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Piecewise quantile autoregressive modeling for nonstationary time series

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We develop a new methodology for the fitting of nonstationary time series that exhibit nonlinearity, asymmetry, local persistence and changes in location scale and shape of the underlying distribution. In order to achieve this goal, we perform model selection in the class of piecewise stationary quantile autoregressive processes. The best model is defined in terms of minimizing a minimum description length criterion derived from an asymmetric Laplace likelihood. Its practical minimization is done with the use of genetic algorithms. If the data generating process follows indeed a piecewise quantile autoregression structure, we show that our method is consistent for estimating the break points and the autoregressive parameters. Empirical work suggests that the proposed method performs well in finite samples.

Keywords: autoregressive time series; change-point; genetic algorithm; minimum description length principle; nonstationary time series; structural break

1. Introduction

Many time series observed in practice display nonstationary behavior, especially if data is collected over long time spans. Nonstationarity can affect the trend, the variance–covariance structure or, more comprehensively, aspects of the underlying distribution. Since estimates and forecasts can be severely biased if nonstationarity is not properly taken into account, identifying and locating structural breaks has become an important issue in the analysis of time series. Over the years, there has been a large amount of research on issues related to testing and estimating structural breaks in sequences of independent random variables, time series and regression models. Most of these focus on considering breaks in the (conditional) mean, while a smaller number of publications are available for breaks in the (conditional) variance. The relevant lines of research are summarized in the monograph [6] and the more recent survey paper [2].

In various situations, however, it may be helpful and more informative to study structural breaks in the (conditional) quantiles. As a case in point, Hughes *et al.* [12] have argued convincingly that the increase in mean surface temperatures recorded at temperature stations across the Antarctic can to a large degree be attributed to an increase in the minimum and lower quantile temperatures. When focusing on the mean, this additional information about the underlying changes in variation is smoothed out and unavailable for a more in-depth analysis. As another example, the Value at Risk, a measure of loss associated with a rare event under normal market conditions, is by definition a quantile and more important for risk managers than information on measures of central tendency such as the mean.

Global estimation procedures for quantiles are often performed in the quantile regression framework described in [13]. There is by now a rich body of literature on the various aspects of quantile regression models. Detecting structural breaks in nonstationary time series over different quantiles, however, is a comparatively new research area. Contributions in a different direction from ours include [5], who considered the estimation of structural breaks in the median of an underlying regression model by means of least absolute deviations. In the quantile regression framework, Aue *et al.* [1] have recently developed a related methodology to perform segmented variable selection that includes break point detection as a special case. The focus of the present paper, however, is more on the aspects of nonlinear time series analysis.

In order to capture nonlinearities such as asymmetries, local persistence, and changes in location, scale and shape, in conjunction with temporal dependence that is frequently observed in applications, and thus to obtain a more complete picture of the distributional evolution of the underlying random processes, we propose in this paper a new method for estimating structural breaks at any single quantile or across multiple quantiles. Our methodology differs from the works above in that it is not based on hypothesis testing. Instead we try to match the observed data with a best fitting piecewise quantile autoregression. These models, introduced by Koenker and Xiao [14], are members of the class of random coefficient autoregressions that allow the autoregressive coefficients to be quantile dependent and, therefore, generalize linear quantile autoregressions as studied by Koul and Saleh [15], and Hallin and Jurečková [10], among others. We discuss quantile autoregression models and their piecewise specifications in Section 2. In particular, we state necessary and sufficient conditions for the existence of stationary solutions and discuss the estimation of the parameters via optimizing a subgradient condition. These results will then be generalized to the piecewise stationary case.

Recognizing the connection between estimation of quantile autoregression parameters and maximum likelihood estimation for asymmetric Laplace random variables [23], we shall apply the minimum description length principle [20] to define the best fitting piecewise quantile autoregression. Details of this are given in Section 3. Minimization of the resulting convex objective function will then yield the best fitting model for the given data. The numerical complexity of this optimization problem is handled via the application of a genetic algorithm [7].

From a technical perspective, our methodology is related to [8], who proposed an automatic procedure termed Auto-PARM. This procedure is designed to detect structural breaks by fitting piecewise stationary, linear autoregressive time series models which are estimated through the minimization of a minimum description length criterion using a normal likelihood. Auto-PARM is defined to mimic the second-order properties of the data but is not always able to adjust to a nonlinear framework and does not provide additional insight into distributional changes other than those affecting the conditional mean and variance of the data given past observations.

The remainder of the paper is organized as follows. In Section 2, quantile autoregressive models are introduced. Estimation and model selection aspects for piecewise quantile autoregressive models are detailed in Section 3. Sections 4 and 5 deal with asymptotic results and implementation details, respectively. Empirical properties of the proposed methodology are evaluated through simulations in Section 6 and real data examples in Section 7. Section 8 concludes and all technical proofs are given in the Appendix.

2. Quantile autoregressions

Linear autoregressive models have played a dominant role in classical time series analysis for at least half a century. The popularity stems partially from their closeness to the linear regression framework with its well-developed theory. They are, however, unable to capture nonlinear dynamics and local persistence. With the objective of dynamically modeling the evolution of location, scale and shape of the underlying processes, Koenker and Xiao [14] have introduced a particular subclass of random coefficient autoregressive models called quantile autoregressions. In this model, autoregressive coefficients are allowed to vary with the quantiles $\tau \in [0, 1]$. In contrast to many of the standard contributions to the random coefficient autoregression area for which independence is a key assumption, the coefficients possess a strong functional relationship; in sequel $\mathbb{Z}$ denotes the set of integers. A real time series $(y_t\colon\ t \in \mathbb{Z})$ is said to follow a quantile autoregression of order p, shortly QAR(p), if

$$y_t = \theta_0(u_t) + \theta_1(u_t)y_{t-1} + \cdots + \theta_p(u_t)y_{t-p}, \qquad t \in \mathbb{Z}, \tag{1}$$

where $(u_t\colon\ t \in \mathbb{Z})$ are independent random variables distributed uniformly on the interval $[0, 1]$, and $\theta_j : [0, 1] \to \mathbb{R}$, $j = 0, 1, \ldots, p$, are the coefficient functions. In order to exhibit the connection to standard random coefficient autoregressions, (1) can also be written more conventionally in the form

$$y_t = \phi_0 + \phi_{1,t}y_{t-1} + \cdots + \phi_{p,t}y_{t-p} + \varepsilon_t, \qquad t \in \mathbb{Z}, \tag{2}$$

where $\phi_0 = E\{\theta_0(u_t)\}$, $\varepsilon_t = \theta_0(u_t) - \phi_0$, and $\phi_{j,t} = \theta_j(u_t)$ for $j = 1, \ldots, p$ and $t \in \mathbb{Z}$. We have in particular that the innovations $(\varepsilon_t\colon\ t \in \mathbb{Z})$ constitute an independent, identically distributed sequence with distribution function $F(\cdot) = \theta_0^{-1}(\cdot + \phi_0)$. Therefore, necessary and sufficient conditions for the existence of a strictly stationary solution to the equations (1) can be derived from the work of Aue *et al.* [3], which also contains statements concerning the finiteness of moments of quantile autoregressions.

The estimation of the quantile autoregression functions $\theta(\tau)$ in stationary quantile autoregressive models (1) is typically achieved [13] by solving the convex optimization problem

$$\min_{\theta(\tau)\in\mathbb{R}^{p+1}} \sum_{t=1}^{n} \rho_\tau\big\{y_t - X_t'\theta(\tau)\big\}, \tag{3}$$

where $\rho_\tau(u) = u\{\tau - I(u < 0)\}$ is the check function. Solutions $\hat{\theta}(\tau)$ of (3) are called autoregression quantiles. Asymptotic properties of the estimation procedure have been derived in [14]. It should be noted that the assumptions under which the following proposition holds require $X_t'\theta(\tau)$ to be monotonic. This will not always be reasonable. However, for the methodology developed in this paper, this is not an issue insofar as we derive asymptotic statements only about the quality of the segmentation procedure but not on the quality of the estimator $\hat{\theta}$.

