

An empirical likelihood approach for symmetric α -stable processes

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Empirical likelihood approach is one of non-parametric statistical methods, which is applied to the hypothesis testing or construction of confidence regions for pivotal unknown quantities. This method has been applied to the case of independent identically distributed random variables and second order stationary processes. In recent years, we observe heavy-tailed data in many fields. To model such data suitably, we consider symmetric scalar and multivariate α -stable linear processes generated by infinite variance innovation sequence. We use a Whittle likelihood type estimating function in the empirical likelihood ratio function and derive the asymptotic distribution of the empirical likelihood ratio statistic for α -stable linear processes. With the empirical likelihood statistic approach, the theory of estimation and testing for second order stationary processes is nicely extended to heavy-tailed data analyses, not straightforward, and applicable to a lot of financial statistical analyses.

Keywords: confidence region; empirical likelihood ratio; heavy tail; normalized power transfer function; self-normalized periodogram; symmetric α -stable process; Whittle likelihood

1. Introduction

Non-parametric methods have been developed for the statistical analysis of univariate and multivariate observations in the area of time series analysis to carry out the problem of inference and hypothesis testing. Rank-based methods and empirical likelihood methods have been introduced in succession in these two decades.

Owen [23] introduced the empirical likelihood approach to independent and identically distributed (i.i.d.) data and he showed that the empirical likelihood ratio statistic is asymptotically χ^2 -distributed. For dependent data, Monti [20], Ogata and Taniguchi [22] derived the limit distribution of the empirical likelihood ratio statistic based on the derivative of the Whittle likelihood with respect to parameters. From these papers, we can construct confidence sets for the coefficients in a predictor and autocorrelation coefficients in multivariate stationary processes, etc.

In the last few decades, heavy-tailed data have been observed in a variety of fields involving electrical engineering, hydrology, finance and physical systems (Nolan [21] and Samorodnitsky and Taqqu [28]). In particular, Fama [10] and Mandelbrot [17] gave economic and financial examples that show such data are poorly grasped by Gaussian model. When we fit a GARCH-model to some financial data and estimate the stable index of the residuals by Hill's estimator $\hat{\alpha}$, we often observe that the tail of the distribution is heavier than that of Gaussian model.

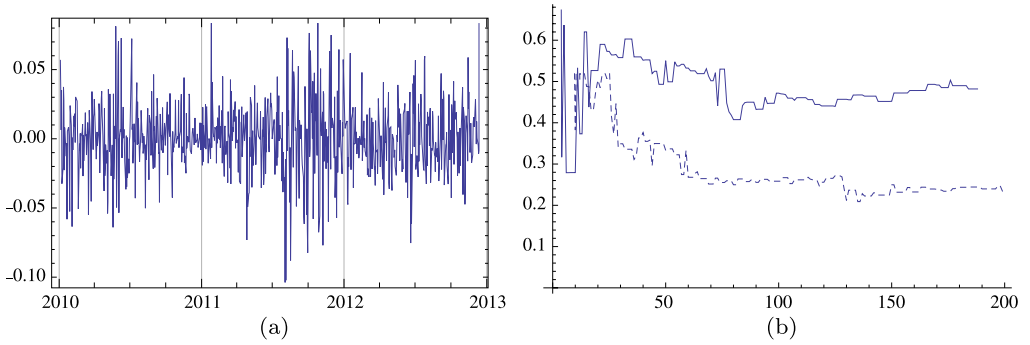


Figure 1. Log return of Hewlett Packard company and the Hill-plot. (a) log return of Hewlett Packard’s stock price (from 1, January, 2010 to 14, December, 2012). (b) Hill-plot for residuals (dashed line is for i.i.d. normal random variables).

Figure 1 shows daily stock returns of Hewlett Packard company and the Hill-plot for the residuals (we used AIC to select the order of GARCH). These graphs imply that it is more suitable to suppose these data are generated from a process with stable innovations rather than to assume these data have finite variances (for discussion of Hill-plot, see Drees, de Haan and Resnick [9], Hall [11], Hsing [13], Resnick and Stărikă [26] and [25]).

To model such heavy-tailed data suitably, we introduce the following linear process generated by stable innovations,

$$X(t) = \sum_{j=0}^{\infty} \psi_j Z(t - j), \quad t \in \mathbb{Z}, \tag{1.1}$$

where $\psi_0 = 1$ and $\{Z(t); t \in \mathbb{Z}\}$ (\mathbb{Z} is the set of all integers) is a sequence of i.i.d. symmetric α -stable random variables (for short *s α s*). In the case of $\alpha = 2$, this process is Gaussian. When α is less than 2, the usual spectral density function of (1.1) cannot be defined.

Davis and Resnick [6,7] and [8] investigated the sample autocorrelation function (ACF) at lag h , and derived the consistency of ACF. Resnick and Stărică [25] gave a consistent estimator of the tail index α . In view of the frequency domain approach, Klüppelberg and Mikosch [14,15] and [16] proposed a self-normalized periodogram because the expectation of the usual periodogram does not exist, and introduced some methods for parameter estimation and hypothesis testing. Then, they showed that for any frequencies, self-normalized periodogram converges to a random variable with finite second moment, and proved the convergence of the functional of the self-normalized periodogram.

In this paper, we apply non-parametric method to the discrete linear process (1.1). It is natural to express the process non-parametrically partly because finite parametric models often cannot describe real data sufficiently, and partly because there is no general solution of probability density function for stable distribution. Recently economists and quantitative analysts have introduced stable stochastic models to asset returns in econometrics and finance. In such situa-

tions, what we are interested in is to test statistical hypothesis on an important pivotal quantity “ $\theta = \theta_0$ ”, such as the correlation between the different realizations. To achieve this goal, Monti [20] and Ogata and Taniguchi [22] employed the empirical likelihood to construct confidence sets for linear processes when innovations have finite variance. A plausible way to define the important index θ_0 is Whittle’s approach, that is, θ_0 minimizes the disparity

$$D(f_\theta, \tilde{g}) = \int_{-\pi}^{\pi} \frac{\tilde{g}(\omega)}{f(\omega; \theta)} d\omega, \tag{1.2}$$

where $\tilde{g}(\omega)$ is called a normalized power transfer function of (1.1), and $f(\omega; \theta)$ is an appropriate score function.

This setting is useful for many situations. For example, let us consider the h -step linear prediction of a scalar stationary process $\{X(t); t \in \mathbb{Z}\}$. We predict $X(t)$ by a linear combination of $\{X(s); s \leq t - h\}$,

$$\hat{X}(t) = \sum_{j=h}^{\infty} \phi_j(\theta) X(t - j).$$

The spectral representations of $X(t)$ and $\hat{X}(t)$ are

$$X(t) = \int_{-\pi}^{\pi} \exp(-it\omega) d\zeta_X(\omega), \quad \hat{X}(t) = \int_{-\pi}^{\pi} \exp(-it\omega) \sum_{j=h}^{\infty} \phi_j(\theta) \exp(ij\omega) d\zeta_X(\omega),$$

where $\{\zeta_X(\omega); -\pi \leq \omega \leq \pi\}$ is an orthogonal increment process satisfying

$$E d\zeta_X(\omega) d\zeta_X(\mu) = \begin{cases} g(\omega) d\omega & (\omega = \mu), \\ 0 & (\omega \neq \mu). \end{cases}$$

Then, the prediction error is

$$E|X(t) - \hat{X}(t)|^2 = \int_{-\pi}^{\pi} \left| 1 - \sum_{j=h}^{\infty} \phi_j(\theta) \exp(ij\omega) \right|^2 g(\omega) d\omega. \tag{1.3}$$

Hence the best h -step predictor is given by $\sum_{j=h}^{\infty} \phi_j(\theta_0) X(t - j)$, where θ_0 minimizes (1.3). Comparing this with (1.2), if we set

$$f(\omega; \theta) = \left| 1 - \sum_{j=h}^{\infty} \phi_j(\theta) \exp(ij\omega) \right|^{-2},$$

this problem is exactly the same as that of seeking θ_0 in their definition. In addition to the linear prediction, the empirical likelihood approach can also be applied to the case of sample autocorrelation estimation, which will be given in Section 2.

The empirical likelihood ratio function for the problem of testing $H: \boldsymbol{\theta} = \boldsymbol{\theta}_0$ is defined as

$$R(\boldsymbol{\theta}) = \max_{w_1, \dots, w_n} \left\{ \prod_{t=1}^n n w_t; \sum_{t=1}^n w_t \mathbf{m}(\lambda_t; \boldsymbol{\theta}) = \mathbf{0}, \sum_{t=1}^n w_t = 1, 0 \leq w_1, w_2, \dots, w_n \leq 1 \right\},$$

and then the estimating function takes the form

$$\mathbf{m}(\lambda_t; \boldsymbol{\theta}) \equiv \frac{\partial}{\partial \boldsymbol{\theta}} \frac{\tilde{I}_{n,X}(\lambda_t)}{f(\lambda_t; \boldsymbol{\theta})}, \quad \lambda_t = \frac{2\pi t}{n} \in (-\pi, \pi],$$

where $\tilde{I}_{n,X}(\omega)$ is called self-normalized periodogram. For our general stable linear process (1.1), we derive the limit distribution of $R(\boldsymbol{\theta}_0)$ with its normalizing factor and construct the confidence interval through a numerical method.

Here it should be noted that our extension to the stable case from the finite variance case requires new asymptotic methods, and we report new aspects of the asymptotics for empirical likelihood approach, which are different from the usual ones. Furthermore, we extend the results to those of the multivariate one with independent innovations. This is extremely important from a viewpoint of practical use. In particular, we can analyze the relationship between two heavy-tailed processes. The way to derive the asymptotics of the multivariate case has also new aspects. We find that the asymptotics for multivariate process need more stronger conditions than what we need in the univariate case. The self-normalizing factor is also difficult to find in that case and we use the norm of the stable series defined in Section 4 instead of the square root matrix.

