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Poisson convergence on the free Poisson algebra

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Based on recent findings by Bourguin and Peccati, we give a *fourth moment* type condition for an element of a free Poisson chaos of arbitrary order to converge to a free (centered) Poisson distribution. We also show that free Poisson chaos of order strictly greater than one do not contain any non-zero free Poisson random variables. We are also able to give a sufficient and necessary condition for an element of the first free Poisson chaos to have a free Poisson distribution. Finally, depending on the parity of the considered free Poisson chaos, we provide a general counterexample to the naive universality of the semicircular Wigner chaos established by Deya and Nourdin as well as a transfer principle between the Wigner and the free Poisson chaos.

Keywords: chaos structure; combinatorics of free Poisson random measures; contractions; diagram formulae; fourth moment theorem; free Poisson distribution; free probability; multiplication formula

1. Introduction and background

1.1. Overview

Let $\{W(t): t \geq 0\}$ be a standard Brownian motion on $\mathbb{R}_+$ and let $q \geq 1$ be an integer. Denote by $I_q^W(f)$ the multiple stochastic Wiener–Itô integral of order q of a deterministic symmetric function $f \in L^2(\mathbb{R}_+^q)$. Denote by $L_s^2(\mathbb{R}_+^q)$ the subset of $L^2(\mathbb{R}_+^q)$ composed of symmetric functions. The collection of random variables $\{I_q^W(f): f \in L_s^2(\mathbb{R}_+^q)\}$ is what is usually called the qth Wiener chaos associated with W. In a seminal paper of 2005, Nualart and Peccati [10] proved that convergence to the standard normal distribution of an element with variance one living inside a fixed Wiener chaos was equivalent to the convergence of the fourth moment of this element to three. This result is now known as the *fourth moment theorem* and can be stated as follows.

Theorem 1.1 (Nualart and Peccati [10]). *Fix an integer $q \geq 2$ and let $\{f_n : n \geq 1\}$ be a sequence of functions in $L_s^2(\mathbb{R}_+^q)$ such that, for each $n \geq 1$, $\mathbb{E}(I_q^W(f_n)) = q!\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = 1$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) $I_q^W(f_n) \xrightarrow{\text{law}} \mathcal{N}(0,1)$;
(ii) $\mathbb{E}(I_q^W(f_n)^4) \to \mathbb{E}(\mathcal{N}(0,1)^4) = 3$.

This result has led to a wide collection of new results and inspired several new research directions – see the book [8], as well as the constantly updated webpage

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https://sites.google.com/site/malliavinstein/home.
In [6], the authors obtained a similar criterion for non-central convergence to the Gamma distribution on a Wiener chaos. In [7], the Malliavin calculus of variations was combined with Stein's method to obtain quantitative versions of these fourth moment theorems. In the framework of non-commutative probability, Kemp *et al.* [4] obtained an analog of Theorem 1.1 for multiple integrals with respect to a free Brownian motion.

Remark 1.2. Observe that Theorem 1.1 is stated for symmetric functions in $L^2(\mathbb{R}_+^q)$. In free probability theory, the symmetry assumption can be weakened to *mirror symmetry* for defining and working with free multiple integrals (see Section 2.2 and definitions therein). We say that an element f of $L^2(\mathbb{R}_+^q)$ (the collection of all complex-valued functions on $\mathbb{R}_+^q$ that are square-integrable with respect to the Lebesgue measure) is *mirror symmetric* if $f(t_1,\dots,t_q) = \overline{f(t_q,\dots,t_2,t_1)}$, for almost every vector $(t_1,\dots,t_q) \in \mathbb{R}_+^q$.

The aim of this paper is to investigate the convergence of sequences of multiple integrals with respect to a free Poisson measure. More precisely, denote by $(\mathcal{A},\varphi)$ a free probability space and let $\{\hat{N}(B) : B \in \mathscr{B}(\mathbb{R}_+)\}$, where $\mathscr{B}(\mathbb{R}_+)$ denotes the Borels sets of $\mathbb{R}_+$, be a centered free Poisson measure on this space. For an integer $q \geq 1$, the free Poisson multiple integral of order q of a mirror-symmetric bounded function with bounded support $f \in L^2(\mathbb{R}_+^q)$ is denoted $I_q^{\hat{N}}(f)$. Random variables of this type compose the so-called free Poisson chaos of order q associated with $\hat{N}$. The above mentioned objects are defined and constructed in Section 2; refer to that section for more details. A first result in this direction has recently been obtained by Bourguin and Peccati in [2], who proved that a fourth moment type theorem (for semicircular limits) holds on the free Poisson algebra.

Theorem 1.3 (Bourguin and Peccati [2]). *Fix an integer $q \geq 1$ and let $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$ be a tamed sequence of mirror symmetric kernels such that $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} \to 1$, as $n \to \infty$. Then, $I_q^{\hat{N}}(f_n)$ converges in law to the semicircular distribution $\mathcal{S}(0,1)$ if and only if $\varphi(I_q^{\hat{N}}(f_n)^4) \to 2$.*

The notion of *tamed sequence* of kernels is introduced in Section 2 below: this additional assumption has been introduced in [2] in order to deal with the complicated combinatorial structures arising from the computation of moments (more precisely, it is a sufficient condition in order to preserve spectral bounds when converging on the free Poisson algebra). Bourguin and Peccati also proved that a *transfer principle* (as the one established in [4], Theorem 1.8) between the classical and the free Poisson chaos cannot hold in full generality by providing an example where, for the same sequence of kernels, the free Poisson multiple integral converges towards a semicircular distribution when the corresponding classical Poisson multiple integral converges towards a classical Poisson limit, instead of a Gaussian distribution as would be expected in this type of transfer results.

A natural question that arises in this context is to find moment conditions (and potentially a unique condition in the form of a linear combination of moments as in other fourth moment type results) to ensure the convergence of free Poisson multiple integrals to a free Poisson distribution. One could be tempted to consider such a question as an analog, in the free case, of the results obtained in [11] or [2] for the classical case (Poisson approximations on the classical Poisson space), but the classical analog of the free Poisson distribution is actually the Gamma distribution as pointed out in [9], Section 1.1 (see also [5], page 203). A recent partial result in the classical framework addressing the convergence of Poisson multiple integrals to the Gamma distribution has been obtained in [13], Theorem 2.6.

In the free case, the closest result in this direction is the fourth moment type characterization obtained by Nourdin and Peccati in [9], Theorem 1.4, for the convergence of multiple Wigner integrals to a free Poisson distribution, this result being itself an analogue of the main result in [6], proved by the same authors. This first free Poisson approximation theorem reads as follows.

Theorem 1.4 (Nourdin and Peccati [9]). *Let $f \in L^2(\mathbb{R}_+^q)$. Let $I_q^S(f)$ denote the multiple Wigner integral of order q of f. Let $Z(\lambda)$ have a free centered Poisson distribution with rate $\lambda > 0$, fix an even integer $q \geq 2$ and let $\{f_n : n \geq 1\}$ be a sequence of mirror-symmetric functions in $L^2(\mathbb{R}_+^q)$ such that, for each $n \geq 1$, $\varphi(I_q^S(f_n)^2) = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) $I_q^S(f_n) \overset{\text{law}}{\to} Z(\lambda)$;
(ii) $\varphi(I_q^S(f_n)^4) - 2\varphi(I_q^S(f_n)^3) \to \varphi(Z(\lambda)^4) - 2\varphi(Z(\lambda)^3) = 2\lambda^2 - \lambda$.

The following theorem, and main result of this paper, is a fourth moment type characterization of the convergence to free Poisson limits on the free Poisson algebra, and could be seen as a free counterpart of [13], Theorem 2.6.