Proposition 2.1. *Let $F_{t-1} = P(y_t < \cdot \mid \mathcal{F}_{t-1})$ be the conditional distribution function of y_t given $\mathcal{F}_{t-1}$, and denote by f_{t-1} its derivative. Under stationarity and if f_{t-1} is uniformly integrable*

on $\mathcal{X} = \{x\colon 0 < F(x) < 1\}$, *then*

$$\Sigma^{-1/2} n^{1/2} \big[\hat{\theta}(\cdot) - \theta(\cdot)\big] \xrightarrow{\mathcal{D}} B_{p+1}(\cdot) \qquad (n \to \infty),$$

where $\Sigma = \Omega_1^{-1}\Omega_0\Omega_1^{-1}$ *with* $\Omega_0 = E(X_t X_t')$ *and* $\Omega_1 = \lim_n \frac{1}{n}\sum_{t=1}^n f_{t-1}\{F_{t-1}^{-1}(\tau)\}X_t X_t'$. *Moreover,* $(B_{p+1}(\tau)\colon \tau \in [0,1])$ *is a standard* $(p+1)$*-dimensional Brownian bridge.*

If the number of break points m is given, then estimating their locations and the $m+1$ piecewise quantile autoregressive models at a specific quantile $\tau \in (0,1)$ can be done via solving

$$\min_{\theta(\tau),\mathcal{K}} \sum_{j=1}^{m+1} \sum_{t=k_{j-1}+1}^{k_j} \rho_\tau\big\{y_t - X_{j,t}'\theta_j(\tau)\big\}. \tag{4}$$

Given that the number of observations in each segment increases as a fraction of the overall sample size, the limit behavior of (4) follows directly from Proposition 2.1. For unknown m, we use a model selection approach to select the numbers of segments. To this end, we discuss the relation between (3) and (4), and optimizing the likelihood obtained from asymmetric Laplace distributions next.

The connection between the asymmetric Laplace distribution and quantile regression has long been recognized and has often been used in the Bayesian context. Yu *et al.* [23] have made this explicit. If we assume that at the τth quantile the innovations $(\varepsilon_t\colon t \in \mathbb{Z})$ in model (2) follow an asymmetric Laplace distribution with parameter τ, then maximizing the likelihood function

$$L\big\{\theta(\tau)\big\} \propto \exp\left[-\sum_{t=1}^n \rho_\tau\big\{y_t - X_t'\theta(\tau)\big\}\right]$$

is equivalent to solving the problem in (3). The equivalent to (4) could be stated in a similar fashion. The use of the asymmetric Laplace likelihood allows us to formulate a minimum description length criterion in order to do model selection with (4).

3. Piecewise quantile autoregressions

3.1. The model

Koenker and Xiao [14] have pointed out that a fitted quantile autoregressive model should serve as a useful local approximation to a potentially more complicated global dynamic. While a single quantile autoregression fit can already adequately and quite explicitly describe local persistence and seemingly explosive behavior (see Sections 6 and 7 for examples), it does not provide us with means to fit nonstationary data. We propose to match a nonstationary time series by blocks of different stationary quantile autoregressions.

The piecewise stationary quantile autoregressive models are defined as follows. Assume that the data $y_1, \ldots, y_n$ can be segmented into $m+1$ stationary pieces, and that, for $\ell = 1, \ldots, m+1$,

the ℓth piece can be modeled by a QAR(p_ℓ) process. For $\ell = 1, \ldots, m+1$, we denote by k_ℓ the ℓth break date, that is, the time lag at which the transition from the ℓth to the $(\ell+1)$th segment occurs. Using the convention $k_0 = 1$ and $k_{m+1} = n$ and letting $u_1, \ldots, u_n$ be independent standard uniform random variables, the ℓth segment is, for $t = k_{\ell-1}+1, \ldots, k_\ell$, given by

$$y_t = \theta_{\ell,0}(u_t) + \theta_{\ell,1}(u_t)y_{t-1} + \cdots + \theta_{\ell,p_\ell}(u_t)y_{t-p_\ell} = X'_{\ell,t}\theta_\ell(u_t), \tag{5}$$

where $X_{\ell,t} = (1, y_{t-1}, \ldots, y_{t-p_\ell})'$ and $\theta_\ell(u_t) = \{\theta_{\ell,0}(u_t), \ldots, \theta_{\ell,p_\ell}(u_t)\}'$. At $\tau \in (0,1)$, model (5) is determined by the parameters m, $\mathcal{K} = (k_1, \ldots, k_m)'$ and $\theta(\tau) = \{\theta_1(\tau)', \ldots, \theta_{m+1}(\tau)'\}'$, where the segment autoregression functions are denoted by $\theta_\ell(\tau) = \{\theta_{\ell,0}(\tau), \theta_{\ell,1}(\tau), \ldots, \theta_{\ell,p_\ell}(\tau)\}'$. Observe that in the case that $m = 0$, (5) reduces to the single QAR(p) model (1). One can fit the model (5) even if it is not the true data generating process and that we can then view the piecewise quantile autoregressive structure as an approximation.

The approach taken in this paper is related to the piecewise AR model fitting technique Auto-PARM developed in [8]. These authors utilized linear time series models, changing the coefficient functions $\theta_{\ell,j}(\cdot)$ in (5) to constants, say, $\phi_{\ell,j}$, and were concerned mainly about matching the second-order structure of the data with stationary AR segments. The present paper focuses on nonlinear aspects of the time series as observed from quantiles, thereby enabling a more comprehensive study of changes in the distribution of the underlying data. The switch from linear to nonlinear time series means in particular that somewhat different arguments are needed in order to prove large-sample results (see Section 4). In terms of practical estimation, the genetic algorithm behind Auto-PARM can be modified for the piecewise quantile autoregression fitting. Details are given in Section 5.

3.2. Model selection at a single quantile

In this section, we derive a minimum description length criterion for choosing the best fitting model from the piecewise quantile autoregressive models defined in (5). As to be seen below, the "best" model is defined as the one that enables the best compression of the observed series $Y = (y_1, \ldots, y_n)'$. For introductory material on this, see, for example, [11,17,20].

There are different versions of the minimum description length principle, and the version adopted here is the so-called two-part code. It begins with splitting Y into two parts. The first part, denoted by $\hat{\mathcal{F}}$, represents the fitted piecewise quantile autoregression, and the second part, denoted by $\hat{\mathcal{E}} = Y - \hat{Y}$, represents the residuals, where $\hat{Y}$ is the fitted value for Y. Notice that once $\hat{\mathcal{F}}$ and $\hat{\mathcal{E}}$ are known, Y can be completely retrieved. The idea of the minimum description length principle is to find the best pair of $\hat{\mathcal{F}}$ and $\hat{\mathcal{E}}$ so that via encoding (or compressing) $\hat{\mathcal{F}}$ and $\hat{\mathcal{E}}$, Y can be transmitted (or stored) with the least amount of codelength (or memory). To quantify this idea, let $\mathrm{CL}_{\mathcal{F}}(Z|\tau)$ denote the codelength of an object Z using model $\mathcal{F}$ at a specific quantile τ. Then we have the decomposition

$$\mathrm{CL}_{\mathcal{F}}(Y|\tau) = \mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{F}}|\tau) + \mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{E}}|\hat{\mathcal{F}}, \tau) \tag{6}$$

for the data Y. In the above $\mathrm{CL}_{\mathcal{F}}(Y|\tau)$ is the codelength for Y, $\mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{F}}|\tau)$ is the codelength for $\hat{\mathcal{F}}$, while $\mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{E}}|\hat{\mathcal{F}}, \tau)$ is the codelength for $\hat{\mathcal{E}}$. The minimum description length principle defines the best fitting $\hat{\mathcal{F}}$ as the one that minimizes $\mathrm{CL}_{\mathcal{F}}(Y|\tau)$.

