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Exchangeable exogenous shock models

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We characterize a comprehensive family of d-variate exogenous shock models. Analytically, we consider a family of multivariate distribution functions that arises from ordering, idiosyncratically distorting, and finally multiplying the arguments. Necessary and sufficient conditions on the involved distortions to yield a multivariate distribution function are given. Probabilistically, the attainable set of distribution functions corresponds to a large class of exchangeable exogenous shock models. Besides, the vector of exceedance times of an increasing additive stochastic process across independent exponential trigger variables is shown to constitute an interesting subclass of the considered distributions and yields a second probabilistic model. The alternative construction is illustrated in terms of two examples.

Keywords: additive process; copula; exogenous shock model; frailty-model; multivariate distribution function

1. Introduction

Fatal shock models are standard tools in reliability theory, insurance, credit risk, and various other fields of application. The present work characterizes a large class of such models, described by functions $C:[0,1]^d \to [0,1]$, $d \geq 2$, of the form

$$C(u_1, \dots, u_d) = \prod_{k=1}^{d} g_k(u_{(k)}), \tag{1}$$

where $g_1 = \mathrm{id}_{[0,1]}$ is the identity on $[0,1]$, the mappings $g_k:[0,1] \to [0,1]$ satisfy $g_k(1) = 1, k = 2, \dots, d$, and where $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(d)}$ denotes the ordered list of $u_1, \dots, u_d$. We derive necessary and sufficient conditions on $g_k, k = 2, \dots, d$, such that the function C defines a distribution function (a so-called copula[1]), and we construct the corresponding random vector $\mathbf{X} = (X_1, \dots, X_d)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having C as distribution function.

It turns out that there is a one-to-one correspondence between copulas of the form (1) and random vectors $(X_1, \dots, X_d)$ with representation

$$X_k = \max\{Z^E : k \in E\}, \qquad k = 1, \dots, d, \tag{2}$$

[1] A d-dimensional function $C:[0,1]^d \to [0,1]$ is a copula if (a) C is the distribution function of a vector $(U_1, \dots, U_d)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and (b) $U_k, k = 1, \dots, d$, is uniformly distributed on $[0,1]$. For the function C in equation (1), (a) implies (b) as for $(U_1, \dots, U_d) \sim C$, one has $\mathbb{P}(U_k \leq u) = C(u, 1, \dots, 1) = u$.

1350-7265

where Z^E, $\varnothing \neq E \subseteq \{1, \dots, d\}$, denote 2^{d-1} independent real-valued random variables whose distribution functions depend only on the cardinality of E. More precisely, we show that (a) any random vector $\mathbf{X}$ defined by (2) with continuous marginal distributions has a (unique) copula of type (1) and that (b) any copula of the form (1) represents the distribution function of a random vector $\mathbf{X}$ that can be constructed as in (2).

Put differently, as the copula of a random vector $\mathbf{X}$ is the survival copula of $-\mathbf{X}$, copulas of type (1) characterize precisely the set of survival copulas of random vectors $(\tilde{X}_1, \dots, \tilde{X}_d)$ with

$$\tilde{X}_k := \min\{\tilde{Z}^E \colon k \in E\}, \qquad k = 1, \dots, d, \tag{3}$$

where the $\tilde{Z}^E$ are identically distributed for subsets E having the same cardinality. Such random vectors $(\tilde{X}_1, \dots, \tilde{X}_d)$ are often referred to as exogenous shock models in the literature (for references see Section 2), because the $\tilde{Z}^E$ can be thought of as the arrival times of shocks affecting one or several constituents in a system of d components, and $\tilde{X}_k$ represents the first time component k is hit by a shock.

Moreover, with the considered vector $(X_1, \dots, X_d)$, respectively, the vector $(\tilde{X}_1, \dots, \tilde{X}_d)$, being exchangeable, that is,

$$(X_1, \dots, X_d) \stackrel{\mathrm{d}}{=} (X_{\sigma(1)}, \dots, X_{\sigma(d)}) \qquad \text{for all permutations } \sigma \text{ on } \{1, \dots, d\},$$

copulas of type (1) coincide with the set of survival copulas corresponding to exchangeable exogenous shock models. Theorem 1.1 is the main contribution to be derived. For notational brevity, we introduce a particular set of distribution functions on $[0, 1]$, denoted $\mathcal{D}$ and defined by[2]

$$\mathcal{D} := \{F \colon [0,1] \to [0,1] \colon F \text{ continuous and increasing}, F(1) = 1, F \text{ strictly positive on } (0,1]\}.$$

Theorem 1.1. *Let $C \colon [0,1]^d \to [0,1]$ have analytical form (1). The following statements are equivalent*:

(i) *C is a copula, that is, a multivariate distribution function.*

(ii) *For all $0 < u < v \leq 1$, $k \in \mathbb{N}_0$, $j \in \mathbb{N}$, with $k + j \leq d$, it holds that*

$$G_{j,k}(u, v) := \sum_{i=0}^{j} \binom{j}{i} (-1)^i \prod_{l=1}^{i} g_{l+k}(u) \prod_{l=i+1}^{j} g_{l+k}(v) \geq 0.$$

(iii) *For all $k \in \mathbb{N}_0$, $j \in \mathbb{N}$, with $k + j \leq d$, it holds that $H_{j,k} \in \mathcal{D}$, where*

$$H_{j,k}(u) := \begin{cases} \displaystyle\prod_{i=0}^{j-1} g_{k+1+i}^{(-1)^i \binom{j-1}{i}}(u), & u \in (0,1], \\ \displaystyle\lim_{v \searrow 0} \prod_{i=0}^{j-1} g_{k+1+i}^{(-1)^i \binom{j-1}{i}}(v), & u = 0. \end{cases}$$

[2]Note that throughout this document, whenever talking about increasing functions, we refer to functions that are non-decreasing.

(iv) *For all* $m \in \{1, \dots, d\}$, *it holds that* $H_{m,d-m} \in \mathcal{D}$.

In this case, C *is the distribution function of* $\mathbf{X}$ *as defined in* (2), *where* $Z^E \sim H_{m,d-m}$ *for all subsets* E *with cardinality* $|E| = m$.

Remark 1.1 (Implications of Theorem 1.1). 1. Theorem 1.1 consists of three crucial achievements: First of all, in (ii), copulas of type (1) are characterized purely analytically and it is shown that the functions g_k, $k = 1, \dots, d$, have to satisfy certain inequality conditions. Second, in (iii) respectively (iv), these conditions are translated to monotonicity requirements, showing that certain functionals of the g_k have to yield distribution functions on [0, 1]. Last but not least, the functionals are interpreted from a probabilistic point of view by introducing a stochastic model which manifests the connection to shock models.

2. It is important to note (see Proposition 2.1) that the $H_{m,d-m}$ defined in Theorem 1.1(iii) can be solved for the g_k, yielding

$$g_k = \prod_{m=1}^{d+1-k} H_{m,d-m}^{\binom{d-k}{m-1}}, \qquad k = 1, \dots, d.$$

Consequently, $H_{m,d-m} \in \mathcal{D}$, $m = 1, \dots, d$, can be arbitrary distribution functions, provided that the normalization constraint

$$g_1 = \prod_{m=1}^{d} H_{m,d-m}^{\binom{d-1}{m-1}} = \mathrm{id}_{[0,1]},$$

which solely stems from the formulation of the theorem in terms of copulas rather than general multivariate distribution functions, is fulfilled. Thus, Theorem 1.1 shows that copulas of type (1) not only comprise, but precisely consist of the set of survival copulas corresponding to exchangeable exogenous shock models.

3. The g_k, $k = 1, \dots, d$ can be interpreted as conditional distribution functions. More precisely, for $u \in (0, 1]$ and $(U_1, \dots, U_d) \sim C$,

$$\begin{aligned} g_k(u) &= \frac{\prod_{i=1}^{k} g_i(u)}{\prod_{i=1}^{k-1} g_i(u)} = \frac{\mathbb{P}(U_1 \le u, \dots, U_k \le u)}{\mathbb{P}(U_1 \le u, \dots, U_{k-1} \le u)} \\ &= \mathbb{P}(U_1 \le u, \dots, U_k \le u | U_1 \le u, \dots, U_{k-1} \le u). \end{aligned}$$

Note that for $k \ge 3$, g_k corresponds to the ratio between two diagonal sections of copulas. As such, it is the ratio between a k-Lipschitz and a $(k-1)$-Lipschitz function.

The remaining sections are organized as follows. Section 2 gives a brief overview on exogenous shock models and analyzes bivariate copulas of the form (1) in more detail. Moreover, a sketch of the proof of Theorem 1.1 is given that is thoroughly carried out in Appendix B. Section 3 provides an alternative stochastic model for a subclass of type (1)-copulas that is based on first-passage time constructions involving additive processes. Furthermore, two examples of corresponding parametric copula families are given. Finally, Section 4 concludes.

