Quenched limit theorems for Fourier transforms and periodogram

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In this paper, we study the quenched central limit theorem for the discrete Fourier transform. We show that the Fourier transform of a stationary ergodic process, suitable centered and normalized, satisfies the quenched CLT conditioned by the past sigma algebra. For functions of Markov chains with stationary transitions, this means that the CLT holds with respect to the law of the chain started at a point for almost all starting points. It is necessary to emphasize that no assumption of irreducibility with respect to a measure or other regularity conditions are imposed for this result. We also discuss necessary and sufficient conditions for the validity of quenched CLT without centering. The results are highly relevant for the study of the periodogram of a Markov process with stationary transitions which does not start from equilibrium. The proofs are based of a nice blend of harmonic analysis, theory of stationary processes, martingale approximation and ergodic theory.

Keywords: central limit theorem; discrete Fourier transform; martingale approximation; periodogram; spectral analysis

1. Introduction

The finite Fourier transform, defined as

$$S_n(t) = \sum_{k=1}^{n} e^{ikt} X_k, \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit, plays an essential role for the study of stationary time series $(X_j)_{j\in\mathbb{Z}}$ of centered random variables with finite second moment, defined on a probability space $(\Omega, \mathcal{K}, \mathbb{P})$.

The periodogram, introduced as a tool by Schuster [35] in 1898, is essential in the estimation of the spectral density of the stationary processes. It is defined by

$$I_n(t) = \frac{1}{2\pi n} |S_n(t)|^2, \qquad t \in [0, 2\pi]. \tag{2}$$

Wiener and Wintner [39] showed that for any stationary sequence $(X_j)_{j\in\mathbb{Z}}$ in $\mathbb{L}^1$ (namely $\mathbb{E}|X_0| < \infty$) there is a set $\Omega' \subset \Omega$ of probability one such that for any $t \in [0, 2\pi]$ and any $\omega \in \Omega'$, $S_n(t)/n$ converges. The speed of this convergence (see Peligrad and Wu [26] and the references therein) is usually given by a central limit theorem for the real and imaginary parts

of $S_n(t)/\sqrt{n}$ under various dependence restrictions. Peligrad and Wu [26] showed that, under a very mild regularity condition and finite second moment, $[\mathrm{Re}(S_n(t))/\sqrt{n}, \mathrm{Im}(S_n(t))/\sqrt{n}]$ are asymptotically independent normal random variables with mean 0 and variance $\pi f(t)$, for almost all t. Here f is the spectral density of $(X_j)_{j \in \mathbb{Z}}$. This result implies that for almost all t, the periodogram $I_n(t)$ converges in distribution to $f(t)\chi^2$ where χ^2 has a chi-square distribution with 2 degrees of freedom. Sufficient conditions for the validity of the law of iterated logarithm were recently pointed out in Cuny *et al.* [8].

An interesting problem with practical applications, is to study the validity of the central limit theorem for Fourier transform and the periodogram for a Markov chain with stationary transitions which is not started from equilibrium but rather started from a point for almost all starting points. This is often the case for simulated data and also for evolutions in random media or particle systems. The problem is difficult, since the Markov chain started at a point is no longer stationary. This type of central limit theorem, started at a point, is known under the name of quenched central limit theorem (CLT) and it is a consequence of a more general result, the almost sure conditional CLT for stationary processes. This means that on a set of measure one the central limit theorem holds when, in the definition of weak convergence, we replace the usual expectation by the conditional expectation with respect to the past σ-algebra. The almost sure conditional CLT implies CLT. Some examples of stationary processes satisfying the CLT but not the almost sure conditional CLT can be found in Volný and Woodroofe [37].

The problem of the quenched CLT for stationary Markov chains or for stationary processes is a subject of intense research. We mention the papers [7,9,10,12,13,27,28,37], among many others. Several of these results were surveyed in [25].

As far as we know, this type of convergence was not yet investigated for the Fourier transforms or the periodograms. In this paper, we show that the quenched CLT holds for almost all frequencies of the properly centered and normalized discrete Fourier transform of any stationary and ergodic sequence. We also provide necessary and sufficient conditions for the validity of quenched CLT without centering and specify a sufficient condition for the validity of quenched CLT for fixed frequencies.

All these results shed additional light on the speed of convergence of the periodogram in approximating the spectral density $f(t)$ of a stationary process. The techniques are a nice blend of martingale approximation, rooted in Gordin [19] and Rootzén [30] and developed by Gordin and Lifšic [20] and Woodroofe [40], and tools from ergodic theory and harmonic analysis.

To allow for flexibility in applications, we introduce a stationary sequence and a filtration in two different ways. First by using a measure preserving transformation, and then, in Section 3, as a function of a Markov chain. We formulate the main results for measure preserving transformations in terms of almost sure conditional CLT. However, in Section 3 we show that, only by a change of language, the results can be formulated for stationary and ergodic Markov chains, where the terminology of a process started at a point becomes natural.

A variety of applications to functions of linear processes, functions of Markov chains, iterated random functions, mixing sequences, are also pointed out. It is remarkable that for the case of a stationary ergodic reversible Markov chain the quenched CLT without centering holds without any other additional assumptions.

Our paper is organized as follows. Section 2 contains the presentation of the results. Several applications are given in Section 3. Section 4 is devoted to the proofs. Section 5 contains several auxiliary results needed for the main proofs.

2. Definitions, background and results

A strictly stationary sequence can be introduced in many equivalent ways. It can be viewed, for instance, as a function of a stationary Markov chain with general state space. This definition will be given in Section 3. For more flexibility in the selection of filtration, in this section, we shall introduce a stationary sequence and a filtration by using a measure preserving transformation.

Let $(\Omega, \mathcal{K}, \mathbb{P})$ be a probability space where, without restricting the generality, we shall assume that $\mathcal{K}$ is countably generated, and let $T : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability $\mathbb{P}$. An element A is said to be invariant if $T(A) = A$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. The transformation T is ergodic with respect to $\mathbb{P}$ if each element of $\mathcal{I}$ has probability 0 or 1. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{K}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$. Define the nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$ and let $\mathcal{F}_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k$. Let X_0 be a $\mathcal{F}_0$-measurable, square integrable and centered random variable. Define the sequence $\mathbf{X} = (X_i)_{i \in \mathbb{Z}}$ by

$$X_i = X_0 \circ T^i. \tag{3}$$

For $p \geq 1$, we denote by $\| \cdot \|_p$ the norm in $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ and for an integrable random variable Y we denote by $\mathbb{E}_0(Y) = \mathbb{E}(Y|\mathcal{F}_0)$.

Since $\mathcal{K}$ is countably generated, there is a regular conditional probability measure $\mathbb{P}^\omega(\cdot)$ with respect to $\mathcal{F}_0$, such that for all $\omega \in \Omega$, $\mathbb{P}^\omega(\cdot)$ is a measure on $\mathcal{K}$ and for each $A \in \mathcal{K}$ we have $\mathbb{P}^\omega(A) = \mathbb{P}(A|\mathcal{F}_0)(\omega)$, $\mathbb{P}$ a.s. For integrable X, the corresponding conditional expectation is denoted by $\mathbb{E}^\omega(X)$ and it is a regular version of $\mathbb{E}(X|\mathcal{F}_0)(\omega)$.

Relevant to our results is the notion of spectral distribution function induced by the covariances. By Herglotz's theorem (see, e.g., Brockwell and Davis [4]), there exists a nondecreasing function G (the spectral distribution function) on $[0, 2\pi]$ such that, for all $j \in \mathbb{Z}$,

$$\operatorname{cov}(X_0, X_j) = \int_0^{2\pi} e^{ij\theta} \, dG(\theta), \qquad j \in \mathbb{Z}.$$

If G is absolutely continuous with respect to the normalized Lebesgue measure λ on $[0, 2\pi]$, then the Radon–Nikodym derivative f of G with respect to the Lebesgue measure is called the spectral density and we have

$$\operatorname{cov}(X_0, X_j) = \int_0^{2\pi} e^{ij\theta} f(\theta) \, d\theta, \qquad j \in \mathbb{Z}.$$

We shall introduce the notations

$$\mathbf{V}_n(t) = \frac{1}{\sqrt{n}} \big[\operatorname{Re}\big(S_n(t)\big), \operatorname{Im}\big(S_n(t)\big) \big].$$

We also denote

$$\mathbf{W}_n(t) = \frac{1}{\sqrt{n}} \big[\operatorname{Re}\big(S_n(t) - \mathbb{E}_0 S_n(t)\big), \operatorname{Im}\big(S_n(t) - \mathbb{E}_0 S_n(t)\big) \big].$$

The central limit theorem for $\mathbf{V}_n(t)$ has a long history. We mention, among many others, Rosenblatt (Theorem 5.3, page 131, [34]) who considered mixing processes; Brockwell and Davis (Theorem 10.3.2, page 347, [4]), Walker [38] and Terrin–Hurvich [36] discussed linear processes; Wu [41] treated mixingales.

Peligrad and Wu [26] established the following result, where, besides a mild regularity assumption (4), no other restriction of dependence is imposed to the stochastic process. Below, by $\Rightarrow$ we denote convergence in distribution.