Using the estimated quantile autoregression structure, we obtain the following expression:

$$\begin{aligned}\mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{F}}|\tau) &= \mathrm{CL}_{\mathcal{F}}(m|\tau) + \mathrm{CL}_{\mathcal{F}}(k_1,\ldots,k_m|\tau) + \mathrm{CL}_{\mathcal{F}}(p_1,\ldots,p_{m+1}|\tau)\\ &\quad + \mathrm{CL}_{\mathcal{F}}\big\{\hat{\theta}_1(\tau),\ldots,\hat{\theta}_{m+1}(\tau)\big\}\\ &= \mathrm{CL}_{\mathcal{F}}(m|\tau) + \mathrm{CL}_{\mathcal{F}}(n_1,\ldots,n_{m+1}|\tau) + \mathrm{CL}_{\mathcal{F}}(p_1,\ldots,p_{m+1}|\tau)\\ &\quad + \mathrm{CL}_{\mathcal{F}}\big\{\hat{\theta}_1(\tau),\ldots,\hat{\theta}_{m+1}(\tau)\big\}.\end{aligned} \tag{7}$$

To proceed further, we need the following coding result: the codelength for an integer T is $\log_2 T$ bits, leading to $\mathrm{CL}_{\mathcal{F}}(m|\tau) = \log_2 m$ and $\mathrm{CL}_{\mathcal{F}}(p_1,\ldots,p_{m+1}|\tau) = \sum_{j=1}^{m+1}\log_2 p_j$. On the other hand, if the upper bound T_U of an integer T is known, the corresponding codelength is $\log_2 T_U$ bits. This gives $\mathrm{CL}_{\mathcal{F}}(n_1,\ldots,n_{m+1}|\tau) = (m+1)\log_2 n$, as each n_ℓ is upper-bounded by n. Lastly, Rissanen [20] has shown that a maximum likelihood estimate computed from n data points can be effectively encoded with $\frac{1}{2}\log_2 n$ bits. Applying this to the $\hat{\theta}_\ell(\tau)$'s, we have $\mathrm{CL}_{\mathcal{F}}\{\hat{\theta}_1(\tau),\ldots,\hat{\theta}_{m+1}(\tau)\} = \sum_{j=1}^{m+1}\frac{p_j+1}{2}\log_2 n_j$. Combining these codelength expressions, (7) becomes

$$\mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{F}}|\tau) = \log_2 m + (m+1)\log_2 n + \sum_{j=1}^{m+1}\log_2 p_j + \sum_{j=1}^{m+1}\frac{p_j+1}{2}\log_2 n_j. \tag{8}$$

Now for the last term in (6). It is shown in [20] that the codelength of the residuals $\hat{\mathcal{E}}$ is the negative of the log likelihood of the fitted model $\hat{\mathcal{F}}$. Utilizing the asymmetric Laplace likelihood this leads to

$$\mathrm{CL}_{\mathcal{F}}(\hat{\mathcal{E}}|\hat{\mathcal{F}},\tau) = -\log L\big\{\theta(\tau)\big\} = \sum_{j=1}^{m+1}\sum_{t=k_{j-1}+1}^{k_j}\rho_\tau(\hat{\varepsilon}_t) - n\log\big\{\tau(1-\tau)\big\}. \tag{9}$$

Combining equations (6), (7) and (9) and dropping the constant term $-n\log\{\tau(1-\tau)\}$, we define the best fitting piecewise quantile autoregressive model at a single quantile $\tau \in (0,1)$ as the one that minimizes the minimum description length criterion

$$\begin{aligned}&\mathrm{MDL}(m,k_1,\ldots,k_m,p_1,\ldots,p_{m+1}|\tau)\\ &= \log_2 m + (m+1)\log_2 n\\ &\quad + \sum_{j=1}^{m+1}\log_2 p_j + \sum_{j=1}^{m+1}\frac{p_j+1}{2}\log_2 n_j + \sum_{j=1}^{m+1}\sum_{t=k_{j-1}+1}^{k_j}\rho_\tau(\hat{\varepsilon}_t).\end{aligned} \tag{10}$$

3.3. Model selection at multiple quantiles

To extend the scope of detecting break points at a single quantile, it is worthwhile to study the joint estimation of, say, L quantiles in order to gain more insight into the global behavior of the

process. To estimate break points for multiple quantiles, it can, for example, be assumed that the true break locations are the same across the different quantiles under consideration. This could lead to a borrowing of strength in the segmentation procedure because information on the behavior of various quantiles is added into the analysis. Instead of summing up the minimum description length function defined in (10) for all L quantiles, one could also use their weighted sums. That is,

$$\begin{aligned}&\mathrm{MDL}(m, k_1, \ldots, k_m, p_1, \ldots, p_{m+1}|\tau_1, \ldots, \tau_L)\\&\quad=\sum_{\ell=1}^{L}\omega_\ell \mathrm{MDL}(m, k_1, \ldots, k_m, p_1, \ldots, p_{m+1}|\tau_\ell).\end{aligned}\tag{11}$$

The weights can either be chosen in advance or data-adaptively. In the latter case it may be worthwhile to read the discussion in Chapter 5.5 of [13], where similar ideas are discussed in a location-shift regression model. For this case the optimal weights $\omega_{\mathrm{opt}} = (\omega_{1,\mathrm{opt}}, \ldots, \omega_{L,\mathrm{opt}})'$ are given by $\omega_{\mathrm{opt}} = W^{-1}v$, where W is the $L \times L$ matrix with entries $A_{\ell,\ell'} = \min\{\tau_\ell, \tau_{\ell'}\} - \tau_\ell\tau_{\ell'}$ and $v = (v_1, \ldots, v_L)'$ with $v_\ell = f(F^{-1}(\tau_\ell))$. For the more complicated model under consideration here, one could use these results as a starting point for a more detailed analysis.

On the other hand, one could also think about a more general version of the segmentation procedure that would not enforce simultaneous breaks across the quantiles under consideration. Such an approach may be useful if it could be coupled with prior information on the effect breaks would have on the underlying distribution; for example, if breaks would propagate in a monotone way from the lower to the upper quantiles. The resulting minimum description length criterion would then be even more complex. While a few issues concerning multiple quantiles are highlighted in the empirical parts of the paper, any detailed analysis of such modeling is, however, beyond the scope of the present paper.

4. Large sample results

To study large sample properties assume that the underlying true model indeed follows the piecewise quantile autoregressive structure in (5). We denote the true number of break points and their locations respectively by m^0 and k_j^0, $j = 1, \ldots, m^0$, where $k_j^0 = \lfloor \lambda_j^0 n \rfloor$ and $0 < \lambda_1^0 < \lambda_2^0 < \cdots < \lambda_{m^0}^0 < 1$. Following standard convention in order to ensure sufficient separation of the break points, we choose an $\epsilon > 0$ such that $\epsilon \ll \min_{j=1,\ldots,m^0+1}(\lambda_j^0 - \lambda_{j-1}^0)$ and set

$$\Lambda_m = \big\{(\lambda_1, \ldots, \lambda_m)\colon\ 0 < \lambda_1 < \cdots < \lambda_m < 1, \lambda_j - \lambda_{j-1} \geq \epsilon,\ j = 1, 2, \ldots, m+1\big\},$$

where $\lambda_0 = 0$ and $\lambda_{m+1} = 1$. Fix $\tau \in (0, 1)$, and set $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $p = (p_1, \ldots, p_{m+1})$. The parameters m, λ and p are estimated by minimizing the minimum description length criterion

$$(\hat{m}, \hat{\lambda}, \hat{p}) = \arg\min_{(m,\lambda,p)\in\mathcal{M}} \frac{1}{n}\mathrm{MDL}(m, \lambda, p|\tau),\tag{12}$$

where the minimum is taken in the set $\mathcal{M} = \{(m, \lambda, p)\colon\ m \le M_0, \lambda \in \Lambda_m, 0 \le p_j \le P_0\}$ with M_0 and P_0 denoting upper bounds for m and p_j, respectively. The large sample behvavior of the minimum description length criterion is given in the next theorem. Its proof can be found in the Appendix.