Theorem 1.5. *Let $Z(\lambda)$ have a free centered Poisson distribution with rate $\lambda > 0$. Fix an integer $q \geq 1$ and let $\{f_n : n \geq 1\}$ be a tamed sequence of mirror-symmetric kernels in $L^2(\mathbb{R}_+^q)$ such that, for each $n \geq 1$, $\varphi(I_q^{\hat{N}}(f_n)^2) = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) $I_q^{\hat{N}}(f_n) \overset{\text{law}}{\to} Z(\lambda)$;
(ii) $\varphi(I_q^{\hat{N}}(f_n)^4) - 2\varphi(I_q^{\hat{N}}(f_n)^3) \to \varphi(Z(\lambda)^4) - 2\varphi(Z(\lambda)^3) = 2\lambda^2 - \lambda$.

In [9], Nourdin and Peccati mention that one cannot expect to have convergence of a Wigner multiple integral of odd order to the free Poisson distribution since these integrals have all odd moments equal to zero, as opposed to the free Poisson distribution. It is worth pointing out that, unlike in Theorem 1.4, even as well as odd orders can be considered in Theorem 1.5. The reason for this difference will become clear in the sequel. Additionally, Theorem 1.5 is given in a way more general setting than the classical case result proved in [13], Theorem 2.6. Indeed, [13], Theorem 2.6, was only proved for multiple integrals of even orders, although convergence of multiple integrals of odd orders is not excluded by the authors who claim that the odd case is more intricate to analyze. Also, the fourth moment characterization in [13], Theorem 2.6, was only obtained for sequences of multiple integrals of order two. The main result in this paper, namely Theorem 1.5, is given for multiple integrals of any order and the validity of the fourth moment characterization is not restricted to the order two.

As a consequence of Theorem 1.5, we prove a corollary giving insights on the (almost completely unknown) structure of the free Poisson algebra.

Corollary 1.6. *The two following statements hold*:

(1) *Let $q \geq 2$ be an integer, and let $F \neq 0$ be in the qth free Poisson chaos. Then, F cannot have a free Poisson distribution.*
(2) *Let $f \in L^2(\mathbb{R}_+)$ be such that $\|f\|^2_{L^2(\mathbb{R}_+)} = \lambda > 0$. Then, $I_1^{\hat{N}}(f)$ has a free Poisson distribution with parameter λ if and only if f takes values in* $\{0, 1\}$.

Remark 1.7. Point (ii) in Corollary 1.6 shows that the first free Poisson chaos does not contain only free Poisson distributions, as opposed to the first Wigner chaos that only consists of semicircular distributed elements. The following example illustrates the situation where an element of the first free Poisson chaos doesn't have a free Poisson distribution. Let A_1 and A_2 be two orthogonal (with respect to μ) Borel sets of $\mathbb{R}_+$ such that $\mu(A_1) = \mu(A_2) = 1$. Let $Z(8)$ be a free Poisson random variable with parameter $\lambda = 8$. Finally, let f be the function of $L^2(\mathbb{R}_+)$ defined by $f = 2(\mathbb{1}_{A_1} + \mathbb{1}_{A_2})$. It is easily checked that $\varphi(I_1^{\hat{N}}(f)^2) = 8 = \lambda$. According to the upcoming Proposition 4.2, it holds that

$$\varphi\big(I_1^{\hat{N}}(f)^3\big) = \big(f \star_1^0 f\big) \overset{1}{\frown} f = \int_{\mathbb{R}_+} f^3 \, \mathrm{d}\mu = 16.$$

As $\varphi(Z(8)^3) = 8$, $I_1^{\hat{N}}(f)$ cannot have a free Poisson distribution.

Finally, we provide a general counterexample to the naive universality of the free Wigner chaos (as stated in [3]), where it is proven that for multilinear homogeneous sums of free random variables, a universality phenomenon happens in the sense that it is sufficient that a multiple Wigner integral with an appropriate "homogeneous" kernel converges to the semicircular distribution for multiple integrals with respect to any free random measure (with the same kernel) to converge as well to the semicircular distribution. The following theorem could also be seen as a transfer principle between the Wigner and free Poisson chaos for multiple integrals of even orders. This theorem should be compared with the counterexample to the naive universality of the Wiener chaos given in [2], Proposition 5.4.

Theorem 1.8. *Let λ be a positive real number. Denote by $\hat{P}(\lambda)$ the free centered Poisson distribution with rate λ and by $\mathcal{S}(0, \lambda)$ the centered semicircular distribution with variance λ. Fix an integer $q \geq 1$ and let $\{f_n : n \geq 1\}$ be a tamed sequence of mirror-symmetric functions in $L^2(\mathbb{R}_+^q)$ such that, for each $n \geq 1$, $\varphi(I_q^{\hat{N}}(f_n)^2) = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda$ and $\varphi(I_q^{\hat{N}}(f_n)^4) - 2\varphi(I_q^{\hat{N}}(f_n)^3) \underset{n\to+\infty}{\longrightarrow} 2\lambda^2 - \lambda$. Then, it holds that*

(i) *$I_q^{\hat{N}}(f_n) \overset{\text{law}}{\to} Z(\lambda)$ and $I_q^S(f_n) \overset{\text{law}}{\to} Z(\lambda)$ if q is even*;
(ii) *$I_q^{\hat{N}}(f_n) \overset{\text{law}}{\to} Z(\lambda)$ and $I_q^S(f_n) \overset{\text{law}}{\to} \mathcal{S}(0, \lambda)$ if q is odd.*

Point (i) in the above theorem implies that it suffices that the free Poisson multiple integral (of even order) of a sequence of functions converges towards a free Poisson limit for the Wigner integral of the same sequence to converge to the same limit. Point (ii) provides a counterexample to the naive universality of the Wigner chaos stated in [3] since we are in a situation where the semicircular multiple integral converges to a semicircular limit but not the free Poisson multiple integral.

Remark 1.9. In order to avoid unnecessary heavy notations, Theorem 1.5, Corollary 1.6 and Theorem 1.8 are stated (and proved) for kernels in $L^2(\mathbb{R}_+^q)$, for some integer $q \geq 1$. These results also hold for kernels in $L^2((\mathbb{R}^d)^q)$, for any integer $d \geq 1$, without changing anything to the proofs.

1.2. Plan

Section 2 provides some preliminaries on non-crossing partitions as well as some basic definitions about the free Poisson algebra, where our main objects of interest live. The proofs of Theorem 1.5, Corollary 1.6 and Theorem 1.8 can be found in Section 3. Finally, Section 4 contains some auxiliary lemmas along with their proofs that are used in the proofs of our main results in Section 3.

2. Preliminaries

The framework used here and the associated notation are the same as in [2]; see that paper for all the definitions that are not explicitly provided here.