This paper is organized as follows: In Section 2, we shall introduce the fundamental setting and a brief overview on the empirical likelihood approach based on the Whittle likelihood. With a different normalizing order for the empirical likelihood ratio function, the main theoretical results, limit distribution of the empirical likelihood ratio statistic for univariate and multivariate stable linear processes, are formulated in Sections 3 and 4, respectively. In Section 5, the numerical results will be given under several settings. We shall demonstrate some effectiveness of the empirical likelihood ratio method. The proofs of theorems in Sections 3 and 4 are relegated to Section 6.

As for notations and symbols used in this paper, the set of all integers, non-negative integers ($= \{0, 1, 2, \dots\}$) and real numbers are denoted by \mathbb{Z} , \mathbb{N} and \mathbb{R} , respectively. For any sequence of random vectors $\{\mathbf{A}(t); t \in \mathbb{Z}\}$, $\mathbf{A}(t) \xrightarrow{\mathcal{P}} \mathbf{A}$ and $\mathbf{A}(t) \xrightarrow{\mathcal{L}} \mathbf{A}$, respectively, denote the convergence to a random (or constant) vector \mathbf{A} in probability and law. Especially, “ p - $\lim_{t \rightarrow \infty} \mathbf{A}(t) = \mathbf{A}$ ” implies “ $\mathbf{A}(t) \xrightarrow{\mathcal{P}} \mathbf{A}$ as $t \rightarrow \infty$ ”. The transpose and conjugate transpose of matrix \mathbf{M} are denoted by \mathbf{M}' and \mathbf{M}^* , and define $\|\mathbf{M}\|_E := \sqrt{\text{tr}[\mathbf{M}^* \mathbf{M}]}$.

2. Fundamental setting

In this section, we state the fundamental setting for the main results. Throughout this paper, we use the following notations. For any sequence $\{A(t); t \in \mathbb{Z}\}$ of random variables,

$$\begin{aligned} \gamma_{n,A}^2 &= n^{-2/\alpha} \sum_{t=1}^n A(t)^2, \\ I_{n,A}(\omega) &= n^{-2/\alpha} \left| \sum_{t=1}^n A(t) \exp(it\omega) \right|^2, \\ \tilde{A}_t &= \frac{A(t)}{\sqrt{A(1)^2 + \dots + A(n)^2}}, \quad t = 1, \dots, n, \end{aligned} \tag{2.1}$$

and

$$\tilde{I}_{n,A}(\omega) = \frac{I_{n,A}(\omega)}{\gamma_{n,A}^2} = \left| \sum_{t=1}^n \tilde{A}_t \exp(it\omega) \right|^2.$$

We call $\tilde{I}_{n,A}(\omega)$ a *self-normalized periodogram* of $A(1), \dots, A(n)$. Mikosch *et al.* [18] studied estimation of the following stable and causal ARMA process:

$$\begin{aligned} X(t) + \phi_1 X(t-1) + \dots + \phi_p X(t-p) &= Z(t) + \theta_1 Z(t-1) + \dots + \theta_q Z(t-q), \\ Z(1) &\in \text{DNA}(\alpha) \text{ (see Mikosch } et al. \text{ [18])}, \end{aligned}$$

where $\text{DNA}(\alpha)$ denotes the set of random variables in the domain of normal attraction of a symmetric α -stable random variable. Letting $\beta = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q)$, define

$$\begin{aligned} C &= \{ \beta \in \mathbb{R}^{p+q} : \phi_p, \theta_q \neq 0, \phi(z) \text{ and} \\ &\quad \theta(z) \text{ have no common zeros, } \phi(z)\theta(z) \neq 0 \text{ for } |z| \leq 1 \}, \end{aligned}$$

where $\phi(z) = 1 + \phi_1 z + \dots + \phi_p z^p$, and $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$. Let $g(\omega; \beta)$ be

$$g(\omega; \beta) = \left| \frac{1 + \theta_1 \exp(i\omega) + \dots + \theta_q \exp(iq\omega)}{1 + \phi_1 \exp(i\omega) + \dots + \phi_p \exp(ip\omega)} \right|^2.$$

They defined the Whittle estimator of β by

$$\hat{\beta}_n \equiv \arg \min_{\beta \in C} \int_{-\pi}^{\pi} \frac{\tilde{I}_{n,X}(\omega)}{g(\omega; \beta)} d\omega.$$

Then Mikosch *et al.* [18] showed that the estimator $\hat{\beta}_n$ is consistent to the true parameter $\beta_0 \in C$.

In many cases, however, we know neither the true stochastic structure of the process nor the true pivotal unknown quantities. In such cases, we can apply the empirical likelihood approach

to the data, without assuming that the data come from a known family of stochastic models. The empirical likelihood approach was introduced as a non-parametric method of inference based on a data-driven likelihood ratio function in the i.i.d. case (e.g., Owen [23]). For dependent data, Monti [20] applied the empirical likelihood approach to a stationary linear process with the finite second moment. She used

$$\mathbf{m}(\lambda_t; \boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \left\{ \log g(\lambda_t; \boldsymbol{\theta}) + \frac{I_{n,X}(\lambda_t)}{g(\lambda_t; \boldsymbol{\theta})} \right\}, \quad t = 1, \dots, n$$

as an *estimating function*. This can be understood as a discretized derivative of the Whittle likelihood

$$\int_{-\pi}^{\pi} \left\{ \log g(\omega; \boldsymbol{\theta}) + \frac{I_{n,X}(\omega)}{g(\omega; \boldsymbol{\theta})} \right\} d\omega.$$

Here $g(\omega; \boldsymbol{\theta})$ and $I_{n,X}(\omega)$ are, respectively, the usual spectral density of a stationary process and the periodogram. Using this estimating function, the empirical likelihood ratio function is defined as

$$R(\boldsymbol{\theta}) = \max_{w_1, \dots, w_n} \left\{ \prod_{t=1}^n n w_t; \sum_{t=1}^n w_t \mathbf{m}(\lambda_t; \boldsymbol{\theta}) = \mathbf{0}, \sum_{t=1}^n w_t = 1, 0 \leq w_1, w_2, \dots, w_n \leq 1 \right\}. \quad (2.2)$$

Under the circular assumption, It is shown that the quantity $-2 \log R(\boldsymbol{\theta})$ converges in distribution to chi-square random variable with degree of freedom q under $H: \boldsymbol{\theta} = \boldsymbol{\theta}_0$ (the pivotal true value of $\boldsymbol{\theta}$) $\in \Theta \subset \mathbb{R}^q$. Ogata and Taniguchi [22] developed the empirical likelihood approach to multivariate non-Gaussian stationary processes without the circular assumption. For a vector-valued process $\{\mathbf{X}(t); t \in \mathbb{Z}\}$,

$$\mathbf{X}(t) = \sum_{j=0}^{\infty} \mathbf{G}(j) \mathbf{e}(t - j), \quad E[\mathbf{e}(t) \mathbf{e}(l)'] = \delta(t, l) \boldsymbol{\Sigma},$$

they introduced the disparity measure

$$D(\mathbf{f}_{\boldsymbol{\theta}}; \mathbf{g}) = \int_{-\pi}^{\pi} [\log \det \mathbf{f}(\omega; \boldsymbol{\theta}) + \text{tr}\{\mathbf{f}(\omega; \boldsymbol{\theta})^{-1} \mathbf{g}(\omega)\}] d\omega$$

on

$$\mathcal{P} = \left\{ \mathbf{f}(\omega; \boldsymbol{\theta}) | \mathbf{f}(\omega; \boldsymbol{\theta}) = \left\{ \sum_{j=0}^{\infty} \mathbf{G}(j; \boldsymbol{\theta}) \exp(ij\omega) \right\} \boldsymbol{\Sigma} \left\{ \sum_{j=0}^{\infty} \mathbf{G}(j; \boldsymbol{\theta}) \exp(ij\omega) \right\}^*, \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^q \right\},$$

where $\mathbf{g}(\omega)$ is the usual spectral density matrix of the s -dimensional stationary linear process. If the innovation variance of the process is independent of unknown parameter $\boldsymbol{\theta}$, we call $\boldsymbol{\theta}$ “*innovation free*”. Then, the first integration of the disparity measure is independent of $\boldsymbol{\theta}$ (e.g., Hannan [12], page 162). Therefore if $\boldsymbol{\theta}$ is innovation-free, the derivative of this measure is

$$\frac{\partial}{\partial \boldsymbol{\theta}} D(\mathbf{f}_{\boldsymbol{\theta}}; \mathbf{g}) = \frac{\partial}{\partial \boldsymbol{\theta}} \int_{-\pi}^{\pi} \text{tr}\{\mathbf{f}(\omega; \boldsymbol{\theta})^{-1} \mathbf{g}(\omega)\} d\omega.$$

They introduced the pivotal true value θ_0 defined by

$$\frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} \text{tr}\{\mathbf{f}(\omega; \theta)^{-1} \mathbf{g}(\omega)\} d\omega \Big|_{\theta=\theta_0} = \mathbf{0}. \tag{2.3}$$

In this case, the estimating function is naturally set to be

$$\mathbf{m}(\lambda_t; \theta) = \frac{\partial}{\partial \theta} \text{tr}\{\mathbf{f}(\lambda_t; \theta)^{-1} \mathbf{I}_{n,X}(\lambda_t)\}, \quad t = 1, \dots, n,$$

where $\mathbf{I}_{n,X}(\omega)$ is the usual periodogram matrix. Under mild conditions on the fourth order cumulant of the process, they showed that $-2 \log R(\theta)$ converges in law to a sum of gamma distributed random variables under $H: \theta = \theta_0$.