Theorem 4.1. *Assume that the conditions of Proposition 2.1 are satisfied and let the number of break points m^0 be known. Then estimating the piecewise quantile autoregressive model specified in (5) at any single quantile $\tau \in (0, 1)$ leads to*

$$\hat{\lambda}_j \to \lambda_j^0 \qquad \textit{with probability one } (n \to \infty)$$

for all $j = 1, 2, \ldots, m^0$, where $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{m^0})$ is the minimizer of the criterion function (10).

The following corollary extends the result of Theorem 4.1 to the multiple quantile case. Its verification is also provided in the Appendix.

Corollary 4.1. *Assume that the conditions of Proposition 2.1 are satisfied. Let the number of break points m^0 be known and assume that the break locations as well as the autoregressive orders are the same across the quantiles under consideration. Then estimating the piecewise quantile autoregressive model specified in (5) at the collection of quantiles $(\tau_1, \ldots, \tau_L) \in (0, 1)^L$ leads to*

$$\hat{\lambda}_j \to \lambda_j^0 \qquad \textit{with probability one } (n \to \infty)$$

for all $j = 1, 2, \ldots, m^0$, where $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{m^0})$ is the minimizer of the criterion function (11).

We remark that in practice the assumption of known m^0 is often unrealistic. However, it is substantially more difficult to establish consistency in the general case of unknown m^0. Even in the simpler univariate change-point frameworks, where independent variables are grouped into segments of identical distributions, only special cases such as normal distributions and exponential families have been thoroughly investigated; for example, [16,22] as well as [4] for image segmentation. The reason for this is that sharp tail estimates for maxima of certain squared Gaussian processes are needed which do not hold for distributions with thicker tails.

5. Practical minimization using genetic algorithms

Practical minimization of the minimum description length criteria (10) and (11) is not a trivial task. We propose using genetic algorithms to solve this minimization problem.

Genetic algorithms are a class of stochastic optimization techniques. They are based on the idea of Darwin's theory of natural selection. Typically a genetic algorithm begins with a random population of possible solutions to the optimization problems. These solutions are known as *chromosomes* and often represented in vector form. These chromosomes are allowed to evolve

over time through the so-called *crossover* and *mutation* operations. The hope is that the evolution process would ultimately lead to a chromosome which represents a good answer to the optimization problem. Successful applications of genetic algorithms for solving various optimization problems can be found, for examples, in [7].

For a similar piecewise AR modeling minimization problem, Davis *et al.* [8] developed a genetic algorithm for approximating the minimizer. We modified their genetic algorithm to solve the present minimization problem. For conciseness, we only describe the major differences between the genetic algorithm for the present piecewise quantile autoregressive model fitting problem and the one from [8]. We refer the reader to [8] for complete details.

Chromosome representation. For the current problem of detecting break points for a nonstationary time series at a specific quantile τ, a chromosome should contain information of all the break points k_j as well as the quantile autoregression orders p_j for any $\mathcal{F} \in \mathcal{M}$, where $\mathcal{M}$ denotes the whole class of piecewise quantile autoregressive models. We express a chromosome as a vector of n integers: a chromosome $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ is of length n with gene values δ_t defined as

$$\delta_t = \begin{cases} -1, & \text{if no break point at time } t, \\ p_j, & \text{if } t = k_{j-1} \text{ and for the } j\text{th piece we choose the QAR}(p_j) \text{ model at quantile } \tau. \end{cases}$$

In practice, we impose an upper bound P_0 on the order p_j of each quantile autoregressive process. For our numerical work, we set $P_0 = 20$. While the algorithm is running, we also impose the following constraint on each δ: in order to have enough observations for parameter estimation, each piecewise quantile autoregressive process is required to have a minimum length m_p, which is chosen as a function of the order p_j of the piecewise process; their values are listed in Table 1.

Island Model and convergence. The Island Model was also applied to speed up the convergence rate. We used 40 islands with subpopulation size 40, performed a migration for every 5 generations, and migrated 2 chromosomes during each migration. And at the end of each migration the overall best chromosome that has the smallest minimum description length value is selected. If this best chromosome does not change for 20 consecutive migrations, or the total number of generations exceeds 100, the genetic algorithm stops and the best chromosome is taken as the solution to the optimization problem.

Table 1. Values of m_p used in the genetic algorithm

	p							
	0–1	2	3	4	5	6	7–10	11–20
m_p	10	12	14	16	18	20	25	50

6. Simulation studies

6.1. Preliminaries

In this section, four sets of simulation experiments are conducted to evaluate the empirical performance of the proposed method for fitting piecewise stationary quantile autoregressions. We shall compare the results from our method with the Auto-PARM method of [8], who developed an automatic procedure for fitting piecewise autoregressive processes. In each set of experiments, the results are based on 500 repetitions. For the proposed method, we estimated the structural changes at individual quantiles $\tau = 0.25$, 0.5 and 0.75, as well as jointly at $(0.25, 0.5, 0.75)$ using equal weights for the three quantiles. For convenience, we will report the *relative* locations of break points defined as $\hat{\lambda}_j = \hat{k}_j/n$ for $j = 1, \ldots, \hat{m}$.

6.2. Piecewise AR(2) processes

This simulation experiment is designed to compare the performance of the proposed method and Auto-PARM in a linear autoregressive process setting favoring the latter. The data generating process is

$$y_t = \begin{cases} 0.5y_{t-1} + 0.3y_{t-2} + \varepsilon_t & (1 \le t \le n/2), \\ -0.5y_{t-1} - 0.7y_{t-2} + \varepsilon_t & (n/2 < t \le 3n/4), \\ 1.3y_{t-1} - 0.5y_{t-2} + \varepsilon_t & (3n/4 < t \le n), \end{cases} \tag{13}$$

where (ε_t) are independent standard normal, and $n = 1024$ and 2048.

For each simulated process we applied both procedures to locate the break points. We recorded the number of break points detected by each method, together with their relative locations. These numbers are summarized in Tables 2 and 3. From Table 2, we observe that, for the case $n = 1024$, the performance of Auto-PARM is slightly better than for the proposed method at the median and is better at the other two quantiles under consideration. However, as n increased to 2048, the performance of the quantile autoregression procedure improved and is comparable with Auto-PARM both in terms of finding the correct number of breaks and their locations, as can be seen from Table 3.

We have repeated the same experiment but with innovations distributed as the t-distribution with 5 degrees of freedom. In this case, our method outperformed Auto-PARM for all quantiles tested. Due to space limitation, tabulated results are omitted.