2.1. Non-crossing partitions

Given an integer $m \geq 1$, we write $[m] = \{1, \dots, m\}$. A *partition* of $[m]$ is a collection of non-empty and disjoint subsets of $[m]$, called *blocks*, such that their union is equal to $[m]$. The set of all partitions of $[m]$ is denoted by $\mathcal{P}(m)$. The cardinality of a block is called *size*. We adopt the convention of ordering the blocks of a given partition $\pi = \{B_1, \dots, B_r\}$ by their least element, that is: $\min B_i < \min B_j$ if and only if $i < j$. A partition π of $[n]$ is said to be *non-crossing* if one cannot find integers p_1, q_1, p_2, q_2 such that: (a) $1 \leq p_1 < q_1 < p_2 < q_2 \leq m$, (b) p_1, p_2 are in the same block of π, (c) q_1, q_2 are in the same block of π, and (d) the p_i's are not in the same block of π as the q_i's. The collection of the non-crossing partitions of $[n]$ is denoted by $\mathrm{NC}(n)$, $n \geq 1$. It is a well-known fact (see, e.g., [5], page 144) that the reversed refinement order (written $\preceq$) induces a lattice structure on $\mathrm{NC}(n)$: we shall denote by $\vee$ and $\wedge$, respectively, the associated *join* and *meet* operations, where $\hat{0} = \{\{1\}, \dots, \{n\}\}$ and $\hat{1} = \{[n]\}$ are the corresponding minimal and maximal partitions of the lattice. Let m and q be two integers such that $m, q \geq 1$. We define the partition $\pi^* = \{B_1, \dots, B_m\} \in \mathrm{NC}(mq)$, where $B_1 = \{1, \dots, q\}$, $B_2 = \{q+1, \dots, 2q\}$ and so on until $B_m = \{(m-1)q, \dots, mq\}$. Such a partition π^* is sometimes called a *block partition*. For any integers $m, q \geq 1$, we define the four following subsets of partitions of $[mq]$:

$$\begin{aligned}
\mathrm{NC}^0\big([mq], \pi^*\big) &= \big\{\sigma \in \mathrm{NC}(mq) : \sigma \wedge \pi^* = \hat{0}\big\};\\
\mathrm{NC}_2^0\big([mq], \pi^*\big) &= \big\{\sigma \in \mathrm{NC}^0\big([mq], \pi^*\big) : |b| = 2, \forall b \in \sigma\big\};\\
\mathrm{NC}_{>2}^0\big([mq], \pi^*\big) &= \big\{\sigma \in \mathrm{NC}^0\big([mq], \pi^*\big) : |b| > 2, \forall b \in \sigma\big\};\\
\mathrm{NC}_{\geq 2}^0\big([mq], \pi^*\big) &= \big\{\sigma \in \mathrm{NC}^0\big([mq], \pi^*\big) : |b| \geq 2, \forall b \in \sigma\big\}.
\end{aligned}$$

Observe that, by definition, it holds that for any $m, q \geq 1$, $\mathrm{NC}_{\geq 2}^0 = \mathrm{NC}_2^0 \cup \mathrm{NC}_{>2}^0$ and $\mathrm{NC}_2^0 \cap \mathrm{NC}_{>2}^0 = \varnothing$.

Let $q, m \geq 1$ be integers, and consider a function f in q variables. Given a partition σ of $[mq]$, we define the function f_σ, in $|\sigma|$ variables, as the mapping obtained by identifying the variables x_i and x_j in the argument of the tensor

$$f \otimes \cdots \otimes f(x_1, \dots, x_{mq}) = \prod_{j=1}^{m} f(x_{(j-1)q+1}, \dots, x_{jq}) \tag{1}$$

if and only if i and j are in the same block of σ.

Definition 2.1. *Let $q \geq 1$ be an integer. We say that the sequence $\{g_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$ is tamed if the following conditions hold: every g_n is bounded and has bounded support and, for every $m \geq 2$ and every $\sigma \in \mathcal{P}(mq)$ such that $\sigma \wedge \pi^* = \hat{0}$, the numerical sequence*

$$\int_{\mathbb{R}_+^{|\sigma|}} |g_n|_\sigma \, \mathrm{d}\mu^{|\sigma|}, \qquad n \geq 1, \tag{2}$$

is bounded, where $\pi^* \in \mathcal{P}(mq)$ is the block partition with m consecutive blocks of size q, and the function $|g_n|_\sigma$, in $|\sigma|$ variables, is defined according to (1) in the case $f_n = |g_n|$.

There exists sufficient conditions in order for a sequence $\{f_n\}$ to be tamed. It basically consists in requiring that $\{f_n\}$ concentrates asymptotically, without exploding, around a hyperdiagonal: fix $q \geq 2$, and consider a sequence $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$. Assume that there exist strictly positive numerical sequences $\{M_n, z_n, \alpha_n : n \geq 1\}$ such that $\alpha_n / z_n \to 0$ and the following properties are satisfied: (a) the support of f_n is contained in the set $(-z_n, z_n) \times \cdots \times (-z_n, z_n)$ (Cartesian product of order q); (b) $|f_n| \leq M_n$; (c) $f_n(x_1, \dots, x_q) = 0$, whenever there exist x_i, x_j such that $|x_i - x_j| > \alpha_n$; (d) for every integer $m \geq q$, the mapping $n \mapsto M_n^m z_n \alpha_n^{m-1}$ is bounded. Then, $\{f_n : n \geq 1\}$ is tamed.

2.2. The free Poisson algebra

Let $(\mathcal{A}, \varphi)$ be a free tracial probability space and let $\mathcal{A}_+$ denote the cone of positive operators in $\mathcal{A}$. Denote by μ the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+)$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel sets of

$\mathbb{R}_+$ and write $\mathscr{B}_\mu(\mathbb{R}_+) = \{B \in \mathscr{B}(\mathbb{R}_+) : \mu(B) < \infty\}$. The following is a brief description the free Poisson algebra, as constructed and studied in [2].

For every integer $q \geq 2$, the space $L^2(\mathbb{R}_+^q)$ is the collection of all complex-valued functions on $\mathbb{R}_+^q$ that are square-integrable with respect to μ^q. Given $f \in L^2(\mathbb{R}_+^q)$, we write $f^*(t_1, t_2, \dots, t_q) = \overline{f(t_q, \dots, t_2, t_1)}$, and we call f^* the *adjoint* of f. As pointed out in Remark 1.2, a mirror-symmetric element of $L^2(\mathbb{R}_+^q)$ is a function f that satisfies $f(t_1, \dots, t_q) = f^*(t_1, \dots, t_q)$, for almost every vector $(t_1, \dots, t_q) \in \mathbb{R}_+^q$. Observe that mirror symmetric functions constitute a Hilbert subspace of $L^2(\mathbb{R}_+^q)$. Let $f \in L^2(\mathbb{R}_+^m)$ and $g \in L^2(\mathbb{R}_+^n)$. We define the *arc* and *star* contractions of f and g: for $1 \leq k \leq m \wedge n$, we set

$$\begin{aligned} & f \overset{k}{\frown} g(t_1, \dots, t_{m+n-2k}) \\ & \quad = f \star_k^k g(t_1, \dots, t_{m+n-2k}) \\ & \qquad \times \int_{\mathbb{R}_+^k} f(t_1, \dots, t_{m-k}, s_k, \dots, s_1) g(s_1, \dots, s_k, t_{m-k+1}, \dots, t_{m+n-2k}) \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_k), \end{aligned}$$

and we set moreover $f \overset{0}{\frown} g = f \star_0^0 g = f \otimes g$. For $k = 1, \dots, m \wedge n$, the star contraction of index k (of f and g) is defined by

$$\begin{aligned} & f \star_k^{k-1} g(t_1, \dots, t_{m+n-2k+1}) \\ & \quad = \int_{\mathbb{R}_+^{k-1}} f(t_1, \dots, t_{m-k+1}, s_{k-1}, \dots, s_1) \\ & \qquad \times g(s_1, \dots, s_{k-1}, t_{m-k+1}, \dots, t_{m+n-2k+1}) \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_{k-1}). \end{aligned}$$

