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Local asymptotic mixed normality property for nonsynchronously observed diffusion processes

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We prove the local asymptotic mixed normality (LAMN) property for a family of probability measures defined by parametrized diffusion processes with nonsynchronous observations. We assume that observation times of processes are independent of processes and we will study asymptotics when the maximum length of observation intervals goes to zero in probability. We also prove that the quasi-maximum likelihood estimator and the Bayes-type estimator proposed in Ogihara and Yoshida (*Stochastic Process. Appl.* **124** (2014) 2954–3008) are asymptotically efficient.

Keywords: asymptotic efficiency; Bayes-type estimators; diffusion processes; local asymptotic mixed normality property; Malliavin calculus; nonsynchronous observations; parametric estimation; quasi-maximum likelihood estimators

1. Introduction

Given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, we consider a two-dimensional $\mathbf{F}$-adapted process $Y = \{Y_t\}_{0\le t\le T} = \{(Y_t^1, Y_t^2)\}_{0\le t\le T}$ satisfying the following stochastic differential equation:

$$\mathrm{d}Y_t = \mu(t, Y_t, \sigma_*)\,\mathrm{d}t + b(t, Y_t, \sigma_*)\,\mathrm{d}W_t, \qquad t \in [0, T], \tag{1.1}$$

where $\{W_t\}_{0\le t\le T}$ is a two-dimensional standard $\mathbf{F}$-Wiener process, $b = (b^{ij})_{1\le i,j\le 2} : [0, T] \times \mathbb{R}^2 \times \Lambda \to \mathbb{R}^2 \otimes \mathbb{R}^2$ is a Borel function, $\mu = (\mu^1, \mu^2)$ is a $\mathbb{R}^2$-valued function, $\sigma_* \in \Lambda$, and Λ is a bounded open subset of $\mathbb{R}^d$.

We will consider the problem of estimating the unknown true value σ_* of the parameter by nonsynchronous observations $\{Y^1_{S^{n,i}}\}_{i=0}^{\ell_{1,n}}$ and $\{Y^2_{T^{n,j}}\}_{j=0}^{\ell_{2,n}}$, where $\{S^{n,i}\}_{i=0}^{\ell_{1,n}}$ and $\{T^{n,j}\}_{j=0}^{\ell_{2,n}}$ are observation times of Y^1 and Y^2, respectively.

The problem of nonsynchronous observations appears when we study statistical inference for high-frequency financial data. Hayashi and Yoshida [12] pointed out that simple '*synchronization*' methods such as linear interpolation or 'previous-tick' interpolation do not work well for covariation estimation. They constructed a consistent estimator of the quadratic covariation of processes. On the other hand, Malliavin and Mancino [16] proposed an estimator based on a Fourier analytic method, and Ogihara and Yoshida [19] constructed a quasi-maximum likelihood estimator and a Bayes-type estimator for a statistical model of nonsynchronously ob-

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served diffusion processes. There are also several studies about covariation estimation under nonsynchronous observations and market microstructure noise. See Barndorff-Nielsen *et al.* [6], Christensen, Kinnebrock and Podolskij [8], Aït-Sahalia, Fan and Xiu [3], Bibinger *et al.* [7], for example.

In this work, we will study the local asymptotic mixed normality (LAMN) property of a statistical model of nonsynchronously observed diffusion processes. The definition of the LAMN property is as follows (Jeganathan [15]).

Definition 1.1. *Let $P_{\sigma,n}$ be a probability measure on some measurable space $(\mathcal{X}_n, \mathcal{A}_n)$ for each $\sigma \in \Lambda$ and $n \in \mathbb{N}$. Then the family $\{P_{\sigma,n}\}_{\sigma,n}$ satisfies the local asymptotic mixed normality (LAMN) property at $\sigma = \sigma_*$ if there exist a sequence $\{b_n\}_{n\in\mathbb{N}}$ of positive numbers, $d \times d$ symmetric random matrices Γ_n, Γ and d-dimensional random vectors $\mathcal{N}_n, \mathcal{N}$ such that Γ is positive definite a.s., $P_{\sigma_*,n}[\Gamma_n$ is positive definite$] = 1$ $(n \in \mathbb{N})$, $b_n \to \infty$,*

$$\log \frac{\mathrm{d}P_{\sigma_*+b_n^{-1/2}u,n}}{\mathrm{d}P_{\sigma_*,n}} - \left(u^\star\sqrt{\Gamma_n}\mathcal{N}_n - \frac{1}{2}u^\star\Gamma_n u\right) \to 0$$

in $P_{\sigma_*,n}$-probability as $n \to \infty$ for any $u \in \mathbb{R}^d$, where $\star$ represents transpose. Moreover, $\mathcal{N}$ follows the d-dimensional standard normal distribution, $\mathcal{N}$ is independent of Γ and $\mathcal{L}(\mathcal{N}_n, \Gamma_n | P_{\sigma_*,n}) \to \mathcal{L}(\mathcal{N}, \Gamma)$ as $n \to \infty$.