6.3. QAR(1) processes exhibiting explosive behavior

The data generating mechanism in this simulation follows the QAR(1) process

$$y_t = (0.85 + 0.25u_t)y_{t-1} + \Phi^{-1}(u_t), \tag{14}$$

where (u_t) is a sequence of independent standard uniform random variables and Φ the standard normal distribution function. Shown in Figure 1 is a typical realization. There is no structural

Table 2. Summary of the estimated number of break points $\hat{m}$ for the proposed procedure for the process (13) with $n = 1024$. Mean (standard deviation (Std)) of the relative break point location is reported where applicable. If mult is specified for the quantile, it refers to the multiple case $\tau = (0.25, 0.50, 0.75)$. The rows labeled Auto-PARM give the results for that method

	$\hat{m}$					
	0	1		2		3
τ	%	%	Mean (Std)	%	Mean (Std)	%
0.25	1.2	23.2	0.759 (0.016)	75.6	0.501 (0.024) 0.747 (0.012)	0
0.50	0	3.6	0.757 (0.012)	96.4	0.504 (0.021) 0.747 (0.011)	0
0.75	0.6	19.8	0.756 (0.014)	79.6	0.501 (0.025) 0.747 (0.013)	0
mult	0	14.2	0.750 (0.013)	85.8	0.503 (0.023) 0.748 (0.012)	0
Auto-PARM	0	0		99.6	0.501 (0.004) 0.751 (0.002)	0.4

break in this series but from the plot one can see that it exhibits explosive behavior in the upper tail. Processes such as this one seem to be capable of modeling certain macroeconomic time series; for example, interest rate data. We will revisit this issue in Section 7 below. While our method does not detect break points at any of the quantiles tested, only about one-third of the

Table 3. Similar to Table 2 except for $n = 2048$

	$\hat{m}$		
	2		3
τ	%	Mean (Std)	%
0.25	99.2	0.503 (0.015) 0.747 (0.008)	0.8
0.50	99.4	0.503 (0.012) 0.744 (0.006)	0.6
0.75	99.6	0.503 (0.015) 0.748 (0.007)	0.4
mult	99.4	0.504 (0.013) 0.748 (0.007)	0.6
Auto-PARM	100	0.501 (0.002) 0.750 (0.001)	0

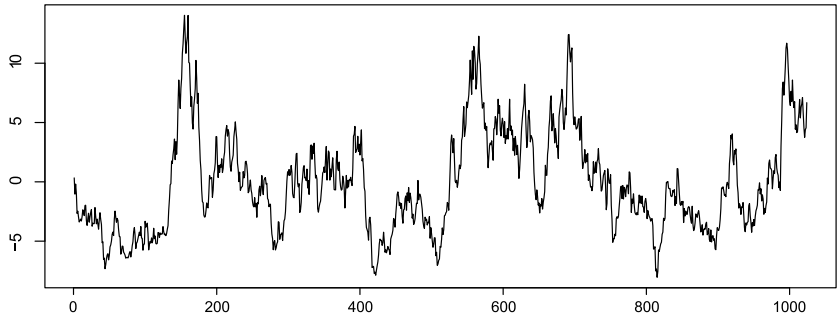

Figure 1. A typical realization for the process in (14).

results from Auto-PARM lead to the correct conclusion; the numbers of break points detected by their method are summarized in Table 4. It is apparent that it is much less tolerant to nonlinearity.

6.4. Piecewise AR(1) processes with changes in certain quantile ranges

In this simulation experiment, the nonstationary time series is generated from the model

$$y_t = \begin{cases} \{0.5I(\tau \le 0.2) + 0.8I(\tau > 0.2)\}y_{t-1} + \varepsilon_t & (1 \le t \le n/2), \\ 0.5y_{t-1} + \varepsilon_t & (n/2 < t \le n), \end{cases} \tag{15}$$

where (ε_t) are independent asymmetric Laplace with parameter 0.4 for $t \le n/2$ and independent asymmetric Laplace with parameter 0.6 for $t > n/2$.

For this process, results from our method and Auto-PARM are reported in Table 5 in a similar manner as in Table 2. Not reported in this table is the fact that, when the coefficients of y_{t-1} in the two pieces are the same (which happens for quantiles $\tau \le 0.2$), then the proposed procedure does not detect any break points even though the residuals of the two pieces are slightly different. For the quantile at $\tau = 0.25$ which is close to the threshold at which the autoregressive coefficient changes, our method detected a (nonexisting) break point in 16% of the simulation runs. On the other hand, when $\tau \ge 0.5$, the quantile autoregression method performs reasonably well, especially at the median where the performance is excellent. Also at $\tau = 0.75$ it outperforms

Table 4. Relative frequencies of the number of break points estimated from Auto-PARM for the process (14) with $n = 1024$. Independent of the specific quantile it was applied to, the proposed methodology always correctly chose $\hat{m} = 0$

	Number of break points					
	0	1	2	3	4	5
Relative frequency	33.8	35.2	23.8	5.6	1.4	0.2

Table 5. Similar to Table 2 except for the process (15) with $n = 1024$

	$\hat{m}$				
	0	1		2	3
τ	%	%	Mean (Std)	%	%
0.25	83.4	16.6	0.527 (0.096)	0	0
0.50	1.5	98.5	0.503 (0.038)	0	0
0.75	24.4	75.6	0.479 (0.055)	0	0
mult	35.2	64.8	0.498 (0.046)	0	0
Auto-PARM	51.0	44.4	0.487 (0.181)	4.0	0.6

Auto-PARM. When estimating jointly at $\tau = (0.25, 0.5, 0.75)$, the percentage of detecting the correct number of break points is not as high as at $\tau = 0.5$ due to the inclusion of the quantiles at $\tau = 0.25$ and $\tau = 0.75$, indicating that care has to be exercised if quantiles are jointly specified. We can also see that the performance of our method is better than that of Auto-PARM in both percentage and accuracy (in terms of smaller standard deviations) for this simulation example. In Table 6, we summarize the proposed procedure's estimates of the quantile autoregression orders for the above process at $\tau = 0.5$, and we can see that most of the segments are correctly modeled as QAR(1) processes.

6.5. Higher-order QAR processes

In this experiment, the data generating process is

$$y_t = \begin{cases} (0.2 + 0.1u_t)y_{t-1} + (0.5 + 0.1u_t)y_{t-2} + \epsilon_t & (1 \le t \le n/2), \\ 0.7u_t y_{t-1} + \epsilon_t & (n/2 < t \le n), \end{cases} \tag{16}$$

where (u_t) is a sequence of independent standard uniform random variables, (ϵ_t) are independent standard normal for $t \le n/2$, and independent asymmetric Laplace with parameter 1 for $t > n/2$. A typical realization is displayed in Figure 2, and break detection results from our method for this process are reported in Table 7. One can see that our method has successfully detected one

Table 6. Relative frequencies of the quantile autoregression orders selected by the proposed method at $\tau = 0.5$ for the realizations from the process (15)

	Order				
	1	2	3	4	5
p_1	80.3	15.7	2.6	1.4	0
p_2	72.4	19.2	6.6	1.4	0.4

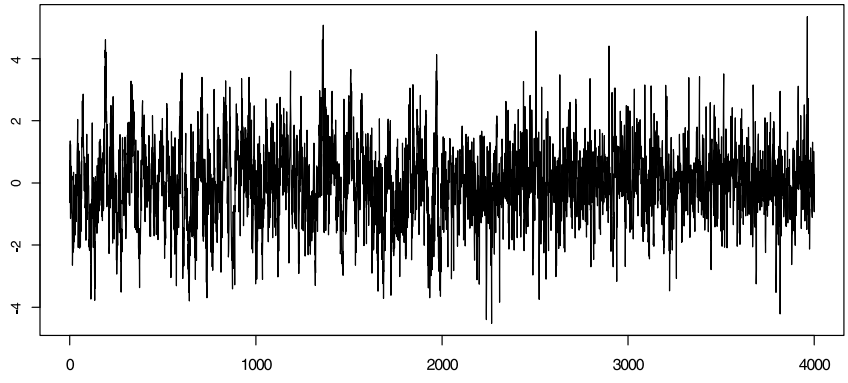

Figure 2. A typical realization for the process in (16).

break with very high probability in most considered cases, and that the detected relative locations are also very close to the true location.

In order to assess the performance of the MDL criterion for order selection in QAR(p) models for $p > 1$, we tabulated the relative frequencies of the order selected by the proposed method for the first piece of process (16) in Table 8. The proposed method never underestimates the order, but only achieves about 50% accuracy. At first sight, these correct estimation rates seem to be relatively low. However, in the break point detection context, the problem of order estimation seems to be hard even for linear AR processes (of higher order), as is seen in Table 3 of [8], where Auto-PARM only gave around 65% correct estimation rates for AR(2) processes. Thus, we believe that a 50% correct rate is not unreasonable for QAR(p) models.

