p)}\int_0^{(2T2^{-n})\wedge t} \mathrm{d}s \mathbb{E}\big(\big\|X_n^{-,m-1}(s) - X_n^{m-1}(s)\big\|_{\gamma,p,K(s)}^p\big). \tag{2.49}$$

Schwarz' inequality implies

$$V_{n,m}^4(t) \leq C\mathbb{E}\big(\big\|\big\|G(\cdot, \bullet - *)Z(\cdot, *)\big\|_{\mathcal{H}_{(2T2^{-n})\wedge t}}\big\|_{\gamma,p,K(t)}^p\big),$$

with $Z(s,y) := D(X_n^{m-1}(t-s,y))1_{L_n(t-s)}$. Therefore, similarly as in the study of the term $V_{n,m}^1(t)$ we obtain

$$V_{n,m}^4(t) \leq C2^{-(np/2)\eta}, \tag{2.50}$$

with $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta)$.

To study $V_{n,m}^5(t)$, we first apply Minkowski's inequality and then the linear growth of b and Lemma 3.3. We obtain (the details are left to the reader),

$$V_{n,m}^5(t) \leq C2^{-np}. \tag{2.51}$$

With (2.43), (2.44), (2.46), (2.49), (2.50), (2.51), we have

$$\mathbb{E}\big(\big\|X_n^m(t) - X^{-,m}(t)\big\|_{\gamma,p,K(t)}^p\big)$$
$$\leq c_1 2^{-(np/2)\eta} + c_2 n^{3p}2^{-(np/2)(\eta+2-\beta-2\eta/p)}$$
$$+ c_3 n^{3p/2}2^{-(np/2)(\eta-1-2\eta/p)}\int_0^{(2T2^{-n})\wedge t} \mathrm{d}s \mathbb{E}\big(\big\|X_n^{-,m-1}(s) - X_n^{m-1}(s)\big\|_{\gamma,p,K(s)}^p\big),$$

with $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta) \in (1,2)$ (see Lemma 2.6).

In Lemma 3.4, we prove that for any $p > \frac{2(4-\beta)}{2-\beta}$, $\eta_1 := \eta - 1 - \frac{2\eta}{p} > 0$. This implies $\eta + 2 - \beta - \frac{2\eta}{p} > 1$.

Set $f_n^m(t) = \mathbb{E}(\|X_n^m(t) - X^{-,m}(t)\|_{\gamma,p,K(t)}^p)$, and $\eta_2 := \inf(\eta, \eta + 2 - \beta - \frac{2\eta}{p})$. We have proved that

$$f_n^m(t) \leq \varphi_n + \psi_n \int_0^{2T2^{-n}\wedge t} \mathrm{d}s f_n^m(s),$$

with $\varphi_n := (c_1 \vee c_2)2^{-(np/2)\eta_2}$, $\psi_n := c_3 n^{3p/2} 2^{-np\eta_1}$, $\eta_1 > 0$, $\eta_2 > 1$. Then, Gronwall's lemma yields

$$f_n^m(t) \leq \varphi_n\big(1 + \exp(T\psi_n)\big).$$

Clearly, $\sup_n \psi_n < \infty$. Thus,

$$f_n^m(t) \leq C\varphi_n, \qquad n \geq 1,$$

and this yields (2.42). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 2.5. Fix p and γ as in the assertion. Using induction on $m \geq 0$, we will first establish a result analogue to (2.17) for the Picard's iterations defined in (2.39) and (2.40). More precisely, we will prove

$$\sup_{t\in[0,T]} \mathbb{E}\big(\big[\|X_n^m(t)\|_{W^{\gamma,p}(K(t))}^p + \|X_n^{-,m}(t)\|_{W^{\gamma,p}(K(t))}^p\big]1_{L_n(t)}\big) \leq C, \qquad (2.52)$$

for some constant C independent of n, m. By Fatou's lemma, and the convergence in the L^p norm of $X_n^m(t,x)1_{L_n(t)}$, and $X_n^{-,m}(t,x)1_{L_n(t)}$ to $X_n(t,x)1_{L_n(t)}$, and $X_n^-(t,x)1_{L_n(t)}$, respectively, for any $(t,x) \in [0,T] \times \mathbb{R}^3$ [see (3.5)], this will imply (2.17).

For $m = 0$, (2.52) is just the property (H2.1), which is a consequence of hypothesis (H2).