2. Exogenous shock models and bivariate copulas of type (1)

2.1. Exogenous shock models

Exogenous shock models have extensively been analyzed in the literature. The most prominent example is given by the multivariate distribution (known as Marshall–Olkin distribution) introduced in [20]. The authors consider a random vector $(\tilde{X}_1, \dots, \tilde{X}_d)$ as constructed in (3), where the shocks $\tilde{Z}^E$ are exponentially distributed. [23] relax this condition in the bivariate case by considering univariate shocks $\tilde{Z}^{\{1\}}, \tilde{Z}^{\{2\}}$ with generalized exponential distribution. [13] extend this idea and study the law of the vector (X_1, X_2), where $X_1 = \max\{Z_1, Z_3\}$, $X_2 = \max\{Z_2, Z_3\}$, and both Z_1, Z_2, and Z_3 are independent random variables with generalized exponential distribution. Proceeding in a similar way, [14] consider a minimum-type construction as in (3) with the shocks $\tilde{Z}^E$ having arbitrary distribution functions on $(0, \infty)$, and study the resulting dependence structure in the bivariate case. Relying on a method of the eponymous authors, [26] introduce the Block and Basu bivariate generalized exponential distribution, which results from decomposing the joint distribution function in its singular and absolutely continuous part and solely considering the latter. Another direction is pursued by [12]. The authors start with the construction in (3) for $d = 2$ and consider Weibull distributed random variables $\tilde{Z}^E$. Denoting by F the resulting joint distribution function of $(\tilde{X}_1, \tilde{X}_2)$, they study the law of $(\tilde{Y}_1, \tilde{Y}_2)$ given by

$$\tilde{Y}_1 = \min\{\tilde{X}^{\{1,1\}}, \dots, \tilde{X}^{\{1,N\}}\}, \qquad \tilde{Y}_2 = \min\{\tilde{X}^{\{2,1\}}, \dots, \tilde{X}^{\{2,N\}}\},$$

where N is a mixing variable with geometric distribution and $(\tilde{X}^{\{1,n\}}, \tilde{X}^{\{2,n\}})_{n\in\mathbb{N}}$ are i.i.d. random vectors with distribution function F.

The ongoing research interest concerning exogenous shock models is explained by means of their suitability for various applications. [20] show that in their setup with exponentially distributed shocks, the joint survival function of $(\tilde{X}_1, \tilde{X}_2)$ can be linked to the first jump times of independent Poisson processes affecting either one or both components of the random vector. This derivation is picked up in [15], where $(\tilde{X}_1, \tilde{X}_2)$ is identified with the first occurrence time of windstorm losses in France, respectively, Germany, that are governed by west (affecting only France), central (affecting only Germany), and pan-European (affecting both countries) windstorms. The idea of modelling insurance events by independent counting processes and mapping the events to one or several claim types is a popular approach in multivariate insurance models (see the extensive overview in [1]). Alternatively, applying exogenous shock models in a credit risk context, one might also think of $\tilde{Z}^E$ as arrival times of economic catastrophes influencing the default of one or several assets in a portfolio. This idea is applied, for example, in [9]. In [2], the authors rely on the Marshall–Olkin setup (i.e., exponentially distributed shocks) and model cross-country dependencies between European obligors in order to compute the systemic risk of the banking sector in the post-crisis era.

While the universe of exogenous shock models in the literature is wide, most of them either focus on specific distribution functions of the shocks $\tilde{Z}^E$ (see [11] and the references therein), or study the vector $(\tilde{X}_1, \dots, \tilde{X}_d)$ in the bivariate case, that is, for $d = 2$. In the present article, we consider an arbitrary dimension $d \geq 2$ and assume that the distribution of $\tilde{Z}^E$ solely depends on the cardinality of E, that is, $\tilde{Z}^E$ and $\tilde{Z}^I$ are identically distributed for $|E| = |I|$. Within this setup, Proposition 2.1 shows that the corresponding set of copulas has form (1). For brevity of the proof, we consider the case of continuous, strictly increasing marginal distribution functions.

Proposition 2.1. *Let* $\mathbf{X} := (X_1, \dots, X_d)$ *be defined as in* (2), *i.e.* $Z^E \stackrel{d}{=} Z^I$ *for* $|E| = |I|$. *Assume that the distribution functions of the random variables* Z^E *are continuous and strictly increasing. The* (*unique*) *copula* C *of* $\mathbf{X}$ *has the form* (1).

Proof. Let Z^E have the distribution function F_m for $|E| = m$, $\varnothing \neq E \subseteq \{1, \dots, d\}$. The marginal distribution functions of $\mathbf{X} = (X_1, \dots, X_d)$ equal

$$\hat{g}_1(x) := \mathbb{P}(X_k \le x) = \prod_{E:\, k \in E} \mathbb{P}\big(Z^E \le x\big) = \prod_{m=1}^{d} F_m^{\binom{d-1}{m-1}}(x), \qquad x \in \mathbb{R},$$

as there are $d-1$ over $m-1$ subsets E with cardinality m and distribution function F_m that appear in the stochastic construction of X_k. By Sklar's theorem, we have to show that the joint distribution function of $\hat{g}_1(X_1), \dots, \hat{g}_1(X_d)$ is a copula C of type (1). To recognize that, note that

$$\hat{g}_1(X_k) = \max\big\{\hat{g}_1\big(Z^E\big):\, k \in E\big\}, \qquad k = 1, \dots, d,$$

where $\hat{Z}^E := \hat{g}_1(Z^E)$ have distribution functions $H_{m,d-m} := F_m \circ \hat{g}_1^{-1} \in \mathcal{D}$. As pointed out in Remark 1.1(2), one can find functions $g_1, \dots, g_d$ such that the distribution function of $\hat{g}_1(X_1), \dots, \hat{g}_1(X_d)$ is of the form (1). Finally, as the required normalization constraint

$$g_1 := \prod_{m=1}^{d} \big(F_m \circ \hat{g}_1^{-1}\big)^{\binom{d-1}{m-1}} = \mathrm{id}_{[0,1]}$$

is valid, the claim follows like in the proof of Theorem 1.1(iv) $\Rightarrow$ (i) in Appendix B. $\square$

The crucial condition for a d-variate function to represent a copula is d-increasingness (see [21], Definition 2.10.1, page 43). However, this property is typically non-trivial to check, with the complexity of the problem increasing exponentially in higher dimensions.[3] Theorem 1.1(ii) indicates that for functions of form (1), the d-increasingness conditions can be massively simplified and reduce to the verification of $G_{j,k}(u, v) \ge 0$ for certain indices $k \in \mathbb{N}_0$, $j \in \mathbb{N}$, and certain pairs $(u, v) \in [0, 1]^2$. To get accustomed to the paper and to develop a deeper understanding of these conditions, the following section analyzes copulas of type (1) in the simpler bivariate case.

2.2. Bivariate copulas of type (1)

In [6], the authors study copulas of type (1) for the special case $g_2 = g_3 = \dots = g_d$. In the bivariate case, such copulas coincide with the more general class considered in the present article.

[3]This explains why many studies are restricted to the case $d = 2$.

By Theorem 1.1, $C:[0,1]^2 \to [0,1]$, $C(u_1,u_2) = g_1(u_{(1)})g_2(u_{(2)})$, is a copula if and only if $G_{j,k}(u,v) \geq 0$ for $(k,j) \in \{(0,1),(0,2),(1,1)\}$ and $u,v \in [0,1], u < v$, that is, if and only if $g_1(v) - g_1(u) \geq 0$, $g_2(v) - g_2(u) \geq 0$, and $g_1(v)g_2(v) - 2g_1(u)g_2(v) + g_1(u)g_2(u) \geq 0$.

While the first two conditions are easy to interpret and imply that g_1 and g_2 have to be increasing (note that $g_1 = \mathrm{id}_{[0,1]}$ is increasing by definition already), the third one is more interesting to analyze. It is shown in [5] and the proof of the main theorem below that, given increasingness of g_1 and g_2, the third condition is equivalent to g_2 being strictly positive and continuous and g_1/g_2 being increasing on $(0,1]$. It is easy to verify that C is the copula of $\mathbf{X} = (X_1, X_2)$, where

$$X_1 = \max\{Z^{\{1\}}, Z^{\{1,2\}}\},$$
$$X_2 = \max\{Z^{\{2\}}, Z^{\{1,2\}}\},$$

and $Z^{\{1\}}, Z^{\{2\}}, Z^{\{1,2\}}$ are independent random variables with distribution functions[4] $Z^{\{1\}}, Z^{\{2\}} \sim g_2$ and $Z^{\{1,2\}} \sim g_1/g_2$.