6.6. Stochastic volatility models

The simulation section concludes with an application of the proposed methodology to stochastic volatility models (SVM) often used to fit financial time series; see [21] for a recent overview. It should be noted that the proposed quantile methodology and Auto-PARM are not designed to deal

Table 7. Similar to Table 2 except for the process (16) with $n = 4000$

	$\hat{m}$			
	0	1		2
τ	%	%	Mean (Std)	%
0.25	4.0	95.5	0.517 (0.049)	0.5
0.50	0	98.5	0.505 (0.039)	1.5
0.75	3.0	97.0	0.508 (0.052)	0
mult	0	100.0	0.509 (0.045)	0.5

Table 8. Relative frequencies of the quantile autoregression orders selected by the proposed method at different τ values ($\tau = 0.25, 0.50, 0.75$, and mult) for the first piece in the process (16). The true order is 2

τ	1	2	3	4	5	6	≥ 7
0.25	0	48.69	31.41	15.71	2.09	1.57	0.52
0.50	0	51.78	26.40	12.18	5.58	2.03	2.03
0.75	0	55.15	22.68	11.86	7.73	1.55	1.05
mult	0	50.50	26.00	14.50	5.00	2.00	2.00

with this type of model as it consists of uncorrelated random variables exhibiting dependence in higher-order moments. However, SVM are used to compare the two on a data generating process different from nonlinear QAR and linear AR time series. Following Section 4.2 of [9], the process

$$y_t = \sigma_t \xi_t = e^{\alpha_t/2} \xi_t, \tag{17}$$

is considered, where $\alpha_t = \gamma + \phi \alpha_{t-1} + \eta_t$. The following two-piece segmentations were compared:

$$\begin{aligned} \text{Scenario A} \quad \text{Piece 1:} \quad & \gamma = -0.8106703, \quad \phi = 0.90, \quad (\eta_t) \sim \text{ i.i.d. } N(0, 0.45560010), \\ \text{Piece 2:} \quad & \gamma = -0.3738736, \quad \phi = 0.95, \quad (\eta_t) \sim \text{ i.i.d. } N(0, 0.06758185), \end{aligned}$$

while $(\xi_t) \sim$ i.i.d. $N(0, 1)$ for both pieces, and

$$\begin{aligned} \text{Scenario B} \quad \text{Piece 1:} \quad & \gamma = -0.8106703, \quad \phi = 0, \quad (\xi_t) \sim \text{ i.i.d. } N(0, 1), \\ \text{Piece 2:} \quad & \gamma = -0.3738736, \quad \phi = 0, \quad (\xi_t) \sim \text{ i.i.d. } N(0, 4), \end{aligned}$$

while $(\eta_t) \sim$ i.i.d. $N(0, 0.5)$ for both pieces. Scenario A corresponds to a change in dynamics of the volatility function σ_t, Scenario B basically to a scale change.

Scenario A was considered in [9]. These authors developed a method tailored to deal with financial time series of SVM and GARCH type. The method, termed Auto-Seg, was able to detect one break in 81.8% of 500 simulation runs and detected no break otherwise. On this data, Auto-PARM tends to use a too fine segmentation as 62.4% of the simulations runs resulted in two or more estimated break points. One (no) breakpoint was detected in 21.2% (16.4%) of the cases. The proposed method failed to detect any changes at any of the tested quantiles ($\tau = 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95$). It should be noted, however, that there is no change at the median and changes in the other quantiles are very hard to find as is evidenced by Figure 3, which displays the averaged (over 50 simulation runs) empirical quantile–quantile plot from the first and the second segment of the two-piece Scenario A process.

The results for Scenario B are summarized in Table 9. It can be seen that, for the proposed method, the scale change, is detected at the more extreme quantiles ($\tau = 0.05, 0.10, 0.90, 0.95$) with very good accuracy and with reasonable accuracy at intermediate quantiles ($\tau = 0.25$ and $\tau = 0.75$), while no change is found (correctly) at the median $\tau = 0.50$, reflecting that the proposed procedure describes the local behavior of the SVM process adequately. Auto-PARM does the same on a global basis.

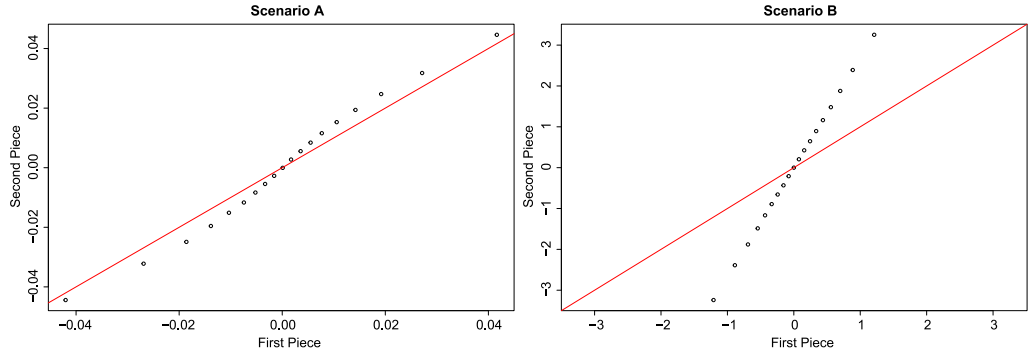

Figure 3. Empirical quantile–quantile plot for the SVM process specified under Scenario A (left panel) and Scenario B (right panel). The x-axis (y-axis) shows the empirical quantiles of Piece 1 (Piece 2). The 45 degree line is given for ease of comparison.

7. Real data applications

7.1. Treasury bill data

Treasury bills are short-term risk-free investments that are frequently utilized by investors to hedge portfolio risks. In this application, the observations are three-month treasury bills from the secondary market rates in the United States, ranging from January 1954 to December 1999. The weekly data can be found at the website http://research.stlouisfed.org/fred2/series/TB3MS and are displayed in Figure 4.

It can be seen from Figure 4 that the time series exhibits obvious explosive behavior in the upper tail. In many instances similar time series would be viewed as random walks and sophisticated testing procedures would have to be applied to either confirm or reject what is known as unit-root hypothesis; see, for example, [18,19] for more. As in Section 6.3, Auto-PARM aims in this case at partitioning the series into segments with structures mimicking linear behavior. In the present case, this leads to 15 segments. On the other hand the proposed procedure does not detect break points at any of the quantiles tested ($\tau = 0.05, 0.10, \ldots, 0.90, 0.95$), thus indicating that with the use of some extra parameters a more parsimonious stationary but nonlinear modeling

Table 9. Summary of the estimated number of break points $\hat{m}$ for the proposed procedure and Auto-PARM for the process (17) with specifications given under Scenario B

	τ							
$\hat{m}$	0.05	0.10	0.25	0.50	0.75	0.90	0.95	Auto-PARM
0	0.4%	0.2%	32.6%	100.0%	29.6%	0.0%	0.6%	0.2%
1	99.6%	99.8%	67.4%	0.0%	70.4%	100.0%	99.4%	99.6%
2	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.2%

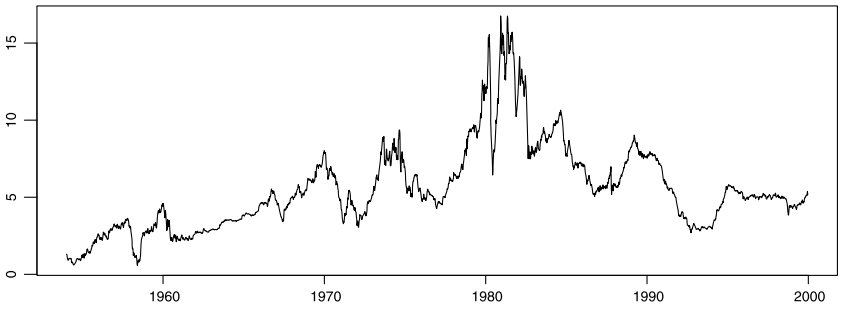

Figure 4. Three-month treasury bills (01/1954 to 12/1999).