Let $m \geq 1$ and assume that (2.52) holds for any Picard iterative of order less or equal to $m-1$. We recall that if $x \in K(t)$ and $|x - y| = t - s$, then $y \in K(s)$. Thus, from (2.39) we see that

$$\mathbb{E}\big(\|X_n^m(t)\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big) \leq C \sum_{i=1}^6 R_{n,m}^i(t),$$

with

$$R_{n,m}^1(t) = \big\|X^0(t)\big\|_{W^{\gamma,p}(K(t))}^p,$$

$$R_{n,m}^2(t) = \mathbb{E}\bigg(\bigg\|\int_0^t \int_{\mathbb{R}^3} G(t-s, \bullet - y)A\big(X_n^{m-1}(s,y)\big)M(\mathrm{d}s, \mathrm{d}y)\bigg\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\bigg),$$

$$R_{n,m}^3(t) = \mathbb{E}\big(\big\|\langle G(t-\cdot, \bullet - *)B\big(X_n^{-,m-1}(\cdot,*)\big), w^n\rangle_{\mathcal{H}_t}\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big),$$

$$R_{n,m}^4(t) = \mathbb{E}\big(\big\|\langle G(t-\cdot, \bullet - *)\big[B\big(X_n^{m-1}(\cdot,*)\big) - B\big(X_n^{-,m-1}(\cdot,*)\big)\big], w^n\rangle_{\mathcal{H}_t}\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big),$$

$$R_{n,m}^5(t) = \mathbb{E}\big(\big\|\langle G(t-\cdot, \bullet - *)D\big(X_n^{m-1}(\cdot,*)\big), h\rangle_{\mathcal{H}_t}\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big),$$

$$R_{n,m}^6(t) = \mathbb{E}\bigg(\bigg\|\int_0^t \big[G(t-s, *) \star b\big(X_n^{m-1}(s,*)\big)\big](\bullet)\,\mathrm{d}s\bigg\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\bigg),$$

where the symbols "·", "∗" denote the time and space variables, respectively, that are relevant for the $\mathcal{H}_t$ norm, while the symbol "•" denotes the argument corresponding to functions in the space $W^{\gamma,p}(K(t))$.

As has been pointed out before, the assumption (H2) implies

$$R_{n,m}^1(t) \le C. \tag{2.53}$$

By the induction hypotheses and (3.8), (3.9) applied to the function $g := A$ and $Z(t,x) := X_n^{m-1}(t,x)1_{L_n(t)}$, we see that the assumptions of [5], Theorem 3.1, hold. Therefore, we have

$$R_{n,m}^2(t) \le C \int_0^t \mathrm{d}s\, \mathbb{E}\big(\big\|A\big(X_n^{m-1}(s)\big)\big\|_{W^{\gamma,p}(K(t)^{t-s})}^p 1_{L_n(s)}\big).$$

From this inequality, the definition (2.9), the Lipschitz continuity of the function A and Lemma 3.3, it follows that

$$R_{n,m}^2(t) \le C_1 + C_2 \int_0^t \mathrm{d}s\, \mathbb{E}\big(\big\|X_n^{m-1}(s)\big\|_{W^{\gamma,p}(K(s))}^p 1_{L_n(s)}\big) \le C. \tag{2.54}$$

As has been mentioned in the analysis of the term $V_{n,m}^2(t)$ in Proposition 2.7, the following identity holds:

$$\big\langle G(t-\cdot, \bullet - \ast) B\big(X_n^{-,m-1}(\cdot, \ast)\big), w^n\big\rangle_{\mathcal{H}_t}$$
$$= \int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n)(G(t-\cdot, \bullet - \ast) B\big(X_n^{-,m-1}(\cdot, \ast)\big))(s,y) M(\mathrm{d}s, \mathrm{d}y).$$

Consequently, $R_{n,m}^3(t) \le C[R_{n,m}^{3,1}(t) + R_{n,m}^{3,2}(t)]$, with

$$R_{n,m}^{3,1}(t)$$
$$= \mathbb{E}\bigg(\bigg\|\int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n)\big(G(t-\cdot, \bullet - \ast) B\big(X_n^{-,m-1}(\cdot, \ast)\big)\big)(s,y) M(\mathrm{d}s, \mathrm{d}y)\bigg\|_{L^p(K(t))}^p 1_{L_n(t)}\bigg),$$

$$R_{n,m}^{3,2}(t)$$
$$= \mathbb{E}\bigg(\bigg\|\int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n)\big(G(t-\cdot, \bullet - \ast) B\big(X_n^{-,m-1}(\cdot, \ast)\big)\big)(s,y) M(\mathrm{d}s, \mathrm{d}y)\bigg\|_{\gamma,p,K(t)}^p 1_{L_n(t)}\bigg).$$