The LAMN property is significantly related to asymptotic efficiency of estimators. Let E_σ denote expectation with respect to $P_{\sigma,n}$. Jeganathan [15] proved the minimax theorem:

$$\lim_{\alpha\to\infty} \liminf_{n\to\infty} \sup_{|u|\leq\alpha} E_{\sigma_*+b_n^{-1/2}u}\left[l\left(\left|b_n^{1/2}\left(V_n - \sigma_* - b_n^{-1/2}u\right)\right|\right)\right] \geq E\left[l\left(\left|\Gamma^{-1/2}\mathcal{N}\right|\right)\right] \tag{1.2}$$

for any estimators $\{V_n\}_n$ and any function $l : [0,\infty) \to [0,\infty)$ which is nondecreasing and $l(0) = 0$, when the family $\{P_{\sigma,n}\}_{\sigma,n}$ has the LAMN property at $\sigma = \sigma_*$. This inequality gives lower bounds of risk functions of estimation errors. In particular, this inequality gives a lower bound of asymptotic variance of estimators if $l(x) = x^2$. When estimators $\{V_n\}_n$ attain the lower bound of (1.2), they are called *asymptotically efficient*.

In a statistical model with independent identically distributed random variables, the maximum likelihood estimator and the Bayes estimator have minimal asymptotic variance under certain regularity conditions. See Chapter I of Ibragimov and Has'minskii [13] for the details. The LAMN property is proved for a statistical model of one-dimensional diffusion process with synchronous, equispaced observations in Dohnal [9], and then the results are extended to a multi-dimensional diffusion in Gobet [10], by using a Malliavin calculus approach. On the other hand, Gobet [11] proved the LAN property (that means the LAMN property with a deterministic Γ) for ergodic diffusion process when the end time T of observations goes to infinity.

The aim of this paper is to show the LAMN property for nonsynchronously observed diffusion processes, and consequently have the minimax theorem (1.2). We also prove that the quasi-maximum likelihood estimator and the Bayes-type estimator proposed in Ogihara and Yoshida [19,20] are asymptotically efficient. Ogihara and Yoshida [19] constructed an estimator of quadratic covariation of the processes based on the quasi-maximum likelihood estimator and verified that the variance of estimation error of the estimator is much smaller than that of the Hayashi–Yoshida estimator in a simple example.

When the observations occur in synchronous manner, the log-likelihood ratio $\log(\mathrm{d}P_{\sigma_*+b_n^{-1/2}u,n}/\mathrm{d}P_{\sigma_*,n})$ is decomposed into differences of logarithms of transition density functions. A Malliavin calculus approach enables us to apply limit theorems to these differences, and consequently to obtain the LAMN property, as seen in Gobet [10]. However, when the sampling scheme is a nonsynchronous one, the log-likelihood ratio does not have such a simple form and we cannot apply the Malliavin calculus approach in Gobet [10] directly. Instead, we will define stochastic processes that '*connect*' the process Y and an Euler–Maruyama approximation process (Section 3), and we prove asymptotic equivalence of the log-likelihood ratio of Y and that of Euler–Maruyama approximation. Since the log-likelihood ratio of Euler–Maruyama approximation is asymptotically equivalent to the quasi-log-likelihood ratio in Ogihara and Yoshida [19] and the quasi-log-likelihood ratio has a LAMN-type property, we obtain the LAMN property of the model.

This paper is organized as follows. Section 2 presents assumptions and main theorems. Section 3 contains some preliminary results. In Section 3.1, we introduce fundamental lemmas, some notation and the result in Ogihara and Yoshida [19] with respect to a LAMN-type property of the quasi-log-likelihood ratio. Section 3.2 gives some results in Malliavin calculus, and Section 3.3 is devoted to prove tightness of some log-likelihood ratios, which is used in the proof of the LAMN property. We complete the proof of the main theorem in Section 4.

2. Main results

We begin with some general conventions. For a real number x, $[x]$ denotes the maximum integer which is not greater than x. Let us denote by $|K|$ the length of interval K. For a matrix M, $\|M\|$ represents the operator norm of M and $M^\star$ represents transpose of M. Let $\mathcal{E}_l$ be the unit matrix of size l and $\delta_{i,j}$ be Kronecker's delta function. We denote $|x|^2 = \sum_{i_1,\dots,i_k} |x_{i_1,\dots,i_k}|^2$ for $x = \{x_{i_1,\dots,i_k}\}_{i_1,\dots,i_k}$. For a vector $x = (x_1, \dots, x_k)$, we denote $\partial_x^l = (\frac{\partial^l}{\partial x_{i_1}\cdots\partial x_{i_l}})_{i_1,\dots,i_l=1}^k$. We use the symbol C for a generic positive constant varying from line to line. We denote by $\to^{s\text{-}\mathcal{L}}$ stable convergence of a random sequence, which is stronger than convergence in distribution and weaker than convergence in probability. See Aldous and Eagleson [4] or Jacod [14] for the definition and fundamental properties of stable convergence.