Theorem A (Peligrad and Wu). *Let $(X_k)_{k\in\mathbb{Z}}$ be a stationary ergodic process, centered, with finite second moments, such that the following regularity assumption is satisfied,*

$$\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0 \qquad \mathbb{P}\text{-a.s.} \tag{4}$$

Then, for almost all $t \in (0, 2\pi)$, the following convergence holds:

$$\lim_{n\to\infty} \frac{\mathbb{E}|S_n(t)|^2}{n} = 2\pi f(t), \tag{5}$$

where $f(t)$ is the spectral density of $(X_k)_{k\in\mathbb{Z}}$. Furthermore

$$\frac{1}{\sqrt{n}}\mathbf{V}_n(t) \Rightarrow \mathbf{N}(t) \qquad under\ \mathbb{P}, \tag{6}$$

where $\mathbf{N}(t) = [N_1(t), N_2(t)]$, with $N_1(t)$ and $N_2(t)$ independent identically distributed normal random variables mean 0 and variance $\pi f(t)$.

The proof of Theorem A is based on the celebrated Carleson's [5] theorem on almost sure convergence of Fourier transforms. A different proof, without using Carleson's result, was recently given in Cohen and Conze [6]. This suggests that the power of Carleson's theorem might lead to a stronger type of limiting distribution, in the almost sure sense.

The goal of this paper is to study a more general form of (6) known under the name of quenched CLT.

By the quenched CLT we shall understand the following almost sure conditional limit theorem: For almost all $t \in [0, 2\pi]$, there is Ω' with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ we have

$$\mathbb{E}^\omega\big[g\big(\mathbf{V}_n(t)\big)\big] \to \mathbb{E}\big[g\big(\mathbf{N}(t)\big)\big] \qquad \text{as } n \to \infty, \tag{7}$$

for any function g which is continuous and bounded. We shall say in this case that the quenched CLT holds for almost all frequencies. In other notation, for almost all $t \in [0, 2\pi]$, there is Ω' with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ we have

$$\mathbf{V}_n(t) \Rightarrow \mathbf{N}(t) \qquad \text{as } n \to \infty \text{ under } \mathbb{P}^\omega.$$

Clearly (7) implies (6) by integration with respect to $\mathbb{P}$.

Our first result gives a quenched CLT under a certain centralization. Note that the next theorem applies to any stationary and ergodic sequence.

Theorem 1. *Let* $(X_k)_{k\in\mathbb{Z}}$ *be a stationary ergodic process,* X_k *defined by* (3) *and let* $S_n(t)$ *be defined by* (1). *Then, for almost all* $t \in [0, 2\pi]$

$$\lim_{n\to\infty} \frac{\mathbb{E}_0|S_n(t) - \mathbb{E}_0 S_n(t)|^2}{n} = \sigma_t^2 \qquad \mathbb{P}\text{-}a.s.$$

and the quenched CLT holds for $\mathbf{W}_n(t)$, *where* $N_1(t)$ *and* $N_2(t)$ *are independent identically distributed normal random variables with mean* 0 *and variance* $\sigma_t^2/2$.

Our second theorem provides a characterization of quenched convergence without centering.

Theorem 2. *Let* $(X_k)_{k\in\mathbb{Z}}$ *be as in Theorem* 1. *Then the following statements are equivalent*:

(a) *For almost all* $t \in [0, 2\pi]$ *the quenched CLT in* (7) *holds, where* $N_1(t)$ *and* $N_2(t)$ *are as in Theorem* 1.
(b) *For almost all* $t \in [0, 2\pi]$ *we have* $\frac{1}{\sqrt{n}}\mathbb{E}_0(S_n(t)) \to 0$, $\mathbb{P}$*-a.s.*

Discussion

An interesting problem is to specify σ_t^2. Note first that if $\mathbb{E}_0(S_n(t))/\sqrt{n}$ converges in $\mathbb{L}^2$ to 0, our proofs show that in this case σ_t^2 can be identified as

$$\lim_{n\to\infty} \frac{\mathbb{E}|S_n(t)|^2}{n} = \sigma_t^2. \tag{8}$$

Note also that in both Theorems 1 and 2 we do not require the sequence to be regular, that is, it may happen that $E(X_0|\mathcal{F}_{-n})$ does not converge to 0 in $\mathbb{L}^2$. The spectral density might not exist. If we assume condition (4) then, as shown in Peligrad and Wu [26], the spectral density $f(t)$ of $(X_k)_{k\in\mathbb{Z}}$ exists and $\sigma_t^2 = 2\pi f(t)$. Furthermore, we have

$$\frac{1}{\sqrt{n}}\mathbb{E}_0(S_n(t)) \to 0 \qquad \text{in } \mathbb{L}^2,$$

and then σ_t^2 can also be identified as

$$\lim_{n\to\infty} \frac{\mathbb{E}|S_n(t)|^2}{n} = \sigma_t^2 = 2\pi f(t). \tag{9}$$

By the mapping theorem (see Theorem 29.2 in [1]), all our results imply corresponding results for the periodogram. As a consequence of Theorem 2 and the discussion above we obtain the following corollary:

Corollary 3. *Assume that the sequence* $(X_k)_{k\in\mathbb{Z}}$ *is as in Theorem* 1 *and in addition satisfies* (4) *and item* (b) *of Theorem* 2. *Then, for almost all* $t \in [0, 2\pi]$ *the periodogram* $I_n = (2\pi n)^{-1}|S_n(t)|^2$ *satisfies a quenched limit theorem with the limit* $f(t)\chi^2(2)$, *where* $\chi^2(2)$ *is a chi-square random variable with* 2 *degrees of freedom, and* $f(t)$ *the spectral density.*

The next corollary provides sufficient conditions for the validity of item (b) of Theorem 2.

Corollary 4. *Assume that* $(X_k)_{k\in\mathbb{Z}}$ *is as in Theorem 1 and in addition that*

$$\sum_{k\geq 1}\frac{|\mathbb{E}_0(X_{k+1}-X_k)|^2}{k}<\infty \qquad \mathbb{P}\text{-}a.s. \tag{10}$$

Then (7) *holds, where* $N_1(t)$ *and* $N_2(t)$ *are as in Theorem 1.*

Clearly (10) is satisfied if

$$\sum_{k\geq 1}\frac{|\mathbb{E}_0(X_k)|^2}{k}<\infty \qquad \mathbb{P}\text{-a.s.}, \tag{11}$$

which is further implied by

$$\sum_{k\geq 1}\frac{\|\mathbb{E}_0(X_k)\|_2^2}{k}<\infty. \tag{12}$$

Moreover, since $\|\mathbb{E}_0(X_k)\|_2$ is decreasing, condition (12) implies condition (4). These remarks justify the following corollary:

Corollary 5. *Condition* (12) *is sufficient for the quenched CLT in* (7) *with* $N_1(t)$ *and* $N_2(t)$ *i.i.d. normal random variables with mean* 0 *and variance* $\pi f(t)$, $f(t)$ *being the spectral density of the process.*

The above results hold for almost all frequencies. Actually it is possible that on a set of measure 0 the behavior be quite different. For the case when $t=0$, there are a variety of examples where the partial sums of a stationary sequence do not satisfy a nondegenerate CLT. One important example of this kind is provided by filters of Gaussian processes with long range dependence, when the covariances are not summable. For example, Rosenblatt [32] proved that for a stationary Gaussian sequence $(X_k)_{k\in\mathbb{Z}}$ of standard normal random variables with $\text{cov}(X_0,X_k)=(1+k^2)^{-\alpha/2}, 0<\alpha<1/2$, the sequence $n^{-1+\alpha}\sum_{k=1}^n(X_k^2-1)$ has a nonnormal limiting distribution as $n\to\infty$. Another interesting example, also for $t=0$, is provided by Herrndorf [22] who constructed a stationary sequence of centered uncorrelated random variables with finite second moment, which is strongly mixing with arbitrary mixing rate and the partial sums do not satisfy a nondegenerate CLT under any normalization converging to infinite. This example satisfies condition (4). Furthermore, Bradley [3] (see Theorem 34.14, Vol. 3) constructed a stationary sequence of centered random variables with finite second moment, satisfying our condition (12) and such that its partial sums normalized by its standard deviation is attracted to a non-Gaussian nondegenerate distribution. Rosenblatt [33] studied the Fourier transform of nonlinear functions of Gaussian processes and established for certain frequencies, on a set of measure 0, non-Gaussian attraction for the Fourier transform properly normalized.

In the spirit of Maxwell and Woodroofe [24] and Cuny and Merlevède [7], we give below a result allowing us to identify frequencies for which the quenched CLT holds.

Theorem 6. *Let $t \in (0, 2\pi)$ be such that e^{-2it} is not an eigenvalue of T. Assume that the sequence $(X_k)_{k \in \mathbb{Z}}$ is as in Theorem* 1 *and in addition that we have*

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \left\| \mathbb{E}_0 \big(S_k(t) \big) \right\|_2 < \infty. \tag{13}$$

Then (7) *holds with $N_1(t)$ and $N_2(t)$ independent identically distributed normal random variables mean* 0 *and variance $\sigma_t^2/2$ where σ_t^2 is identified by* (8)*.*

3. Applications

3.1. Functions of Markov chains

Let $(\xi_n)_{n \in \mathbb{Z}}$ be a stationary and ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(S, \mathcal{A})$. The marginal distribution is denoted by $\pi(A) = \mathbb{P}(\xi_0 \in A)$ and we assume that there is a regular conditional distribution for ξ_1 given ξ_0 denoted by $Q(x, A) = \mathbb{P}(\xi_1 \in A | \xi_0 = x)$. In addition Q denotes the Markov operator acting via $(Qh)(x) = \int_S h(s) Q(x, ds)$. Next, let $\mathbb{L}_0^2(\pi)$ be the set of measurable functions on S such that $\int h^2 \, d\pi < \infty$ and $\int h \, d\pi = 0$. For a function $h \in \mathbb{L}_0^2(\pi)$ let

$$X_i = h(\xi_i). \tag{14}$$

Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i with $i \leq k$. For any integrable random variable X we denote $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$ and $\mathbb{P}_k(A) = \mathbb{P}(A|\mathcal{F}_k)$. In our notation $\mathbb{E}_0(X_1) = (Qh)(\xi_0) = \mathbb{E}(X_1|\xi_0)$.