By developing the $L^p(K(t))$ norm, and using Fubini's theorem, Burkholder's inequality and the boundedness of the operator $\pi_n \circ \tau_n$, we have

$$R_{n,m}^{3,1}(t) \le C\mathbb{E}\bigg(\bigg(\int_{K(t)} \mathrm{d}x\, \big\|\big(G(t-\cdot, x-\ast) B\big(X_n^{-,m-1}\big)(\cdot, \ast)\big)\big\|_{\mathcal{H}_t}^p\bigg) 1_{L_n(t)}\bigg). \tag{2.55}$$

Now we apply the usual estimates on the $\mathcal{H}_t$ norm along with the property (3.8) and the induction assumption to conclude that $R_{n,m}^{3,1}(t) \le C$.

Using the definition of the fractional norm (see (2.8)), Fubini's theorem, along with Burkholder's inequality and the boundedness of the operator $\pi_n \circ \tau_n$ yields,

$$
\begin{aligned}
R_{n,m}^{3,2}(t) = \int_{K(t)} dx \int_{K(t)} dz \frac{1}{|x-z|^{3+\gamma p}} \\
\times \mathbb{E}\bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n)\big([G(t-\cdot, x-*) - G(t-\cdot, z-*)] \\
\times B\big(X_n^{-,m-1}(\cdot, *)\big)(s, y) M(ds, dy)\bigg|^p 1_{L_n(t)}\bigg) \\
\leq C \int_{K(t)} dx \int_{K(t)} dz \frac{1}{|x-z|^{3+\gamma p}} \\
\times \mathbb{E}\big(\big\|[G(t-\cdot, x-*) - G(t-\cdot, z-*)] \\
\times B\big(X_n^{-,m-1}(\cdot, *)\big)\big\|_{\mathcal{H}_t}^p 1_{L_n(t)}\big).
\end{aligned}
\tag{2.56}
$$

Consider the stochastic processes $\{B(X_n^{-,m-1}(t, x))1_{L_n(t)}, (t, x) \in [0, T] \times \mathbb{R}^3\}$. First, we apply Lemma 3.3 to $g := B$, $Z(t, x) := X_n^{-,m-1}(t, x)1_{L_n(t)}$. Then, by the induction hypothesis we see that the assumptions of Lemma 2.6 are satisfied. This yields $R_{n,m}^{3,2}(t) \leq C$. Hence, we have proved

$$
R_{n,m}^3(t) \leq C.
\tag{2.57}
$$

To study the term $R_n^5(t)$, we first apply Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
R_{n,m}^5(t) \leq \|h\|_{\mathcal{H}_t}^p \mathbb{E}\big(\big\|\|G(t-\cdot, \bullet-*)D\big(X_n^{m-1}(\cdot, *)\big)\|_{\mathcal{H}_t}\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big) \\
\leq C\big(R_{n,m}^{5,1}(t) + R_{n,m}^{5,2}(t)\big),
\end{aligned}
$$

with

$$
\begin{aligned}
R_{n,m}^{5,1}(t) = \mathbb{E}\big(\big\|\|G(t-\cdot, \bullet-*)D\big(X_n^{m-1}(\cdot, *)\big)\|_{\mathcal{H}_t}\big\|_{L^p(K(t))}^p 1_{L_n(t)}\big), \\
R_{n,m}^{5,2}(t) = \mathbb{E}\big(\big\|\|G(t-\cdot, \bullet-*)D\big(X_n^{m-1}(\cdot, *)\big)\|_{\mathcal{H}_t}\big\|_{\gamma, p, K(t)}^p 1_{L_n(t)}\big).
\end{aligned}
$$

Notice that $R_{n,m}^{5,1}(t)$ is similar to the right-hand side of (2.55), with $B(X_n^{-,m-1})$ replaced by $D(X_n^{m-1})$. Then, by analogue arguments as for $R_{n,m}^{3,1}(t)$, we obtain $R_{n,m}^{5,1}(t) \leq C$.