Several dependence properties of C can be derived in closed form. For instance, the lower and upper tail dependence coefficients λ_L and λ_U equal

$$\lambda_L := \lim_{u \searrow 0} \frac{C(u,u)}{u} = \lim_{u \searrow 0} g_2(u),$$
$$\lambda_U := \lim_{u \nearrow 1} \frac{C(u,u) - 2u + 1}{1-u} = 1 - g_2'(1-),$$

where $g_2'(1-)$ denotes the left-sided derivative of g_2 at $u = 1$, which exists by monotonicity of g_2. Further dependence properties are derived in [4,6], including measures of association and extremal dependence coefficients. As an example for C, define $g_2:[0,1] \to [0,1]$ by

$$g_2(u) := \min\{au + b, 1 - c + cu\}, \qquad a > 1, b > 0, c > 0, b + c \leq 1.$$

Specified in that way, g_2 starts at b with slope a, has a kink at $u = (1-(b+c))/(a-c)$, continues to increase with slope c, and ends at $g_2(1) = 1$. Applying the tail dependence formulas, it follows that $\lambda_L = \lim_{u \searrow 0} g_2(u) = b$ and $\lambda_U = 1 - g_2'(1-) = 1 - c$. Thus, copulas of type (1) can admit both positive upper and lower tail dependence with arbitrary values in $[0,1]$.

2.3. Extreme-value copulas of type (1)

One may determine the intersection between copulas having form (1) and extreme-value copulas. A d-variate extreme-value copula C satisfies

$$C(u_1^t, \ldots, u_d^t) = C^t(u_1, \ldots, u_d) \qquad \text{for all } t > 0, u_1, \ldots, u_d \in [0,1]. \tag{4}$$

Proposition 2.2 shows that extreme-value copulas of type (1) correspond to choosing power functions for g_k, where the sequence of exponents must be d-monotone.

[4]Strictly speaking, in order for g_2 and g_1/g_2 to be proper distribution functions on $[0,1]$, one has to consider their right-continuous extensions at zero as considered in the definition of $H_{j,k}$ in Theorem 1.1(iii).

***Definition 2.1** (**d-monotone sequence**). A real-valued sequence $\{a_0, \dots, a_{d-1}\}$ is called d-monotone if*

$$\sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} a_{k+i} \geq 0 \qquad \text{for all } k \in \mathbb{N}_0,\ j \in \mathbb{N}\colon\ k+j \leq d.$$

Proposition 2.2 (Extreme-value copulas of type (1)). *Let C have the form (1). C is an extreme-value copula if and only if $g_k(u) = u^{a_{k-1}}$, $k = 1, \dots, d$, for $u \in (0, 1]$ and a d-monotone sequence $\{a_0, \dots, a_{d-1}\}$ with $a_0 = 1$.*

For a proof, see Appendix A. The corresponding class of extreme-value copulas is well-known in the literature: It is precisely the exchangeable family of Marshall–Olkin survival copulas (see, e.g., [22]). In Section 3.2, we will investigate this example in more detail.

2.4. Strategy to prove Theorem 1.1

Apart from some technical lemmata, the proof of Theorem 1.1 provides valuable insights into the structure of the objects $G_{j,k}$ and $H_{j,k}$ and their relation to the stochastic model in equation (2). We are going to show that (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). The central ideas can be summarized as follows. The rigorous proof of Theorem 1.1 is given in Appendix B.

Remark 2.1 (Structure of the proof of Theorem 1.1). (iv) $\Rightarrow$ (i) Starting with the random vector $\mathbf{X} = (X_1, \dots, X_d)$ given in the theorem, one can compute that each X_k, $k = 1, \dots, d$, is uniformly distributed on $[0, 1]$ and that C is the distribution function of $\mathbf{X}$, hence a copula.

(i) $\Rightarrow$ (ii) Being a copula, C induces a probability measure dC on $[0, 1]^d$. It can be deduced that $G_{j,k}(u, v)$ corresponds to the mass assigned by dC to certain subsets of $[0, 1]^d$. Therefore, it has to be greater than or equal to zero.

(ii) $\Rightarrow$ (iii) This is the most difficult and lengthy part of the proof. Besides minor technical conditions, the central task is to show that non-negativity of $G_{j,k}(u, v)$, $u < v$, implies increasingness of $H_{j,k}$. The underlying proof idea is to split up $G_{j,k}(u, v)$ into two summands, one involving the difference $H_{j,k}(v) - H_{j,k}(u)$, the other one corresponding to the probability mass $dC(I)$ induced by a copula C of type (1) for a subset $I \subset [0, 1]^d$. For a sufficiently "small" subset I, it is shown that the sign of $G_{j,k}(u, v)$ is dominated by the first part, that is, the difference $H_{j,k}(v) - H_{j,k}(u)$. Thus, for $G_{j,k}(u, v)$ to be non-negative, $H_{j,k}(v) - H_{j,k}(u)$ has to be greater than or equal to zero, which establishes the claimed increasingness of $H_{j,k}$.

(iii) $\Rightarrow$ (iv) This is trivial as (iv) is a special case of (iii).

3. Alternative construction via additive processes

3.1. Additive frailty construction

Theorem 1.1 has shown that any d-dimensional copula of the form (1) arises from the stochastic construction in (2), involving $2^d - 1$ random variables Z^E, $\varnothing \neq E \subseteq \{1, \dots, d\}$. Consequently, due to the exponentially increasing number of random objects that have to be sampled, simulation becomes practically impossible in large dimensions. This section provides an alternative construction for a subclass of type (1)-copulas based on a first-passage time construction with additive processes.

According to [24], Definition 1.6, page 3, a stochastic process $\{\Lambda_t\}_{t\geq 0}$ on $\mathbb{R}$ is called additive if it starts at zero almost surely, is stochastically continuous, admits independent increments, and is càdlàg almost surely. Increasing additive processes, called additive subordinators in the sequel, are closely connected with Bernstein functions. A function $\Psi : [0,\infty) \to [0,\infty)$ is called Bernstein function[5] if $\Psi(0) = 0$, Ψ is infinitely often differentiable on $(0,\infty)$ with $(-1)^{n-1}\Psi^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x > 0$. The law of an additive subordinator $\{\Lambda_t\}_{t\geq 0}$ can be described by a family $\{\Psi_t\}_{t\geq 0}$ of Bernstein functions subject to certain consistency conditions (see (i)–(iii) below). Denoting the Laplace transform operator by $\mathcal{L}$, it follows from [24], page 47 ff., that

(i) $\Psi_0(x) \equiv 0$ for all $x \geq 0$,
(ii) $\Psi_t - \Psi_s$ is a Bernstein function for all $0 \leq s \leq t$,
(iii) $\mathcal{L}(\Lambda_t - \Lambda_s) = \exp(-\Psi_t + \Psi_s)$ for all $0 \leq s \leq t$.

It is well known that Ψ_t admits a Lévy–Khintchine representation, that is,

$$\Psi_t(x) = a_t \mathbb{1}_{\{x>0\}} + b_t x + \int_{(0,\infty)} \bigl(1 - \mathrm{e}^{-xs}\bigr)\nu_t(\mathrm{d}s), \qquad x \geq 0, \tag{5}$$

with a Lévy measure ν_t on $(0,\infty)$ and parameters $a_t, b_t \geq 0$. One of the most prominent examples for increasing additive processes are Lévy subordinators, which not only exhibit independent, but even stationary increments. For a Lévy subordinator, the corresponding family $\{\Psi_t\}_{t\geq 0}$ satisfies $\Psi_t = t\Psi_1$, $t \geq 0$. Another example is given by so-called self-similar additive subordinators (see [24], page 99 ff.), which are characterized by $\Psi_t(x) = \Psi_1(xt^H)$, $x, t \geq 0$, for an $H > 0$ and a specific Bernstein function Ψ_1.

Now consider an additive subordinator $\Lambda = \{\Lambda_t\}_{t\geq 0}$ with $\lim_{t\to\infty} \Lambda_t = \infty$ and define a sequence $\{X_k\}_{k\in\mathbb{N}}$ of random variables by

$$X_k := \inf\{t \geq 0 \colon \ \Lambda_t \geq E_k\}, \qquad k \in \mathbb{N}, \tag{6}$$

where E_k, $k \in \mathbb{N}$, are i.i.d. unit exponentially distributed random variables that are independent of $\{\Lambda_t\}_{t\geq 0}$. By construction, $\{X_k\}_{k\in\mathbb{N}}$ is an exchangeable sequence of random variables. The following proposition outlines that the (unique) survival copula of $(X_1, \ldots, X_d)$, denoted $C_{\Lambda,d}$ in the sequel, is of type (1) for any $d \geq 2$.

Proposition 3.1. *Define a sequence $\{X_k\}_{k\in\mathbb{N}}$ of random variables as in (6). Let $\{\Psi_t\}_{t\geq 0}$ be the family of Bernstein functions corresponding to the increasing additive process $\Lambda = \{\Lambda_t\}_{t\geq 0}$ and denote by $\bar{F}_1$ the survival function of X_1. The survival copula $C_{\Lambda,d}$ of $(X_1, \ldots, X_d)$ has the form (1) for any $d \geq 2$, with*

$$g_k(u) := \exp\bigl(-\Psi_{\bar{F}_1^{-1}(u)}(k) + \Psi_{\bar{F}_1^{-1}(u)}(k-1)\bigr), \qquad k = 1, \ldots, d.$$

[5] For more details on Bernstein function, see [25].

The proof is to be found in Appendix C. Referring to the numerical motivation for the first-passage time construction at the beginning of this section, a generic sampling algorithm for $C_{\Lambda,d}$ can be stated as follows.