Let us start with some definitions and assumptions. The end time $T > 0$ of observations is assumed to be a fixed constant. We assume that the parameter space Λ satisfies Sobolev's inequality, that is, for any $p > d$, there exists $C > 0$ such that

$$\sup_{x\in\Lambda}\bigl|u(x)\bigr| \le C \sum_{k=0,1} \bigl\|\partial_x^k u(x)\bigr\|_p \qquad \bigl(u \in C^1(\Lambda)\bigr).$$

It is the case if Λ has a Lipschitz boundary (see Adams [1], Adams and Fournier [2]).

Let $\{\ell_{1,n}\}_{n\in\mathbb{N}}$ and $\{\ell_{2,n}\}_{n\in\mathbb{N}}$ be sequences of positive integer-valued random variables, the observation times $\Pi_n = ((S^{n,i})_{i=0}^{\ell_{1,n}}, (T^{n,j})_{j=0}^{\ell_{2,n}})$ satisfy $S^{n,0} = T^{n,0} = 0$, $S^{n,\ell_{1,n}} = T^{n,\ell_{2,n}} = T$

and random times $\{S^{n,i}\}_i, \{T^{n,j}\}_j$ be monotone increasing with respect to i, j. Moreover, we assume that $\sigma(\{\Pi_n\}_n)$ is independent of $\{(Y_t, W_t)\}_{0\leq t\leq T}$. We assume that Π_n and Y_0 do not depend on σ_*.

Let $b^k = (b^{k1}, b^{k2})$ for $k = 1, 2$, where $\{b^{ij}\}_{i,j}$ are elements of the diffusion coefficient b. Let $I^i = [S^{n,i-1}, S^{n,i})$, $J^j = [T^{n,j-1}, T^{n,j})$, $r_n = \max_{i,j}(|I^i| \vee |J^j|)$, $\mathcal{E}^1(t) = \{\delta_{i,i'} 1_{\{I^i\cap[0,t)\neq\varnothing\}}\}_{i,i'=1}^{\ell_{1,n}}$, $\mathcal{E}^2(t) = \{\delta_{j,j'} 1_{\{J^j\cap[0,t)\neq\varnothing\}}\}_{j,j'=1}^{\ell_{2,n}}$ for $t \in (0, T]$ and G be an $\ell_{1,n} \times \ell_{2,n}$ matrix with the elements $G_{ij} = |I^i \cap J^j||I^i|^{-1/2}|J^j|^{-1/2}$. Moreover, let

$$\mathcal{U} = \big\{\bar{u} = \big((s^i)_{i=0}^{L^1}, (t^j)_{j=0}^{L^2}\big); L^1, L^2 \in \mathbb{N},$$
$$0 = s^0 < s^1 < \cdots < s^{L^1} = T, 0 = t^0 < t^1 < \cdots < t^{L^2} = T\big\},$$

and we denote $\mathsf{X}_{\bar{u}} = ((\mathsf{X}^1_{s^i})_{i=0}^{L^1}, (\mathsf{X}^2_{t^j})_{j=0}^{L^2})$ and $\mathsf{X}_{\bar{v}} = ((\mathsf{X}^1_{v^i})_{i=0}^{L}, (\mathsf{X}^2_{v^j})_{j=0}^{L})$ for a two-dimensional stochastic process $\mathsf{X} = \{(\mathsf{X}^1_t, \mathsf{X}^2_t)\}_{0\leq t\leq T}$, $\bar{u} = ((s^i)_{i=0}^{L_1}, (t^j)_{j=0}^{L_2}) \in \mathcal{U}$ and $\bar{v} = (v^i)_{i=0}^{L}$ satisfying $0 = v^0 < \cdots < v^L = T$. Let $Y^{(\sigma)} = \{Y^{(\sigma)}_t\}_{0\leq t\leq T}$ denote the two-dimensional diffusion process satisfying (1.1) with a parameter σ and $Y^{(\sigma)}_0 = Y_0$. Let $P_{\sigma,n}$ be the distribution of $(\Pi_n, Y^{(\sigma)}_{\Pi_n})$.

Our purpose is to obtain the LAMN property of probability measures $\{P_{\sigma,n}\}_{\sigma\in\Lambda,n\in\mathbb{N}}$ of nonsynchronous observations $(\Pi_n, Y^{(\sigma)}_{\Pi_n})$. For this purpose, we will introduce several assumptions. First, we consider conditions for the process Y.

[$A1$]

1. For $0 \leq i + j \leq 3$ and $0 \leq k \leq 4$, the derivatives $\partial_t^i \partial_x^j \partial_\sigma^k b$ and $\partial_t^i \partial_x^j \partial_\sigma^k \mu$ exist and are continuous with respect to (t, x, σ). Moreover, $\partial_x \mu$, $\partial_x b$ are bounded uniformly in $[0, T] \times \mathbb{R}^2 \times \Lambda$.
2. A matrix $(rb(t_1, x_1, \sigma) + (1 - r)b(t_2, x_2, \sigma))(rb(t_1, x_1, \sigma) + (1 - r)b(t_2, x_2, \sigma))^\star$ is positive definite for any $r \in [0, 1]$, $t_1, t_2 \in [0, T]$, $x_1, x_2 \in \mathbb{R}^2$ and $\sigma \in \Lambda$.
3. $E[|Y_0|^2] < \infty$.