To guarantee that the regular transitions exist, we shall assume that $\mathcal{A}$ is countably generated.

The Markov chain is usually constructed in a canonical way on $\Omega = S^\infty$ endowed with sigma algebra $\mathcal{A}^\infty$, and ξ_n is the nth projection on S. The shift $T : \Omega \to \Omega$ is defined by $\xi_n(T\omega) = \xi_{n+1}(\omega)$ for every $n \geq 0$.

For any probability measure υ on $\mathcal{A}$ the law of $(\xi_n)_{n \in \mathbb{Z}}$ with transition operator Q and initial distribution υ is the probability measure $\mathbb{P}^\upsilon$ on $(S^\infty, \mathcal{A}^\infty)$ such that

$$\mathbb{P}^\upsilon(\xi_{n+1} \in A | \xi_n = x) = Q(x, A) \quad \text{and} \quad \mathbb{P}^\upsilon(\xi_0 \in A) = \upsilon(A).$$

For $\upsilon = \pi$, we denote $\mathbb{P} = \mathbb{P}^\pi$. For $\upsilon = \delta_x$, the Dirac measure, denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the regular probability and conditional expectation for the process started at x. Note that for each x fixed $\mathbb{P}^x(\cdot)$ is a measure on $\mathcal{F}^\infty$, the sigma algebra generated by $\bigcup_k \mathcal{F}_k$. Furthermore $\mathbb{P}^x(\cdot)$ is a version of the conditional probability on $\mathcal{F}^\infty$ given ξ_0 and, by Markov property, $\mathbb{P}^x(\cdot)$ is also the regular measure on $\mathcal{F}^\infty$ given $\mathcal{F}_0$.

We mention that any stationary sequence $(Y_k)_{k \in \mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_j; j \leq k)$ with the function $g(\xi_k) = Y_k$. Therefore the theory of stationary processes can be embedded in the theory of Markov chains.

For a Markov chain, by the quenched CLT for the Fourier transform we shall understand the following convergence: for almost all $t \in [0, 2\pi]$ there is a set $S' \subset S$ with $\pi(S') = 1$ such that for $x \in S'$

$$\mathbf{V}_n(t) \Rightarrow \mathbf{N}(t) \qquad \text{under } \mathbb{P}^x. \tag{15}$$

In other words for almost all $t \in [0, 2\pi]$, there is a set $S' \subset S$ with $\pi(S') = 1$ such that for $x \in S'$

$$\mathbb{E}^x\big[g\big(\mathbf{V}_n(t)\big)\big] \to \mathbb{E}\big[g\big(\mathbf{N}(t)\big)\big] \qquad \text{as } n \to \infty,$$

for any function g continuous and bounded. When the stationary process is viewed as a function of Markov chain, then $\xi_0 = (Y_j; \, j \leq 0)$, and therefore a fixed value of ξ_0 means a fixed past trajectory up to the moment of time 0.

All our results hold in the setting of Markov chains. In this case, the transformation T is the shift. The Markov property allows for the formulation (15).

It is remarkable that for ergodic reversible Markov chains the quenched CLT holds without centering and without any additional assumptions.

Corollary 7. *Assume that* $(X_k)_{k \in \mathbb{Z}}$ *is defined by* (14) *and in addition that the Markov chain* $(\xi_k)_{k \in \mathbb{Z}}$ *is reversible (i.e.* $Q = Q^*$*). Let* $t \in (0, 2\pi) \setminus \{\pi, \pi/2, 3\pi/2\}$*. Then,* (15) *holds where* $N_1(t)$ *and* $N_2(t)$ *are as in Theorem* 6.

Proof. We shall verify the conditions of Theorem 6. Since the spectrum of Q is contained in $[-1, 1]$ and for $t \in (0, 2\pi) \setminus \{\pi\}$ we have that e^{it} is not real, the operator $I - e^{it} Q$ is invertible, and therefore there exists $g \in \mathbb{L}^2(S, \mathcal{A}, \pi)$ such that $h = g - e^{it} Qg$. We obtain

$$\mathbb{E}_0\big(S_n(t)\big) = \sum_{k=1}^{n} \mathbb{E}_0\big(e^{itk} g(\xi_k) - e^{it(k+1)} g(\xi_{k+1})\big) = e^{it} \mathbb{E}_0\big(g(\xi_1)\big) - e^{it(n+1)} \mathbb{E}_0\big(g(\xi_{n+1})\big).$$

Then clearly

$$\big\|\mathbb{E}_0\big(S_n(t)\big)\big\|_2 \leq 2\|g\|_2 < \infty,$$

and therefore condition (13) is satisfied. Furthermore, since T is the shift operator, under our hypotheses, cannot have eigenvalues other than ± 1 (see page 15 in Cuny *et al.* [8]). □

3.2. Iterated random functions

Let (Γ, d) be a complete and separable metric space and let $\xi_n = F_{\varepsilon_n}(\xi_{n-1})$, where $F_\varepsilon(\cdot) = F(\cdot, \varepsilon)$ is the ε-section of a jointly measurable function $F : \Gamma \times \Upsilon \to \Gamma$ and $\varepsilon, \varepsilon_n, \, n \in \mathbb{Z}$ are i.i.d. random variables taking values in a second measurable space Υ. Define $L_\varepsilon = \sup_{x \neq x'} d(F_\varepsilon(x); F_\varepsilon(x'))/d(x, x')$. Diaconis and Freedman [15] proved that (ξ_n) admits a unique stationary distribution π provided that for some $\alpha > 0$ and $x_0 \in \Gamma$,

$$\mathbb{E}\big(L_\varepsilon^\alpha\big) < \infty, \qquad \mathbb{E}(\log L_\varepsilon) < 0 \quad \text{and} \quad \mathbb{E}\big(d^\alpha\big(x_0, F_\varepsilon(x)\big)\big) < \infty. \tag{16}$$

Let h be a function and let $X_k = h(\xi_k)$. Assume $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1|^2 < \infty$. To analyze this example, we shall use the coupling function introduced by Wu [41]:

$$\Delta_h(t) = \sup \left\| \left(h(\xi) - h(\xi') \right) I \left(d(\xi, \xi') < t \right) \right\|_2,$$

where the supremum is taken over all ξ, ξ' independent distributed as π. We shall establish the following:

Corollary 8. *Assume condition* (16) *is satisfied and*

$$\int_0^{1/2} \frac{\Delta_h^2(t)}{t |\log t|} \, dt < \infty. \tag{17}$$

Then, for almost all frequencies, the quenched CLT (15) *holds with* $N_1(t)$ *and* $N_2(t)$ *i.i.d. normal random variables with mean* 0 *and variance* $\pi f(t)$, $f(t)$ *being the spectral density of the process.*

Proof. We shall verify condition (11). By Lemma 3 in Wu and Woodroofe [42], condition (16) implies that there is $\beta > 0$, $C > 0$ and $0 < r < 1$ such that

$$\mathbb{E}\left(d^\beta(\xi_n, \xi'_n) \right) \leq C r^n, \tag{18}$$

where ξ_n, ξ'_n are i.i.d. Since $\mathbb{E}(h(\xi'_n)|\xi_0) = 0$ a.s.

$$\begin{aligned}
\left| \mathbb{E}\left(h(\xi_n)|\xi_0 \right) \right| &\leq \left| \mathbb{E}\left(\left[h(\xi_n) - h(\xi'_n) \right] I \left(d(\xi_n, \xi'_n) \leq \delta_n \right) |\xi_0 \right) \right| \\
&\quad + \left| \mathbb{E}\left(\left[h(\xi_n) - h(\xi'_n) \right] I \left(d(\xi_n, \xi'_n) > \delta_n \right) |\xi_0 \right) \right| \\
&= I_n + II_n.
\end{aligned}$$

To establish (11), it is enough to prove that

$$\sum_{n \geq 1} \frac{\mathbb{E}(I_n^2)}{n} < \infty \tag{19}$$

and

$$\sum_{n \geq 1} \frac{II_n^2}{n} < \infty \qquad \text{a.s.} \tag{20}$$

By Cauchy–Schwartz inequality and Markov inequality

$$\mathbb{E}(II_n) \leq 2^{1/2} \|X_0\|_2 \mathbb{P}^{1/2}\left(d(\xi_n, \xi'_n) > \delta_n \right) \leq \|X_0\|_2 \left[2\mathbb{E}\left(d^\beta(\xi_n, \xi'_n) \right) / \delta_n^\beta \right]^{1/2}.$$

By selecting now $\delta_n = r^{n/2\beta}$ we obtain $\mathbb{E}(II_n) \leq r^{n/4}$. Therefore $\mathbb{P}(II_n > r^{n/8}) \leq r^{n/4}$, and (20) follows by the Borel–Cantelli lemma.

Next, note that $I_n^2 \leq \Delta_h^2(\delta_n)$ and for the selection of $\delta_n = r^{n/2\beta}$, the convergence of the series in (19) holds under the integral condition (17).

Furthermore, the above computations also show that $\mathbb{E}|\mathbb{E}(h(\xi_n)|\xi_0)| \to 0$ as $n \to \infty$ which proves (4). $\qquad \square$

3.3. Linear processes

Next, we give an application to linear processes.