Algorithm 3.1 (Simulation of $C_{\Lambda,d}$ in Proposition 3.1). 1. *Simulate d independent, unit exponentially distributed random variables $E_1, \dots, E_d$.*

2. *Simulate one path of $\{\Lambda_t\}_{t\geq 0}$ until $\Lambda_t \geq \max\{E_1, \dots, E_d\}$.*
3. *Compute $X_k := \inf\{t \geq 0 \colon \Lambda_t \geq E_k\}, k = 1, \dots, d$.*
4. *Set $U_k := \bar{F}_1(X_k), k = 1, \dots, d$, and return $(U_1, \dots, U_d)$.*

Clearly, the central task is to simulate the path of the additive process Λ. Provided this can be accomplished efficiently, the algorithm provides a fast sampling routine even in large dimensions. For Lévy processes of (compound) Poisson type, see [24], page 17 ff. The path generation of self-similar additive processes can be accomplished via more general results in [19]. Last but not least, the construction of Dirichlet processes (which might be viewed as an elementary transform of a special class of additive subordinators) is discussed in [8].

Extending $(X_1, \dots, X_d)$ to larger dimensions solely requires the simulation of further i.i.d. exponentially distributed triggers $E_{d+1}, E_{d+2}, \dots,$ possibly supplemented by simulating additional increments of Λ until the largest trigger is exceeded (it can be shown that $\lim_{d\to\infty} \mathbb{E}[E_{(d)}]/\log d = 1$). Besides, we consider it interesting to study the coherence between properties of the additive process in the first-passage time setup (6) and the distribution functions of Z^E, respectively, $\tilde{Z}^E$, in the maximum construction in (2), respectively, (3).

3.2. Examples of tractable families

(1) *Extendible Marshall–Olkin copulas*

The first example assigns a very special meaning to the survival copula of the random vector $(X_1, \dots, X_d)$ defined in (6) when choosing $\{\Lambda_t\}_{t\geq 0}$ to be a Lévy subordinator. If C in (1) is given by the functions $g_k(u) = u^{a_{k-1}}, k = 1, \dots, d$, and defines a copula for any $d \geq 2$, it can be constructed in two quite different ways. On the one hand, C arises as the survival copula of the additive frailty construction when plugging in a Lévy subordinator for $\{\Lambda_t\}_{t\geq 0}$. On the other hand, C is the survival copula of $(\tilde{X}_1, \dots, \tilde{X}_d)$ in (3) with exponentially distributed shocks $\tilde{Z}^E$. Interestingly, it can be deduced that the stationary increments of the Lévy process in the frailty setup translate to the characterizing lack-of-memory property of the exponential distribution in the shock construction.

To recognize this coherence, consider a copula C of type (1) with $g_k(u) = u^{a_{k-1}}, k = 1, \dots, d$, for a real-valued sequence $\{a_0, \dots, a_{d-1}\}$ with $a_0 = 1$. In [18], Theorem 2.3, the authors show that C corresponds to the exchangeable subclass of the multivariate distribution function introduced in [20], which is why C is called an exchangeable Marshall–Olkin copula. Put differently, C is the survival copula of $(\tilde{X}_1, \dots, \tilde{X}_d)$ in (3), where the shocks $\tilde{Z}^E$ are exponentially distributed, with rates depending on the cardinality of E.

By Theorem 1.1, a function C of type (1) with $g_k(u) = u^{a_{k-1}}, k = 1, \dots, d$, and a real-valued sequence $\{a_0, \dots, a_{d-1}\}$, $a_0 = 1$, defines a copula for any $d \geq 2$ if and only if $H_{j,k} \in \mathcal{D}$ for all $j \in \mathbb{N}, k \in \mathbb{N}_0$. As discussed in the proof of Proposition 2.2, this is naturally equivalent to $\{a_0, \dots, a_{d-1}\}$ being d-monotone for any $d \geq 2$. A sequence $\{a_k\}_{k \in \mathbb{N}_0}$ that is d-monotone for any $d \geq 2$ is called completely monotone. Combining [17], Lemma 2.6 with [10], Corollary 4.2, $\{a_k\}_{k \in \mathbb{N}_0}$ with $a_0 = 1$ is completely monotone if and only if there exists a Lévy subordinator $\Lambda = \{\Lambda_t\}_{t \geq 0}$ characterized by $\{\Psi_t\}_{t \geq 0}$, $\Psi_t = t\Psi_1$, such that $\Psi_1(1) = 1$ and $a_{k-1} = \Psi_1(k) - \Psi_1(k-1)$ for all $k \in \mathbb{N}$. Applying Proposition 3.1, $C_{\Lambda,d}$ is a survival copula of type (1) for all $d \geq 2$, with

$$g_k(u) := \exp\bigl(-\Psi_{\bar{F}_1^{-1}(u)}(k) + \Psi_{\bar{F}_1^{-1}(u)}(k-1)\bigr), \qquad k \in \mathbb{N},$$

$$\bar{F}_1(x) = e^{-\Psi_x(1)} \overset{(\Psi_x = x\Psi_1)}{=} e^{-x\Psi_1(1)} \overset{(\Psi_1(1)=1)}{=} e^{-x}.$$

Substituting $\bar{F}_1^{-1}(u) = -\log(u)$ in g_k, it follows that

$$\begin{aligned} g_k(u) &= \exp\bigl(-\Psi_{-\log(u)}(k) + \Psi_{-\log(u)}(k-1)\bigr) \\ &= \exp\bigl(\log(u)\Psi_1(k) - \log(u)\Psi_1(k-1)\bigr) = u^{\Psi_1(k)-\Psi_1(k-1)} = u^{a_{k-1}}. \end{aligned}$$

(2) *Sato-frailty copulas*

The second example concerns the survival copula of $(X_1, \dots, X_d)$ in (6) for self-similar additive processes $\{\Lambda_t\}_{t \geq 0}$, which admits a nice simple form and might be attractive for applications and further analyses. A self-similar additive subordinator $\Lambda = \{\Lambda_t\}_{t \geq 0}$, sometimes referred to as increasing Sato process in the literature, can be identified with a family $\{\Psi_t\}_{t \geq 0}$ of Bernstein functions satisfying $\Psi_t(x) = \Psi_1(xt^H)$, $x, t \geq 0$, for an $H > 0$ and a so-called self-decomposable Bernstein function Ψ_1. The attribute "self-decomposable" means that Ψ_1 admits a representation as in (5) with $a_1 = 0$ and with ν_1 possessing a density $\nu_1(\mathrm{d}s) = k(s)/s\,\mathrm{d}s$ such that $s \mapsto k(s)$ is decreasing on $(0, \infty)$. By Proposition 3.1, the copula $C_{\Lambda,d}$ has the form (1) for all $d \geq 2$, with $g_k, k = 1, \dots, d$, given by

$$g_k(u) := \exp\bigl(-\Psi_{\bar{F}_1^{-1}(u)}(k) + \Psi_{\bar{F}_1^{-1}(u)}(k-1)\bigr), \qquad k \in \mathbb{N},$$

$$\bar{F}_1(x) = e^{-\Psi_x(1)} \overset{(\Psi_x(1)=\Psi_1(x^H))}{=} e^{-\Psi_1(x^H)}.$$

Plugging $\bar{F}^{-1}(u) = (\Psi_1^{-1}(-\log u))^{1/H}$ into g_k yields

$$\begin{aligned} g_k(u) &= \exp\bigl(-\Psi_{(\Psi^{-1}(-\log u))^{1/H}}(k) + \Psi_{(\Psi^{-1}(-\log u))^{1/H}}(k-1)\bigr) \\ &= \exp\bigl(-\Psi_1\bigl(k\Psi_1^{-1}(-\log u)\bigr) + \Psi_1\bigl((k-1)\Psi_1^{-1}(-\log u)\bigr)\bigr) \\ &= \frac{\psi(k\psi^{-1}(u))}{\psi((k-1)\psi^{-1}(u))} \end{aligned}$$

for $\psi : [0, \infty) \to (0, 1]$, $\psi(x) := \exp(-\Psi_1(x))$. The resulting class $C_{\Lambda,d}$ is analytically tractable and is analyzed in detail and illustrated in [16]. As a concluding example, consider the self-decomposable Bernstein function

$$\Psi_1(x) := \int_0^\infty \left(1 - \mathrm{e}^{-xt}\right) \underbrace{\beta \frac{\exp(-\eta t)}{t}\,\mathrm{d}t}_{:=\nu(\mathrm{d}t)} = \beta \log\left(1 + \frac{x}{\eta}\right), \qquad x, \beta, \eta > 0,$$

which obviously satisfies the required property that $\nu(\mathrm{d}s) = k(s)/s\,\mathrm{d}s$ for a decreasing function $s \mapsto k(s)$. Defining $\psi : [0, \infty) \to (0, 1]$, $\psi(x) := \exp(-\Psi_1(x))$, and choosing $\Lambda = \{\Lambda_t\}_{t \geq 0}$ to be an increasing Sato process with $\Psi_t(x) = \Psi_1(xt^H)$, $x, t \geq 0$, for an $H > 0$, yields

$$\begin{aligned} C_{\Lambda,d}(u_1, \dots, u_d) &= \prod_{k=1}^d \frac{\psi(k\psi^{-1}(u_{(k)}))}{\psi((k-1)\psi^{-1}(u_{(k)}))} \\ &= \left(\prod_{k=1}^d \frac{1 + (k-1)(u_{(k)}^{-1/\beta} - 1)}{1 + k(u_{(k)}^{-1/\beta} - 1)}\right)^\beta, \end{aligned}$$

which in the bivariate case admits the form $C_{\psi,2}(u_1, u_2) = u_{(1)}/(2 - u_{(2)}^{1/\beta})^\beta$. Summarizing, the construction in (6) can be used to derive interesting new parametric families of copulas.