Let $\hat{N}$ and S be a *free centered Poisson random measure* and a *semicircular random measure*, respectively. For $f \in L^2(\mathbb{R}_+^q)$, we denote by $I_q^{\hat{N}}(f)$ (respectively, $I_q^S(f)$) the multiple integral of f with respect to $\hat{N}$ (respectively, S). The space $L^2(\mathscr{X}(\hat{N}), \varphi) = \{I_q^{\hat{N}}(f) : f \in L^2(\mathbb{R}_+^q), q \geq 0\}$ is a unital $*$-algebra, with product rule given, for any $m, n \geq 1$, $f \in L^2(\mathbb{R}_+^m)$, $g \in L^2(\mathbb{R}_+^n)$, by

$$I_m^{\hat{N}}(f) I_n^{\hat{N}}(g) = \sum_{k=0}^{m \wedge n} I_{m+n-2k}^{\hat{N}}(f \overset{k}{\frown} g) + \sum_{k=1}^{m \wedge n} I_{m+n-2k+1}^{\hat{N}}\big(f \star_k^{k-1} g\big) \tag{3}$$

and involution $I_q^{\hat{N}}(f)^* = I_q^{\hat{N}}(f^*)$. Denote by $L_b(\mathscr{X}(\hat{N}), \varphi)$ the collection of all objects of the type $I_q^{\hat{N}}(f)$, where f is a bounded function with bounded support. $L_b(\mathscr{X}(\hat{N}), \varphi)$ is a subalgebra of $L^2(\mathscr{X}(\hat{N}), \varphi)$.

The space $L^2(\mathscr{X}(S), \varphi) = \{I_q^S(f) : f \in L^2(\mathbb{R}_+^q), q \geq 0\}$ is a unital $*$-algebra, with product rule given, for any $m, n \geq 1$, $f \in L^2(\mathbb{R}_+^m)$, $g \in L^2(\mathbb{R}_+^n)$, by

$$I_m^S(f) I_n^S(g) = \sum_{k=0}^{m \wedge n} I_{m+n-2k}^S(f \overset{k}{\frown} g) \tag{4}$$

and involution $I_q^{\hat{N}}(f)^* = I_q^{\hat{N}}(f^*)$.

Observe that it follows from the definition of the involution on the algebras $L^2(\mathcal{X}(\hat{N}),\varphi)$ and $L^2(\mathcal{X}(S),\varphi)$ that operators of the type $I_q^{\hat{N}}(f)$ or $I_q^S(f)$ are self-adjoint if and only if f is mirror symmetric.

The following diagram formulas were proved in [2], Theorem 3.15, and provide an explicit combinatorial way of computing moments of multiple integrals with respect to either a free Poisson or a free semicircular measure: for any $f \in L^2(\mathbb{R}_+^q)$ and any integer $q \geq 1$ and $m \geq 2$, it holds that

$$\varphi\big(I_q^{\hat{N}}(f)^m\big) = \sum_{\sigma \in \mathrm{NC}^0_{\geq 2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\sigma|}} f_\sigma \,\mathrm{d}\mu^{|\sigma|}, \tag{5}$$

$$\varphi\big(I_q^{S}(f)^m\big) = \sum_{\sigma \in \mathrm{NC}^0_{2}([mq],\pi^*)} \int_{\mathbb{R}_+^{mq/2}} f_\sigma \,\mathrm{d}\mu^{mq/2}. \tag{6}$$

Remark 2.2. These diagram formulae are free analogues of the classical diagram formulae for multiple integrals with respect to a Gaussian or Poisson random measure (see, e.g., [12], Theorem 7.1.3). The main difference between the free and classical diagram formulae is the kind of partitions of $[mq]$ on which to compute the sums appearing in the right-hand sides of the formulae. In the free case, the sets of partitions on which to sum are subset of those of the classical case where only non-crossing partitions are considered.

3. Proof of the main results

Let $i < m$ be two non-negative integers. For $0 \leq i \leq m-1$, define the multisets $M_i^m = \{1,\dots,1,0,\dots,0\}$ where the element 1 has multiplicity i and the element 0 has multiplicity $m-i-1$. Such a set is sometimes denoted $\{(1,i),(0,m-i-1)\}$. We denote the group of permutations of the multiset M_i^m by $\mathfrak{S}_i^m$ and its cardinality is given by the multinomial coefficient $\binom{m-1}{i,m-i-1} = \frac{(m-1)!}{i!(m-i-1)!} = \binom{m-1}{i}$. Observe that in the definition of the group of permutations of a multiset, each permutation yields a different ordering of the elements of the multiset, which is why the cardinality of $\mathfrak{S}_i^m$ is $\binom{m-1}{i}$ and not $(m-1)!$.

For any $m \geq 1$, $\mathfrak{S}_0^m$ and $\mathfrak{S}_m^m$ each only have exactly one element that we denote by σ_0 and σ_1 respectively, (σ_0 and σ_1 are in fact the identity maps on the sets M_0^m and M_m^m respectively). Furthermore, for a given $\sigma \in \mathfrak{S}_i^m$, $0 \leq i \leq m$, we define the sets

$$\mathfrak{A}_m^\sigma = \Bigg\{(r_1,\dots,r_{m-1}) \in (0,1,\dots,q)^{m-1} : \forall 1 \leq p \leq m-1,$$
$$\sigma(p) \leq r_p \leq pq + \sum_{k=1}^{p-1}\big(\sigma(k) - 2r_k\big)\Bigg\},$$

$$\mathfrak{B}_m^\sigma = \Bigg\{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma : 2r_1 + \cdots + 2r_{m-1} = mq + \sum_{p=1}^{m-1}\sigma(p)\Bigg\},$$

$$\mathfrak{D}_m^\sigma = \left\{ (r_1, \dots, r_{m-1}) \in \mathfrak{B}_m^\sigma \cap \left\{0, \frac{q+1}{2}, q\right\} : \forall 1 \le j \le m-1, \right.$$
$$\left. r_j \in \{0, q\} \Leftrightarrow \sigma(j) = 0 \text{ and } r_j = \frac{q+1}{2} \Leftrightarrow \sigma(j) = 1 \right\},$$
$$\mathfrak{E}_m^\sigma = \mathfrak{B}_m^\sigma \setminus \mathfrak{D}_m^\sigma.$$

In the upcoming proofs, we will drop the superscript $\hat{N}$ on free Poisson multiple integral whenever there is only this one kind of multiple integrals involved. Whenever a proof deals with different sorts of multiple integrals, we will resume using the appropriate superscripts to avoid confusion. Finally, in order to avoid more than necessarily heavy notations, we will write

$$\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f = \bigl(\cdots\bigl(\bigl(f \star_{r_1}^{r_1-\sigma(1)} f\bigr) \star_{r_2}^{r_2-\sigma(2)} f\bigr)\cdots f\bigr) \star_{r_{m-1}}^{r_{m-1}-\sigma(m-1)} f,$$

where the $\sigma(p)$ are integers equal to either 0 or 1. Using the notation introduced in Section 2.2, we write

$$\mathop{\frown}_{p=1}^{m-1}{}^{r_p} f := \mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p} f = \bigl(\cdots\bigl((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f\bigr)\cdots f\bigr) \overset{r_{m-1}}{\frown}$$

whenever all the $\sigma(p)$, $1 \le p \le m-1$, are zero.

Remark 3.1. The reason why the sequences of functions appearing in the main results of the paper are required to be tamed and mirror symmetric does not appear explicitly in the upcoming proofs. This condition ensures that the spectral radius of free Poisson multiple integrals is bounded (see [2], Theorem 3.15, for details) and that the diagram formula (5) holds, hence guarantying the validity of the convergence in distribution results.