Corollary 9. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a sequence of stationary and ergodic square integrable martingale differences. Define*

$$X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, \qquad where \sum_{j=0}^{\infty} a_j^2 < \infty. \tag{21}$$

Then, under the condition

$$\sum_{j\geq 3} (a_j - a_{j+1})^2 \log j < \infty,$$

the conclusion of Corollary 4 holds.

Proof. We shall verify the conditions of Corollary 4.

Clearly for $k \geq 1$, by the orthogonality of the martingale differences

$$\left\| \mathbb{E}_0(X_{k+1} - X_k) \right\|_2^2 = \left\| \mathbb{E}_0 \left(\sum_{j\geq -1} a_{j+1} \xi_{k-j} - \sum_{j\geq 0} a_j \xi_{k-j} \right) \right\|_2^2$$

$$= \left\| \sum_{j\geq k} a_{j+1} \xi_{k-j} - \sum_{j\geq k} a_j \xi_{k-j} \right\|_2^2 = \sum_{j\geq k} (a_{j+1} - a_j)^2 \|\xi_0\|_2^2.$$

Now

$$\sum_{k\geq 1} \frac{1}{k} \sum_{j\geq k} (a_{j+1} - a_j)^2 \leq \sum_{j\geq 1} (a_{j+1} - a_j)^2 \log j,$$

and the conclusion follows by Corollary 4. $\qquad\square$

Remark 10. In the case when the sequence a_j is positive and decreasing, then the natural condition $\sum_{j=0}^{\infty} a_j^2 < \infty$ is necessary and sufficient for the conclusion of Corollary 9.

3.4. Functions of linear processes

In this section, we shall focus on functions of real-valued linear processes. Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of square summable real numbers and $(\xi_i)_{i\in\mathbb{Z}}$ is a sequence of i.i.d. random variables in $\mathbb{L}^2$ with mean 0 and variance σ^2. Define X_k by (21) and let h be a real valued function and define

$$Y_k = h(X_k) - \mathbb{E}h(X_k). \tag{22}$$

As in [8] we shall give sufficient conditions for the validity of (7) in terms of the modulus of continuity of the function h on the interval $[-M, M]$, defined by

$$w_h(u, M) = \sup\{|h(x) - h(y)|, |x - y| \le u, |x| \le M, |y| \le M\}. \tag{23}$$

Corollary 11. *Assume that h is γ-Hölder on any compact set, with $w_h(u, M) \le Cu^\gamma M^\beta$, for some $C > 0$, $\gamma \in (0, 1]$ and $\beta \ge 0$. Assume that $\mathbb{E}(h^2(X_k)) < \infty$ and*

$$\sum_{k \ge 3} a_k^2 \log k < \infty \quad \text{and} \quad \mathbb{E}|\xi_0|^{2 \vee 2\gamma \vee 2\beta} < \infty. \tag{24}$$

Then (7) *holds with $N_1(t)$ and $N_2(t)$ i.i.d. normal random variables, mean 0 and variance $\pi f(t)$, $f(t)$ being the spectral density of the process.*

Proof. We shall apply Corollary 5. Define $\mathcal{F}_k = \sigma(\xi_l, l \le k)$. Since $\mathcal{F}_{-\infty}$ is trivial, (4) holds. We write

$$Y_0 = \sum_{l \ge 0} P_{-l}(Y_0),$$

where P_{-l} denotes the projector operator

$$P_{-l}(\cdot) = \mathbb{E}_{-l}(\cdot) - \mathbb{E}_{-l-1}(\cdot). \tag{25}$$

By the orthogonality of the projections,

$$\left\| \mathbb{E}_0(Y_k) \right\|_2^2 = \sum_{l \ge 0} \left\| P_{-l}(Y_k) \right\|_2^2 = \sum_{j \ge k} \left\| P_0(Y_j) \right\|_2^2 < \infty.$$

Therefore, condition (12) follows from

$$\sum_{j \ge 2} \left\| P_0(Y_j) \right\|_2^2 \log j < \infty. \tag{26}$$

So it remains to verify (26). We estimate $\|P_0(Y_j)\|_2^2$ as in [8]. We give here the argument for completeness. Let ξ' be an independent copy of ξ, and denote by $\mathbb{E}_\xi(\cdot)$ the conditional expectation with respect to ξ. Clearly

$$P_0(Y_k) = \mathbb{E}_\xi \left[h \left(\sum_{j=0}^{k-1} a_j \xi'_{k-j} + a_k \xi_0 + \sum_{j>k} a_j \xi_{k-j} \right) - h \left(\sum_{j=0}^{k-1} a_j \xi'_{k-j} + a_k \xi'_0 + \sum_{j>k} a_j \xi_{k-j} \right) \right].$$

By using definition (23),

$$\left| P_0(Y_k) \right| \le C \mathbb{E}_\xi \left| a_k \left(\xi_0 - \xi'_0 \right) \right|^\gamma \left(\left| X'_k \right| \vee \left| X''_k \right| \right)^\beta,$$

where $X'_k = \sum_{j=0}^{k-1} a_j \xi'_{k-j} + a_k \xi_0 + \sum_{j>k} a_j \xi_{k-j}$ and $X''_k = \sum_{j=0}^{k-1} a_j \xi'_{k-j} + a_k \xi'_0 + \sum_{j>k} a_j \times \xi_{k-j}$. Therefore, by taking the expected value, noticing that X'_k and X''_k are identically distributed

as $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, and then applying the Cauchy–Schwarz inequality, for a positive constant C', we obtain

$$\left\| P_0(X_k) \right\|_2^2 \leq C' a_k^2 \mathbb{E}(|\xi_0|^{2\gamma}) \mathbb{E}(|X_0|^{2\beta}).$$

We estimate now $\mathbb{E}(|X_0|^{2\beta})$. If $\beta < 1$ then $\mathbb{E}(|X_0|^{2\beta}) \leq (\mathbb{E}|X_0|^2)^\beta \leq (\sum_{l \geq 0} a_l^2)^\beta \sigma^{2\beta}$. In case $\beta \geq 1$, by the Rosenthal inequality (see Theorem 1.5.9 in [14]), for some positive constant C_β, $\mathbb{E}(|X_0|^{2\beta}) \leq C_\beta ((\sum_{l \geq 0} a_l^2)^\beta \sigma^{2\beta} + \sum_{l \geq 0} a_l^{2\beta} \mathbb{E}(|\xi_0|^{2\beta}))$. Since we assume that $\sum_{l \geq 0} a_l^2 < \infty$, it follows that we can find a constant K such that

$$\left\| P_0(Y_k) \right\|_2^2 \leq K a_k^2 \mathbb{E}(|\xi_0|^{2\gamma})(\mathbb{E}|\xi_0|^{2\beta} \vee \sigma^{2\beta}).$$

The result follows by (26) and by taking into account condition (24). $\qquad \square$

3.5. Application to mixing stationary sequences

Mixing coefficients are important for quantifying the strength of dependence in a stochastic process. They have proven essential for analyzing Markov chains, Gaussian processes, dynamical systems and other dependent structures.

We shall introduce the following strong mixing coefficient: For a σ-algebra $\mathcal{A}$ and a random variable X the strong mixing coefficient is defined as

$$\tilde{\alpha}(\mathcal{A}, X) = \sup\{|\mathbb{P}(A \cap \{X > x\}) - \mathbb{P}(A)\mathbb{P}(X > x)|; x \in R\}.$$

This coefficient was introduced by Rosenblatt [31] and also analyzed by Rio [29]. It is weaker than those involving all the future of the process which are usually used in the literature and they are estimable for a variety of examples from dynamical systems.

For a stationary sequence of random variables $(X_k)_{k \in \mathbb{Z}}$, we denote by $\mathcal{F}_m$ the σ-field generated by X_l with indices $l \leq m$. Notice that $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ defined in this way is a minimal filtration such that $(X_k)_{k \in \mathbb{Z}}$ is adapted to $(\mathcal{F}_k)_{k \in \mathbb{Z}}$. The sequences of coefficients $\tilde{\alpha}(n)$ are then defined by

$$\tilde{\alpha}(n) = \tilde{\alpha}(\mathcal{F}_0, X_n).$$

We refer to the book by Bradley [3] for classical mixing coefficients and to Dedecker *et al.* [11] for specific estimates of coefficients of type $\tilde{\alpha}$ for certain dynamical systems generated by intermittent maps.

For integrable random variable X_0, define the "upper tail" quantile function Q by

$$Q(u) = \inf\{t \geq 0 : \mathbb{P}(|X_0| > t) \leq u\}.$$

By relation (1.11c) in Rio [29] notice that

$$\left\| \mathbb{E}_0(X_k) \right\|_2^2 = \mathbb{E}(X_k \mathbb{E}_0(X_k)) \leq 2 \int_0^{\tilde{\alpha}(k)} Q^2(u) \, du. \tag{27}$$

By using this inequality, condition (10) is verified provided

$$\sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\tilde{\alpha}(k)} Q^2(u) \, du < \infty. \tag{28}$$

Denoting $\tilde{\alpha}^{-1}(x) = \min\{k \in \mathbb{N} : \tilde{\alpha}(k) \leq x\}$ we can write relation (28) in the equivalent formulation

$$\int_0^1 \log\left(1 + \tilde{\alpha}^{-1}(u)\right) Q^2(u) \, du < \infty.$$

In particular, if $E(|X_0|^{2+\delta}) < \infty$ for some positive $\delta > 0$, by decoupling the above integral via the Cauchy–Schwarz inequality, we obtain that a sufficient condition for (10) is

$$\int_0^1 \left[\log\left(1 + \tilde{\alpha}^{-1}(u)\right)\right]^{(2+\delta)/\delta} \, du < \infty,$$

which requires a logarithmic rate of decay of the coefficients $\tilde{\alpha}(k)$. If $\|X_0\|_\infty < \infty$, condition (28) is implied by

$$\sum_{k=1}^{\infty} \frac{1}{k} \tilde{\alpha}(k) < \infty.$$

Since $(\tilde{\alpha}(k))_{k \geq 1}$ is decreasing, by (27), condition (28) implies the regularity condition (4). Therefore it is a sufficient condition for (7) which holds with $N_1(t)$ and $N_2(t)$ i.i.d. normal random variables, mean 0 and variance $\pi f(t)$, $f(t)$ being the spectral density of the process.