Using the triangular inequality, we see that $R_{n,m}^{5,2}(t)$ is similar to the last term in (2.56), with $B(X_n^{-,m-1})(t, x)$ in the latter expression replaced by $D(X_n^{m-1})(t, x)$ in the former. Therefore, $R_{n,m}^{5,2}(t) \leq C$, and consequently,

$$
R_{n,m}^5(t) \leq C.
\tag{2.58}
$$

By writing explicitly the convolution operator and then using Minkowski's inequality, we have

$$R^6_{n,m}(t) \le C \mathbb{E} \left(\int_0^t \mathrm{d}s \int_{\mathbb{R}^3} G(t-s, \mathrm{d}y) \big\| b \big(X_n^{m-1}(s, \bullet - y) \big) \big\|_{W^{\gamma, p}(K(t))} 1_{L_n(s)} \right)^p.$$

Next, we apply Hölder's inequality with respect to the finite measure on $[0, T] \times \mathbb{R}^3$ given by $\mathrm{d}s G(t-s, \mathrm{d}y)$ along with (3.8), (3.10) with $g := b$ and $Z(t, x) := X_n^{m-1}(t, x) 1_{L_n(t)}$. We obtain

$$R^6_{n,m}(t) \le C_1 + C_2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} G(t-s, \mathrm{d}y) \mathbb{E} \big(\big\| X_n^{m-1}(s, \bullet) \big\|^p_{W^{\gamma, p}(K(s))} 1_{L_n(s)} \big).$$

Since $\sup_{s \in [0, T]} \int_{\mathbb{R}^3} G(s, \mathrm{d}y) < \infty$, this yields

$$R^6_{n,m}(t) \le C_1 + C_2 \int_0^t \mathrm{d}s \mathbb{E} \big(\big\| X_n^{m-1}(s) \big\|^p_{W^{\gamma, p}(K(s))} 1_{L_n(s)} \big) \le C. \tag{2.59}$$

We now study the contribution of $R^4_{n,m}(t)$. As we did for $R^5_n(t)$, first we apply Cauchy–Schwarz inequality along with (2.12) to obtain

$$
\begin{aligned}
& R^4_{n,m}(t) \\
& \le C n^{3p/2} 2^{np/2} \\
& \quad \times \mathbb{E} \big(\big\| \big\| G(t - \cdot, \bullet - *) \big[B \big(X_n^{m-1}(\cdot, *) \big) - B \big(X_n^{-, m-1}(\cdot, *) \big) \big] \big\|_{\mathcal{H}_t} \big\|^p_{W^{\gamma, p}(K(t))} 1_{L_n(t)} \big).
\end{aligned}
\tag{2.60}
$$

There are two contributions coming from the right-hand side of (2.60) – the L^p norm and the fractional norm. They will be studied separately (see the terms below denoted by $R^{4,1}_{n,m}(t)$, $R^{4,2}_{n,m}(t)$, resp.).

We start with the contribution of the L^p norm. From Fubini's theorem and the Lipschitz continuity of B, it follows that

$$
\begin{aligned}
R^{4,1}_{n,m}(t) & := C n^{3p/2} 2^{np/2} \\
& \quad \times \mathbb{E} \left(\left(\int_{K(t)} \mathrm{d}x \big\| G(t - \cdot, x - *) \big[B \big(X_n^{m-1}(\cdot, *) \big) - B \big(X_n^{-, m-1}(\cdot, *) \big) \big] \big\|^2_{\mathcal{H}_t} \right)^{p/2} 1_{L_n(t)} \right) \\
& \le C n^{3p/2} 2^{np/2} \\
& \quad \times \int_{K(t)} \mathrm{d}x \mathbb{E} \left(\left(\int_0^t \mathrm{d}s \big\| G(t - s, x - *) \right. \right. \\
& \qquad\qquad\qquad \left. \left. \times \big[B \big(X_n^{m-1}(s, *) \big) - B \big(X_n^{-, m-1}(s, *) \big) \big] \big\|^2_{\mathcal{H}} \right)^{p/2} 1_{L_n(t)} \right) \\
& \le C n^{3p/2} 2^{np/2} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E} \big(\big| X_n^{m-1}(t, x) - X_n^{-, m-1}(t, x) \big|^p 1_{L_n(t)} \big).
\end{aligned}
$$