Condition [$A1$] is similar conditions to that for the LAMN property of the statistical model with synchronous, equispaced observations in Gobet [10]. We do not need further conditions for the process Y. If the diffusion coefficient b is symmetric and positive definite, we have [$A1$] 2.

Second, we give assumptions of observation times. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $b_n \geq 1$ and $b_n \to \infty$ as $n \to \infty$.

[$A2$] There exist positive constants $\{\delta_j\}_{j=1}^3$ such that $(5\delta_1 + 4\delta_3) \vee (3\delta_1 + 2\delta_2 + 2\delta_3) \vee (3\delta_1/2 + 3\delta_2) < 1/2$ and the following conditions hold true:

1. $r_n = O_p(b_n^{-1+\delta_1})$.
2.

$$\lim_{n\to\infty} b_n^2 \sup_{j_1,j_2\in\mathbb{N},|j_1-j_2|\geq b_n^{\delta_2}} P\left[\ell_{1,n} \geq j_1 \vee j_2 \text{ and } \frac{|S^{n,j_2} - S^{n,j_1}|}{|j_2 - j_1|} \leq b_n^{-1-\delta_3}\right] = 0, \quad (2.1)$$

$$\lim_{n\to\infty} b_n^2 \sup_{j_1,j_2\in\mathbb{N},|j_1-j_2|\geq b_n^{\delta_2}} P\left[\ell_{2,n} \geq j_1 \vee j_2 \text{ and } \frac{|T^{n,j_2} - T^{n,j_1}|}{|j_2 - j_1|} \leq b_n^{-1-\delta_3}\right] = 0. \quad (2.2)$$

Condition $[A2]$ 2. controls the probability that too many observations occur in some local interval. For example, if we set $S^{n,i} = iT/n^2$ for $0 \le i \le n$, $S^{n,i} = (i+1-n)T/n$ for $n+1 \le i \le 2n-1$ and $T^{n,j} = jT/n$ for $0 \le j \le n$, then we can easily see that $[A2]$ 2. is not satisfied for $b_n \equiv n$. In this setting, extremely many observations of Y^1 occur in the interval $[0, T/n]$ compared to other intervals. Condition $[A2]$ is a condition to exclude observations with such extremely different frequency. This condition is necessary to obtain asymptotic equivalence between the true log-likelihood ratios and the quasi-log-likelihood ratios defined later (Lemmas 4.3, 4.7 and 4.8), and to obtain convergence results of the quasi-log-likelihood ratios (Theorem 3.1).

We need one more condition for observation times.

$[A3]$ There exist $\sigma(\{\Pi_n\}_n)$-measurable left-continuous processes $a_0(t)$ and $c_0(t)$ such that $\int_0^T a_0(t)\,\mathrm{d}t \vee \int_0^T c_0(t)\,\mathrm{d}t < \infty$ almost surely, $b_n^{-1}\operatorname{tr}(\mathcal{E}^1(t)) \to^p \int_0^t a_0(s)\,\mathrm{d}s$ and $b_n^{-1}\operatorname{tr}(\mathcal{E}^2(t)) \to^p \int_0^t c_0(s)\,\mathrm{d}s$ as $n \to \infty$ for $t \in (0, T]$. Moreover, at least one of the following conditions holds true:

1. There exist $\eta \in (0,1)$ and a $\sigma(\{\Pi_n\}_n)$-measurable process $a(z,t)$ such that a is continuous with respect to z, left-continuous with respect to t, $\int_0^T a(z,t)\,\mathrm{d}t < \infty$ a.s. and

$$b_n^{-1}\operatorname{tr}\big(\mathcal{E}^1(t)\big(\mathcal{E}_{\ell_{1,n}} - z^2 GG^\star\big)^{-1}\big) \to^p \int_0^t a(z,s)\,\mathrm{d}s$$

as $n \to \infty$ for $t \in (0,T]$ and $z \in \mathbb{C}$, $|z| < \eta$.

2. There exist $\eta \in (0,1)$ and a $\sigma(\{\Pi_n\}_n)$-measurable process $c(z,t)$ such that c is continuous with respect to z, left-continuous with respect to t, $\int_0^T c(z,t)\,\mathrm{d}t < \infty$ a.s. and

$$b_n^{-1}\operatorname{tr}\big(\mathcal{E}^2(t)\big(\mathcal{E}_{\ell_{2,n}} - z^2 G^\star G\big)^{-1}\big) \to^p \int_0^t c(z,s)\,\mathrm{d}s$$

as $n \to \infty$ for $t \in (0,T]$ and $z \in \mathbb{C}$, $|z| < \eta$.

In particular, $[A3]$ implies tightness of $\{b_n^{-1}(\ell_{1,n} + \ell_{2,n})\}_n$.