It is worth mentioning that some more restrictive mixing conditions make possible to obtain (7) directly from (6). One of these conditions is called ϕ-mixing. A stationary sequence of random variables $(X_k)_{k \in \mathbb{Z}}$ is called ϕ-mixing if

$$\phi(n) = \sup\left\{\left|\mathbb{P}(B|A) - \mathbb{P}(B)\right|; A \in \mathcal{F}_0, B \in \mathcal{F}^n\right\} \to 0.$$

Here $\mathcal{F}^n$ is the σ-field generated by X_l with indices $l \geq n$. It is equivalent to saying that (see [3], Vol. 1)

$$\phi(n) = \sup\left\{\left|\mathbb{P}(B|\mathcal{F}_0) - \mathbb{P}(B)\right|; B \in \mathcal{F}^n\right\} \to 0 \qquad \text{a.s.}$$

If we fix now $m > 0$, we have $S_m(t)/\sqrt{n} \to 0$ $\mathbb{P}$-a.s. and it is enough to study the asymptotic behavior of

$$\mathbf{V}_{n,m}(t) = \left(\mathrm{Re}\left[S_n(t) - S_m(t)\right]/\sqrt{n}, \, \mathrm{Im}\left[S_n(t) - S_m(t)\right]/\sqrt{n}\right).$$

By the definition of ϕ-mixing coefficients, for h continuous and bounded (see again [3], Vol. 1)

$$\left|\mathbb{E}\left(h\left(\mathbf{V}_{n,m}(t)\right)|\mathcal{F}_0\right) - \mathbb{E}\left(h\left(\mathbf{V}_{n,m}(t)\right)\right)\right| \leq \phi(m) \qquad \text{a.s.},$$

and the claim follows easily by Theorem 3.2 in [2].

4. Proofs

Proof of Theorem 1. The proof of Theorem 1 is based on the following approximation lemma for Fourier transforms. Recall the definition of projection operator (25).

Lemma 12. *Under the conditions of Theorem 1, for almost all $t \in [0, 2\pi]$, the martingale difference*

$$D_k(t, \omega) = \sum_{j > k} e^{ijt} P_k X_j(\omega) = e^{ikt} \sum_{j \geq 1} e^{ijt} P_0 X_j(\omega) \circ T^k$$

is well defined in the almost sure sense and in $\mathbb{L}^2(\Omega, \mathcal{K}, \mathbb{P})$. Denote by $M_n(t)(\omega) = \sum_{k=1}^{n} D_k(t, \omega)$. Then, for almost all $t \in [0, 2\pi]$,

$$\frac{1}{n} \mathbb{E}_0 \big| S_n(t) - \mathbb{E}_0(S_n(t)) - M_n(t) \big|^2 \to 0 \qquad \mathbb{P}\text{-a.s. and in } \mathbb{L}_1.$$

Proof. The convergence in $\mathbb{L}_1$ was established in Peligrad and Wu [26]. We shall prove here the almost sure convergence. It is convenient to work on the product space, $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}) = ([0, 2\pi] \times \Omega, \mathcal{B} \otimes \mathcal{A}, \lambda \otimes \mathbb{P})$ where λ is the normalized Lebesgue measure on $[0, 2\pi]$, and $\mathcal{B}$ is the Borel σ-algebra on $[0, 2\pi]$, $\widetilde{\mathbb{P}} = \lambda \otimes \mathbb{P}$. Consider also the filtration $(\widetilde{\mathcal{F}}_n)_{n \in \mathbb{Z}}$ given by $\widetilde{\mathcal{F}}_n := \mathcal{B} \otimes \mathcal{F}_n$. Denote by $\widetilde{\mathbb{E}}$, the integral with respect to $\widetilde{\mathbb{P}}$, by $\widetilde{\mathbb{E}}_0$ the conditional expectation with respect to $\widetilde{\mathcal{F}}_0$, $\widetilde{P}_k(\cdot) = \widetilde{\mathbb{E}}_k(\cdot) - \widetilde{\mathbb{E}}_{k-1}(\cdot)$.

Let $t \in [0, 2\pi)$ be a real number, fixed for the moment. Clearly, the transformation $\widetilde{T}_t$ from $\widetilde{\Omega}$ to $\widetilde{\Omega}$ given by

$$\widetilde{T}_t : (u, \omega) \mapsto \big(u + t \text{ modulo } 2\pi, T(\omega) \big),$$

is invertible, bi-measurable and preserves $\widetilde{\mathbb{P}}$. For every $(u, \omega) \in \widetilde{\Omega}$ define the variable $\widetilde{X}_0$ on $\widetilde{\Omega}$ by $\widetilde{X}_0(u, \omega) = e^{iu} X_0(\omega)$ and for any $n \in \mathbb{Z}$, $\widetilde{X}_n(t; u, \omega) = \widetilde{X}_0(u, \omega) \circ \widetilde{T}_t^n$. For simplicity, in the sequel, we shall drop from the notation the variables u and ω in $\widetilde{X}_k(t; u, \omega)$ and we shall write instead $\widetilde{X}_k(t)$ and $\widetilde{S}_n(t) = \sum_{k=1}^{n} \widetilde{X}_k(t)$. Notice that $(\widetilde{X}_n(t))_{n \in \mathbb{Z}}$ is a stationary sequence of complex random variables adapted to the nondecreasing filtration $(\widetilde{\mathcal{F}}_n)$. Moreover $e^{iu} e^{ikt} X_k(\omega) = \widetilde{X}_k(t; u, \omega)$. We shall construct a martingale $\widetilde{M}_n(t)$, adapted to $(\widetilde{\mathcal{F}}_n)$, with stationary differences, such that for almost all $t \in [0, 2\pi]$

$$\frac{1}{n} \widetilde{\mathbb{E}}_0 \big[\widetilde{S}_n(t) - \widetilde{\mathbb{E}}_0(\widetilde{S}_n(t)) - \widetilde{M}_n(t) \big]^2 \to 0 \qquad \widetilde{\mathbb{P}}\text{-a.s.}$$

With this aim we shall apply Proposition 15, given in the Section 5. In order to verify the conditions of this proposition, we have to show that for almost all t in $[0, 2\pi]$

$$\widetilde{P}_0(\widetilde{S}_n(t)) \to \widetilde{D}_0(t) \qquad \widetilde{\mathbb{P}}\text{-a.s.} \tag{29}$$

and

$$\widetilde{\mathbb{E}} \Big[\sup_n \big| \widetilde{P}_0(\widetilde{S}_n(t)) \big|^2 \Big] < \infty. \tag{30}$$

In order to prove (29), note that by the orthogonality of the projections and the fact that the sequence $\|E_{-n}X_0\|_2$ is decreasing, it follows that

$$\sum_{k\geq 0}\|P_{-k}X_0\|_2^2 = \lim_{n\to\infty}\sum_{k=0}^{n}\|P_{-k}X_0\|_2^2 = \lim_{n\to\infty}\left\|\sum_{k=0}^{n}P_{-k}X_0\right\|_2^2$$

$$= \lim_{n\to\infty}\|X_0 - E_{-n}X_0\|_2^2 = \|X_0\|_2^2 - \|E_{-\infty}X_0\|_2^2 \leq \|X_0\|_2^2 < \infty.$$

Clearly this implies

$$\sum_{k\geq 0}|P_0X_k|^2 < \infty \qquad \mathbb{P}\text{-a.s.}$$

Now for ω such that $\sum_{k\geq 0}|P_0X_k|^2(\omega) < \infty$, by Carleson's [5] theorem, $P_0S_n(t) = \sum_{1\leq k\leq n}e^{ikt}(P_0X_k)(\omega)$ converges λ-almost surely. Denote the limit by $D_0 = D_0(t)$. We now consider the set

$$A = \left\{(t,\omega) \subset [0,2\pi]\times\Omega, \text{ where } \left[P_0S_n(t)\right]_n \text{ does not converge}\right\}$$

and note that almost all sections for ω fixed have Lebesgue measure 0. So by Fubini's theorem the set A has measure 0 in the product space and therefore, again by Fubini's theorem, almost all sections for t fixed have probability 0. It follows that for almost all t in $[0,2\pi]$, $P_0(S_n(t)) \to D_0(t)$ almost surely under $\mathbb{P}$. This shows that, after multiplying by e^{iu}, we get, for almost all t, that condition (29) is verified with

$$\tilde{D}_0(t) = e^{iu}\sum_{j\geq 1}e^{ijt}P_0X_j(\omega).$$

Note that

$$\tilde{D}_k(t) = \tilde{D}_0(t)\circ\widetilde{T}_t^k = e^{iu}\sum_{j>k}e^{ijt}P_kX_j(\omega).$$