4. Conclusion

The main object of our study are copulas of functional form (1), which are parameterized by d functions $g_k, k = 1, \dots, d$. Three major findings constitute the core of the present article. First, necessary and sufficient conditions in terms of inequalities for the g_k are derived. Second, these conditions are shown to relate to monotonicity conditions for multiplicative conjunctions of the g_k. Third, the monotonicity restrictions are embedded into a stochastic model for the corresponding copula. As a result, it turns out that the considered class of copulas is interrelated with the set of exchangeable exogenous shock models, which have been analyzed in the literature on the level of specific models, however, to the best of our knowledge not on a unified level as in the present work. In addition, it is outlined how a subclass of the considered set of copulas can alternatively be constructed via additive processes. The latter finding seems to be promising in two regards. On the one hand, it provides an alternative sampling approach that may be beneficial for high-dimensional simulation purposes. On the other hand, by combining the characterization properties in Theorem 1.1 with the alternative construction, interesting theoretical results – as illustrated by the Marshall–Olkin example in Section 3.2 – may be derived.

Appendix A

Proof of Proposition 2.2. If $g_k(u) = u^{a_{k-1}}, k = 1, \dots, d$, for a sequence $\{a_0, \dots, a_{d-1}\}$, the functions $H_{j,k}$ in Theorem 1.1(iii) are given by

$$H_{j,k}(u) = \prod_{i=0}^{j-1} g_{k+1+i}^{(-1)^i \binom{j-1}{i}}(u) = \prod_{i=0}^{j-1} u^{(-1)^i \binom{j-1}{i} a_{k+i}} = u^{\sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} a_{k+i}}.$$

Thus, the $H_{j,k}$ are distribution functions in $\mathcal{D}$ (i.e. C is a copula) if and only if the sequence $\{a_0, \dots, a_{d-1}\}$ is d-monotone. Moreover, it is apparent that due to the power function structure of the g_k, C satisfies the extreme-value property.

It remains to show that any extreme-value copula of type (1) implies a power function structure for the g_k. By setting $u_1 = u_2 = u \in [0, 1]$ and $u_3 = \dots = u_d = 1$ in (4),

$$g_2(u^t) = g_2(u)^t \qquad \text{for all } u \in (0, 1], t > 0.$$

Defining $\Theta : [0, \infty) \to \mathbb{R}$, $\Theta(x) := g_2(\exp(-x))$, this is equivalent to

$$\Theta(tx) = \Theta(x)^t \qquad \text{for all } x, t > 0. \tag{A.1}$$

Thus, setting $t = n \in \mathbb{N}$ and $x = 1/n$, it holds that

$$\Theta(1) = \Theta\left(\frac{1}{n}\right)^n \quad \Longrightarrow \quad \Theta\left(\frac{1}{n}\right) = \Theta(1)^{1/n}. \tag{A.2}$$

Consequently, for all $x \in \mathbb{Q} \cap (0, \infty)$, $x = p/q$, $p, q \in \mathbb{N}$, we have

$$\Theta\left(\frac{p}{q}\right) \overset{\text{(A.1)}}{=} \Theta\left(\frac{1}{q}\right)^p \overset{\text{(A.2)}}{=} \Theta(1)^{p/q}.$$

By Theorem 1.1(iii), it holds that $H_{1,1} = g_2 \in \mathcal{D}$, implying that $g_2(u) > 0$ for $u \in (0, 1]$. Therefore, one deduces that $0 < \Theta(1) = g_2(\exp(-1)) \leq 1$ and the previous equation yields $\Theta(x) = \exp(-a_1 x)$ for all $x \in \mathbb{Q} \cap (0, \infty)$, where $a_1 = -\log(\Theta(1)) \geq 0$. As $\mathbb{Q} \cap (0, \infty)$ is dense in $\mathbb{R}^+$, it follows that

$$g_2(\mathrm{e}^{-x}) = \Theta(x) = \mathrm{e}^{-a_1 x} = (\mathrm{e}^{-x})^{a_1}, \qquad x \in \mathbb{R}^+.$$

By continuity of g_2 at zero, $g_2(u) = u^{a_1}$ for all $u \in [0, 1]$. Iteratively, by setting $u_1 = \dots = u_l = u \in [0, 1]$, $u_{l+1} = \dots = u_d = 1$, and subsequently raising l, we conclude that $g_k(u) = u^{a_{k-1}}$, $k = 1, \dots, d$, for parameters $a_0, \dots, a_{d-1} \geq 0$, $a_0 = 1$.[6] □

Appendix B

Proof of Theorem 1.1. (iv) ⇒ (i): If (iv) holds, the functions $H_{m,d-m}$ defined in Theorem 1.1(iii) are valid distribution functions on $[0, 1]$, and we can consider the corresponding

[6] An alternative way to constitute the power function structure of the g_k for extreme-value copulas of type (1) is via Pickand's theorem as considered in [7]. Comparing the Pickands representation of C with the functional form in (1) on the diagonal $u = u_1 = \dots = u_d$, it can be shown by induction that $g_k(u) = u^{a_{k-1}}$ for a parameter

$$a_{k-1} = kP\Big(\underbrace{\frac{1}{k}, \dots, \frac{1}{k}}_{k \text{ times}}, 0, \dots, 0\Big) - (k-1)P\Big(\underbrace{\frac{1}{k-1}, \dots, \frac{1}{k-1}}_{(k-1) \text{ times}}, 0, \dots, 0\Big)$$

depending on the Pickand dependence function P for fixed values. By the first part of the proof, the claim follows. We thank the referee for pointing us to the idea for this alternative proof.

stochastic model given in the theorem. We will show that the resulting distribution function of $(X_1, \dots, X_d)$ is a copula given by C. First of all, recognize that each $X_k, k = 1, \dots, d$, has a uniform marginal distribution due to

$$\begin{aligned}
\mathbb{P}(X_k \le u) &= \prod_{E:\, k \in E} \mathbb{P}\big(Z^E \le u\big) \overset{(*)}{=} \prod_{m=1}^{d} H_{m,d-m}^{\binom{d-1}{m-1}}(u) \\
&= \prod_{m=1}^{d} \left(\prod_{i=0}^{m-1} g_{d-m+1+i}^{(-1)^i \binom{m-1}{i}}(u) \right)^{\binom{d-1}{m-1}} \\
&= \prod_{m=1}^{d} \prod_{i=0}^{m-1} g_{d-m+1+i}^{(-1)^i \binom{m-1}{i} \binom{d-1}{m-1}}(u) \\
&\overset{(**)}{=} \prod_{k=1}^{d} g_k^{\sum_{m=d+1-k}^{d} (-1)^{k+m-d-1} \binom{m-1}{k+m-d-1} \binom{d-1}{m-1}}(u) \\
&= u.
\end{aligned}$$

The equality in $(*)$ stems from the fact that there are $d-1$ over $m-1$ shocks Z^E with cardinality m and distribution function $H_{m,d-m}$ that appear in the stochastic construction of X_k. The equality in $(**)$ is yielded by grouping the g_k, i.e. regarding all $g_{d-m+1+i}$ with $d-m+1+i = k$, which is the same as setting $i = k+m-1-d$. For $k = 1$, it is apparent that the exponent in the second last line is equal to one. For $k \ge 2$, due to

$$\binom{m-1}{k+m-d-1} \binom{d-1}{m-1} = \frac{(d-1)!}{(d-k)!(k-1)!} \binom{k-1}{d-m},$$

it follows that the exponent equals

$$\begin{aligned}
&\frac{(d-1)!}{(d-k)!(k-1)!} \sum_{m=d+1-k}^{d} (-1)^{k+m-d-1} \binom{k-1}{d-m} \\
&\quad = \frac{(d-1)!}{(d-k)!(k-1)!} \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} = 0.
\end{aligned}$$

Second, consider the joint distribution function of $(X_1, \dots, X_d)$. For $u_1, \dots, u_d \in (0, 1]$, it is given by

$$\mathbb{P}(X_k \le u_k, k = 1, \dots, d) = \prod_{\emptyset \ne E \subseteq \{1,\dots,d\}} \mathbb{P}\big(Z^E \le \min\{u_k \colon k \in E\}\big). \tag{B.1}$$