Lemma 4 in Ogihara and Yoshida [19] shows that *both* 1. and 2. in $[A3]$ hold true if $r_n \to^p 0$ and $[A3]$ holds true, that is, the first statement of $[A3]$ and *either* 1. or 2. in $[A3]$ hold true. Moreover, a and c are analytic with respect to z and $a(z,t) - a(0,t) = c(z,t) - c(0,t)$ for any $z \in \mathbb{C}$, $|z| < \eta$ and $t \in [0,T]$ almost surely, assuming that $r_n \to^p 0$ and $[A3]$ (Lemmas 3 and 4 and Proposition 2 in [19]). We will give tractable sufficient conditions of $[A2]$ and $[A3]$ in Lemmas 2.1 and 2.2.

The intuitive meaning of $[A3]$ is as follows. If $\mu \equiv 0$ and $b(t,x,\sigma)$ does not depend on (t,x), then Y is a Wiener process and we obtain

$$\log(\mathrm{d}P_{\sigma_* + b_n^{-1/2}u,n}/\mathrm{d}P_{\sigma_*,n}) = H_n\big(\sigma_* + b_n^{-1/2}u\big) \circ (\Pi, Y_\Pi) - H_n(\sigma_*) \circ (\Pi, Y_\Pi),$$

where $H_n(\sigma)$ is defined in (2.4). Roughly speaking, $H_n(\sigma) \circ (\Pi, Y_\Pi)$ is asymptotically equivalent to

$$\begin{aligned} &E\big[H_n(\sigma) \circ (\Pi, Y_\Pi)|\Pi\big] \\ &\quad = -\frac{|b^1|^2(\sigma_*)}{2|b^1|^2(\sigma)} \operatorname{tr}\big(\big(\mathcal{E}_{\ell_{1,n}} - \rho^2 GG^\star\big)^{-1}\big) - \frac{|b^2|^2(\sigma_*)}{2|b^2|^2(\sigma)} \operatorname{tr}\big(\big(\mathcal{E}_{\ell_{2,n}} - \rho^2 G^\star G\big)^{-1}\big) \\ &\qquad + \frac{b^1 \cdot b^2(\sigma_*)}{|b^1||b^2|(\sigma)} \operatorname{tr}\big(\rho\big(\mathcal{E}_{\ell_{1,n}} - \rho^2 GG^\star\big)^{-1} GG^\star\big) - \frac{1}{2} \log\det S(\sigma), \end{aligned}$$

where $\rho = \rho(\sigma) = b^1 \cdot b^2 |b^1|^{-1}|b^2|^{-1}(\sigma)$. Therefore, it is natural to assume conditions about asymptotic behaviors of $\operatorname{tr}((\mathcal{E}_{\ell_{1,n}} - \rho^2 GG^\star)^{-1})$ and $\operatorname{tr}((\mathcal{E}_{\ell_{2,n}} - \rho^2 G^\star G)^{-1})$ in this special case of μ and b. Since the diffusion coefficient of the diffusion process Y in general is locally approximated by a constant and asymptotic contribution of drift coefficient μ is negligible, $[A3]$ is suitable for specifying asymptotic behaviors of log-likelihood ratios in general cases.

Let $B_t^k = B_t^k(\sigma) = |b^k(t, Y_t, \sigma_*)|/|b^k(t, Y_t, \sigma)|$ for $k = 1, 2$, $\rho_t = \rho_t(\sigma) = b^1 \cdot b^2 |b^1|^{-1} \times |b^2|^{-1}(t, Y_t, \sigma)$ and

$$\begin{aligned} \Gamma = \int_0^T \bigg\{ &\partial_z a\big(\rho_t(\sigma_*), t\big) \frac{(\partial_\sigma \rho_t(\sigma_*))^2}{\rho_t(\sigma_*)} 1_{\{\rho_t(\sigma_*) \neq 0\}} \\ &+ 2a\big(\rho_t(\sigma_*), t\big)\big(\partial_\sigma B_t^1(\sigma_*)\big)^2 + 2c\big(\rho_t(\sigma_*), t\big)\big(\partial_\sigma B_t^2(\sigma_*)\big)^2 \\ &- \big(a\big(\rho_t(\sigma_*), t\big) - a(0, t)\big)\bigg(\frac{\partial_\sigma \rho_t(\sigma_*)}{\rho_t(\sigma_*)} 1_{\{\rho_t(\sigma_*) \neq 0\}} - \partial_\sigma B_t^1(\sigma_*) - \partial_\sigma B_t^2(\sigma_*)\bigg)^2 \bigg\} \,\mathrm{d}t. \end{aligned} \tag{2.3}$$

We also assume the following condition.

$[H]$ The $d \times d$ random matrix Γ is positive definite almost surely.

We can now formulate our main theorem.