Next, we prove (30). By the maximal inequality in Hunt and Young [23], there is a constant C such that

$$\int_0^{2\pi}\left[\sup_n|P_0(S_n(t))|^2\right]\lambda(dt) \leq C\sum_{k\geq 1}|P_0X_k|_2^2.$$

Then we integrate with respect to $\mathbb{P}$ and use Fubini theorem to obtain

$$\int_0^{2\pi}\mathbb{E}\left[\sup_{n\geq 0}|P_0(S_n(t))|^2\right]\lambda(dt) \leq C\|X_0\|_2^2 < \infty.$$

It follows that

$$\mathbb{E}\left[\sup_{n\geq 1}\left|P_0\big(S_n(t)\big)\right|^2\right] < \infty \qquad \text{for almost all } t.$$

Therefore, we obtain that condition (30) is satisfied. We apply now Proposition 15 to obtain for almost all t in $[0, 2\pi]$

$$\frac{1}{n}\tilde{\mathbb{E}}_0\left|\tilde{S}_n(t) - \tilde{\mathbb{E}}_0\big(\tilde{S}_n(t)\big) - \tilde{M}_n(t)\right|^2 \to 0 \qquad \tilde{\mathbb{P}}\text{-a.s.,} \tag{31}$$

where

$$\tilde{M}_n(t) = \sum_{k=1}^{n}\tilde{D}_k(t).$$

Now fix t in $[0, 2\pi]$ such that (31) holds. Clearly

$$\frac{1}{n}\mathbb{E}_0\left|S_n(t) - \mathbb{E}_0\big(S_n(t)\big) - M_n(t)\right|^2 \to 0 \qquad \mathbb{P}\text{-a.s.}$$

The result follows. □

We study next the behavior of $M_n(t)/\sqrt{n}$. We shall do it in general in the context of stationary and ergodic complex valued martingale differences. Below, the martingale difference D may depend on t.

Proposition 13. *Let T and $\mathcal{F}_0$ be as in Section 2. Assume that $t \in (0, 2\pi)$ be such that e^{-2it} is not an eigenvalue of T. Let $D = D_0$ be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_0$ measurable and such that $E(D \circ T | \mathcal{F}_0) = 0$ a.s. For any $k \in \mathbb{Z}$, let*

$$\mathbf{D}_k(t) = \big(\mathrm{Re}\big(e^{ikt}D \circ T^k\big), \mathrm{Im}\big(e^{ikt}D \circ T^k\big)\big).$$

Let $\mathbf{M}_n(t) = \sum_{k=1}^{n}\mathbf{D}_k(t)$. Then, there is a set Ω' with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$

$$\frac{1}{\sqrt{n}}\mathbf{M}_n(t) \Rightarrow \mathbf{N} \qquad \text{under } \mathbb{P}^{\omega}, \tag{32}$$

where $\mathbf{N} = (N_1, N_2)$, with N_1, N_2 are two independent centered normal random variables with variance $\mathbb{E}|D|^2/2$.

Proof. Fix $t \in (0, 2\pi)$ such that e^{-2it} is not an eigenvalue of T. Denote $R_k(t) = \mathrm{Re}(e^{ikt}D \circ T^k)$ and $I_k(t) = \mathrm{Im}(e^{ikt}D \circ T^k)$.

The proof is based on Theorem 16 and the following two convergence results: for any real constants a and b

$$\frac{1}{\sqrt{n}}\mathbb{E}_0\Big(\max_{1\leq k\leq n}\big|aR_k(t) + bI_k(t)\big|\Big) \to 0 \qquad \mathbb{P}\text{-a.s.} \tag{33}$$

and

$$\mathbb{P}_0\left(\left|\frac{1}{n}\sum_{k=1}^{n}\left|aR_k(t)+bI_k(t)\right|^2 - \frac{1}{2}(a^2+b^2)\mathbb{E}|D|^2\right| > \varepsilon\right) \to 0 \qquad \mathbb{P}\text{-a.s.} \qquad (34)$$

Before proving (33) and (34) let us show how they lead to the result.

Let a and b be two rational numbers and let $\Omega_{a,b}$ be the set of probability 1 where (33) and (34) hold. Construct $\Omega_1 = \bigcap \Omega_{a,b}$, where the intersection is taken over all the rationals a and b. Clearly $\mathbb{P}(\Omega_1) = 1$. Then, by Theorem 16 in Section 5, we get via (33) and (34) that for all $\omega \in \Omega_1$

$$\sum_{k=1}^{n}\left(aR_k(t)+bI_k(t)\right)\Big/\sqrt{n} \Rightarrow N(a,b,t) \qquad \text{under } \mathbb{P}^\omega, \qquad (35)$$

where $N(a,b,t)$ is a centered normal random variable with variance $(a^2+b^2)\mathbb{E}|D|^2/2$.

Because $\mathbb{E}_0$ is regular, by Hopf ergodic theorem

$$\frac{1}{n}\mathbb{E}_0|\mathbf{M}_n(t)|^2 = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_0|\mathbf{D}_k(t)|^2 \to \mathbb{E}|\mathbf{D}_0(t)|^2 \qquad \text{as } n \to \infty \; \mathbb{P}\text{-a.s.}$$

By Markov inequality it follows that there is a set Ω_2 such that for all $\omega \in \Omega_2$ the sequence $(\mathbf{M}_n(t)/\sqrt{n})_{n\geq 1}$ is tight under $\mathbb{P}^\omega$.

Now construct $\Omega' = \Omega_1 \cap \Omega_2$. For $\omega \in \Omega'$, we apply Lemma 17 in Section 5 and obtain (32).

It remains to prove (33) and (34). To prove the convergence in (34) we shall use relation (16) in Cuny–Merlevède–Peligrad [8], with $u = 0$, which gives

$$\frac{1}{n}\sum_{k=1}^{n}\left|aR_k(t)+bI_k(t)\right|^2 \to \frac{1}{2}(a^2+b^2)\mathbb{E}|D|^2 \qquad \mathbb{P}\text{-a.s.}$$

This convergence was obtained by trigonometric computations along with Dunford–Schwartz ergodic theorem from Sections VIII.5 and VIII.6 of [16], which requires that $t \in (0, 2\pi)$ be such that e^{-2it} is not an eigenvalue of T (see Proposition 30 in [8]).

This last convergence implies that, for every $\varepsilon > 0$

$$I\left(\left|\frac{1}{n}\sum_{k=1}^{n}\left|aR_k(t)+bI_k(t)\right|^2 - \frac{1}{2}(a^2+b^2)\mathbb{E}|D|^2\right| > \varepsilon\right) \to 0 \qquad \mathbb{P}\text{-a.s.,}$$

whence (34) follows by Theorem 34.2(v) in Billingsley [1].

We verify now relation (33). Note that

$$\left|aR_k(t)+bI_k(t)\right| \leq (|a|+|b|)|D| \circ T^k = (|a|+|b|)|D_k|.$$

It is enough to verify that

$$\frac{1}{n}\mathbb{E}_0\left(\max_{1\leq k\leq n}|D_k|^2\right) \to 0 \qquad \mathbb{P}\text{-a.s.}$$

We shall use a truncation argument. Let $\varepsilon > 0$ and $c > 0$ be fixed for the moment. Let n be sufficiently large such that $\varepsilon\sqrt{n} \geq c$. For this selection of n, we have

$$\frac{1}{n}\mathbb{E}_0\Big(\max_{1\leq k\leq n}|D_k|^2\Big) \leq \frac{1}{n}\mathbb{E}_0\Big(\max_{1\leq k\leq n}|D_k|^2 I\big(|D_k|\leq\varepsilon\sqrt{n}\big)\Big)$$

$$+ \frac{1}{n}\mathbb{E}_0\Big(\max_{1\leq k\leq n}|D_k|^2 I\big(|D_k|>\varepsilon\sqrt{n}\big)\Big)$$

$$\leq \varepsilon^2 + \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_0\big(|D_k|^2 I\big(|D_k|>c\big)\big).$$

Now, by the Hopf theorem for Dunford–Schwartz operators (see [16] or [17]),

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_0\big(|D_k|^2 I\big(|D_k|>c\big)\big) \to \mathbb{E}\big(|D_0|^2 I\big(|D_0|>c\big)\big) \qquad \text{as } n\to\infty \ \mathbb{P}\text{-a.s.}$$

Then we have

$$\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}_0\Big(\max_{1\leq k\leq n}|D_k|^2\Big) \leq \varepsilon^2 + \mathbb{E}\big(|D_0|^2 I\big(|D_0|>c\big)\big).$$

The result follows by letting $\varepsilon \to 0$ and $c \to \infty$. $\qquad\square$

Remark 14. Note that because $\mathcal{K}$ is countably generated then $\mathbb{L}^2(\Omega,\mathcal{K},\mathbb{P})$ is separable and by Lemma 32 in Cuny *et al.* [8], T can admit only a countable number of eigenvalues. Therefore the quenched CLT in Proposition 13 holds for almost all $t \in [0,2\pi]$.