The approach has been discussed for stationary processes with the “finite second moments”. In this paper, we consider a linear process $\{X(t); t \in \mathbb{Z}\}$ generated by (1.1) with $\{Z(t); t \in \mathbb{Z}\}$, a sequence of i.i.d. symmetric α -stable random variables with scale $\sigma > 0$, and the characteristic function of $Z(1)$ is given as

$$E \exp\{iZ(1)\xi\} = \exp\{-\sigma|\xi|^\alpha\}, \quad \xi \in \mathbb{R}.$$

Generally, we can define the stable process for $\alpha \in (0, 2]$. However, we assume that $\alpha \in [1, 2)$ to guarantee probability convergence of important terms which will appear in proofs of theorems in this paper. This restriction is not quite strict, since the process (1.1) with $\alpha \in [1, 2)$ still does not have the finite second moment. To guarantee the a.s. absolute convergence of (1.1), we make the following assumption.

Assumption 2.1. For some δ satisfying $0 < \delta < 1$,

$$\sum_{j=0}^{\infty} |j| |\psi_j|^\delta < \infty.$$

Under this assumption, the series (1.1) converges almost surely. This is an easy consequence of the three-series theorem (c.f. Petrov [24]). Furthermore, the process (1.1) has the normalized power transfer function

$$\tilde{g}(\omega) = \frac{1}{\psi^2} \left| \sum_{j=0}^{\infty} \psi_j \exp(ij\omega) \right|^2, \quad \psi^2 = \sum_{j=0}^{\infty} \psi_j^2.$$

From the property of stable random variables,

$$X(t) =_d \left\{ \sum_{j=0}^{\infty} |\psi_j|^\alpha \right\}^{1/\alpha} Z(1),$$

which implies that this process does not have the finite second moment when $\alpha < 2$, so we cannot use the method of moments. The empirical likelihood approach is still useful when we deal with

the stable process. Hereafter, we define a pivotal true value θ_0 of the process (1.1) as the solution of

$$\frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} \frac{\tilde{g}(\omega)}{f(\omega; \theta)} d\omega \Big|_{\theta=\theta_0} = \mathbf{0}, \tag{2.4}$$

where $\theta = (\theta_1, \dots, \theta_p)' \in \Theta \subset \mathbb{R}^p$. Note that the score function does not necessarily coincide with the true normalized power transfer function $\tilde{g}(\omega)$, and we can choose various important quantities θ_0 by choosing the form of $f(\omega; \theta)$. For example, for fixed $l \in \mathbb{N}$, set

$$f(\omega; \theta) = |1 - \theta \exp(i l \omega)|^{-2}.$$

Solving (2.4), we have

$$\theta_0 = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+l}}{\sum_{j=0}^{\infty} \psi_j^2} \equiv \rho(l) \quad (\text{say}).$$

On the other hand, a sample autocorrelation function

$$\hat{\rho}(l) \equiv \frac{\sum_{t=1}^{n-l} X(t)X(t+l)}{\sum_{t=1}^n X(t)^2}, \quad l \in \mathbb{N}$$

for the stable process (1.1) is weakly consistent to the autocorrelation function of the process in the stable case; namely, for fixed l , $p\text{-}\lim_{n \rightarrow \infty} \hat{\rho}(l) = \rho(l)$ (e.g., Davis and Resnick [8]).

By these motivation, we consider the empirical likelihood ratio function (2.2) with

$$\mathbf{m}(\lambda_t; \theta) = \frac{\partial \tilde{I}_{n,X}(\lambda_t)}{\partial \theta} f(\lambda_t; \theta), \quad \lambda_t = \frac{2\pi t}{n}, t = 1, \dots, n.$$

Hereafter, we make the following assumptions on $f(\omega; \theta)$.

Assumption 2.2.

- (i) Θ is a compact subset of \mathbb{R}^q and $f(\omega; \theta)$ has an parametrized representation as an element of \mathcal{P} , where \mathcal{P} is defined by

$$\mathcal{P} = \left\{ f(\omega; \theta) \mid f(\omega; \theta) = \left| \sum_{j=0}^{\infty} \eta_j(\theta) \exp(ij\omega) \right|^2, \theta \in \Theta \subset \mathbb{R}^q \right\}.$$

- (ii) For any $\theta \in \text{Int } \Theta$, $f(\omega; \theta)$ is continuously twice differentiable with respect to θ .
- (iii) There exists an unique $\theta_0 \in \Theta$ satisfying (2.4).

3. Main results

In this section, we introduce the limit distribution of the empirical likelihood statistic for the scalar stable process (1.1). Our main purpose is to make an accurate confidence region of θ_0

based on the empirical likelihood approach. Because of the properties of stable random variables, it is difficult to use the method of moments. To overcome this problem, we frequently make use of the self-normalized periodogram defined in Section 2. Klüppelberg and Mikosch [15] or Mikosch *et al.* [18] introduced the self-normalized periodogram, and Klüppelberg and Mikosch [16] showed some limit theorems of integrated self-normalized periodogram. Under the settings in Section 2, we derive the asymptotic distribution of the empirical likelihood ratio statistic, and construct a confidence region for θ_0 .

We impose an assumption to describe the asymptotics of the empirical likelihood ratio statistic.

Assumption 3.1. For some $\mu \in (0, \alpha)$ and all $k = 1, \dots, q$,

$$\sum_{l=1}^{\infty} \left| \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta_k} \frac{\tilde{g}(\omega)}{f(\omega; \boldsymbol{\theta})} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \cos(t\omega) \, d\omega \right|^{\mu} < \infty.$$

Assumption 3.1 is used for Proposition 3.5 of Klüppelberg and Mikosch [16]. It is easy to see that stable AR(p) processes satisfying Assumption 2.2 satisfy this assumption.

In order to control the rate of convergence of the empirical likelihood ratio statistic, we introduce the normalizing sequence

$$x_n = \left(\frac{n}{\log n} \right)^{1/\alpha}, \quad n = 2, 3, \dots$$

The next theorem gives the asymptotics of $R(\theta_0)$. The proof will be given in Section 6.

Theorem 3.1. Suppose that $\alpha \in [1, 2)$, and Assumptions 2.1, 2.2 and 3.1 hold. Then,

$$-\frac{2x_n^2}{n} \log R(\theta_0) \xrightarrow{\mathcal{L}} \mathbf{V}'\mathbf{W}^{-1}\mathbf{V} \quad \text{under } H : \boldsymbol{\theta} = \boldsymbol{\theta}_0, \tag{3.1}$$

where \mathbf{V} and \mathbf{W} are $q \times 1$ random vector and $q \times q$ constant matrix, respectively, whose j th and (k, l) -elements are expressed as

$$V_j = \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{S_l}{S_0} \left\{ \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_j} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \tilde{g}(\omega) \cos(t\omega) \, d\omega \right\},$$

$$W_{kl} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_k} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_l} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} 2\tilde{g}(\omega)^2 \, d\omega$$

with independent random variables S_0, S_1, S_2, \dots ; S_0 is a positive $\alpha/2$ -stable random variable and $\{S_j; j = 1, 2, \dots\}$ is a sequence of symmetric α -stable random variables.

Remark 3.1. The limit distribution (3.1) depends on the characteristic exponent α and unknown normalized power transfer function $\tilde{g}(\omega)$. We can construct appropriate consistent estimators of

them. It is shown that Hill’s estimator

$$\hat{\alpha}_{\text{Hill}} = \left\{ \frac{1}{k} \sum_{t=1}^k \log \frac{|X|_{(t)}}{|X|_{(k+1)}} \right\}^{-1}$$

is a consistent estimator of α , where $|X|_{(1)} > \dots > |X|_{(n)}$ is the order statistic of $|X(1)|, \dots, |X(n)|$ and $k = k(n)$ is an integer satisfying some conditions (e.g., Resnick and Stărică [26] and [25]). Next, it is known that the smoothed self-normalized periodogram by an appropriate weighting function $W_n(\cdot)$ is weekly consistent to the normalized power transfer function. That is,

$$\tilde{J}_{n,X}(\omega) = \sum_{|k| \leq m} W_n(k) \tilde{I}_{n,X}(\lambda_k) \xrightarrow{\mathcal{P}} \tilde{g}(\omega), \quad \lambda_k = \omega + \frac{k}{n}, |k| \leq m$$

for any $\omega \in [-\pi, \pi]$ (Klüppelberg and Mikosch [14], Theorem 4.1), where the integer $m = m(n)$ satisfies $m \rightarrow \infty$ and $m/n \rightarrow 0$ as $n \rightarrow \infty$. One possible choice of the weighting function $W_n(\cdot)$ and $m = m(n)$ are $W_n(k) = (2m + 1)^{-1}$ and $m = \lfloor \sqrt{n} \rfloor$ ($\lfloor x \rfloor$ denotes the integer part of x). We use this weighting function in the section of numerical studies. Then, by Slutsky’s lemma and continuous mapping theorem, we obtain consistent estimator $\hat{\mathbf{W}}$ of \mathbf{W} . So if we choose a proper threshold value γ_p , which is p -percentile corresponding to $\mathbf{V}^T \mathbf{W} \mathbf{V}$, $C_{\alpha,p}$ below is an approximate $p/100$ level confidence region of θ_0 .

$$C_{\alpha,p} = \left\{ \theta \in \Theta; -\frac{2x_n^2}{n} \log R(\theta) < \gamma_p \right\}. \tag{3.2}$$

4. Vector α -stable processes

So far we focused on the scalar case for clarity. In this section, we extend the empirical likelihood analysis to the case of vector α -stable processes. Consider a d -dimensional vector-valued linear process $\{\mathbf{X}(t); t \in \mathbb{Z}\}$ generated by

$$\mathbf{X}(t) = \sum_{j=0}^{\infty} \Psi(j) \mathbf{Z}(t - j), \tag{4.1}$$

where $\Psi(0)$ is the identity matrix and $\{\Psi(j); j \in \mathbb{N}\}$ is a sequence of $d \times d$ real matrices, and $\{\mathbf{Z}(t); t \in \mathbb{Z}\}$ is an independently and identically distributed sequence of symmetric α -stable random vectors whose elements are also independent.