Theorem 2.1. *Assume $[A1]$–$[A3]$ and $[H]$. Then the family $\{P_{\sigma,n}\}_{\sigma,n}$ defined by nonsynchronous observations (Π_n, Y_{Π_n}) has the LAMN property at $\sigma = \sigma_*$, where $\mathcal{N}$ in Definition 1.1 is a random variable on an extension of $(\Omega, \mathcal{F}, P)$, $\mathcal{N}$ is independent of $\mathcal{F}$ and Γ in Definition 1.1 is defined by (2.3). Moreover, $\mathcal{N}_n$ and Γ_n can be taken so that $(\mathcal{N}_n, \Gamma_n) \circ (\Pi_n, Y_{\Pi_n}) \to^{s\text{-}\mathcal{L}} (\mathcal{N}, \Gamma)$.*

Conditions $[A2]$, $[A3]$ and $[H]$ are often not easy to check for practical settings. So we see some easily tractable sufficient conditions for these conditions.

$[B1]$ There exists exponential α-mixing simple point process $\{\bar{N}_t\}_{t\geq 0} = \{(\bar{N}_t^1, \bar{N}_t^2)\}_{t\geq 0}$ such that $\bar{N}_0 = 0$, $S^{n,i} = \inf\{t \geq 0; \bar{N}^1_{b_n t} \geq i\} \wedge T$, $T^{n,j} = \inf\{t \geq 0; \bar{N}^2_{b_n t} \geq j\} \wedge T$ and the distribution of $(\bar{N}^i_{t+t_k} - \bar{N}^i_{t+t_{k-1}})_{k=1}^M$ does not depend on $t \geq 0$ for $M \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_M$ and $i = 1, 2$. Moreover, $E[|\bar{N}_1|^q] < \infty$ and $\limsup_{u\to\infty} \max_{i=1,2} u^q P[\bar{N}_u^i = 0] < \infty$ for any $q > 0$.

[H'] There exists a constant $\varepsilon > 0$ such that $|bb^\star(t,x,\sigma_1) - bb^\star(t,x,\sigma_2)| \geq \varepsilon|\sigma_1 - \sigma_2|$ for any $t \in [0,T]$, $x \in \mathbb{R}^2$ and $\sigma_1, \sigma_2 \in \Lambda$.

For example, we can easily see that condition [$B1$] is satisfied if the processes $\{\bar{N}^1\}_{t\geq 0}$ and $\{\bar{N}^2\}_{t\geq 0}$ are two independent homogeneous Poisson processes.

The following lemma is proved in Section 6, Proposition 4 and Remark 2 in Ogihara and Yoshida [19]. (We also use some localization techniques.)

Lemma 2.1. 1. *Condition* [$B1$] *implies* [$A3$].

2. *Assume* [$A1$], [$B1$] *and* [H']. *Then* [H] *holds true.*

Let $\mathbf{N}_t^1 = \sum_{i=1}^{\ell_{1,n}} 1_{\{S^{n,i}\leq t\}}$ and $\mathbf{N}_t^2 = \sum_{j=1}^{\ell_{2,n}} 1_{\{T^{n,j}\leq t\}}$. Then we also have the following. The proof is left in the Appendix.

Lemma 2.2. *Let $q > 0$. Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{n\geq n_0} \max_{1\leq i\leq 2} \sup_{0\leq t\leq T-b_n^{-1}} E\big[\big(\mathbf{N}^i_{t+b_n^{-1}} - \mathbf{N}^i_t\big)^q\big] < \infty.$$

Then (2.1) *and* (2.2) *hold true for any $\delta_2 > 3/q$ and $\delta_3 > 3/q$. In particular,* [$B1$] *implies* [$A2$].

Remark 2.1. Conditions [$B1$] and [H'] are the simplest sufficient conditions of [$A2$], [$A3$] and [H]. More detailed discussion about sufficient conditions of [$A3$] and [H] can be found in Sections 4 and 6 in Ogihara and Yoshida [19] and Section 4 in Uchida and Yoshida [22].

By Theorem 2.1, we obtain the minimax theorem (1.2) under the conditions in Theorem 2.1. In the rest of this section, we will prove that the quasi-maximum likelihood estimator and the Bayes-type estimator defined in Ogihara and Yoshida [19] attain the lower bound in (1.2) under certain conditions. So these estimators are asymptotically efficient in this sense. For these purposes, we use the scheme of Yoshida [23] which leads to convergence of moments of estimators.

We will make the assumptions for asymptotic efficiency of estimators. We denote $\omega_\alpha(g) = \sup_{t\neq s}|g(t) - g(s)|/|t-s|^\alpha$ for $\alpha \in (0,1/2)$ and an α-Hölder continuous function $g : [0,T] \to \mathbb{R}$. Let $\mathbf{K}(\bar{u}) = \{[s^{i-1}, s^i)\}_{i=1}^{L^1} \cup \{[t^{j-1}, t^j)\}_{j=1}^{L^2}$ and $\{\theta(p,l;\bar{u})\}_{1\leq l\leq L^1+L^2, p\in\mathbb{Z}_+}$ be defined by $\theta(0,l;\bar{u}) = [s^{l-1}, s^l)$ $(1 \leq l \leq L^1)$, $\theta(0,l;\bar{u}) = [t^{l-L^1-1}, t^{l-L^1})$ $(L^1 < l \leq L^1 + L^2)$ and