is possible for this data set. Using a QAR(2) model with cubic polynomial coefficients in the uniform random variables (u_t), the data can be approximated via the following model with 12 parameters:

$$y_t = \theta_0(u_t) + \theta_1(u_t)y_{t-1} + \theta_2(u_t)y_{t-2}, \tag{18}$$

where

$$\begin{aligned}
\theta_0(u_t) &= -0.0144 + 0.2264u_t - 0.5448u_t^2 + 0.3848u_t^3,\\
\theta_1(u_t) &= 1.3721 - 0.9635u_t + 1.5312u_t^2 - 0.6939u_t^3,\\
\theta_2(u_t) &= -0.4394 + 1.3154u_t - 2.1945u_t^2 + 1.1353u_t^3.
\end{aligned}$$

Figure 5 depicts several realizations generated by the estimated model (18), which all show a pattern closely resembling the data in Figure 4. This example illustrates that quantile autoregressions can expand the modeling options available to the applied statistician as it accurately captures temporary explosive behavior and nonlinearity.

7.2. Monthly minimum temperature data

In this section the monthly mean minimum temperature at Melbourne in Australia is considered. The data set is obtainable from the Bureau of Meteorology of the Australian Government (http://www.bom.gov.au/climate/data/). The plots for the original series and its deseasonalized version are shown in Figure 6. This data set has been investigated by [13] who pointed out that, due to the quantile dependent behavior visible in the scatter plots, linear autoregressive models are insufficient to describe the data. Our method was applied to this data set at various quantiles and for all cases one break point was found near the year 1960. This agrees with a visual inspection of Figure 6.

It can be seen from Table 10 that the break point location estimated with the multiple quantile procedure, set up with equal weights for the three quantiles under consideration, is between the break point locations estimated at the individual quantiles. This should always be the case, as

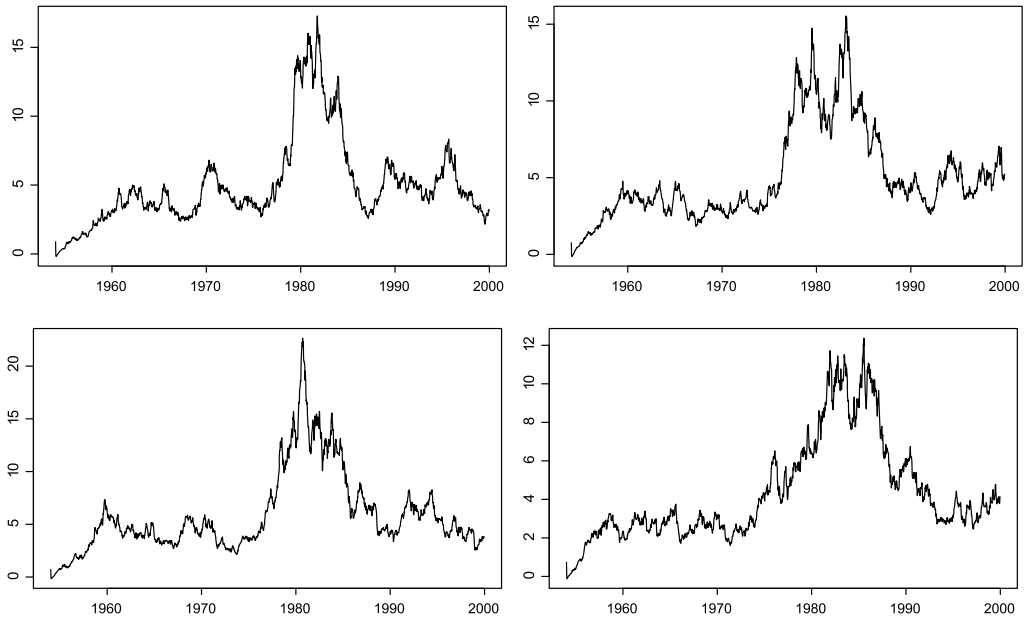

Figure 5. Four typical realizations of the process in (18).

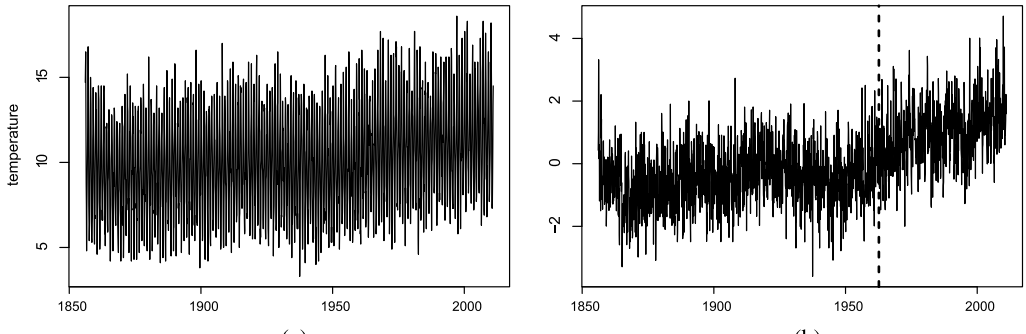

Figure 6. (a) Monthly minimum air temperature in Melbourne, Australia from January 1856 to December 2010. (b) Deseasonalized series. The dashed line represents the estimated break point in August 1962.

Table 10. Estimated break points at different quantiles for the Australian temperature data

	Quantiles			
	0.25	0.5	0.75	mult
Estimated break point	December 1960	August 1963	December 1958	August 1962

the requirement of simultaneous occurrence of breaks automatically leads to a weighted average interpretation. In general, one would ideally find weights that prefer quantiles which stronger exhibit the structural break and attenuate the impact of quantiles that are only marginally subjected to the break. This would mean to more closely evaluate properties of the (piecewise) density and distribution function of the underlying random process.

8. Conclusions

This article proposes a new segmentation procedure that helps breaking down a given nonstationary time series into a number of stationary pieces by means of quantile autoregression modeling. In contrast to most of the existing literature, this is done either for individual quantiles or across a collection of quantiles. The proposed method utilizes the minimum description length principle and a genetic algorithm to obtain the best segmentation. It has been proved that this method is asymptotically consistent, and simulation results have demonstrated that the finite sample performance of the proposed procedure is quite good. Data applications are also provided with satisfactory results. It can be seen in particular that our method can add to second-order time series modeling by enriching the statistician's tool box via the inclusion of nonlinearity, asymmetry, local persistence and other distributional aspects. An interesting problem for future research that shows some potential is the investigation of the properties of the multiple quantile segmentation procedure for the case of quantile-dependent break point locations, thereby loosening the assumption of simultaneous breaks utilized in this paper.

Appendix: Proofs

Lemma A.1. *If* $(y_t\colon\ t \in \mathbb{Z})$ *follow a stationary* QAR(p) *model such that the assumptions of Proposition* 2.1 *are satisfied, then with probability one and for all* $\tau \in (0, 1)$,

$$\frac{1}{n}\sum_{t=1}^{n} \rho_\tau(\hat{\varepsilon}_t) \to E\big\{\rho_\tau(\varepsilon_1)\big\} \qquad (n \to \infty),$$

where ρ_τ *is the check function defined below* (3).