3.1. Proof of Theorem 1.5

Proving the implication (i) $\Rightarrow$ (ii) is trivial as in the free probability setting, convergence in distribution is equivalent to the convergence of moments. Hence, $\varphi(I_q(f_n)^4) - 2\varphi(I_q(f_n)^3) \underset{n\to+\infty}{\longrightarrow} \varphi(Z(\lambda)^4) - 2\varphi(Z(\lambda)^3) = 2\lambda^2 - \lambda$. The rest of the proof will be devoted to proving the implication (ii) $\Rightarrow$ (i). As convergence in distribution is equivalent to the convergence of moments, we will prove that for any integer $m \ge 2$, we have $\varphi(I_q(f_n)^m) \to \varphi(Z(\lambda)^m)$. The proof will consist in two steps, depending on whether q is even or odd. We start with the case where q is even.

Step 1: *q is even*. Using Lemma 4.2, one can write

$$\begin{aligned} \varphi\bigl(I_q(f_n)^m\bigr) = {} & \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma_0}} \bigl(\cdots\bigl((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n\bigr)\cdots f\bigr) \overset{q}{\frown} f_n \\ & + \sum_{i=1}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}_{2i}^{m-1}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma}} \Bigl(\mathop{\bigstar}_{p=1}^{m-2}{}_{r_p}^{r_p-\sigma(p)} f_n\Bigr) \overset{q}{\frown} f_n. \end{aligned} \tag{7}$$

The first sum has already been addressed in [9], Proof of Theorem 1.4. Under condition (ii) and by Lemma 4.3 (note that it is here that the special behaviour of the contraction of order $q/2$ comes into play. See [9], Proof of Theorem 1.4, for details), it holds that

$$\sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma}} \Big(\cdots\big((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n\big)\cdots f_n\Big) \overset{q}{\frown} f_n \underset{n\to+\infty}{\longrightarrow} \varphi\big(Z(\lambda)^m\big).$$

Therefore, it remains to prove that

$$\sum_{i=1}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}_{2i}^{m-1}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma}} \Big(\bigstar_{p=1}^{m-2}{}_{r_p}^{r_p-\sigma(p)} f_n\Big) \overset{q}{\frown} f_n \to 0. \tag{8}$$

Recalling that $\mathrm{NC}_{\geq 2}^{0}$ is the disjoint union of NC_2^0 and $\mathrm{NC}_{>2}^0$ and using the diagram formula (5), we can write

$$\varphi\big(I_q(f_n)^m\big) = \sum_{\tau\in\mathrm{NC}_2^0([mq],\pi^*)} \int_{\mathbb{R}_+^{mq/2}} (f_n)_\tau \,\mathrm{d}\mu^{mq/2} + \sum_{\tau\in\mathrm{NC}_{>2}^0([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|}. \tag{9}$$

Observe that, on one hand, the diagram formula for semicircular multiple integrals (6) states that the mth moment of a semicircular multiple integral is equal to

$$\sum_{\tau\in\mathrm{NC}_2^0([mq],\pi^*)} \int_{\mathbb{R}_+^{mq/2}} (f_n)_\tau \,\mathrm{d}\mu^{mq/2}$$

and that on the other hand, [9], Proof of Theorem 1.4, provides the following expression for it:

$$\sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma_0}} \Big(\cdots\big((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n\big)\cdots f\Big) \overset{q}{\frown} f_n.$$

Using (7) and (9), we get the following identification:

$$\sum_{i=1}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}_{2i}^{m-1}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{B}_{m-1}^{\sigma}} \Big(\bigstar_{p=1}^{m-2}{}_{r_p}^{r_p-\sigma(p)} f_n\Big) \overset{q}{\frown} f_n = \sum_{\tau\in\mathrm{NC}_{>2}^0([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|}.$$

Using the same argument as in [2], Proof of Theorem 4.3, it holds that the condition $\| f_n \star_r^{r-1} f_n\|_{L^2(\mathbb{R}_+^{2q-2r+1})} \underset{n\to+\infty}{\longrightarrow} 0$ for all $r\in\{1,\dots,q\}$, implied by (ii) as stated in Lemma 4.3, along with the fact that the sequence $\{f_n : n\geq 1\}$ is tamed, is a sufficient condition in order to have

$$\sum_{\tau\in\mathrm{NC}_{>2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|} \underset{n\to+\infty}{\longrightarrow} 0,$$

and hence (8), which concludes this step of the proof.

Step 2: *q is odd*. Recall that $\mathfrak{B}_m^\sigma$ is the disjoint union of $\mathfrak{D}_m^\sigma$ and $\mathfrak{E}_m^\sigma$. Using Lemma 4.2, we now have

$$\varphi\big(I_q(f_n)^m\big) = \sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{D}^\sigma_{m-1}} \Big(\bigstar_{p=1}^{m-2}{}^{r_p-\sigma(p)}_{r_p} f_n\Big) \overset{q}{\frown} f_n$$
$$+ \sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{E}^\sigma_{m-1}} \Big(\bigstar_{p=1}^{m-2}{}^{r_p-\sigma(p)}_{r_p} f_n\Big) \overset{q}{\frown} f_n.$$

Condition (ii), along with Lemma 4.3, implies that $\| f_n \star^{(q-1)/2}_{(q+1)/2} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \underset{n\to+\infty}{\longrightarrow} 0$ and Lemma 4.4 ensures that, given these facts,

$$\sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{D}^\sigma_{m-1}} \Big(\bigstar_{p=1}^{m-2}{}^{r_p-\sigma(p)}_{r_p} f_n\Big) \overset{q}{\frown} f_n \underset{n\to+\infty}{\longrightarrow} \varphi\big(Z(\lambda)^m\big).$$

It remains to show that

$$\sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{E}^\sigma_{m-1}} \Big(\bigstar_{p=1}^{m-2}{}^{r_p-\sigma(p)}_{r_p} f_n\Big) \overset{q}{\frown} f_n \underset{n\to+\infty}{\longrightarrow} 0.$$

Observe that in the decomposition

$$\sum_{\tau\in \mathrm{NC}^0_{>2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|} = \sum_{\tau\in \mathcal{C}^{0,1}_{>2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|}$$
$$+ \sum_{\tau\in \mathcal{C}^{0,2}_{>2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|},$$

where $\mathcal{C}^{0,1}_{>2}([mq],\pi^*) = \{\tau \in \mathrm{NC}^0_{>2}([mq],\pi^*) : \forall b_1, b_2 \in \tau, \sharp(b_1\cap b_2) \in \{0, \frac{q+1}{2}, q\}\}$ and $\mathcal{C}^{0,2}_{>2}([mq],\pi^*) = \mathrm{NC}^0_{>2}([mq],\pi^*) \setminus \mathcal{C}^{0,1}_{>2}([mq],\pi^*)$, we have

$$\begin{aligned}&\sum_{\tau\in \mathcal{C}^{0,2}_{>2}([mq],\pi^*)} \int_{\mathbb{R}_+^{|\tau|}} (f_n)_\tau \,\mathrm{d}\mu^{|\tau|} \\ &\quad= \sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{E}^\sigma_{m-1}} \Big(\bigstar_{p=1}^{m-2}{}^{r_p-\sigma(p)}_{r_p} f_n\Big) \overset{q}{\frown} f_n.\end{aligned} \tag{10}$$

Condition (ii) implies (through Lemma 4.3), that $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \underset{n\to+\infty}{\longrightarrow} 0$ for all $r \in \{1,\dots,q-1\}$ and $\|f_n \star_r^{r-1} f_n\|_{L^2(\mathbb{R}_+^{2q-2r+1})} \underset{n\to+\infty}{\longrightarrow} 0$ for all $r \in \{1,\dots,q\} \setminus \{\frac{q+1}{2}\}$. As there is at least one of these contractions appearing in each summand of the left-hand side of (10) (the fact that contractions appear in the left-hand side is a direct consequence of the definition (1) of the quantity $(f_n)_\tau$), the argument in [2], Proof of Theorem 4.3, applies once more and concludes the proof.