End of the Proof of Theorem 1. By using Theorem 3.1 in Billingsley [2], Lemma 12 shows that for almost all $t \in [0,2\pi]$, there is a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega')=1$ such that for all $\omega \in \Omega'$, the limiting behavior $S_n(t) - \mathbb{E}_0(S_n(t))$ is the same as of the martingale $M_n(t)$ under $\mathbb{P}^\omega$. Then, by Proposition 13 and Remark 14, for almost all $t \in [0,2\pi]$ the quenched CLT holds for $\mathbf{W}_n(t)$, with the limit $\mathbf{N}(t) = (N_1(t), N_2(t))$, where $N_1(t), N_2(t)$ are two independent centered normal random variables with variance $\mathbb{E}|D(t)|^2/2$. For an alternative characterization of $\mathbb{E}|D(t)|^2$, it remains to note that by Lemma 12, for almost all $t \in [0,2\pi]$

$$\Big(\frac{1}{n}\mathbb{E}_0\big|S_n(t) - \mathbb{E}_0\big(S_n(t)\big)\big|^2\Big)^{1/2} - \Big(\frac{1}{n}\mathbb{E}_0\big|M_n(t)\big|^2\Big)^{1/2} \to 0 \qquad \mathbb{P}\text{-a.s.}$$

Furthermore, by the Hopf ergodic theorem

$$\frac{1}{n}\mathbb{E}_0\big|M_n(t)\big|^2 = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_0\big|D_k(t)\big|^2 \to \mathbb{E}\big|D(t)\big|^2 \qquad \mathbb{P}\text{-a.s.} \qquad\square$$

Proof of Theorem 2. Clearly (b) implies (a) via Theorem 1. To prove that (a) implies (b), we shall use again Theorem 1 along with the Theorem of types. This latter theorem states that if

$V_n = a_n U_n + b_n$ and $V_n \Rightarrow V$ and $U_n \Rightarrow U$ with U nondegenerate then $a_n \to a$, $b_n \to b$ and $V = aU + b$.

Under conditions of Theorem 2, for λ-almost all $t \in [0, 2\pi]$ there is a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$

$$\frac{1}{\sqrt{n}} \mathrm{Re}\big[S_n(t) - \mathbb{E}_0\big(S_n(t)\big)\big] \Rightarrow N_1(t) \qquad \text{under } \mathbb{P}^\omega.$$

By the properties of conditional expectations and measure theoretical arguments (see Lemma 18), we know that for every function g continuous and bounded and random variables X and Y, such that Y is $\mathcal{F}_0$-measurable,

$$\mathbb{E}^\omega\big(g(X, Y)|\mathcal{F}_0\big) = \mathbb{E}^\omega\big(g\big(X, Y(\omega)\big)|\mathcal{F}_0\big)$$

for ω in a set of probability 1. By this observation along with the definition of convergence in distribution, we derive that for λ-almost all $t \in [0, 2\pi]$ there is a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$, such that for all $\omega \in \Omega'$

$$\frac{1}{\sqrt{n}} \mathrm{Re}\big[S_n(t) - \mathbb{E}_0\big(S_n(t)\big)(\omega)\big] \Rightarrow N_1(t) \qquad \text{under } \mathbb{P}^\omega,$$

and by (a) there is a set $\Omega'' \subset \Omega$ with $\mathbb{P}(\Omega'') = 1$, such that for all $\omega \in \Omega''$

$$\frac{1}{\sqrt{n}} \mathrm{Re}\big(S_n(t)\big) \Rightarrow N_1(t) \qquad \text{under } \mathbb{P}^\omega.$$

Now assume that $N_1(t)$ is nondegenerate. For $\omega \in \Omega' \cap \Omega''$, by the Theorem of types we have $\mathrm{Re}\,\mathbb{E}_0(S_n(t))(\omega)/\sqrt{n} \to 0$. A similar argument gives $\mathrm{Im}\,\mathbb{E}_0(S_n(t))(\omega)/\sqrt{n} \to 0$ and (b) follows for this case. If $N_1(t)$ is degenerate, then both $\mathrm{Re}[S_n(t)]/\sqrt{n} \to 0$ under $\mathbb{P}^\omega$ and $\mathrm{Re}[S_n(t) - \mathbb{E}_0(S_n(t))(\omega)]/\sqrt{n} \to 0$ under $\mathbb{P}^\omega$, and the result follows. $\qquad \square$

Proof of Corollary 4. In order to prove this result, we shall verify the item (b) of Theorem 2. We have then to show that for almost all t

$$\frac{\mathbb{E}_0(S_n(t))}{\sqrt{n}} \to 0 \qquad \mathbb{P}\text{-a.s.} \tag{36}$$

Note that it is enough to show instead that for almost all $t \in [0, 2\pi]$

$$\frac{(1 - \mathrm{e}^{\mathrm{i}t})\mathbb{E}_0(S_n(t))}{\sqrt{n}} \to 0 \qquad \mathbb{P}\text{-a.s.}$$

With this aim note that

$$\frac{(1 - \mathrm{e}^{\mathrm{i}t})\mathbb{E}_0(S_n(t))}{\sqrt{n}} = \frac{\mathbb{E}_0(S_n(t)) - \mathrm{e}^{\mathrm{i}t}\mathbb{E}_0(S_n(t))}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}}\mathrm{e}^{\mathrm{i}t}\mathbb{E}_0(X_1) - \mathrm{e}^{\mathrm{i}t(n+1)}\frac{1}{\sqrt{n}}\mathbb{E}_0(X_n) + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}\mathrm{e}^{\mathrm{i}t(k+1)}\mathbb{E}_0(X_{k+1} - X_k).$$

We shall analyze each term in the last sum separately. The first term, $e^{it}\mathbb{E}_0(X_1)/\sqrt{n}$ in the above expression is trivially convergent to 0, $\mathbb{P}$-a.s. By Jensen's inequality the second one is dominated as follows:

$$\left| e^{it(n+1)} \frac{1}{\sqrt{n}} \mathbb{E}_0(X_n) \right|^2 \leq \frac{1}{n} \mathbb{E}_0(X_n^2).$$

We write

$$\frac{1}{n} \mathbb{E}_0(X_n^2) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_0(X_j^2) - \frac{1}{n} \sum_{j=1}^{n-1} \mathbb{E}_0(X_j^2),$$

which convergence to 0, $\mathbb{P}$-a.s., by the Hopf ergodic theorem for Dunford–Schwartz operators (see again [17]).

To prove the convergence of the third term, since we assumed (10), it follows that

$$\sum_{k \geq 1} \frac{|\mathbb{E}_0(X_{k+1} - X_k)|^2}{k} < \infty \qquad \mathbb{P}\text{-a.s.}$$

By Carleson theorem (see [5]) it follows that for almost all t

$$\sum_{k \geq 1} \frac{e^{itk} \mathbb{E}_0(X_{k+1} - X_k)}{k^{1/2}} \text{ converges } \mathbb{P}\text{-a.s.}$$

which implies by Kronecker lemma

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} e^{it(k+1)} \mathbb{E}_0(X_{k+1} - X_k) \to 0 \qquad \mathbb{P}\text{-a.s.}$$

which completes the proof of (36) and of this corollary. $\qquad\square$

Proof of Theorem 6. With the notations from the proof of Lemma 12, we note that under condition (13) we also have

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \left(\tilde{\mathbb{E}} \left| \tilde{\mathbb{E}}_0(\tilde{S}_k(t)) \right|^2 \right)^{1/2} < \infty. \tag{37}$$

Then, we can apply directly the martingale approximation in Theorem 2.7 in Cuny and Merlevède [7] which also remains valid for complex valued variables. It follows that

$$\frac{1}{\sqrt{n}} \tilde{\mathbb{E}}_0 \left| \tilde{S}_n(t) - \tilde{M}_n(t) \right| \to 0 \qquad \tilde{\mathbb{P}}\text{-a.s. and in } \tilde{\mathbb{L}}^2,$$

where $\tilde{M}_n$ has stationary complex martingale differences defined by

$$\tilde{D}_j(t) = \sum_{n \geq 0} \sum_{k \geq n} \frac{\tilde{P}_0(\tilde{X}_k(t)) \circ T^j}{k+1}.$$

Whence we obtain

$$\frac{1}{\sqrt{n}}\mathbb{E}_0\big|\big(S_n(t)\big) - M_n(t)\big)\big| \to 0 \qquad \mathbb{P}\text{-a.s. and in } \mathbb{L}^2, \tag{38}$$

where the differences of the martingale $M_n(t)$ are

$$D_j(t) = \sum_{n\geq 0}\sum_{k\geq n} \frac{e^{itk} P_0(X_k) \circ T^j}{k+1}.$$

It follows that for $\mathbb{P}$-almost all ω, under $\mathbb{P}^\omega$, the behavior of $S_n(t)/\sqrt{n}$ is equivalent to $M_n(t)/\sqrt{n}$. We have just to apply Proposition 13 to obtain the quenched CLT, where the limiting independent normal variables have the variance $\|D_0(t)\|_2^2/2$. It remains to note that by (38) we can identify $\|D_0(t)\|_2^2$ as

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\big|S_n(t)\big|^2 = \big\|D_0(t)\big\|_2^2. \qquad \square$$

5. Technical results

First, we prove the following martingale approximation for complex valued random variables. It is similar to Proposition 7 in Cuny and Peligrad [9] but we do not assume ergodicity and the variables are complex valued. Because there are various changes in the proof we give the proof for completeness.

Proposition 15. *Assume that $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence of complex valued random variables and let $S_n = \sum_{k=1}^n X_k$. Assume*

$$P_0(S_n) \to D_0 \qquad converges\ a.s.\quad and\quad \mathbb{E}\Big[\sup_m \big|P_0(S_m)\big|^2\Big] < \infty \tag{39}$$

(where P_0 is defined by (25)). Then, D_0 is a martingale difference and

$$\frac{1}{n}\mathbb{E}_0\big(\big|S_n - \mathbb{E}_0(S_n) - M_n\big|^2\big) \to 0 \qquad \mathbb{P}\text{-a.s.,}$$

where $M_n = \sum_{k=1}^n D_k$ with $D_k = D_0 \circ T^k$.