Proof. The assertion follows as in the proof of Lemma A.1 in [1]. □

Lemma A.2. *Let* $(y_t\colon\ t \in \mathbb{Z})$ *be a piecewise stationary* QAR(p) *model that satisfies the assumptions of Proposition* 2.1 *on each of the segments. Let* $\lambda^0 = (\lambda_1^0, \ldots, \lambda_{m_0}^0)$ *denote the true segmentation and choose* $K = \lfloor \kappa n \rfloor$, $M = \lfloor \mu n \rfloor$ *with* $0 \le \kappa < \mu \le 1$. *Then, with probability one for all* $\tau \in (0, 1)$,

$$\frac{1}{M-K}\sum_{t=K+1}^{M} \rho_\tau(\hat{\varepsilon}_t) \to L_\tau(\kappa, \mu).$$

The limit $L_\tau(\kappa, \mu)$ *is the sum of two components,* $A_\tau(\kappa, \mu)$ *and* $B_\tau(\kappa, \mu)$, *both of which are given in the proof.*

Proof. There are two cases to consider, namely (1) K and M are contained in the same segment and (2) K and M are in different segments.

For the case (1), Lemma A.1 implies immediately that

$$\frac{1}{M-K}\sum_{t=K+1}^{M}\rho_\tau(\hat{\varepsilon}_t)\to\rho_{\tau,j}=A_\tau(\kappa,\mu).$$

With $B_\tau(\kappa,\mu)=0$, one can set $L_\tau(\kappa,\mu)=A_\tau(\kappa,\mu)$ and the limit is determined.

For the case (2), there are $1\le j<J\le m^0+1$ such that $\kappa\in[\lambda^0_{j-1},\lambda^0_j)$ and $\mu\in(\lambda^0_{J-1},\lambda^0_J]$. In addition to the residuals $\hat{\varepsilon}_t$ obtained from fitting a QAR model to the observations $y_{K+1},\dots,y_M$, one also defines residuals $\hat{\varepsilon}_{t,\ell}$ obtained from fitting a QAR model on the ℓth underlying (true) segment. If now $t\in\{k^0_{\ell-1}+1,\dots,k^0_\ell\}$ with $k^0_\ell=\lfloor\lambda^0_\ell n\rfloor$, then one gets the decomposition $\rho_\tau(\hat{\varepsilon}_t)=\{\rho_\tau(\hat{\varepsilon}_t)-\rho_\tau(\hat{\varepsilon}_{t,\ell})\}+\rho_\tau(\hat{\varepsilon}_{t,\ell})$. The sum over the first terms on the right-hand side leads to a positive bias term $B_\tau(\kappa,\mu)$ determined by the almost sure limit relation

$$\begin{aligned}&\frac{1}{M-K}\Bigg[\sum_{t=K+1}^{k^0_j}\big\{\rho_\tau(\hat{\varepsilon}_t)-\rho_\tau(\hat{\varepsilon}_{t,j})\big\}\\&\qquad+\sum_{\ell=j+1}^{J-1}\sum_{t=k_{\ell-1}+1}^{k^0_\ell}\big\{\rho_\tau(\hat{\varepsilon}_t)-\rho_\tau(\hat{\varepsilon}_{t,\ell})\big\}+\sum_{t=k^0_{J-1}+1}^{M}\big\{\rho_\tau(\hat{\varepsilon}_t)-\rho_\tau(\hat{\varepsilon}_{t,J})\big\}\Bigg]\\&\to B_\tau(\kappa,\mu).\end{aligned}$$

The remaining segment residuals $\hat{\varepsilon}_{t,\ell}$ allow for an application of Lemma A.1 to each of the underlying (true) segments, so that, with probability one,

$$\begin{aligned}&\frac{1}{M-K}\Bigg\{\sum_{t=K+1}^{k^0_j}\rho_\tau(\hat{\varepsilon}_{t,j})+\sum_{\ell=j+1}^{J-1}\sum_{t=k^0_{\ell-1}+1}^{k^0_\ell}\rho_\tau(\hat{\varepsilon}_{t,\ell})+\sum_{t=k^0_{J-1}+1}^{M}\rho_\tau(\hat{\varepsilon}_{t,J})\Bigg\}\\&\to\frac{1}{\mu-\kappa}\Bigg\{(\lambda^0_j-\kappa)\rho_{\tau,j}+\sum_{\ell=j+1}^{J-1}(\lambda^0_\ell-\lambda^0_{\ell-1})\rho_{\tau,\ell}+(\mu-\lambda^0_{J-1})\rho_{\tau,J}\Bigg\}\\&=A_\tau(\kappa,\mu),\end{aligned}$$

where $\rho_{\tau,j}=E\{\rho_\tau(\varepsilon_{k^0_j})\}$. Setting $L_\tau(\kappa,\mu)=A_\tau(\kappa,\mu)+B_\tau(\kappa,\mu)$ completes the proof. □

Proof of Theorem 4.1. Denote by $\hat{\lambda}=(\hat{\lambda}_1,\dots,\hat{\lambda}_{m^0})$ and $\lambda^0=(\lambda^0_1,\dots,\lambda^0_{m^0})$ the segmentation chosen by the minimum description length criterion (10) and the true segmentation, respectively. The proof is obtained from a contradiction argument, assuming that $\hat{\lambda}$ does not converge almost surely to λ^0. If that was the case, then the boundedness of $\hat{\lambda}$ would imply that, almost surely along a subsequence, $\hat{\lambda}\to\lambda^*=(\lambda^*_1,\dots,\lambda^*_{m^0})$ as $n\to\infty$, where λ^* is different from λ^0. Two

cases for neighboring λ^*_{j-1} and λ^*_j have to be considered, namely (1) $\lambda^0_{j'} \le \lambda^*_{j-1} < \lambda^*_j \le \lambda^0_{j'}$ and (2) $\lambda^0_{j'-1} \le \lambda^*_{j-1} < \lambda^0_{j'} < \cdots < \lambda^0_{j'+J} < \lambda^*_j \le \lambda^0_{j'+J+1}$ for some positive integer J.

For the case (1), Lemma A.1 implies that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=\hat{k}_{j-1}+1}^{\hat{k}_j} \rho_\tau(\hat{\varepsilon}_t) \ge \big(\lambda^*_j - \lambda^*_{j-1}\big)\rho_{\tau,j'},$$

where $\rho_{\tau,j'} = E\{\rho_\tau(\varepsilon_{k^0_{j'}})\}$. For the case (2), Lemma A.2 gives along the same lines of argument that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=\hat{k}_{j-1}+1}^{\hat{k}_j} \rho_\tau(\hat{\varepsilon}_t) > \frac{1}{\lambda^*_j - \lambda^*_{j-1}} \Bigg\{ \big(\lambda^0_{j'} - \lambda^*_{j-1}\big)\rho_{\tau,j'} + \sum_{\ell=j'+1}^{j'+J+1} \big(\lambda^0_\ell - \lambda^0_{\ell-1}\big)\rho_{\tau,\ell} + \big(\lambda^*_j - \lambda^0_{j'+J}\big)\rho_{\tau,j'+J+1} \Bigg\}.$$

Taken together, these two inequalities, combined with the fact that asymptotically all penalty terms in the definition of the MDL in (12) vanish, give, almost surely,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \mathrm{MDL}\big(m^0, \hat{\lambda}, \hat{p}|\tau\big) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{m^0+1} \sum_{t=\hat{k}_{j-1}+1}^{\hat{k}_j} \rho_\tau(\hat{\varepsilon}_t) \\ &> \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{m^0+1} \sum_{t=k^0_{j-1}+1}^{k^0_j} \rho_\tau(\varepsilon_t) = \lim_{n\to\infty} \mathrm{MDL}\big(m^0, \lambda^0, p^0|\tau\big), \end{aligned}$$

which is a contradiction to the definition of the MDL minimizer. □

Proof of Corollary 4.1. Recall that the minimum description length criterion for multiple quantiles $(\tau_1, \ldots, \tau_L)$ is given in (11). If follows from Theorem 4.1 that at any individual quantile τ_ℓ, the minimizer, say, $(\hat{\lambda}_\ell, \hat{p}_\ell)$ of the minimum description length criterion (10) is consistent for (λ^0, p^0). It follows that the minimizer $(\hat{\lambda}, \hat{p})$ of (11) is consistent as it is a weighted sum of several criteria in the form of (10). □

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