By Lemma 3.2, $\sup_m \sup_{t \in [0,T]} R_{n,m}^{4,1}(t) \leq C n^{3p} 2^{-np[(2-\beta)/2]}$. Since $\beta \in (0,2)$, this implies

$$\sup_{n,m} \sup_{t \in [0,T]} R_{n,m}^{4,1}(t) \leq C. \tag{2.61}$$

Next, we study the contribution of the fractional norm of the right-hand side of (2.60):

$$R_{n,m}^{4,2}(t) := C n^{3p/2} 2^{np/2}$$
$$\times \mathbb{E}\big(\big\| \|G(t-\cdot, \bullet - *)[B(X_n^{m-1}(\cdot,*)) - B(X_n^{-,m-1}(\cdot,*))]\|_{\mathcal{H}_t} \big\|_{\gamma,p,K(t)}^p 1_{L_n(t)} \big).$$

Set

$$Z_n^m(s,y) = [B(X_n^{m-1}(s,y)) - B(X_n^{-,m-1}(s,y))] 1_{L_n(s)}.$$

For any $0 \leq s \leq t$, we have

$$\sup_{s \in [0,T]} \mathbb{E}\big(\|Z_n^m(s)\|_{W^{\gamma,p}(K(s)^{t-s})}^p \big) < \infty.$$

Indeed, this holds for $Z_n^m(s)$ replaced by $B(X_n^{m-1}(s,\cdot)) 1_{L_n(s)}$ and $B(X_n^{-,m-1}(s,\cdot)) 1_{L_n(s)}$, separately by the following arguments. We rely on the induction assumption, and for the contribution of the L^p norm, we use (3.8). For the contribution of the fractional norm, we apply (3.9).

Thus, (2.22) implies

$$\mathbb{E}\big(\big\| \|G(t-\cdot, \bullet - *)[B(X_n^{m-1}(\cdot,*)) - B(X_n^{-,m-1}(\cdot,*))]\|_{\mathcal{H}_t} \big\|_{\gamma,p,K(t)}^p 1_{L_n(t)} \big)$$
$$\leq C \int_0^t ds\, \mathbb{E}\big(\|[B(X_n^{m-1}(s)) - B(X_n^{-,m-1}(s))]\|_{W^{\gamma,p}(K(s))}^p 1_{L_n(s)} \big) \tag{2.62}$$
$$= C \int_0^t ds\, \mathbb{E}\big(\|[X_n^{m-1}(s) - X_n^{-,m-1}(s)]\|_{W^{\gamma,p}(K(s))}^p 1_{L_n(s)} \big),$$

where in the last equality we have used that B is affine.

By the definition (2.9), we see that

$$\int_0^t ds\, \mathbb{E}\big(\|X_n^{m-1}(s) - X_n^{-,m-1}(s)\|_{W^{\gamma,p}(K(s))}^p 1_{L_n(s)} \big)$$
$$\leq C \Big\{ \Big[\int_0^t ds\, \mathbb{E}\big(\|X_n^{m-1}(s) - X_n^{-,m-1}(s)\|_{\gamma,p,K(s)}^p 1_{L_n(s)} \big) \tag{2.63}$$
$$+ \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big(|X_n^{m-1}(t,x) - X_n^{-,m-1}(t,x)|^p \big) \Big\}.$$

Using Proposition 2.7 and Lemma 3.2, the right-hand side of this inequality is bounded by $\tilde{g}_n := C(2^{-n\bar{\eta}p/2} + n^{3p/2} 2^{-np(3-\beta)/2})$, with $\bar{\eta} > 1$. Notice that

$$\sup_n n^{3p/2} 2^{np/2} \tilde{g}_n \leq C.$$

With all these results, we conclude $\sup_{n,m} \sup_{t\in[0,T]} R_{n,m}^{4,2}(t) \leq C$. Along with (2.61), this yields

$$\sup_{n,m} \sup_{t\in[0,T]} R_{n,m}^4(t) \leq C. \tag{2.64}$$

Bringing together (2.53), (2.54), (2.57), (2.58), (2.59), (2.64), we obtain

$$\mathbb{E}\big(\big\|X_n^m(t)\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big) \leq C. \tag{2.65}$$

By the same arguments, and using that $t_n \leq t$, we also have

$$\mathbb{E}\big(\big\|X_n^{-,m}(t)\big\|_{W^{\gamma,p}(K(t))}^p 1_{L_n(t)}\big) \leq C. \tag{2.66}$$

From (2.65), (2.66), we obtain (2.52). $\square$

3. Auxiliary results

This section gathers some technical results that are used throughout the paper.