Now, we set down the following assumptions for the general result. Almost all assumptions are similar to those of the 1-dimensional stable processes.

Assumption 4.1. For some δ satisfying $0 < \delta < 1$ and all $k, l = 1, \dots, d$,

$$\sum_{j=0}^{\infty} j |\Psi(j)_{kl}|^{\delta} < \infty. \tag{4.2}$$

The sample autocovariance and the periodogram matrices are defined as

$$\hat{\Gamma}_{n,X}(h) = n^{-2/\alpha} \sum_{t=1}^{n-|h|} \mathbf{X}(t)\mathbf{X}(t+h)',$$

$$\mathbf{I}_{n,X}(\omega) = d_{n,X}(\omega)d_{n,X}(\omega)^*, \quad d_{n,X}(\omega) = n^{-1/\alpha} \sum_{t=1}^n \mathbf{X}(t) \exp(i\omega t),$$

respectively. We define the true power transfer function $\mathbf{g}(\omega)$ by

$$\mathbf{g}(\omega) = \Psi(\omega)\Psi(\omega)^*,$$

where $\Psi(\omega) = \sum_{j=0}^{\infty} \Psi(j) \exp(ij\omega)$. Similarly as in the previous section, we use the empirical likelihood ratio with the estimating function

$$\mathbf{m}(\lambda_t; \theta) = \frac{\partial}{\partial \theta} \text{tr}\{\mathbf{f}(\lambda_t; \theta)^{-1} \mathbf{I}_{n,X}(\lambda_t)\},$$

where $\mathbf{f}(\lambda_t; \theta)$ satisfies the following assumptions.

Assumption 4.2.

- (i) Θ is a compact subset of \mathbb{R}^q and $\mathbf{f}(\omega; \theta)$ has an parametrized representation as an element of \mathcal{P} , where \mathcal{P} is defined by

$$\mathcal{P} = \left\{ \mathbf{f}(\omega; \theta) \mid \mathbf{f}(\omega; \theta) = \left(\sum_{j=0}^{\infty} \Xi(j; \theta) \exp(ij\omega) \right) \left(\sum_{j=0}^{\infty} \Xi(j; \theta) \exp(ij\omega) \right)^*, \theta \in \Theta \subset \mathbb{R}^q \right\}.$$

- (ii) For any $\theta \in \text{Int } \Theta$, $f(\omega; \theta)$ is continuously twice differentiable with respect to θ .
- (iii) There exists an unique $\theta_0 \in \Theta$ satisfying (2.4).

Assumption 4.3 below guarantees the convergence of the functional of periodogram by inequality of an application of Theorem 3.1 in Rosinski and Woyczynski [27].

Assumption 4.3. For some $\mu \in (0, \alpha)$ and all $k = 1, \dots, q$,

$$\sum_{t=1}^{\infty} \left\| \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta_k} \Psi(\omega)^* \mathbf{f}(\omega; \theta) \Psi(\omega) \exp(it\omega) \, d\omega \right\|_E^{\mu} < \infty.$$

Theorem 4.1. Suppose that $\alpha \in [1, 2)$, and Assumptions 4.1–4.3 hold for the process (4.1). If

$$\frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} \Psi(\omega)^* \mathbf{f}(\omega; \theta)^{-1} \Psi(\omega) \, d\omega \Big|_{\theta=\theta_0} = \mathbf{0},$$

then

$$-2 \frac{x_n^2}{n} \log R(\boldsymbol{\theta}_0) \xrightarrow{\mathcal{L}} \mathbf{V}' \mathbf{W}^{-1} \mathbf{V} \quad \text{under } H: \boldsymbol{\theta} = \boldsymbol{\theta}_0,$$

where

$$\mathbf{V} = \frac{1}{2\pi} \sum_{i,j=1}^d \sum_{h=1}^{\infty} \frac{S(h)_{ij}}{S_{\alpha/2}} \begin{pmatrix} \int_{-\pi}^{\pi} (B_1(\omega) + \overline{B_1(\omega)})_{ij} d\omega \\ \int_{-\pi}^{\pi} (B_2(\omega) + \overline{B_2(\omega)})_{ij} d\omega \\ \vdots \\ \int_{-\pi}^{\pi} (B_q(\omega) + \overline{B_q(\omega)})_{ij} d\omega \end{pmatrix}$$

with $S(h)_{ij}$ a matrix whose all elements are stable with index α , $S_{\alpha/2}$ a random variable with index $\alpha/2$ and

$$B_k(\omega) = \Psi(\omega)^* \frac{\partial}{\partial \theta_k} \mathbf{f}(\omega; \boldsymbol{\theta})^{-1} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \Psi(\omega) \exp(ih\omega), \quad k = 1, \dots, q,$$

and the (a, b) -component of \mathbf{W} can be expressed as

$$W_{ab} = \frac{1}{2\pi d^2} \int_{-\pi}^{\pi} \left(\text{tr} \left[\mathbf{g}(\omega) \frac{\partial \mathbf{f}(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_a} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathbf{g}(\omega) \frac{\partial \mathbf{f}(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_b} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right] + \text{tr} \left[\mathbf{g}(\omega) \frac{\partial \mathbf{f}(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_a} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right] \text{tr} \left[\mathbf{g}(\omega) \frac{\partial \mathbf{f}(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_b} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right] \right) d\omega.$$

Proof. The proof of Theorem 4.1 is given in the supplemental article (Akashi *et al.* [1]), since it is more technical. □

Remark 4.1. This extension is not straightforward, and contains some novel aspects. We take up an appealing example for Theorem 4.1. Consider whether the wave structures of the spectra between all components are “close” to each other or not. For simplicity, we formulate this idea in 2-dimensional case and assume the true power transfer function $\mathbf{g}(\omega)$ is

$$\mathbf{g}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{\mathbf{R}}(k) \exp(-ik\omega),$$

where $\tilde{\mathbf{R}}(k)$, a symmetric matrix, denotes the k th autocorrelation function. Then the null hypothesis can be written as

$$H: \tilde{\mathbf{R}}(k) = \theta_0 \tilde{\mathbf{R}}(j) \quad \text{or} \quad \tilde{\mathbf{R}}(k) = \theta_0 \tilde{\mathbf{R}}(j)' \quad \text{for some } k \text{ and } j.$$

To test this hypothesis, we set the estimating function $\mathbf{m}(\lambda_r; \boldsymbol{\theta})$ with an inverse correlation function $\mathbf{f}(\lambda_r; \boldsymbol{\theta})^{-1}$, which was first introduced in Cleveland [5], and deeply discussed by Bhansali [2].

Let

$$\mathbf{f}(\omega; \theta)^{-1} = (\exp(k\omega) + \exp(-k\omega)) \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} + (\exp(j\omega) + \exp(-j\omega)) \begin{pmatrix} \frac{1}{2}\theta^2 & 0 \\ 0 & \frac{1}{2}\theta^2 \end{pmatrix}.$$

Then under the hypothesis, we have

$$\frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} \Psi(\omega)^* \mathbf{f}(\omega; \theta)^{-1} \Psi(\omega) \, d\omega \Big|_{\theta=\theta_0} = \mathbf{0},$$

which satisfies the assumption in Theorem 4.1.

5. Numerical studies

In this section, we carry out some simulation studies for Theorems 3.1 and 4.1. Suppose that the observations $X(1), \dots, X(n)$ are generated from the following scalar-valued stable MA(100) model:

$$X(t) = \sum_{j=0}^{100} \psi_j Z(t - j), \tag{5.1}$$

where $\{Z(t); t \in \mathbb{Z}\}$ is a sequence of i.i.d. α s random variables with scale $\sigma = 1$ and coefficients $\{\psi_j; j \in \mathbb{N}\}$ are defined as

$$\psi_j = \begin{cases} 1 & (j = 0), \\ b^j / j & (1 \leq j \leq 100), \\ 0 & (\text{otherwise}). \end{cases}$$

Since this process can not be expressed as AR or ARMA models with finite dimension, it is suitable to apply the empirical likelihood approach to estimate pivotal unknown quantities. We first discuss the estimation of the autocorrelation with lag 2

$$\rho(2) = p\text{-}\lim_{n \rightarrow \infty} \frac{\sum_{t=1}^{n-2} X(t)X(t+2)}{\sum_{t=1}^n X(t)^2}. \tag{5.2}$$

It is seen that the normalized power transfer function of the process (5.1) is given by

$$\tilde{g}(\omega) = \frac{|\sum_{j=0}^{100} \psi_j \exp(ij\omega)|^2}{\sum_{j=0}^{100} \psi_j^2}.$$

If we set the score function as $f(\omega; \theta) = |1 - \theta \exp(2i\omega)|^{-2}$, we obtain

$$\theta_0 = \frac{\sum_{j=0}^{100} \psi_j \psi_{j+2}}{\sum_{j=0}^{100} \psi_j^2}.$$

Table 1. 90% confidence intervals (and length) for the autocorrelation with lag 2. Sample size is 300 and $\alpha = 1.5$

	$\theta_0 \approx$	E.L.			SAC		
Case 1	0.1168	-0.0761	0.1930	(0.2691)	-0.0676	0.2481	(0.3157)
Case 2	0.3603	0.1320	0.4765	(0.3445)	0.1388	0.5304	(0.3916)

On the other hand, from Davis and Resnick [8], the right-hand side limit of (5.2) exists, and is equal to this θ_0 . So it is natural that we define the estimating function $m(\lambda_t; \theta)$ by this $f(\omega; \theta)$ to estimate $\rho(2)$. The autocorrelation can also be estimated by sample autocorrelation (SAC) method. From Theorem 12.5.1 of Brockwell and Davis [4], for fixed $l \in \mathbb{N}$,

$$x_n \{ \hat{\rho}(l) - \rho(l) \} \xrightarrow{\mathcal{L}} \frac{\tilde{S}_1}{\tilde{S}_0} \left\{ \sum_{j=1}^{\infty} | \rho(l+j) + \rho(l-j) - 2\rho(j)\rho(l) |^\alpha \right\}^{1/\alpha},$$

where $\hat{\rho}(l) = \sum_{t=1}^{n-l} X(t)X(t+l) / \sum_{t=1}^n X(t)^2$, \tilde{S}_0 and \tilde{S}_1 are $\alpha/2$ and α -stable random variables, respectively. Under this setting, we construct confidence intervals of $\theta_0 = \rho(2)$ by calculating $R(\theta)$ at numerous point over $(-1, 1)$, and compare confidence intervals constructed by the empirical likelihood method with the SAC method.