Proof. Starting from condition (39), we notice that this condition implies $P_0(S_n) \to D_0$ in $\mathbb{L}^2(\mathbb{P})$. Since $\mathbb{E}_{-1}[P_0(S_n)] = 0$ a.s. we conclude that $\mathbb{E}_{-1}[D_0] = 0$ a.s. and therefore $(D_k)_{k\geq 1}$ is a sequence of martingale differences adapted to $\mathcal{F}_k$. We shall approximate S_n by $M_n + \mathbb{E}_0(S_n)$. We use now a traditional decomposition of S_n in martingale differences by using the projections on consecutive sigma algebras:

$$S_n - \mathbb{E}_0(S_n) = \big[S_n - \mathbb{E}_{n-1}(S_n)\big] + \big[\mathbb{E}_{n-1}(S_n) - \mathbb{E}_{n-2}(S_n)\big] + \cdots + \big[\mathbb{E}_1(S_n) - \mathbb{E}_0(S_n)\big].$$

So, we have the martingale decomposition

$$S_n - \mathbb{E}_0(S_n) - M_n = \sum_{k=1}^{n} \big[P_k(S_n - S_{k-1}) - D_k \big].$$

We write now

$$P_k(S_n - S_{k-1}) - D_k = \big[P_0(S_{n-k}) \big] \circ T^k - D_0 \circ T^k,$$

and so

$$\sum_{k=1}^{n} \big[P_k(S_n - S_{k-1}) - D_k \big] = \sum_{k=1}^{n} \big(P_0(S_{n-k}) - D_0 \big) \circ T^k = \sum_{k=0}^{n-1} \big(P_0(S_k) - D_0 \big) \circ T^{n-k}.$$

With the notation

$$P_0(S_k) - D_0 = G_k,$$

we have

$$S_n - \mathbb{E}_0(S_n) - M_n = \sum_{j=0}^{n-1} G_j \circ T^{n-j} = \sum_{j=1}^{n} G_{n-j} \circ T^j.$$

By the orthogonality of $G_j \circ T^{n-j}$, we have

$$\mathbb{E}_0 \big| S_n - \mathbb{E}_0(S_n) - M_n \big|^2 = \sum_{j=0}^{n-1} \mathbb{E}_0 \big(|G_j|^2 \circ T^{n-j} \big).$$

Let N be fixed. For n sufficiently large, we decompose the last sum into a sum from 1 to N and one from $N+1$ to n. Then

$$\mathbb{E}_0 \big| S_n - \mathbb{E}_0(S_n) - M_n \big|^2 = \sum_{j=0}^{N} \mathbb{E}_0 \big(|G_j|^2 \circ T^{n-j} \big) + \sum_{j=N+1}^{n-1} \mathbb{E}_0 \big(|G_j|^2 \circ T^{n-j} \big)$$
$$= A_n(N) + B_n(N). \tag{40}$$

It is then well known that we have for all j fixed

$$\frac{1}{n-j} \sum_{u=0}^{n-j} \mathbb{E}_0 \big(|G_j|^2 \circ T^u \big) \text{ converges as } n \to \infty \text{ almost surely and in } \mathbb{L}_1.$$

By writing for all j fixed, $0 \leq j \leq N$,

$$\mathbb{E}_0 \big(|G_j|^2 \circ T^{n-j} \big) = \sum_{j=0}^{n-j} \mathbb{E}_0 \big(|G_j|^2 \circ T^j \big) - \sum_{j=0}^{n-j-1} \mathbb{E}_0 \big(|G_j|^2 \circ T^j \big),$$

it follows easily that

$$\frac{A_n(N)}{n} \to 0 \qquad \text{as } n \to \infty \ \mathbb{P}\text{-a.s. and in } \mathbb{L}_1. \tag{41}$$

Now we treat $B_n(N)$. We bound this term in the following way,

$$\frac{B_n(N)}{n} \le \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_0 \left[\sup_{m>N} |G_m|^2 \circ T^j \right].$$

By the Hopf ergodic theorem and the specification in Section (7) in Dedecker *et al.* [12] we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left[\sup_{m>N} |G_m|^2 \circ T^j \right] = \mathbb{E} \left[\sup_{m>N} |G_m|^2 | \mathcal{I} \right]$$

$$\mathbb{P}\text{-a.s. and in } L_1,$$

where $\mathcal{I}$ is the invariant sigma field. Since by (39) $\sup_{m>N} |G_m|^2 \to 0$ a.s. as $N \to \infty$ and $\sup_{m>N} |G_m|^2 \le \sup_m |G_m|^2 \in \mathbb{L}_1$, by Billingsley [2], Theorem 34.2(v) we also have

$$\lim_{N\to\infty} \mathbb{E} \left[\sup_{m>N} |G_m|^2 | \mathcal{I} \right] = 0, \qquad \mathbb{P}\text{-a.s. and in } L_1,$$

and therefore

$$\lim_{N\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left[\sup_{m>N} |G_m|^2 \circ T^j \right] = 0, \qquad \mathbb{P}\text{-a.s. and in } \mathbb{L}_1.$$

It follows that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{B_n(N)}{n} = 0, \qquad \mathbb{P}\text{-a.s. and in } \mathbb{L}_1. \tag{42}$$

$\square$

We give below a well-known Raikov type central limit theorem for nonstationary martingales.

The following theorem is a variant of Theorem 3.2 in Hall and Heyde [21] (see also Gänssler and Häusler [18]).

Theorem 16. *Assume* $(D_{n,i})_{1\le i\le n}$ *is an array of square integrable martingale differences adapted to an array* $(\mathcal{F}_{n,i})_{1\le i\le n}$ *of nested sigma fields. Suppose*

$$\mathbb{E} \left(\max_{1\le j\le n} |D_{n,j}| \right) \to 0 \qquad \text{as } n \to \infty \tag{43}$$

and

$$\sum_{j=1}^{n} D_{n,j}^2 \to^{\mathbb{P}} \sigma^2 \qquad \text{as } n \to \infty. \tag{44}$$

Then $S_n = \sum_{j=1}^{n} D_{n,j}$ *converges in distribution to a centered normal variable with variance* σ^2.

We give now a result on weak convergence needed for the proof of Proposition 13.

Lemma 17. *Assume that the sequence of random variables $(Y_n, Z_n)_{n \geq 1}$ is tight and for every rational numbers a and b we have*

$$aY_n + bZ_n \Rightarrow aN_1 + bN_2.$$

Then $(Y_n, Z_n) \Rightarrow (N_1, N_2)$.

Proof. Because (Y_n, Z_n) is tight, from any subsequence (n') we can extract another subsequence (n'') convergent in distribution to (L_1, L_2) say. By the Cramér–Wold device, it follows that for all real numbers a and b we have

$$aY_n + bZ_n \Rightarrow aL_1 + bL_2.$$

Therefore for all rational numbers a and b we have

$$\mathbb{E}e^{i(aN_1+bN_2)} = \mathbb{E}e^{i(aL_1+bL_2)}. \tag{45}$$

Now, for any reals (c, d) we take sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ of rational numbers such that $a_n \to c$ and $b_n \to d$. By the Lebesgue dominated convergence theorem, we pass to the limit in (45) (written for a_n and b_n), and obtain that the equality in (45) holds for all real numbers. Since the Fourier transform determines the measure we obtain (L_1, L_2) is distributed as (N_1, N_2). □

The next lemma is a step in the proof of Theorem 2.

Lemma 18. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\mathcal{F}$ countably generated, $\mathcal{G} \subset \mathcal{F}$ a sigma algebra, Y a $\mathcal{G}$-measurable integrable random variable, X integrable and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous and bounded function. Let $\mathbb{P}^\omega$ be a regular version of $\mathbb{P}$ given $\mathcal{F}$. Then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that, for all $\omega \in \Omega_1$*

$$\mathbb{E}^\omega\big[g\big(X, Y(\omega)\big)\big] = \mathbb{E}^\omega\big[g(X, Y)\big]. \tag{46}$$

Proof. It is easy to see that for a simple function V we can find $\Omega_V \subset \Omega$ with $\mathbb{P}(\Omega_V) = 1$ such that, for all $\omega \in \Omega_V$

$$\mathbb{E}^\omega\big[g\big(X, V(\omega)\big)\big] = \mathbb{E}^\omega\big[g(X, V)\big].$$

Indeed, if $V = \sum_{j=1}^m a_j I(B_j)$ with $B_j \in \mathcal{G}$ we have for every $B \in \mathcal{G}$

$$\mathbb{E}\big(I(B)\big(g(X, V)|\mathcal{G}\big)\big) = \sum_{j=1}^m \mathbb{E}\big(I(B \cap B_j)g(X, V)\big) = \sum_{j=1}^m \mathbb{E}\big(I(B \cap B_j)g(X, a_j)\big)$$

$$= \mathbb{E}\left(I(B) \sum_{j=1}^m I(B_j)\mathbb{E}\big(g(X, a_j)|\mathcal{G}\big)\right).$$

Let $V_n \to Y$ a sequence of simple functions. Then we can find a set $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$, namely $\Omega_1 = \bigcap_n \Omega_{V_n}$, such that for all $\omega \in \Omega_1$

$$\mathbb{E}^\omega\big[g\big(X, V_n(\omega)\big)\big] = \mathbb{E}^\omega\big[g(X, V_n)\big].$$

Now, for ω fixed in Ω_1, by Lesbegue dominated convergence theorem we get (46) by passing to the limit. □

Acknowledgements

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