3.2. Proof of Corollary 1.6

Point (i) can be proved in the same way as [9], Proposition 1.5, by using the contraction $f \star_2^1 f$ instead of the contraction $f \overset{q-1}{\frown} f$ in the case $q = 2$.

Point (ii) can be proved by observing that: (a) if f is valued in $\{0, 1\}$, then by definition of a Poisson random measure, $I_1(f)$ has a free centered Poisson distribution with parameter $\|f\|^2_{L^2(\mathbb{R}_+)}$, (b) if $I_1(f)$ has a free centered Poisson distribution with parameter $\lambda > 0$, then by Theorem 1.5 and Lemma 4.3, it holds that $\|f \star_{(q+1)/2}^{(q-1)/2} f - f\|_{L^2(\mathbb{R}_+^q)} = \|f \star_1^0 f - f\|_{L^2(\mathbb{R}_+)} = \|f^2 - f\|_{L^2(\mathbb{R}_+)} = 0 \Leftrightarrow f^2 = f$, μ-a.s, which concludes the proof. □

3.3. Proof of Theorem 1.8

Point (i) is a direct consequence of Theorem 1.5 and Theorem 1.4 in [9]. Point (ii) follows from the observation that when q is odd, the condition

$$\varphi\big(I_q^{\hat{N}}(f_n)^4\big) - 2\varphi\big(I_q^{\hat{N}}(f_n)^3\big) \underset{n\to+\infty}{\longrightarrow} 2\lambda^2 - \lambda$$

along with Lemma 4.3 implies that $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \underset{n\to+\infty}{\longrightarrow} 0$ for all $r \in \{1,\dots,q-1\}$. Applying Theorems 1.3 and 1.6 in [4] ensures that $I_q^S(f_n) \overset{\text{law}}{\to} \mathcal{S}(0,\lambda)$. The fact that $I_q^{\hat{N}}(f_n) \overset{\text{law}}{\to} Z(\lambda)$ follows once again from Theorem 1.5. □

4. Auxiliary lemmas

Lemma 4.1. *Let $q \geq 1$ and $m \geq 2$ be integers. Let $f \in L^2(\mathbb{R}_+^q)$. Then, it holds that*

$$I_q(f)^m = \sum_{i=0}^{m-1} \sum_{\sigma\in\mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1})\in\mathfrak{A}_m^\sigma} I_{mq+i-2\sum_{k=1}^{m-1} r_k}\Big(\bigstar_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big). \tag{11}$$

Proof. The proof is done by induction on m. The initialization for $m = 2$ is precisely the free Poisson multiplication formula (3). Assume (11) holds for all $p \le m$. Then, we have

$$I_q(f)^{m+1} = \sum_{i=0}^{m-1} \sum_{\sigma \in \mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma} I_{mq+i-2\sum_{k=1}^{m-1} r_k}\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big) I_q(f).$$

We use the multiplication formula (3) once again to obtain

$$\begin{aligned}
&I_q(f)^{m+1} \\
&= \sum_{i=0}^{m-1} \sum_{\sigma \in \mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma} \sum_{r_m=0}^{q \wedge [mq+i-2\sum_{k=1}^{m-1} r_k]} I_{(m+1)q+i-2\sum_{k=1}^{m} r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big) \overset{r_p}{\frown} f\Big) \\
&\quad + \sum_{i=0}^{m-1} \sum_{\sigma \in \mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma} \sum_{r_m=1}^{q \wedge [mq+i-2\sum_{k=1}^{m-1} r_k]} \\
&\qquad\qquad I_{(m+1)q+(i+1)-2\sum_{k=1}^{m} r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big) \star_{r_p}^{r_p-1} f\Big)
\end{aligned}$$

We now write the first summand of the first sum and the last summand of the second sum separately, as they have to be treated differently from the other terms. This yields

$$\begin{aligned}
&I_q(f)^{m+1} \\
&= \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^{\sigma_0}} \sum_{r_m=0}^{q \wedge [mq-2\sum_{k=1}^{m-1} r_k]} I_{(m+1)q-2\sum_{k=1}^{m} r_k}\Big(\mathop{\overset{r_p}{\frown}}_{p=1}^{m} f\Big) \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma \in \mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma} \sum_{r_m=0}^{q \wedge [mq+\sum_{k=1}^{m-1}(\sigma(k)-2r_k)]} \\
&\qquad\qquad I_{(m+1)q+i-2\sum_{k=1}^{m} r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big) \overset{r_p}{\frown} f\Big) \\
&\quad + \sum_{i=0}^{m-2} \sum_{\sigma \in \mathfrak{S}_i^m} \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^\sigma} \sum_{r_m=1}^{q \wedge [mq+\sum_{k=1}^{m-1}(\sigma(k)-2r_k)]} \\
&\qquad\qquad I_{(m+1)q+(i+1)-2\sum_{k=1}^{m} r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big) \star_{r_p}^{r_p-1} f\Big) \\
&\quad + \sum_{(r_1,\dots,r_{m-1}) \in \mathfrak{A}_m^{\sigma_1}} \sum_{r_m=1}^{q \wedge [mq-2\sum_{k=1}^{m-1} 2r_k+(m-1)]} I_{(m+1)q+m-2\sum_{k=1}^{m} r_k}\Big(\mathop{\bigstar}_{p=1}^{m}{}_{r_p}^{r_p-1} f\Big).
\end{aligned}$$

Also remark that the i appearing in the upper limit of the sum on r_m has been replaced by $\sum_{k=1}^{m-1}\sigma(k)$ (according to the definition of the sets $\mathfrak{A}_m^\sigma$). The next step consists of shifting the index up in the third sum above. We obtain

$$\begin{aligned}
&I_q(f)^{m+1}\\
&\quad= \sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^{\sigma_0}} \sum_{r_m=0}^{q\wedge[mq-2\sum_{k=1}^{m-1} r_k]} I_{(m+1)q-2\sum_{k=1}^m r_k}\Big(\overset{r_p}{\underset{p=1}{\overset{m}{\frown}}} f\Big)\\
&\qquad+\sum_{i=1}^{m-1}\sum_{\sigma\in\mathfrak{S}_i^m}\sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^\sigma}\sum_{r_m=0}^{q\wedge[mq+\sum_{k=1}^{m-1}(\sigma(k)-2r_k)]}\\
&\qquad\qquad I_{(m+1)q+i-2\sum_{k=1}^m r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big)\overset{r_p}{\frown} f\Big)\\
&\qquad+\sum_{i=1}^{m-1}\sum_{\sigma\in\mathfrak{S}_{i+1}^m}\sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^\sigma}\sum_{r_m=1}^{q\wedge[mq+\sum_{k=1}^{m-1}(\sigma(k)-2r_k)]}\\
&\qquad\qquad I_{(m+1)q+i-2\sum_{k=1}^m r_k}\Big(\Big(\mathop{\bigstar}_{p=1}^{m-1}{}_{r_p}^{r_p-\sigma(p)} f\Big)\star_{r_p}^{r_p-1} f\Big)\\
&\qquad+\sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^{\sigma_1}}\sum_{r_m=1}^{q\wedge[mq-2\sum_{k=1}^{m-1}2r_k+(m-1)]} I_{(m+1)q+m-2\sum_{k=1}^m r_k}\Big(\mathop{\bigstar}_{p=1}^{m}{}_{r_p}^{r_p-1} f\Big).
\end{aligned}$$