The results of our simulations are as follows. First, we generate 300 samples from (5.1). Note that in this case, the characteristic exponent $\alpha = 1.5$ is known. Then using the weighting function W_n which is mentioned in Section 3, we calculate the consistent estimator $\tilde{J}_{n,X}(\omega)$ of $\tilde{g}(\omega)$ and construct an approximate 90% confidence interval of θ_0 defined as (3.2). We also use the Monte Carlo simulation to calculate γ_{90} which is 90 percentile of V/WV for 10^5 times. Table 1 shows the values of θ_0 and confidence intervals by the empirical likelihood method and the sample autocorrelation method for $b = 0.5$ (case 1) and 0.9 (case 2). By this simulation, it is shown that the length of intervals obtained by the empirical likelihood method is seems to be shorter than that by the sample autocorrelation method.

Next, we fix $b = 0.5$ and $n = 300$, and construct confidence intervals for cases of $\alpha = 1.0$ (Cauchy), 1.5 and 1.9 (near Gaussian). The larger α becomes, the better performance both methods show (see Table 2). In particular, the empirical likelihood method provides better inferences than those by the SAC method when α is nearly 1.

Table 2. 90% confidence intervals (and length) for the autocorrelation with lag 2. Sample size is 300, $b = 0.5$ and $\theta_0 \approx 0.1168$

	α	E.L.			SAC		
Case 3	1.0	-0.1583	0.3335	(0.4918)	-0.1342	0.3891	(0.5233)
Case 4	1.5	-0.0761	0.1930	(0.2691)	-0.0676	0.2481	(0.3157)
Case 5	1.9	-0.0465	0.1329	(0.1794)	-0.0450	0.1365	(0.1815)

Table 3. 90% confidence intervals (and length) for the autocorrelation with lag 2. $b = 0.5$, $\alpha = 1.5$ and $\theta_0 \approx 0.1168$

	n	E.L.			SAC		
Case 6	50	-0.2397	0.4313	(0.6710)	-0.2477	0.5629	(0.8106)
Case 7	100	-0.3125	0.2228	(0.5353)	-0.3476	0.2218	(0.5694)

Moreover, we investigate the length of intervals when $b = 0.5$ and $\alpha = 1.5$ for small samples. Table 3 shows the result for $n = 50$ and 100. Even though sample size is small, the empirical likelihood method also works well.

Also, we give an example for multivariate case. Suppose that the observations $\mathbf{X}(1), \dots, \mathbf{X}(n)$ are generated from the 2-dimensional VMA(100) model with innovations $\{\mathbf{Z}(t); t \in \mathbb{Z}\}$ whose marginal distributions are i.i.d. $s\alpha s$ with scale 1, and the coefficient matrices $A(j)$, $j = 1, \dots, 100$ are assumed to be

$$A(j) = \begin{pmatrix} 0.7^j & j^{-2}b^j \\ 0 & 0.5^j \end{pmatrix}.$$

To this model, we use the following score function $\mathbf{f}(\omega; \theta)$ defined by

$$\mathbf{f}(\omega; \theta) = (I - B_\theta \exp(i\omega))^{-1} (I - B_\theta \exp(i\omega))^{-1*}, \quad \text{where } B_\theta = \begin{pmatrix} 0.5 & \theta \\ 0.4 & 0.2 \end{pmatrix}.$$

In this case, the asymptotic distribution of $-2(x_n^2/n) \log R(\theta_0)$ can be simply represented by $(S_1/S_0)^2(V^2/W)$, where S_0 and S_1 are the same as in Theorem 3.1, W is the same as in Theorem 4.1 and

$$V = \frac{1}{\pi} \left[\left| \int_{-\pi}^{\pi} F_{12}(\omega) \cos(\omega) d\omega + \sum_{t=1}^{\infty} (F_{11}(\omega) + F_{22}(\omega) + 2F_{12}(\omega)) \cos(t\omega) \right|^\alpha \right]^{1/\alpha},$$

if we write

$$\mathbf{F}(\omega) = \frac{\partial}{\partial \theta} \Psi(\omega)^* \mathbf{f}(\omega; \theta)^{-1} \Big|_{\theta=\theta_0} \Psi(\omega).$$

The confidence intervals for θ are summarized in the following Table 4.

We also focus on the one-sided coverage error to evaluate the performances of the confidence intervals. Let θ^U and θ^L be the endpoints of a confidence interval. The one-sided coverage error is given by

$$|\Pr[\{\theta_0 < \theta^L\} \cup \{\theta^U < \theta_0\}] - 0.1|.$$

In this time, we calculated the confidence intervals constructed by both methods for univariate case, and by the empirical likelihood approach for multivariate case by 1000 times of Monte

Table 4. 90% confidence intervals (and length) for true parameter. Sample size is 300 and $\alpha = 1.5$

	b	$\theta_0 \approx$	E.L.		(Length)
Case 8	0	0.0000	-0.1685	0.1690	(0.3375)
Case 9	0.3	0.1755	0.0467	0.3208	(0.2741)
Case 10	0.6	0.3669	0.2601	0.4920	(0.2320)
Case 11	0.9	0.5787	0.5046	0.6641	(0.1596)

Carlo simulations. Namely, we made 1000 confidence intervals $(\theta_l^L, \theta_l^U), l = 1, \dots, 1000$, independently, and calculate the quantity

$$\left| \frac{\sum_{l=1}^{1000} \mathbb{I}\{\theta_0 \notin (\theta_l^L, \theta_l^U)\}}{1000} - 0.1 \right|$$

for each case, where \mathbb{I} denotes the indicator function. Empirical coverage errors are shown in Table 5. From this table, the empirical likelihood confidence intervals are more accurate than those by the existing method. Especially, it seems that both methods give the close coverage probabilities to the nominal level when α is nearly 2.0. On the other hand, we can see that both methods give the close coverage probabilities to the nominal level as n increases (case 1, case 6 and case 7).

Furthermore, our results also apply in the multivariate case. Although the coverage error becomes worse as the pseudo true value gets larger, it can be seen that the confidence intervals correspondingly becomes smaller in Table 4.

Table 5. Coverage errors of confidence intervals for the parameter θ_0

	Coverage errors	
	E.L.	SAC
Case 1	0.082	0.087
Case 2	0.089	0.096
Case 3	0.094	0.098
Case 4	0.082	0.087
Case 5	0.053	0.056
Case 6	0.092	0.095
Case 7	0.086	0.090
Case 8	0.011	—
Case 9	0.027	—
Case 10	0.032	—
Case 11	0.049	—

6. Proofs

This section provides the proofs of theorems. The following notation will be used throughout this section.

$$\mathbf{P}_n(\boldsymbol{\theta}_0) \equiv \frac{1}{n} \sum_{t=1}^n \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \quad \text{and} \quad \mathbf{S}_n(\boldsymbol{\theta}_0) \equiv \frac{1}{n} \sum_{t=1}^n \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)'$$

6.1. Proof for Theorem 3.1

We start with some auxiliary results. Recalling (2.1), let

$$\rho_{n,A}(h) = \sum_{t=1}^{n-h} \tilde{A}_t \tilde{A}_{t+h}, \quad h = 1, \dots, n-1 \quad \text{and} \quad T_{n,A}(\omega) = 2 \sum_{h=1}^{n-1} \rho_{n,A}(h) \cos(h\omega).$$

Lemma 6.1.

$$ET_{n,Z}(\omega) = 0, \quad ET_{n,Z}(\omega)^2 \rightarrow \begin{cases} 1 & (\omega \not\equiv 0 \pmod{\pi}), \\ 2 & (\omega \equiv 0 \pmod{\pi}), \end{cases}$$

as $n \rightarrow \infty$ uniformly in $\alpha \in (0, 2]$ and $\sigma > 0$.