$$\theta(p,l;\bar{u}) = \bigcup\big\{K_{2p};\, K_1,\dots,K_{2p} \in \mathbf{K}(\bar{u}),\\ K_1 \cap \theta(0,l;\bar{u}) \neq \varnothing,\, K_j \cap K_{j-1} \neq \varnothing\ (2 \leq j \leq 2p)\big\}$$

for $p \in \mathbb{N}$, $\bar{u} = ((s^i)_{i=0}^{L^1}, (t^j)_{j=0}^{L^2}) \in \mathcal{U}$ and $1 \leq l \leq L^1 + L^2$. That is, the interval $\theta(p,l;\bar{u})$ is the union of intervals which are reached by $2p$ *transfers* from $\theta(0,l;\bar{u})$. Let $\theta_{p,l} = \theta(p,l;\Pi)$.

Let $q > 2$, $\delta \in (0,1)$, $\delta' \geq 1$ and $\eta \in (0,1)$.

[$C1$]

1. The functions b and μ have continuous derivatives $\partial_t^i \partial_x^j \partial_\sigma^k b$, $\partial_t^{i'} \partial_x^{j'} \partial_\sigma^{k'} \mu$ and satisfy

$$\sup_{t\in[0,T],\sigma\in\Lambda} \left|\partial_t^i \partial_x^j \partial_\sigma^k b(t,x,\sigma)\right| \le C\big(1+|x|\big)^C \quad \text{and}$$

$$\sup_{t\in[0,T],\sigma\in\Lambda} \left|\partial_t^{i'} \partial_x^{j'} \partial_\sigma^{k'} \mu(t,x,\sigma)\right| \le C\big(1+|x|\big)^C$$

for $0 \le i + j \le 3$, $0 \le k \le 4$, $0 \le i' + j' + k' \le 1$ and $x \in \mathbb{R}^2$.

2. The derivatives $\partial_x \mu$ and $\partial_x b$ are bounded uniformly in $[0,T] \times \mathbb{R}^2 \times \Lambda$.
3. $\inf_{t,x,\sigma} \det bb^\star(t,x,\sigma) > 0$.
4. $\sup_\sigma \sup_{0\le t\le T} E[|Y_t^{(\sigma)}|^q] < \infty$ for any $q > 0$.
5. The function $\partial_\sigma^k b$ can be continuously extended to $[0,T] \times \mathbb{R}^2 \times \mathrm{clos}(\Lambda)$ for $0 \le k \le 4$, where $\mathrm{clos}(\Lambda)$ represents the closure of Λ.

[$C2$-q,δ] $E[r_n^q] = \mathrm{O}(b_n^{-\delta q})$.

[$C3$-q,η] There exist $n_0 \in \mathbb{N}$, $\alpha \in (0, 1/2 - 1/q)$ and $\sigma(\{\Pi_n\}_n)$-measurable left-continuous processes $\{a_p(t)\}_{p\in\mathbb{Z}_+}$ and $\{c_p(t)\}_{p\in\mathbb{Z}_+}$ such that $\int_0^T (a_p \vee c_p)(t)\,\mathrm{d}t \in L^q(\Omega)$ for $p \in \mathbb{Z}_+$, $E[(\ell_{1,n} + \ell_{2,n})^q] < \infty$ for $n \in \mathbb{N}$ and

$$\begin{aligned} &E\left[\left(b_n^\eta\left|b_n^{-1}\sum_{i=1}^{\ell_{1,n}} g\big(S^{n,i-1}\big)\big((GG^\star)^p\big)_{ii} - \int_0^T g(t)a_p(t)\,\mathrm{d}t\right|\right)^q\right] \\ &\quad\vee E\left[\left(b_n^\eta\left|b_n^{-1}\sum_{j=1}^{\ell_{2,n}} g\big(T^{n,j-1}\big)\big((G^\star G)^p\big)_{jj} - \int_0^T g(t)c_p(t)\,\mathrm{d}t\right|\right)^q\right] \\ &\le C(p+1)^C\Big(\sup_t\big|g(t)\big|^q + \omega_\alpha(g)^q\Big)\end{aligned}$$

for $n \ge n_0$, $p \in \mathbb{Z}_+$ and any α-Hölder continuous function g on $[0,T]$.

[$C4$-q,δ']

$$\begin{aligned} &\lim_{n\to\infty}\left\{E\left[\big(b_n^{-q/2} \vee r_n^q\big)\sum_{p=0}^\infty \frac{(\sum_{l=1}^{\ell_{1,n}+\ell_{2,n}} |\theta_{2p+2,l}|)^q}{(p+1)^{q\delta'}}\right]\right. \\ &\qquad\left.\vee E\left[\left(b_n^{-1}\sum_{p_1,p_2=0}^\infty \frac{\sum_{l_1,l_2=1}^{\ell_{1,n}+\ell_{2,n}} |\theta_{2p_1+3,l_1} \cap \theta_{2p_2+3,l_2}|}{(p_1+1)^{\delta'}(p_2+1)^{\delta'}}\right)^{q/2}\right]\right\} \\ &= 0.\end{aligned}$$

Condition [$C3$-q,η] is a stronger condition than [$A3$] and is required to obtain moment convergence of estimation errors. For any $q > 2$ and $\eta \in (0,1)$, we can prove that [$C3$-q,η] implies [$A3$]. See Section 3.1 in Ogihara and Yoshida [19] for the details.