Recalling the definitions of $\mathfrak{S}_i^m$ and $\mathfrak{A}_m^\sigma$, one can combine the two middle sums into a single one in the following way:

$$\begin{aligned}
&I_q(f)^{m+1}\\
&\quad= \sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^{\sigma_0}} \sum_{r_m=0}^{q\wedge[mq-2\sum_{k=1}^{m-1} r_k]} I_{(m+1)q-2\sum_{k=1}^m r_k}\Big(\overset{r_p}{\underset{p=1}{\overset{m}{\frown}}} f\Big)\\
&\qquad+\sum_{i=1}^{m-1}\sum_{\sigma\in\mathfrak{S}_i^{m+1}}\sum_{(r_1,\ldots,r_m)\in\mathfrak{A}_{m+1}^\sigma} I_{(m+1)q+i-2\sum_{k=1}^m r_k}\Big(\mathop{\bigstar}_{p=1}^{m}{}_{r_p}^{r_p-\sigma(p)} f\Big)\\
&\qquad+\sum_{(r_1,\ldots,r_{m-1})\in\mathfrak{A}_m^{\sigma_1}}\sum_{r_m=1}^{q\wedge[mq-2\sum_{k=1}^{m-1}2r_k+(m-1)]} I_{(m+1)q+m-2\sum_{k=1}^m r_k}\Big(\mathop{\bigstar}_{p=1}^{m}{}_{r_p}^{r_p-1} f\Big).
\end{aligned}$$

It remains to combine the three final sums to conclude the proof. $\square$

Lemma 4.2. *Let $q \geq 1$ and $m \geq 2$ be integers. Let $f \in L^2(\mathbb{R}_+^q)$. Then, it holds that*

$$\varphi\big(I_q(f)^m\big) = \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} \sum_{\sigma \in \mathfrak{S}^{m-1}_{2i+\pi(qm)}} \sum_{(r_1,\dots,r_{m-2}) \in \mathfrak{B}^{\sigma}_{m-1}} \big(\bigstar_{p=1}^{m-2}{}_{r_p}^{r_p-\sigma(p)} f\big) \overset{q}{\frown} f, \tag{12}$$

where π is the parity function defined on $\mathbb{N}$ by $\pi(x) = 0$ if x is even and 1 otherwise.

Proof. When using Lemma 4.1 to evaluate $\varphi(I_q(f)^m)$, one has to determine when $mq - 2\sum_{k=1}^{m-1} r_k + \sum_{k=1}^{m-1} \sigma(k)$ is zero. In order to do so, we will study the quantity $r_1 + \cdots + r_{m-1}$ and determine the sufficient and necessary conditions for it to be equal to $mq + \sum_{k=1}^{m-1} \sigma(k)$. Recall that $(r_1, \dots, r_{m-1}) \in \mathfrak{A}^{\sigma}_m$ and set $\zeta_p = (p+1)q + \sum_{k=1}^{p-1} \sigma(k)$. We will proceed by induction to prove that, for all $p \leq m-1$,

$$2 \max_{(r_1,\dots,r_p) \in \mathfrak{A}^{\sigma}_{p+1}} (r_1 + \cdots + r_p) = \begin{cases} \zeta_p & \text{if } \zeta_p \text{ is even,} \\ \zeta_p - 1 & \text{if } \zeta_p \text{ is odd.} \end{cases} \tag{13}$$

For $p = 1$, it is obvious that $2\max_{\sigma(1) \leq r_1 \leq q} r_1 = 2q$. Fix $p \leq m-2$ and assume (13) is verified up to rank p. Using the induction hypothesis, it is easy to verify that, for $q \geq 2$,

$$2 \max_{(r_1,\dots,r_p) \in \mathfrak{A}^{\sigma}_{p+1}} (r_1 + \cdots + r_p) \geq pq + \sum_{k=1}^{p} \sigma(k).$$

We know that, on r_{p+1}, we have the restriction

$$\sigma(p+1) \leq r_{p+1} \leq q \wedge \left((p+1)q - 2\sum_{k=1}^{p} r_k + \sum_{k=1}^{p} \sigma(k) \right).$$

Hence, if $q \geq (p+1)q - 2\sum_{k=1}^{p} r_k + \sum_{k=1}^{p} \sigma(k) \Leftrightarrow r_1 + \cdots + r_p \geq \frac{pq + \sum_{k=1}^{p} \sigma(k)}{2}$, then, if $pq + \sum_{k=1}^{p} \sigma(k) = \zeta_{p+1}$ is even,

$$\begin{aligned} &2 \max_{(r_1,\dots,r_{p+1}) \in \mathfrak{A}^{\sigma}_{p+2}} (r_1 + \cdots + r_{p+1}) \\ &\quad = (p+1)q - 2 \min_{r_1+\cdots+r_p} \sum_{k=1}^{p} r_k + \sum_{k=1}^{p} \sigma(k) \\ &\quad = 2(p+1)q - pq - \sum_{k=1}^{p} \sigma(k) + 2\sum_{k=1}^{p} \sigma(k) \\ &\quad = (p+2)q + \sum_{k=1}^{p} \sigma(k). \end{aligned}$$

If $pq + \sum_{k=1}^{p} \sigma(k) = \zeta_{p+1}$ is odd, then

$$\begin{aligned}2 \max_{(r_1,\dots,r_{p+1}) \in \mathfrak{A}^{\sigma}_{p+2}} (r_1 + \cdots + r_{p+1}) &= (p+1)q - 2 \min_{r_1+\cdots+r_p} \sum_{k=1}^{p} r_k + \sum_{k=1}^{p} \sigma(k) \\ &= 2(p+1)q - pq - \sum_{k=1}^{p} \sigma(k) - 1 + 2\sum_{k=1}^{p} \sigma(k) \\ &= (p+2)q + \sum_{k=1}^{p} \sigma(k) - 1.\end{aligned}$$

It now remains to consider the case where $q \le (p+1)q - 2\sum_{k=1}^{p} r_k + \sum_{k=1}^{p} \sigma(k) \Leftrightarrow r_1 + \cdots + r_p \le \frac{pq + \sum_{k=1}^{p} \sigma(k)}{2}$.

If $pq + \sum_{k=1}^{p} \sigma(k) = \zeta_{p+1}$ is even,

$$2 \max_{(r_1,\dots,r_{p+1}) \in \mathfrak{A}^{\sigma}_{p+2}} (r_1 + \cdots + r_{p+1}) = 2q + 2 \max_{r_1+\cdots+r_p} \sum_{k=1}^{p} r_k = (p+2)q + \sum_{k=1}^{p} \sigma(k).$$

If $pq + \sum_{k=1}^{p} \sigma(k) = \zeta_{p+1}$ is odd, then

$$2 \max_{(r_1,\dots,r_{p+1}) \in \mathfrak{A}^{\sigma}_{p+2}} (r_1 + \cdots + r_{p+1}) = 2q + 2 \max_{r_1+\cdots+r_p} \sum_{k=1}^{p} r_k = (p+2)q + \sum_{k=1}^{p} \sigma(k) - 1.$$

This completes the induction.

Coming back to finding the necessary and sufficient conditions in order to have $2\sum_{k=1}^{m-1} r_k = mq + \sum_{k=1}^{m-1} \sigma(k)$, the above result for $p = m-1$ shows that it is necessary that ζ_m is even and that $\sigma(m-1) = 0$ for this equality to hold.