Proof. By symmetry and boundedness of \tilde{Z}_t 's, $E\tilde{Z}_1$ exists and is equal to 0. Furthermore, from the definition of $\tilde{Z}_1, \dots, \tilde{Z}_n$, we can see that $\sum_{t=1}^n \tilde{Z}_t^2 = 1$ almost surely, so $E\tilde{Z}_1^2 = 1/n$. Using Chebyshev's inequality, we can see

$$\Pr\{|\tilde{Z}_1| < \varepsilon^{-1/2} n^{-1/2}\} > 1 - \varepsilon$$

for any $\varepsilon > 0$. This inequality means $\sqrt{n}\tilde{Z}_1^2$ is $O_p(n^{-1/2})$, hence $\sqrt{n}\tilde{Z}_1^2$ converges to 0 in distribution uniformly in $\alpha \in (0, 2]$. Therefore, by Taylor's theorem there exists a constant c such that

$$\begin{aligned} E \exp\{i\xi \sqrt{n}\tilde{Z}_1^2\} &= 1 - \frac{\xi^2}{2} n E\tilde{Z}_1^4 + \frac{\xi^3 \sin(\xi c)}{6} n^{3/2} E\tilde{Z}_1^6 + i \operatorname{Im}[E \exp\{i\xi \sqrt{n}\tilde{Z}_1^2\}] \\ &\rightarrow 1 \end{aligned}$$

uniformly in $\xi \in \mathbb{R}$ by Lévy's continuity theorem, where $\operatorname{Im}(z)$ means the imaginary part of a complex number z . So we can conclude $nE\tilde{Z}_1^4$ converges to 0 as n tends to ∞ . We also find that $n(n-1)E\tilde{Z}_1^2\tilde{Z}_2^2$ converges to 1 by taking expectations on both sides of following identical equation:

$$1 = \sum_{t=1}^n \tilde{Z}_t^4 + \sum_{t \neq s} \tilde{Z}_t^2 \tilde{Z}_s^2. \tag{6.1}$$

Remembering the facts above, let us evaluate the expectations. First, from symmetry of \tilde{Z}_1 , it is easy to see that $ET_{n,Z}(\omega)$ is exactly equal to 0. Next, we expand $T_{n,Z}(\omega)^2$ and obtain that

$$\begin{aligned}
 ET_{n,Z}(\omega)^2 &= n(n-1)E\tilde{Z}_1^2\tilde{Z}_2^2 + 2nE\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} \cos(2h\omega) - 2E\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} h \cos(2h\omega). \tag{6.2}
 \end{aligned}$$

The first term of (6.2) converges to 1 as $n \rightarrow \infty$. Suppose that $\omega \equiv 0 \pmod{\pi}$, then

$$2nE\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} \cos(2h\omega) - 2E\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} h \cos(2h\omega) = n(n-1)E\tilde{Z}_1^2\tilde{Z}_2^2 \rightarrow 1.$$

Next, for $\omega \not\equiv 0 \pmod{\pi}$, the following two identical equations hold;

$$\begin{aligned}
 \sum_{h=1}^{n-1} \cos(2h\omega) &= \frac{\cos(2(n-1)\omega) + \cos(2\omega) - \cos(2n\omega)}{2(1 - \cos(2\omega))}, \\
 \sum_{h=1}^{n-1} h \cos(2h\omega) &= \frac{n \cos(2(n-1)\omega) - (n-1) \cos(2n\omega) - 1}{2(1 - \cos(2\omega))}.
 \end{aligned}$$

Using these equations, we obtain that

$$2nE\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} \cos(2h\omega) - 2E\tilde{Z}_1^2\tilde{Z}_2^2 \sum_{h=1}^{n-1} h \cos(2h\omega) \rightarrow 0.$$

Hence we get desired result. □

Lemma 6.2. $\sum \sum_{k \neq l} \text{Cov}\{\tilde{I}_{n,Z}(\lambda_k)^2, \tilde{I}_{n,Z}(\lambda_l)^2\} = O(n)$.

Proof. From Brillinger [3],

$$\text{Cov}\{\tilde{I}_{n,Z}(\lambda_k)^2, \tilde{I}_{n,Z}(\lambda_l)^2\} = \sum_{\mathbf{v}: p=1}^8 \prod_{j=1}^p \text{cum}\{d_{n,Z}(\lambda_{k_j}); k_j \in v_j\},$$

where the summation is taken over all indecomposable partitions $\mathbf{v} = v_1 \cup \dots \cup v_p$, $p = 1, \dots, 8$ of a table

$$\begin{array}{cccc}
 k & | & k & | & -k & | & -k \\
 \hline
 l & | & l & | & -l & | & -l \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array} \tag{6.3}$$

(see Brillinger [3]), and $d_{n,Z}(\lambda_k) = \sum_{t=1}^n \tilde{Z}_t \exp(it\lambda_k)$. Note that $\text{cum}\{d_{n,Z}(\lambda_{k_1}), \dots, d_{n,Z}(\lambda_{k_m})\}$ is 0 for odd m . Let us consider following five partitions;

$$\begin{aligned}
 p = 1, & \quad (k, k, -k, -k, l, l, -l, -l), \\
 p = 2, & \quad (k, -k, l, -l) \cup (k, -k, l, -l), \\
 & \quad (k, -k) \cup (k, -k, l, l, -l, -l), \\
 & \quad (l, -l) \cup (k, k, -k, -k, l, -l) \\
 \text{and } p = 3, & \quad (k, -k) \cup (l, -l) \cup (k, -k, l, -l).
 \end{aligned} \tag{6.4}$$

First, we show that with different k and l in \mathbf{v} ,

$$\sum_{k \neq l} \sum_{\mathbf{v}': p=1}^8 \prod_{j=1}^p \text{cum}\{d_{n,Z}(\lambda_{k_j}); k_j \in v_j\} = O(n). \tag{6.5}$$

for indecomposable decompositions $\mathbf{v}' = \mathbf{v} \setminus (6.4)$. However, the proof for (6.5) contains lengthy and complex algebra, so we confine to giving a representative example here.

Let us consider partitions for $p = 4$. We can evaluate the second order cumulant as

$$\begin{aligned}
 \text{cum}\{d_{n,Z}(\lambda_k), d_{n,Z}(\lambda_l)\} &= E\tilde{Z}_1^2 \sum_{t=1}^n \exp(it(\lambda_k - \lambda_l)) \\
 &= \frac{1}{n} \sum_{t=1}^n \exp\left(it \frac{2\pi(k-l)}{n}\right) \\
 &= \begin{cases} 1 & (k-l \equiv 0 \pmod{n}), \\ 0 & (k-l \not\equiv 0 \pmod{n}), \end{cases}
 \end{aligned}$$

therefore

$$\begin{aligned}
 &\text{cum}\{d_{n,Z}(\lambda_{k_1}), d_{n,Z}(\lambda_{k_2})\} \cdots \text{cum}\{d_{n,Z}(\lambda_{k_7}), d_{n,Z}(\lambda_{k_8})\} \\
 &= \begin{cases} 1 & (k_1 - k_2, \dots, k_7 - k_8 \equiv 0 \pmod{n}), \\ 0 & (\text{otherwise}). \end{cases}
 \end{aligned}$$

So when $p = 4$, we obtain

$$\sum_{k \neq l} \prod_{j=1}^p \text{cum}\{d_{n,Z}(\lambda_{k_j}); k_j \in v_j\} = O(n) \tag{6.6}$$

for any indecomposable partition (6.3). Similarly, we can check (6.6) for $p = 2$ and 3. Next, we need to check the cumulants on partitions (6.4). For simplicity, we introduce generic residual terms $R_n^{(1)}(k, l), \dots, R_n^{(4)}(k, l)$ such that $\sum \sum_{k \neq l} R_n^{(\eta)}(k, l)^\gamma = O(n)$ for $\gamma = 1, 2, \eta = 1, 2, 3$ and 4. A simple example of $R_n^{(\eta)}(k, l)$ is given as

$$R_n^{(\eta)}(k, l) = \begin{cases} \exists(\text{constant}) & (k-l \equiv 0 \pmod{n}), \\ 0 & (k-l \not\equiv 0 \pmod{n}), \end{cases}$$

and these will appear when we expand the cumulants concerned. The fourth order joint cumulant on $(\lambda_k, -\lambda_k, \lambda_l, -\lambda_l)$ is represented as

$$\begin{aligned} \text{cum}\{d_{n,Z}(\lambda_k), d_{n,Z}(-\lambda_k), d_{n,Z}(\lambda_l), d_{n,Z}(-\lambda_l)\} \\ = nE\tilde{Z}_1^4 + n(n-1)E\tilde{Z}_1^2\tilde{Z}_2^2 - 1 + R_n^{(1)}(k, l). \end{aligned} \tag{6.7}$$

From (6.1), (6.7) becomes

$$\text{cum}\{d_{n,Z}(\lambda_k), d_{n,Z}(-\lambda_k), d_{n,Z}(\lambda_l), d_{n,Z}(-\lambda_l)\} = R_n^{(1)}(k, l).$$