Among all subsets E with cardinality $|E| = m$, there are $d-1$ choose $m-1$ elements where $\min\{u_k \colon k \in E\} = u_{(1)}$. Analogously, there are $d-k$ choose $m-1$ elements where $\min\{u_l \colon l \in$

$E\} = u_{(k)}$, $k \in \{2, \dots, d-m+1\}$. Thus, (B.1) is equal to

$$\begin{aligned}
&\prod_{m=1}^{d} \prod_{k=1}^{d-m+1} \mathbb{P}\big(Z^E \le u_{(k)}, |E| = m\big)^{\binom{d-k}{m-1}} \\
&\quad = \prod_{m=1}^{d} \prod_{k=1}^{d-m+1} H_{m,d-m}^{\binom{d-k}{m-1}}(u_{(k)}) \\
&\quad = \prod_{m=1}^{d} \prod_{k=1}^{d-m+1} \prod_{i=0}^{m-1} g_{d-m+1+i}^{(-1)^i \binom{m-1}{i}\binom{d-k}{m-1}}(u_{(k)}) \\
&\quad = \prod_{k=1}^{d} \prod_{m=1}^{d-k+1} \prod_{i=0}^{m-1} g_{d-m+1+i}^{(-1)^i \binom{m-1}{i}\binom{d-k}{m-1}}(u_{(k)}) \\
&\quad \overset{(*)}{=} \prod_{k=1}^{d} \underbrace{\prod_{n=k}^{d} g_n^{\sum_{m=d+1-n}^{d+1-k} (-1)^{m-1+n-d} \binom{m-1}{m-1+n-d}\binom{d-k}{m-1}}(u_{(k)})}_{\text{should be equal to } g_k(u_{(k)})}.
\end{aligned}$$

Now $(*)$ can be derived by sorting all $g_{d-m+1+i}$ with $d-m+1+i = n$, that is, $i = m-1+n-d$. For $n = k$, it becomes obvious that the exponent of g_n is equal to one. For $n \in \{k+1, \dots, d\}$, by using the same deliberations as for the derivation of the marginal distributions, the exponent of g_n is given by

$$\begin{aligned}
&\frac{(d-k)!}{(d-n)!(n-k)!} \sum_{m=d+1-n}^{d+1-k} (-1)^{m-1+n-d} \binom{n-k}{m-1+n-d} \\
&\quad = \frac{(d-k)!}{(d-n)!(n-k)!} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} = 0.
\end{aligned}$$

Summing up, we have

$$\mathbb{P}(X_k \le u_k, \forall k = 1, \dots, d) = \prod_{k=1}^{d} g_k(u_{(k)}) = C(u_1, \dots, u_d),$$

and we can conclude that C is a copula.

(i) $\Rightarrow$ (ii): Let $(U_1, \dots, U_d)$ be a random vector with copula C in (1) as distribution function. Moreover, assume for a moment that g_k, $k = 2, \dots, d$, is strictly positive on $(0, 1]$. Then, for $u, v \in (0, 1]$, $u < v$, $G_{j,k}$ has the representation

$$G_{j,k}(u, v) = \frac{1}{\prod_{m=1}^{k} g_m(u)} \Bigg(\mathbb{P}(A_\varnothing) - \sum_{i=1}^{j} (-1)^{i+1} \sum_{\substack{L \subseteq \{k+1, \dots, k+j\}: \\ |L| = i}} \mathbb{P}\Big(\bigcap_{l \in L} A_l\Big)\Bigg),$$

$$A_l := \Big(\bigcap_{m\in\{1,\dots,k,l\}} \{U_m \le u\}\Big) \cap \Big(\bigcap_{m\in\{k+1,\dots,k+j\}\setminus\{l\}} \{U_m \le v\}\Big),$$
$$A_\varnothing := \{U_1 \le u, \dots, U_k \le u, U_{k+1} \le v, \dots, U_{k+j} \le v\}.$$

Applying the principle of inclusion and exclusion (see [3], page 24), we have

$$\begin{aligned} G_{j,k}(u,v) &= \frac{1}{\prod_{m=1}^k g_m(u)}\Bigg(\mathbb{P}(A_\varnothing) - \mathbb{P}\Bigg(\bigcup_{l=k+1}^{k+j} A_l\Bigg)\Bigg) \\ &= \frac{1}{\prod_{m=1}^k g_m(u)}\mathbb{P}(A) \ge 0, \end{aligned}$$

where

$$A := \big\{U_1 \le u, \dots, U_k \le u, U_{k+1} \in [u,v], \dots, U_{k+j} \in [u,v]\big\}.$$

Strict positivity of g_k (which we have assumed so far) as well as continuity on $(0,1]$ for $k = 2,\dots,d$ can be shown by induction. To begin with, assume that there is a $u^* := \sup\{u \ge 0\colon\ g_2(u) = 0\} > 0$. As C is a copula and hence continuous, it follows that $g_2(u^*) = 0$, such that for $v > u^*$,

$$\mathbb{P}\big(U_1 \in [u^*, v], U_2 \in [u^*, v]\big) = G_{2,0}\big(u^*, v\big) = vg_2(v) - 2u^* g_2(v) < 0$$

for v sufficiently close to u^*. This is a contradiction and hence $g_2(u) > 0$ for $u \in (0,1]$. Similarly, to show continuity, assume that there is a $v^* \in (0,1]$ such that $g_2(v^*-) := \lim_{u\nearrow v^*} g_2(u) < g_2(v^*)$. Then

$$\begin{aligned} 0 \le \lim_{u\nearrow v^*} G_{2,0}\big(u, v^*\big) &= \lim_{u\nearrow v^*} \big(v^* g_2\big(v^*\big) - 2ug_2\big(v^*\big) + ug_2(u)\big) \\ &= -v^* g_2\big(v^*\big) + v^* g_2\big(v^*-\big) < 0, \end{aligned}$$

which is a contradiction. Hence, there is no such v^* and g_2 is left-continuous on $(0,1]$. Analogously, if $g_2(u^*+) := \lim_{v\searrow u^*} > g_2(u^*)$ for an $u^* \in (0,1)$, $G_{2,0}(u^*, v)$ becomes negative for sufficiently small $v > u^*$. Consequently, g_2 is continuous on $(0,1]$.

For the induction step $k-1 \mapsto k$, note that

$$G_{2,k-1}(u,v) = \frac{1}{\prod_{m=1}^{k-1} g_m(u)}\mathbb{P}\big(U_1 \le u, \dots, U_{k-1} \le u, U_k \in [u,v], U_{k+1} \in [u,v]\big)$$

induces $0 \le G_{2,k-1}(u,v) = g_{k-1}(v)g_k(v) - 2g_{k-1}(u)g_k(v) + g_{k-1}(u)g_k(u)$. By the same arguments as for the induction start, this implies that g_k is both strictly positive and continuous on $(0,1]$.

Remark B.1 (Alternative interpretation of $G_{j,k}$). In a very similar way, one can show that if $H_{m,k+j-m}$ in Theorem 1.1(iii) is an element of $\mathcal{D}$ for all $m \in \{1,\dots,j\}$, $G_{j,k}$ can be expressed as

$$G_{j,k}(u,v) = \mathbb{P}\big(X_{k+1} \in [u,v], \dots, X_{k+j} \in [u,v]\big),$$

where

$$X_l := \max\{Z^E \colon k \in E\}, l = k+1, \dots, k+j \text{ and } Z^E \sim H_{m,k+j-m} \text{ for } |E| = m,$$

with independent random variables Z^E, $\varnothing \neq E \subseteq \{k+1, \dots, k+j\}$. However, note that in the proof of "(i) $\Rightarrow$ (ii)" above, we solely require that C is a copula and do not assume increasingness of $H_{m,k+j-m}$, which is why we cannot apply the alternative interpretation in the present case.

(ii) $\Rightarrow$ (iii) Let $G_{j,k}(u, v) \geq 0$ for all $0 < u < v \leq 1$, $k \in \mathbb{N}_0$, $j \in \mathbb{N}$ with $k + j \leq d$. By the proof of "(i) $\Rightarrow$ (ii)" above, this implies g_k, $k = 2, \dots, d$ to be strictly positive and continuous on $(0, 1]$. The idea of this part of the proof is to establish a connection between $G_{j,k}$ and a related stochastic model similar to Remark B.1 in order to derive reasonable estimates that help to derive the required conditions in (iii). For readability, we are going to proceed by induction.

Suppose that we have already shown that for a $j-1 \in \{1, \dots, d\}$, the conditions $G_{i,k}(u, v) \geq 0$ for all $1 \leq i \leq j-1$ and $k+i \leq d$ imply that $H_{i,k}$ is increasing for all $1 \leq i \leq j-1$ and $k+i \leq d$. For $j = 2$, this is obviously satisfied and the induction basis is established. In order to carry out the induction step, we need to show that $H_{j,k}$ is increasing for all $k + j \leq d$ ($H_{i,k}$ for $i \leq j-1$ are increasing by induction hypothesis). This is shown in several steps.

Step 1 (Main observation): There is a useful decomposition of $G_{j,k}$ that we require below.