Note that if q is even, then it suffices that $\sum_{k=1}^{m-2} \sigma(k)$ be even as well for ζ_m to be even. In this case, as $\sigma(m-1)$ has to be zero, it implies that $\sum_{k=1}^{m-1} \sigma(k)$ is even as well. This only happens on the groups of permutations with an even index such as $\mathfrak{S}_{2i}$. Finally, because $2\max_{(r_1,\dots,r_p) \in \mathfrak{A}^{\sigma}_{p+1}} (r_1 + \cdots + r_{m-2}) = \zeta_{m-1}$, it forces r_{m-1} to always be equal to q. As $\sigma(m-1)$ is always 0 and r_{m-1} is always q, there is in fact no sum on r_{m-1} anymore and the groups of permutations that have to appear in (12) need only be the ones on sets of size $m-2$. Combining these conditions yields the desired result.

It remains to examine the case where q is odd. In this case, if m is even, then $\sum_{k=1}^{m-2} \sigma(k)$ has to be even as well in order for ζ_m to be even and the same arguments as in the previous case apply. The only (slightly) different case is whenever q and m are odd. In this case, $\sum_{k=1}^{m-2} \sigma(k)$ has to be odd as well in order for ζ_m to be even, and one has to consider the groups of permutations with an odd index such as $\mathfrak{S}_{2i+1}$ instead of the groups $\mathfrak{S}_{2i}$ for the announced result to follow. This proves that the parity of the groups of permutations to consider has to be the same as the parity of the product qm. $\square$

Lemma 4.3. *Let $q \geq 1$ be an integer, and consider a sequence of functions $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$ such that $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$ for every $n \geq 1$. Then,*

$$\varphi\big(I_q(f_n)^4\big) - 2\varphi\big(I_q(f_n)^3\big) \underset{n\to+\infty}{\longrightarrow} 2\lambda^2 - \lambda$$

(i) *if and only if* $\|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \underset{n\to+\infty}{\longrightarrow} 0$, $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \underset{n\to+\infty}{\longrightarrow} 0$ *for all* $r \in \{1,\dots,q-1\} \setminus \{\frac{q}{2}\}$, *and* $\|f_n \star_r^{r-1} f_n\|_{L^2(\mathbb{R}_+^{2q-2r+1})} \underset{n\to+\infty}{\longrightarrow} 0$ *for all* $r \in \{1,\dots,q\}$ *if q is even*;

(ii) *if and only if* $\|f_n \star_{(q+1)/2}^{(q-1)/2} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \underset{n\to+\infty}{\longrightarrow} 0$, $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \underset{n\to+\infty}{\longrightarrow} 0$ *for all* $r \in \{1,\dots,q-1\}$, *and* $\|f_n \star_r^{r-1} f_n\|_{L^2(\mathbb{R}_+^{2q-2r+1})} \underset{n\to+\infty}{\longrightarrow} 0$ *for all* $r \in \{1,\dots,q\} \setminus \{\frac{q+1}{2}\}$ *if q is odd*;

Proof. Compared to the proof of [9], Lemma 5.1, only the case where q is odd differs slightly. In that case, the product formula (3) and orthogonality in $L^2(\mathscr{A},\varphi)$ of multiple integrals of different orders yield

$$\varphi\big(I_q(f_n)^2 - I_q(f_n)\big) = 2\lambda^2 + \big\|f_n \star_{(q+1)/2}^{(q-1)/2} f_n - f_n\big\|^2_{L^2(\mathbb{R}_+^q)} + \sum_{r=1}^{q-1} \|f_n \overset{r}{\frown} f_n\|^2_{L^2(\mathbb{R}_+^{2q-2r})} + \sum_{\substack{1\leq r\leq q\\ r\neq (q+1)/2}} \big\|f_n \star_r^{r-1} f_n\big\|^2_{L^2(\mathbb{R}_+^{2q-2r+1})}.$$

The conclusion is obtained as in the proof of [9], Lemma 5.1. □

Lemma 4.4. *Let $q \geq 1$ be an odd integer and let $m \geq 2$ be an integer. Let $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$ be a sequence of tamed mirror symmetric functions such that $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$ for every n. Then, if $\|f_n \star_{(q+1)/2}^{(q-1)/2} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \underset{n\to+\infty}{\longrightarrow} 0$, then*

$$\sum_{i=0}^{\lfloor (m-2)/2\rfloor} \sum_{\sigma\in\mathfrak{S}_{2i+\pi(qm)}^{m-1}} \sum_{(r_1,\dots,r_{m-2})\in\mathfrak{D}_{m-1}^{\sigma}} \big(\bigstar_{p=1}^{m-2}{}_{r_p}^{r_p-\sigma(p)} f_n\big) \overset{q}{\frown} f_n \underset{n\to+\infty}{\longrightarrow} \varphi\big(Z(\lambda)^m\big) = \sum_{j=1}^{m} \lambda^j R_{m,j},$$

where $R_{m,j}$ is the number of non-crossing partitions of $[m]$ with exactly j blocks and with no singletons. Notice that, when m is even, one has that $R_{m,j} = 0$ for every $j > m/2$ and when m is odd, then $R_{m,j} = 0$ for every $j > (m-1)/2$. The numbers $R_{m,j}$ are related to the so-called Riordan numbers $\{R_m : m \geq 1\}$ (for a detailed combinatorial analysis of these numbers, see [1]) by $R_m = \sum_{j=1}^m R_{m,j}$ for all $m \geq 1$).

Proof. The same arguments as in the proof of [9], Lemma 5.2, can be used by replacing the case $q = 2$ in the last part (where the argument of two polynomials coinciding on a countable set being necessarily equal is used) by the case where $q = 1$ with a sequence $f_n = f = \sum_{i=1}^{p} \mathbb{1}_{A_i}$, where $\{A_i : i = 1, \dots, p\}$ are disjoint Borel sets with measure 1. $\square$

References

[1] Bernhart, F.R. (1999). Catalan, Motzkin, and Riordan numbers. *Discrete Math*. **204** 73–112. MR1691863
[2] Bourguin, S. and Peccati, G. (2014). Semicircular limits on the free Poisson chaos: Counterexamples to a transfer principle. *J. Funct. Anal*. **267** 963–997. MR3217054
[3] Deya, A. and Nourdin, I. (2014). Invariance principles for homogeneous sums of free random variables. *Bernoulli* **20** 586–603. MR3178510
[4] Kemp, T., Nourdin, I., Peccati, G. and Speicher, R. (2012). Wigner chaos and the fourth moment. *Ann. Probab*. **40** 1577–1635. MR2978133
[5] Nica, A. and Speicher, R. (2006). *Lectures on the Combinatorics of Free Probability*. *London Mathematical Society Lecture Note Series* **335**. Cambridge: Cambridge Univ. Press. MR2266879
[6] Nourdin, I. and Peccati, G. (2009). Noncentral convergence of multiple integrals. *Ann. Probab*. **37** 1412–1426. MR2546749
[7] Nourdin, I. and Peccati, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122
[8] Nourdin, I. and Peccati, G. (2012). *Normal Approximations with Malliavin Calculus*. *From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge: Cambridge Univ. Press. MR2962301
[9] Nourdin, I. and Peccati, G. (2013). Poisson approximations on the free Wigner chaos. *Ann. Probab*. **41** 2709–2723. MR3112929
[10] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab*. **33** 177–193. MR2118863
[11] Peccati, G. (2012). The Chen–Stein method for Poisson functionals. Available at arXiv:1112.5051.
[12] Peccati, G. and Taqqu, M.S. (2011). *Wiener Chaos*: *Moments*, *Cumulants and Diagrams*. *A survey With Computer Implementation*. Milan: Springer. MR2791919
[13] Peccati, G. and Thäle, C. (2013). Gamma limits and U-statistics on the Poisson space. *ALEA Lat. Am. J. Probab. Math. Stat*. **10** 525–560. MR3083936

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