Lemma 3.1. *Consider the Picard iterations defined in* (2.39), (2.40), *respectively. Let* $p \in [1, \infty)$. *For any* $n \geq 1$, $m \geq 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\big(\big[\big|X_n^m(t,x)\big|^p + \big|X_n^{-,m}(t,x)\big|^p\big]1_{L_n(t)}\big) \leq C, \tag{3.1}$$

where the constant C *does not depend on* n, m.
    Consequently,

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\big(\big[\big|X_n(t,x)\big|^p + \big|X_n^-(t,x)\big|^p\big]1_{L_n(t)}\big) < \infty, \tag{3.2}$$

$$\sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big(\big[\big\|X_n^m(t)\big\|_{L^p(K(t))} + \big\|X_n^{-,m}(t)\big\|_{L^p(K(t))}\big]1_{L_n(t)}\big) < \infty, \tag{3.3}$$

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big(\big[\big\|X_n(t)\big\|_{L^p(K(t))} + \big\|X_n^-(t)\big\|_{L^p(K(t))}\big]1_{L_n(t)}\big) < \infty. \tag{3.4}$$

Proof. To establish (3.1), we follow the arguments of the proof of (4.9) in [7] with $X_n(t, x)$, $X_n^-(t, x)$ in this reference replaced by $X_n^m(t, x)$, $X_n^{-,m}(t, x)$, respectively, and we use induction on m.
    For the proof of (3.2), we use the convergences

$$\lim_{m\to\infty} \sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|X_n^m(t,x) - X_n(t,x)\big|^p 1_{L_n(t)}\big) = 0,$$
$$\lim_{m\to\infty} \sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|X_n^{-,m}(t,x) - X_n^-(t,x)\big|^p 1_{L_n(t)}\big) = 0, \tag{3.5}$$

along with Fatou's lemma.

Property (3.3) follows easily from (3.1), and (3.4) is proved by applying (3.3), (3.5) and Fatou's lemma. □

Lemma 3.2. *Let* $p \in [1, \infty)$. *For any* $n \geq 1$, $m \geq 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|X_n^m(t,x) - X_n^{-,m}(t,x)\big|^p 1_{L_n(t)}\big) \leq Cn^{3p/2}2^{-np(3-\beta)/2}, \tag{3.6}$$

where the constant C *does not depend neither on* n *nor on* m. *Consequently,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|X_n(t,x) - X_n^-(t,x)\big|^p 1_{L_n(t)}\big) \leq Cn^{3p/2}2^{-np(3-\beta)/2}. \tag{3.7}$$

Proof. To establish (3.6), we follow the arguments of the proof of (4.10) in [7] with $X_n(t,x)$, $X_n^-(t,x)$ in this reference replaced by $X_n^m(t,x)$, $X_n^{-,m}(t,x)$, respectively, and we use induction on m. Then we obtain (3.7) by applying (3.5) and Fatou's lemma. □

In the next lemma, we establish some results that have been shown in the proof of [5], Theorem 4.6, in a particular context.

Lemma 3.3. *Let* $p \in [1, \infty)$, $\gamma \in (0, 1)$. *Consider a measurable stochastic process* $Z = \{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^3\}$ *such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|Z(t,x)\big|^p\big) < \infty.$$

Let g *be a real-valued Lipschitz continuous function, and* $K \subset \mathbb{R}^3$ *a compact set. The following properties are satisfied*:
(i)

$$\sup_{t\in[0,T]} \mathbb{E}\big(\big\|g\big(Z(t)\big)\big\|_{L^p(K)}^p\big) \leq C\Big(1 + \sup_{t\in[0,T]} \mathbb{E}\big(\big\|Z(t)\big\|_{L^p(K)}^p\big)\Big). \tag{3.8}$$

(ii) *For any* $0 \leq s \leq t \leq T$,

$$\mathbb{E}\big(\big\|g\big(Z(s)\big)\big\|_{\gamma,p,K(t)^{t-s}}^p\big) \leq C\mathbb{E}\big(\big\|Z(s)\big\|_{\gamma,p,K(s)}^p\big). \tag{3.9}$$