Lemma B.1 (Decomposition of $G_{j,k}$). *Instead of $G_{j,k}$, write $G_{g_{k+1},\dots,g_{k+j}}$ to emphasize the dependence of $G_{j,k}$ on the functions $g_{k+1}, \dots, g_{k+j}$. It holds that*

$$\begin{aligned} G_{g_{k+1},\dots,g_{k+j}}(u, v) &= \tilde{g}_{k+1}(v) g_{k+2}(v) \dots g_{k+j}(v) \left(\frac{g_{k+1}(v)}{\tilde{g}_{k+1}(v)} - \frac{g_{k+1}(u)}{\tilde{g}_{k+1}(u)} \right) \\ &\quad + \frac{g_{k+1}(u)}{\tilde{g}_{k+1}(u)} G_{\tilde{g}_{k+1}, g_{k+2},\dots,g_{k+j}}(u, v), \qquad 0 < u < v \leq 1, \end{aligned} \tag{B.2}$$

for an arbitrary function $\tilde{g}_{k+1}$ that is unequal to zero on $(0, 1)$.

Proof. The decomposition consists of nothing else than changing the last summand of $G_{g_{k+1},\dots,g_{k+j}}$ (ending up with the last line in equation (B.2)) and adding the resulting difference as an extra term (corresponding to the first line in equation (B.2)). The non-zero condition for $\tilde{g}_{k+1}$ is required for well-defined quotients. $\square$

Define $\tilde{g}_{k+1} := g_{k+1}/H_{j,k}$, which is continuous and strictly positive on $(0, 1]$ as seen earlier, and note that Lemma B.1 then yields

$$\begin{aligned} 0 &\leq G_{g_{k+1},\dots,g_{k+j}}(u, v) \\ &= \tilde{g}_{k+1}(v) g_{k+2}(v) \cdots g_{k+j}(v) \big(H_{j,k}(v) - H_{j,k}(u) \big) + H_{j,k}(u) G_{\tilde{g}_{k+1}, g_{k+2},\dots,g_{k+j}}(u, v). \end{aligned} \tag{B.3}$$

We want to conclude that $H_{j,k}(v) \geq H_{j,k}(u)$. Therefore, we have to prove that the second summand is not responsible for non-negativity of $G_{g_{k+1},\dots,g_{k+j}}$. The crucial consequence of (B.3) is that $G_{\tilde{g}_{k+1},g_{k+2},\ldots,g_{k+j}}$ can be related to a stochastic model. To this end, we want to apply Remark B.1 to $G_{\tilde{g}_{k+1},g_{k+2},\ldots,g_{k+j}}$. In order to do so, one has to make sure that the corresponding functions $\tilde{H}_{m,k+j-m}$ (which are defined just like $H_{m,k+j-m}$, however with replacing g_{k+1} by $\tilde{g}_{k+1}$) are distribution functions in $\mathcal{D}$ for all $m = 1, \ldots, j$. Due to the definition of $\tilde{g}_{k+1}$, it holds that

$$\tilde{H}_{m,k+j-m} = \begin{cases} H_{m,k+j-m}, & \text{for } m = 1, \ldots, j-1, \\ 1, & \text{for } m = j. \end{cases}$$

As $H_{m,k+j-m}$ are distribution functions for $m = 1, \ldots, j-1$ by induction hypothesis and $\tilde{H}_{m,k+j-m}$ is a degenerated distribution function for $m = j$, the requirements of Remark B.1 are satisfied.

Step 2 (Stochastic model): As a consequence of Remark B.1,

$$G_{\tilde{g}_{k+1},g_{k+2},\ldots,g_{k+j}}(u,v) = \mathbb{P}\big(X_{k+1}, \ldots, X_{k+j} \in [u,v]\big),$$

where

$$X_l := \max\big\{Z^E, E \subset \{k+1, \ldots, k+j\}, E \cap \{l\} \neq \varnothing\big\}, l = k+1, \ldots, k+j,$$
$$Z_E \sim H_{m,k+j-m} \text{ for } |E| = m,$$

with independent random variables $Z^E, \varnothing \neq E \subset \{k+1, \ldots, k+j\}$. Thus,

$$\mathbb{P}\big(X_{k+1}, \ldots, X_{k+j} \in [u,v]\big) = \mathbb{P}\Bigg(\underbrace{\bigcap_{l=k+1}^{k+j} \big\{\max\big\{Z^E, E \cap \{l\} \neq \varnothing\big\} \in [u,v]\big\}}_{:=A}\Bigg)$$

requires that all Z^E are less than or equal to v and – as there is no common shock with $|E| = j$ due to $\tilde{H}_{j,k}(x) \equiv 1$ – at least two $Z^I, Z^J, I, J \subset \{k+1, \ldots, k+j\}, I \neq J$, need to be in the interval $[u,v]$. This implies that

$$A \subset \bigcup_{\substack{\varnothing \neq I, J \subset \{k+1,\ldots,k+j\} \\ I \neq J}} \big\{u \leq Z^I, Z^J \leq v\big\}.$$

Moreover, as $\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i)$ for arbitrary $A_i \in \mathcal{F}$, and as there are $(2^j - 2)$ choose 2 possibilities to pick $Z^I, Z^J \in [u,v]$, we have

$$\begin{aligned} G_{\tilde{g}_{k+1},g_{k+2},\ldots,g_{k+j}}(u,v) = \mathbb{P}(A) &\leq \sum_{\substack{\varnothing \neq I, J \subset \{k+1,\ldots,k+j\} \\ I \neq J}} \mathbb{P}\big(\big\{u \leq Z^I, Z^J \leq v\big\}\big) \\ &\leq \underbrace{\binom{2^j - 2}{2}}_{:=b} \max_{m=1,\ldots,j-1} \Big\{\big(H_{m,k+j-m}(v) - H_{m,k+j-m}(u)\big)^2\Big\}. \end{aligned} \tag{B.4}$$

Step 3 (Lipschitz-continuity): Using equation (B.4), we are going to derive Lipschitz-continuity-type results for $G_{\tilde{g}_{k+1},g_{k+2},\dots,g_{k+j}}$. In order to do so, the following lemma is helpful.

Lemma B.2. *For $k \in \mathbb{N}_0$, $j \geq 2$, let $H_{1,k},\dots,H_{j,k}:(0,1]\to(0,1]$ and $H_{1,k+1},\dots,H_{j-1,k+1}:[0,1]\to[0,1]$ be increasing functions with $H_{l,k}=H_{l-1,k}/H_{l-1,k+1}$ for $l\in\{2,\dots,j\}$. Then it holds that*

$$0 \leq H_{j,k}(v) - H_{j,k}(u) \leq \left(\prod_{l=1}^{j-1} \frac{1}{H_{l,k+1}(u)}\right)\big(H_{1,k}(v) - H_{1,k}(u)\big).$$

Proof. *For $j=2$ and $k\in\mathbb{N}_0$, we have*

$$\begin{aligned}0 \leq H_{2,k}(v) - H_{2,k}(u) &= \frac{1}{H_{1,k+1}(u)}\left(\underbrace{\frac{H_{1,k+1}(u)}{H_{1,k+1}(v)}}_{\leq 1} H_{1,k}(v) - H_{1,k}(u)\right)\\ &\leq \frac{1}{H_{1,k+1}(u)}\big(H_{1,k}(v) - H_{1,k}(u)\big).\end{aligned}$$

For $j \mapsto j+1$, the claim follows by simple induction. □

Applying Lemma B.2 to equation (B.4), $G_{\tilde{g}_{k+1},\dots,g_{k+j}}(u,v)$ has an upper bound

$$b \cdot \max_{m=1,\dots,j-1}\left\{\left(\prod_{l=1}^{m-1}\frac{1}{H_{l,k+j-m+1}(u)}\right)^2 \underbrace{\big(g_{k+j-m+1}(v) - g_{k+j-m+1}(u)\big)^2}_{=H_{k+j-m}(v)-H_{k+j-m}(u)}\right\}. \tag{B.5}$$

This can be further simplified as

$$\begin{aligned}0 \leq G_{2,0}(u,v) &= g_1(v)g_2(v) - 2g_1(u)g_2(v) + g_1(u)g_2(u)\\ &= g_2(v)\big(g_1(v)-g_1(u)\big) - g_1(u)\big(g_2(v)-g_2(u)\big)\\ \iff \quad g_2(v) - g_2(u) &\leq \frac{g_2(v)}{g_1(u)}\big(g_1(v)-g_1(u)\big) = \frac{g_2(v)}{u}(v-u)\end{aligned}$$

and since g_1 is the identity by definition. Analogously, one has

$$0 \leq G_{2,k}(u,v) \quad \iff \quad g_{k+2}(v) - g_{k+2}(u) \leq \frac{g_{k+2}(v)}{g_{k+1}(u)}\big(g_{k+1}(v) - g_{k+1}(u)\big).$$

By induction over k, one can conclude that

$$g_k(v) - g_k(u) \leq \left(\prod_{l=1}^{k-1}\frac{g_{l+1}(v)}{g_l(u)}\right)(v-u) \qquad \text{for all } k \text{ in concern.}$$