By the same argument as above, and using identical equations

$$\begin{aligned} \left\{ \sum_{t=1}^n \tilde{Z}_t^2 \right\} \left\{ \sum_{t,s}^{(*)} \tilde{Z}_t^2 \tilde{Z}_s^2 \right\} &= \sum_{t,s}^{(*)} \tilde{Z}_t^2 \tilde{Z}_s^2 = 2 \sum_{t,s}^{(*)} \tilde{Z}_t^4 \tilde{Z}_s^2 + \sum_{t,s,u}^{(*)} \tilde{Z}_t^2 \tilde{Z}_s^2 \tilde{Z}_u^2, \\ \left\{ \sum_{t=1}^n \tilde{Z}_t^2 \right\} \left\{ \sum_{t=1}^n \tilde{Z}_t^4 \right\} &= \sum_{t=1}^n \tilde{Z}_t^4 = \sum_{t,s}^{(*)} \tilde{Z}_t^4 \tilde{Z}_s^2 + \sum_{t=1}^n \tilde{Z}_t^6 \end{aligned}$$

and

$$\begin{aligned} 1 &= \sum_{t=1}^n \tilde{Z}_t^8 + 4 \sum_{t,s}^{(*)} \tilde{Z}_t^6 \tilde{Z}_s^2 + 3 \sum_{t,s}^{(*)} \tilde{Z}_t^4 \tilde{Z}_s^4 \\ &\quad + 6 \sum_{t,s,u}^{(*)} \tilde{Z}_t^4 \tilde{Z}_s^2 \tilde{Z}_u^2 + \sum_{t,s,u,v}^{(*)} \tilde{Z}_t^2 \tilde{Z}_s^2 \tilde{Z}_u^2 \tilde{Z}_v^2, \end{aligned} \tag{6.8}$$

we obtain that

$$\begin{aligned} \text{cum}\{d_{n,Z}(\lambda_k), d_{n,Z}(-\lambda_k), d_{n,Z}(\lambda_l), d_{n,Z}(\lambda_l), d_{n,Z}(-\lambda_l), d_{n,Z}(-\lambda_l)\} \\ = R_n^{(2)}(k, l), \\ \text{cum}\{d_{n,Z}(\lambda_k), \dots, d_{n,Z}(-\lambda_l)\} \quad (\text{the eighth order joint cumulant}) \\ = 2n^2E\tilde{Z}_1^4\tilde{Z}_2^4 - 6n^3E\tilde{Z}_1^4\tilde{Z}_2^2\tilde{Z}_3^2 \\ + n^4E\tilde{Z}_1^2\tilde{Z}_2^2\tilde{Z}_3^2\tilde{Z}_4^2 - \{n^2E\tilde{Z}_1^2\tilde{Z}_2^2\}^2 + R_n^{(3)}(k, l), \end{aligned} \tag{6.9}$$

where $\sum_{t_1, \dots, t_m}^{(*)}$ is a summation taken over all t_1, \dots, t_m are different from each other.

According to the same argument as that in Lemma 6.1, the first and second terms in (6.9) converge to 0 as $n \rightarrow \infty$, and the fourth term converges to 1.

Finally, from (6.8), the third term converges to 1. Hence the eighth order joint cumulant becomes

$$\text{cum}\{d_{n,Z}(\lambda_k), \dots, d_{n,Z}(-\lambda_l)\} = R_n^{(4)}(k, l),$$

so we have

$$\sum_{k \neq l} \sum_{v:p=1}^8 \prod_{j=1}^p \text{cum}\{d_{n,Z}(\lambda_{k_j}); k_j \in v_j\} = O(n). \quad \square$$

Lemma 6.3. Under Assumption 2.2,

$$\mathbf{S}_n(\boldsymbol{\theta}_0) \xrightarrow{\mathcal{P}} \mathbf{W}$$

as $n \rightarrow \infty$. Here \mathbf{W} is defined in Theorem 3.1.

Proof. We first make use of the decomposition of the periodogram in Klüppelberg and Mikosch [15] as follows, that is,

$$\begin{aligned} \tilde{I}_{n,X}(\omega)^2 &= \tilde{g}(\omega)^2 \tilde{I}_{n,Z}(\omega)^2 + o_p(1) \\ &= \tilde{g}(\omega)^2 \left\{ 1 + 2 \sum_{h=1}^{n-1} \rho_{n,Z}(h) \cos(h\omega) \right\}^2 + o_p(1) \\ &= \tilde{g}(\omega)^2 \{1 + 2T_{n,Z}(\omega) + T_{n,Z}(\omega)^2\} + o_p(1). \end{aligned} \quad (6.10)$$

Then from Lemma 6.1, we obtain that

$$\begin{aligned} \mathbb{E}[\mathbf{S}_n(\boldsymbol{\theta}_0)] &= \frac{1}{n} \sum_{t=1}^n \frac{\partial f(\lambda_t; \boldsymbol{\theta})^{-1}}{\partial \boldsymbol{\theta}} \frac{\partial f(\lambda_t; \boldsymbol{\theta})^{-1}}{\partial \boldsymbol{\theta}'} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathbb{E} \tilde{I}_{n,X}(\lambda_t)^2 \\ &\rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \boldsymbol{\theta}} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \boldsymbol{\theta}'} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} 2\tilde{g}(\omega)^2 d\omega = \mathbf{W}. \end{aligned}$$

From Lemma 6.2, Assumption 2.2 and (6.10), if we define

$$h_{\boldsymbol{\theta}_0}(\omega)_{ab} = \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_a} \frac{\partial f(\omega; \boldsymbol{\theta})^{-1}}{\partial \theta_b} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \tilde{g}(\omega)^2,$$

then

$$\begin{aligned} \text{Cov}\{\mathbf{S}_n(\boldsymbol{\theta}_0)_{ab}, \mathbf{S}_n(\boldsymbol{\theta}_0)_{cd}\} &= \frac{1}{n^2} \sum_{t=1}^n \sum_{s=1}^n h_{\boldsymbol{\theta}_0}(\lambda_t)_{ab} h_{\boldsymbol{\theta}_0}(\lambda_s)_{cd} \text{Cov}\{\tilde{I}_{n,Z}(\lambda_t)^2, \tilde{I}_{n,Z}(\lambda_s)^2\} \\ &= \frac{1}{n^2} \sum_{t=1}^n h_{\boldsymbol{\theta}_0}(\lambda_t)_{ab} h_{\boldsymbol{\theta}_0}(\lambda_t)_{cd} \text{Var} \tilde{I}_{n,Z}(\lambda_t)^2 \\ &\quad + \frac{1}{n^2} \sum_{t \neq s} h_{\boldsymbol{\theta}_0}(\lambda_t)_{ab} h_{\boldsymbol{\theta}_0}(\lambda_s)_{cd} \text{Cov}\{\tilde{I}_{n,Z}(\lambda_t)^2, \tilde{I}_{n,Z}(\lambda_s)^2\} + o(1) \\ &\rightarrow 0 \end{aligned}$$

for $a, b, c, d = 1, \dots, q$. These facts imply the convergence of $\mathbf{S}_n(\boldsymbol{\theta}_0)$ in probability. \square

Proof of Theorem 3.1. By Lagrange’s multiplier method, w_1, \dots, w_n which maximize the objective function in $R(\theta)$ are given by

$$w_t = \frac{1}{n} \frac{1}{1 + \phi' \mathbf{m}(\lambda_t; \theta_0)}, \quad t = 1, \dots, n,$$

where $\phi \in \mathbb{R}^q$ is the Lagrange multiplier which is defined as the solution of q -restrictions

$$J_{n, \theta_0}(\phi) = \frac{1}{n} \sum_{t=1}^n \frac{\mathbf{m}(\lambda_t; \theta_0)}{1 + \phi' \mathbf{m}(\lambda_t; \theta_0)} = \mathbf{0}. \tag{6.11}$$

First of all, let us derive the order of ϕ . Set $Y_t \equiv \phi' \mathbf{m}(\lambda_t; \theta_0)$ and from (6.11),

$$\begin{aligned} \mathbf{0} &= \frac{1}{n} \sum_{t=1}^n \frac{\mathbf{m}(\lambda_t; \theta_0)}{1 + Y_t} \\ &= \frac{1}{n} \sum_{t=1}^n \left\{ 1 - Y_t + \frac{Y_t^2}{1 + Y_t} \right\} \mathbf{m}(\lambda_t; \theta_0) \\ &= \mathbf{P}_n(\theta_0) - \mathbf{S}_n(\theta_0)\phi + \frac{1}{n} \sum_{t=1}^n \frac{\mathbf{m}(\lambda_t; \theta_0)Y_t^2}{1 + Y_t}. \end{aligned}$$

Hence,

$$\phi = \mathbf{S}_n(\theta_0)^{-1} \left\{ \mathbf{P}_n(\theta_0) + \frac{1}{n} \sum_{t=1}^n \frac{\mathbf{m}(\lambda_t; \theta_0)Y_t^2}{1 + Y_t} \right\} \equiv \mathbf{S}_n(\theta_0)^{-1} \mathbf{P}_n(\theta_0) + \varepsilon \quad (\text{say}). \tag{6.12}$$

Next, we introduce $M_n \equiv \max_{1 \leq k \leq n} \|\mathbf{m}(\lambda_k; \theta_0)\|_E$. The order of M_n is given by

$$\begin{aligned} M_n &= \max_{1 \leq t \leq n} \left\| \frac{\partial f(\lambda_t; \theta)^{-1}}{\partial \theta} \Big|_{\theta=\theta_0} \tilde{I}_{n,X}(\lambda_t) \right\|_E \\ &\leq \max_{1 \leq t \leq n} \left\| \frac{\partial f(\lambda_t; \theta)^{-1}}{\partial \theta} \Big|_{\theta=\theta_0} \right\|_E \max_{1 \leq t \leq n} |I_{n,X}(\lambda_t)| \frac{1}{\gamma_{n,X}^2} \\ &\leq \max_{\omega \in [-\pi, \pi]} \left\| \frac{\partial f(\omega; \theta)^{-1}}{\partial \theta} \Big|_{\theta=\theta_0} \right\|_E \max_{\omega \in [-\pi, \pi]} |I_{n,X}(\omega)| \frac{1}{\gamma_{n,X}^2} \\ &= \max_{\omega \in [-\pi, \pi]} \left\| \frac{\partial f(\omega; \theta)^{-1}}{\partial \theta} \Big|_{\theta=\theta_0} \right\|_E \max_{\omega \in [-\pi, \pi]} |g(\omega)| \frac{\max_{\omega \in [-\pi, \pi]} |I_{n,X}(\omega)|}{\max_{\omega \in [-\pi, \pi]} |g(\omega)|} \frac{1}{\gamma_{n,X}^2} \\ &\leq \max_{\omega \in [-\pi, \pi]} \left\| \frac{\partial f(\omega; \theta)^{-1}}{\partial \theta} \Big|_{\theta=\theta_0} \right\|_E \max_{\omega \in [-\pi, \pi]} |g(\omega)| \max_{\omega \in [-\pi, \pi]} \left| \frac{I_{n,X}(\omega)}{g(\omega)} \right| \frac{1}{\gamma_{n,X}^2} \\ &= \exists c_0 \max_{\omega \in [-\pi, \pi]} \left| \frac{I_{n,X}(\omega)}{g(\omega)} \right| \quad (\because \text{Assumption 2.2}). \end{aligned}$$