(iii) *For any* $0 \leq s \leq t \leq T$, $|y| \leq t - s$,

$$\mathbb{E}\big(\big\|g\big(Z(s, \bullet - y)\big)\big\|_{\gamma,p,K(t)}^p\big) \leq C\mathbb{E}\big(\big\|Z(s)\big\|_{\gamma,p,K(s)}^p\big). \tag{3.10}$$

Proof. The assertion (i) follows by using the linear growth of the function g.
Fix $0 \leq s \leq t \leq T$. Since $K(t)^{t-s} \subset K(s)$, we have

$$\mathbb{E}\big(\big\|g\big(Z(s)\big)\big\|_{\gamma,p,K(t)^{t-s}}^p\big) = \mathbb{E}\bigg(\int_{K(t)^{t-s}} dx \int_{K(t)^{t-s}} dy \frac{|g(Z(s,x)) - g(Z(s,y))|^p}{|x-y|^{3+\gamma p}}\bigg)$$

$$\leq C\mathbb{E}\bigg(\int_{K(t)^{t-s}} dx \int_{K(t)^{t-s}} dy \frac{|Z(s,x) - Z(s,y)|^p}{|x-y|^{3+\gamma p}}\bigg) \tag{3.11}$$

$$= C\mathbb{E}\big(\|Z(s)\|^p_{\gamma,p,K(t)^{t-s}}\big)$$

$$\le C\mathbb{E}\big(\|Z(s)\|^p_{\gamma,p,K(s)}\big),$$

which proves (ii).

Let $0 \le s \le t \le T$ and $|y| \le t - s$. By definition,

$$\mathbb{E}\big(\|g(Z(s, \bullet - y))\|^p_{\gamma,p,K(t)}\big) = \mathbb{E}\bigg(\int_{K(t)} dx \int_{K(t)} dz \frac{|g(Z(s, x - y)) - g(Z(s, z - y))|^p}{|x - z|^{3+\gamma p}}\bigg).$$

Consider the change of variables $\bar{x} \mapsto x - y$ and $\bar{z} \mapsto z - y$. Since $x, z \in K(t)$ and $|y| \le t - s$, then $\bar{x}, \bar{z} \in K(s)$. Thus,

$$\mathbb{E}\big(\|g(Z(s, \bullet - y))\|^p_{\gamma,p,K(t)}\big) \le \mathbb{E}\bigg(\int_{K(s)} d\bar{x} \int_{K(s)} d\bar{z} \frac{|g(Z(s, \bar{x})) - g(Z(s, \bar{z}))|^p}{|\bar{x} - \bar{z}|^{3+\gamma p}}\bigg)$$

$$= \mathbb{E}\big(\|g(Z(s))\|^p_{\gamma,p,K(s)}\big)$$

$$\le C\mathbb{E}\big(\|Z(s)\|^p_{\gamma,p,K(s)}\big). \qquad \square$$

Lemma 3.4. *Let* $\eta = \inf(\frac{4-\beta}{2}, 3 - 2\gamma - \frac{6}{p} - \beta)$*, with* $\gamma \in (0, \frac{2-\beta}{2} - \frac{3}{p})$*. Let* $p > \frac{2(4-\beta)}{2-\beta}$*. Then*

$$p > \frac{2\eta}{\eta - 1}, \tag{3.12}$$

equivalently $\eta_1 := \frac{\eta-1}{2} - \frac{\eta}{p} > 0$.

Proof. Consider first the case $\eta = \frac{4-\beta}{2}$. Then $\frac{2\eta}{\eta-1} = \frac{2(4-\beta)}{2-\beta}$, and the conclusion is obvious. Next, we suppose that $\eta = 3 - 2\gamma - \frac{6}{p} - \beta$. Then $\frac{2\eta}{\eta-1} = \frac{6-4\gamma-12/p-2\beta}{2-2\gamma-6/p-\beta}$. Fix γ. The function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{6 - 4\gamma - 2x - 2\beta}{2 - 2\gamma - x - \beta},$$

is increasing and $f(x) \le \lim_{x\to\infty} f(x) = 2$. Choose $p > 2$. Then

$$p > 2 = \sup_{x \ge 0} f(x) > \frac{2\eta}{\eta - 1}.$$

Notice that $\frac{2(4-\beta)}{2-\beta} > 2$. Hence, (3.12) holds. $\qquad \square$

Acknowledgements

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