Applying this result, the expression in (B.5) is less than or equal to

$$\begin{aligned} & b \cdot \max_{m=1,\dots,j-1} \left\{ \left(\prod_{l=1}^{m-1} \frac{1}{H_{l,k+j-m+1}(u)} \right)^2 \left(\prod_{l=1}^{k+j-m} \frac{g_{l+1}(v)}{g_l(u)} \right)^2 (v-u)^2 \right\} \\ & \quad = p_{j,k}(u,v)(v-u)^2, \end{aligned}$$

with

$$p_{j,k}(u,v) := b \cdot \max_{m=1,\dots,j-1} \left\{ \left(\prod_{l=1}^{m-1} \frac{1}{H_{l,k+j-m+1}(u)} \right)^2 \left(\prod_{l=1}^{k+j-m} \frac{g_{l+1}(v)}{g_l(u)} \right)^2 \right\}.$$

Additionally, due to the monotonicity of the g_k and $H_{j,k}$ appearing in $p_{j,k}$, we can conclude that for any $u_0, v_0 \in (0,1]$, $u_0 < v_0$, it holds that $p_{j,k}(u,v) \leq p_{j,k}(u_0,v_0)$ for all $u, v \in [u_0, v_0]$, $u \leq v$. Combining all those observations, one ends up with

$$0 \leq G_{\tilde{g}_{k+1}, g_{k+2}, \dots, g_{k+j}}(u,v) \leq p_{j,k}(u_0,v_0)(v-u)^2, \qquad u, v \in [u_0, v_0], u \leq v. \tag{B.6}$$

Step 4 (Proof by contradiction): Finally, we can proceed similarly to the proof in the bivariate case depicted in [5], page 67. Assume that $H_{j,k}$ is not increasing and that there exist $u_0, v_0 \in (0,1]$, $u_0 < v_0$, such that

$$H_{j,k}(v_0) - H_{j,k}(u_0) = -a(u_0,v_0)(v_0-u_0), \qquad a(u_0,v_0) > 0.$$

Consequently, by continuity of the g_k and hence $H_{j,k}$, for every $\varepsilon \in (0, v_0 - u_0]$, there are $u_\varepsilon, v_\varepsilon \in [u_0, v_0]$, $u_\varepsilon = v_\varepsilon - \varepsilon$, such that

$$H_{j,k}(v_\varepsilon) - H_{j,k}(u_\varepsilon) \leq -a(u_0,v_0)(v_\varepsilon - u_\varepsilon) = -a(u_0,v_0)\varepsilon. \tag{B.7}$$

Independently of this assumption, we can split the positive and negative powers in $H_{j,k}$, yielding

$$H_{j,k}(u) = \prod_{i=0}^{j-1} g_{k+1+i}^{(-1)^i \binom{j-1}{i}}(u) = \frac{\prod_{i=0}^{\lfloor (j-1)/2 \rfloor} g_{k+1+2i}^{\binom{j-1}{2i}}(u)}{\prod_{i=0}^{\lfloor (j-2)/2 \rfloor} g_{k+1+2i+1}^{\binom{j-1}{2i+1}}(u)}, \qquad u > 0,$$

with "$\lfloor \cdot \rfloor$" denoting the floor function, such that for $u \in [u_0, v_0]$, it holds by the monotonicity of the g_k that

$$\begin{aligned} p_{\min}(u_0,v_0) &:= \frac{\prod_{i=0}^{\lfloor (j-1)/2 \rfloor} g_{k+1+2i}^{\binom{j-1}{2i}}(u_0)}{\prod_{i=0}^{\lfloor (j-2)/2 \rfloor} g_{k+1+2i+1}^{\binom{j-1}{2i+1}}(v_0)} \leq H_{j,k}(u) \\ &\leq \frac{\prod_{i=0}^{\lfloor (j-1)/2 \rfloor} g_{k+1+2i}^{\binom{j-1}{2i}}(v_0)}{\prod_{i=0}^{\lfloor (j-2)/2 \rfloor} g_{k+1+2i+1}^{\binom{j-1}{2i+1}}(u_0)} =: p_{\max}(u_0,v_0). \end{aligned}$$

Plugging $u_\varepsilon, v_\varepsilon$ into equation (B.3) and using all previous results yields

$$\begin{aligned} 0 \;\leq\; & G_{g_{k+1},\dots,g_{k+j}}(u_\varepsilon, v_\varepsilon) = \underbrace{\tilde{g}_{k+1}(v_\varepsilon)}_{=\frac{g_{k+1}(v_\varepsilon)}{H_{j,k}(v_\varepsilon)}} g_{k+2}(v_\varepsilon)\cdots g_{k+j}(v_\varepsilon)\underbrace{\bigl(H_{j,k}(v_\varepsilon) - H_{j,k}(u_\varepsilon)\bigr)}_{\leq -a(u_0,v_0)\varepsilon} \\ & + H_{j,k}(u_\varepsilon) G_{\tilde{g}_{k+1},\dots,g_{k+j}}(u_\varepsilon, v_\varepsilon) \\ \overset{(B.6)}{\leq}\; & \frac{g_{k+1}(u_0)}{p_{\max}(u_0, v_0)} g_{k+2}(u_0)\cdots g_{k+j}(u_0)\bigl(-a(u_0, v_0)\varepsilon\bigr) \\ & + p_{\max}(u_0, v_0) p_{j,k}(u_0, v_0)\varepsilon^2. \end{aligned}$$

Thus, for sufficiently small ε, $G_{j,k}$ becomes negative and yields a contradiction. Consequently, $H_{j,k}$ has to be increasing and the induction is complete.

Remark B.2. The proofs "(i) ⇒ (ii)" and "(ii) ⇒ (iii)" generalize parts of the proof ideas for Theorem 1.1 in [16]. The major generalization consists in omitting differentiability of the g_k, which substantially complicates the calculations and requires the alternative proof techniques picked up in "(ii) ⇒ (iii)" above.

(iii) ⇒ (iv): Trivial, as (iv) is a special case of (iii) for $j = m$, $k = d - m$. □

Appendix C

Proof of Proposition 3.1. The survival function of each X_k is given by

$$\begin{aligned} \bar{F}_1(x) &:= \mathbb{P}(X_k > x) = \mathbb{P}(E_k > \Lambda_x) = \mathbb{E}\bigl[\mathbb{P}(E_k > \Lambda_x | \Lambda_x)\bigr] \\ &= \mathbb{E}\bigl[\mathrm{e}^{-\Lambda_x}\bigr] = \mathrm{e}^{-\Psi_x(1)}, \qquad x \geq 0. \end{aligned}$$

The joint survival function of $(X_1, \dots, X_d)$ can be derived analogously. For $\mathbf{x} := (x_1, \dots, x_d) \geq 0$, with the convention $x_0 := 0$, it is given by

$$\begin{aligned} \bar{F}_d(\mathbf{x}) &:= \mathbb{P}(X_1 > x_1, \dots, X_d > x_d) = \mathbb{E}\bigl[\mathrm{e}^{-\sum_{k=1}^d \Lambda_{x_k}}\bigr] \\ &= \mathbb{E}\bigl[\mathrm{e}^{-\sum_{k=1}^d (d-k+1)(\Lambda_{x_{(k)}} - \Lambda_{x_{(k-1)}})}\bigr] = \prod_{k=1}^d \mathbb{E}\bigl[\mathrm{e}^{-(d-k+1)(\Lambda_{x_{(k)}} - \Lambda_{x_{(k-1)}})}\bigr] \\ &= \prod_{k=1}^d \exp\bigl(-\Psi_{x_{(k)}}(d-k+1) + \Psi_{x_{(k-1)}}(d-k+1)\bigr) \\ &= \prod_{k=1}^d \exp\bigl(-\Psi_{x_{(k)}}(d-k+1) + \Psi_{x_{(k)}}(d-k)\bigr). \end{aligned}$$

Due to the stochastic continuity of Λ, $x \mapsto \Psi_x(1)$ is continuous. Thus, the unique survival copula $C_{\Lambda,d}$ of $(X_1, \dots, X_d)$ is defined as

$$\begin{aligned} C_{\Lambda,d}(u_1, \dots, u_d) &:= \bar{F}_d\big(\bar{F}_1^{-1}(u_1), \dots, \bar{F}_1^{-1}(u_d)\big) \\ &= \prod_{k=1}^{d} \exp\big(-\Psi_{\bar{F}_1^{-1}(u_{(k)})}(k) + \Psi_{\bar{F}_1^{-1}(u_{(k)})}(k-1)\big) \qquad \text{(C.1)} \\ &= \prod_{k=1}^{d} g_k(u_{(k)}), \end{aligned}$$

where

$$g_k(u) := \exp\big(-\Psi_{\bar{F}_1^{-1}(u)}(k) + \Psi_{\bar{F}_1^{-1}(u)}(k-1)\big).$$

By construction, $g_1 = \mathrm{id}_{[0,1]}$ and $g_2(1) = \cdots = g_d(1) = 1$. $\square$

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