On the other hand, it is not difficult to check that Assumption 2.1 is sufficient condition for Corollary 3.3 of Mikosch, Resnick and Samorodnitsky [19], so we have $M_n = O_p(\beta_n^2)$, where

$$\beta_n = \begin{cases} (\log n)^{1-1/\alpha} & (1 < \alpha < 2), \\ \log \log n & (\alpha = 1). \end{cases}$$

Henceforth, let $1 < \alpha < 2$. In the case of $\alpha = 1$, the same argument as follows will go on. By Ogata and Taniguchi [22], there exists a unit vector \mathbf{u} in \mathbb{R}^q such that the following inequality holds:

$$\|\phi\|_E \{ \mathbf{u}' \mathbf{S}_n(\boldsymbol{\theta}_0) \mathbf{u} - \mathbf{u}' M_n \mathbf{P}_n(\boldsymbol{\theta}_0) \} \leq \mathbf{u}' \mathbf{P}_n(\boldsymbol{\theta}_0).$$

Lemma P5.1 of Brillinger [3] allows us to write $x_n \mathbf{P}_n(\boldsymbol{\theta}_0)$ as

$$\begin{aligned} x_n \mathbf{P}_n(\boldsymbol{\theta}_0) &= \frac{1}{2\pi} x_n \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \tilde{I}_{n,X}(\omega) d\omega + O_p\left(\frac{x_n}{n}\right) \\ &= \frac{1}{2\pi} \frac{1}{\gamma_{n,X}^2} x_n \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \{ I_{n,X}(\omega) - T_n \psi^2 \tilde{g}(\omega) \} d\omega + O_p\left(\frac{x_n}{n}\right), \end{aligned}$$

where

$$T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{n,X}(\omega)}{\psi^2 \tilde{g}(\omega)} d\omega.$$

Then, by Proposition 3.5 of Klüppelberg and Mikosch [16] and Cramér–Wold device, we have

$$\begin{aligned} &\begin{pmatrix} \gamma_{n,X}^2 \\ x_n \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \theta_1} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \{ I_{n,X}(\omega) - T_n \psi^2 \tilde{g}(\omega) \} d\omega \\ \vdots \\ x_n \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \theta_q} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \{ I_{n,X}(\omega) - T_n \psi^2 \tilde{g}(\omega) \} d\omega \end{pmatrix} \\ &\xrightarrow{\mathcal{L}} \begin{pmatrix} \psi^2 S_0 \\ 2 \sum_{t=1}^{\infty} S_t \left\{ \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \theta_1} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \psi^2 \tilde{g}(\omega) \cos(t\omega) d\omega \right\} \\ \vdots \\ 2 \sum_{t=1}^{\infty} S_t \left\{ \int_{-\pi}^{\pi} \frac{\partial f(\omega; \boldsymbol{\theta})}{\partial \theta_q} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \psi^2 \tilde{g}(\omega) \cos(t\omega) d\omega \right\} \end{pmatrix}. \end{aligned}$$

Therefore

$$x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \xrightarrow{\mathcal{L}} \mathbf{V} \tag{6.13}$$

for $\alpha \in [1, 2)$ as $n \rightarrow \infty$, where \mathbf{V} is defined in Theorem 3.1. So we obtain

$$O_p(\|\phi\|_E) [O_p(1) - \underline{O}_p\{(\log n)^{2-2/\alpha}\} \cdot O_p(x_n^{-1})] \leq O_p(x_n^{-1}). \tag{6.14}$$

Because as $n \rightarrow \infty$,

$$\begin{aligned} (\log n)^{2-2/\alpha} x_n^{-1} &= (\log n)^{2-2/\alpha} \left(\frac{\log n}{n}\right)^{1/\alpha} \\ &= \frac{1}{(\log n)^{1/\alpha}} \frac{(\log n)^2}{n^{1/\alpha}} \\ &\rightarrow 0, \end{aligned}$$

the underlined part in (6.14) is $O_p(1)$. Therefore, we obtain

$$O_p(\|\phi\|_E) \leq O_p(x_n^{-1}). \tag{6.15}$$

On the other hand,

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \|\mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)\|_E^3 &= \frac{1}{n} \sum_{t=1}^n \|\mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)\|_E \|\mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)\|_E^2 \\ &\leq \frac{1}{n} \sum_{t=1}^n M_n \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)' \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \\ &= M_n \text{tr}\{\mathbf{S}_n(\boldsymbol{\theta}_0)\} \\ &= O_p\{(\log n)^{2-2/\alpha}\}. \end{aligned} \tag{6.16}$$

From (6.15) and (6.16), $\boldsymbol{\varepsilon}$ in (6.12) satisfies

$$\|\boldsymbol{\varepsilon}\|_E \leq \frac{1}{n} \sum_{t=1}^n \|\mathbf{m}(\lambda_t; \boldsymbol{\theta})\|_E^3 \|\phi\|_E^2 |1 + Y_t|^{-1}. \tag{6.17}$$

Thus, we have

$$O_p(\|x_n \boldsymbol{\varepsilon}\|_E) = O_p\left\{\frac{(\log n)^{2-1/\alpha}}{n^{1/\alpha}}\right\} \xrightarrow{\mathcal{P}} 0.$$

Now let us show the convergence of the empirical likelihood ratio statistic. Under $H: \boldsymbol{\theta} = \boldsymbol{\theta}_0$, $-2(x_n^2/n) \log R(\boldsymbol{\theta}_0)$ can be expanded as

$$\begin{aligned} -2\frac{x_n^2}{n} \log R(\boldsymbol{\theta}_0) &= -2\frac{x_n^2}{n} \sum_{t=1}^n \log n w_t \\ &= 2\frac{x_n^2}{n} \sum_{t=1}^n \log(1 + Y_t) \\ &= 2\frac{x_n^2}{n} \sum_{t=1}^n Y_t - \frac{x_n^2}{n} \sum_{t=1}^n Y_t^2 + 2\frac{x_n^2}{n} \sum_{t=1}^n O_p(Y_t^3), \end{aligned}$$

where

$$\begin{aligned}
 2\frac{x_n^2}{n} \sum_{t=1}^n Y_t &= 2\frac{x_n^2}{n} \sum_{t=1}^n \phi' \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \\
 &= 2\frac{x_n^2}{n} \{ \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} \mathbf{P}_n(\boldsymbol{\theta}_0) + \boldsymbol{\varepsilon} \}' \sum_{t=1}^n \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \\
 &= 2x_n^2 \{ \mathbf{P}_n(\boldsymbol{\theta}_0)' \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} + \boldsymbol{\varepsilon}' \} \mathbf{P}_n(\boldsymbol{\theta}_0) \\
 &= 2 \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \}' \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \} + 2(x_n \boldsymbol{\varepsilon})' \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \}, \\
 \frac{x_n^2}{n} \sum_{t=1}^n Y_t^2 &= \frac{x_n^2}{n} \sum_{t=1}^n \{ \phi' \mathbf{m}(\lambda_t; \boldsymbol{\theta}_0) \}^2 \\
 &= x_n^2 \phi' \mathbf{S}_n(\boldsymbol{\theta}_0) \phi \\
 &= x_n^2 \{ \mathbf{P}_n(\boldsymbol{\theta}_0)' \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} + \boldsymbol{\varepsilon}' \} \mathbf{S}_n(\boldsymbol{\theta}_0) \{ \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} \mathbf{P}_n(\boldsymbol{\theta}_0) + \boldsymbol{\varepsilon} \} \\
 &= \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \}' \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \} \\
 &\quad + (x_n \boldsymbol{\varepsilon})' \mathbf{S}_n(\boldsymbol{\theta}_0) (x_n \boldsymbol{\varepsilon}) + 2(x_n \boldsymbol{\varepsilon})' \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{x_n^2}{n} \left| \sum_{t=1}^n O_p(Y_t^3) \right| &\leq \frac{x_n^2}{n} \exists c \sum_{t=1}^n |Y_t|^3 \\
 &= \frac{x_n^2}{n} c \|\phi\|_E^3 \sum_{t=1}^n \|\mathbf{m}(\lambda_t; \boldsymbol{\theta}_0)\|_E^3 \\
 &= \frac{x_n^2}{n} O_p(x_n^{-3}) \cdot O_p\{n(\log n)^{2-2/\alpha}\} \\
 &= O_p\left\{ \frac{(\log n)^{2-1/\alpha}}{n^{1/\alpha}} \right\} \\
 &\xrightarrow{\mathcal{P}} 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

Hence, using (6.13) and Lemma 6.3,

$$\begin{aligned}
 -\frac{2x_n^2}{n} \log R(\boldsymbol{\theta}_0) &= \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \}' \mathbf{S}_n(\boldsymbol{\theta}_0)^{-1} \{ x_n \mathbf{P}_n(\boldsymbol{\theta}_0) \} + o_p(1) \\
 &\xrightarrow{\mathcal{L}} \mathbf{V}' \mathbf{W}^{-1} \mathbf{V}
 \end{aligned}$$

for $\alpha \in [1, 2)$.

□

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Supplementary Material

Proof of Theorem 4.1 (DOI: [10.3150/14-BEJ636SUPP](https://doi.org/10.3150/14-BEJ636SUPP); .pdf). We provide additional supporting material for the proof of Theorem 4.1.

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