Condition $[C4\text{-}q, \delta']$ is a technical condition to obtain the asymptotic properties of quasi-likelihood ratios and its derivatives. This condition together with Lemma 13 in [19] enable us to apply martingale limit theorems to the quasi-likelihood ratios, and hence it is essential to obtain asymptotic properties of quasi-likelihood ratios. See Propositions 3 and 10 in [19] and their proofs for the details.

Let $B_t^k(\sigma_1; \sigma_2) = |b^k(t, Y_t^{(\sigma_2)}, \sigma_2)|/|b^k(t, Y_t^{(\sigma_2)}, \sigma_1)|$ for $k = 1, 2$, $\rho_t(\sigma_1; \sigma_2) = b^1 \cdot b^2 |b^1|^{-1} \times |b^2|^{-1}(t, Y_t^{(\sigma_2)}, \sigma_1)$, and

$$\begin{aligned}\mathcal{Y}(\sigma_1; \sigma_2) = \int_0^T \bigg\{ & -\frac{(B_t^1(\sigma_1; \sigma_2))^2}{2} a\big(\rho_t(\sigma_1; \sigma_2), t\big) - \frac{(B_t^2(\sigma_1; \sigma_2))^2}{2} c\big(\rho_t(\sigma_1; \sigma_2), t\big) \\ & + B_t^1 B_t^2(\sigma_1; \sigma_2)\big(a\big(\rho_t(\sigma_1; \sigma_2), t\big) - a_0(t)\big) \frac{\rho_t(\sigma_2; \sigma_2)}{\rho_t(\sigma_1; \sigma_2)} 1_{\{\rho_t(\sigma_1;\sigma_2) \neq 0\}} \\ & + \frac{a_0(t)}{2} + \frac{c_0(t)}{2} + a_0(t) \log B_t^1(\sigma_1; \sigma_2) \\ & + c_0(t) \log B_t^2(\sigma_1; \sigma_2) + \int_{\rho_t(\sigma_2;\sigma_2)}^{\rho_t(\sigma_1;\sigma_2)} \frac{a(\rho, t) - a_0(t)}{\rho} \, \mathrm{d}\rho \bigg\} \, \mathrm{d}t,\end{aligned}$$

where $\{a(z, t)\}$ and $\{c(z, t)\}$ are in $[A3]$.

$[C5]$ There exist a family $\{\tilde{c}_q\}_{q>0}$ of positive constants and an open set Λ' satisfying $\sigma_* \in \Lambda' \subset \Lambda$ such that $\sup_{\sigma_2 \in \Lambda'} P[\inf_{\sigma_1 \in \Lambda \setminus \{\sigma_2\}} (-\mathcal{Y}(\sigma_1; \sigma_2)/|\sigma_1 - \sigma_2|^2) \leq r^{-1}] \leq \tilde{c}_q / r^q$ for $r > 0$ and $q > 0$.

We see that $[C5]$ implies $[H]$, by using the relations $\mathcal{Y}(\sigma_*; \sigma_*) = \partial_{\sigma_1} \mathcal{Y}(\sigma_1; \sigma_*)|_{\sigma_1 = \sigma_*} = 0$, $\Gamma = -\partial_{\sigma_1}^2 \mathcal{Y}(\sigma_1; \sigma_*)|_{\sigma_1 = \sigma_*}$, and hence $\inf_{\sigma \neq \sigma_*}(-\mathcal{Y}(\sigma; \sigma_*)/|\sigma - \sigma_*|^2) \leq \inf_{u \neq 0} u^\star \Gamma u/(2|u|^2)$. Condition $[C5]$ and $[H]$ are conditions about identifiability of statistical models. We only need $[H]$ to have Theorem 2.1. However, we need $[C5]$ to obtain asymptotic efficiency of estimators.

Ogihara and Yoshida [19] proposed a quasi-log-likelihood function H_n defined by

$$H_n(\sigma) \circ (\Pi, Y_\Pi) = -\tfrac{1}{2} Z^\star S^{-1}(\sigma) Z - \tfrac{1}{2} \log \det S(\sigma), \tag{2.4}$$

where

$$Z = \Big(\big(\big(Y^1_{S^{n,i}} - Y^1_{S^{n,i-1}}\big)/\sqrt{|I^i|}\big)_i^\star, \big(\big(Y^2_{T^{n,j}} - Y^2_{T^{n,j-1}}\big)/\sqrt{|J^j|}\big)_j^\star\Big)^\star, \tag{2.5}$$

$\bar{b}^1_{(i)} = \bar{b}^1_{(i)}(\sigma) = b^1(S^{n,i-1}, Y^1_{S^{n,i-1}}, Y^2_{T^{n,j'}}, \sigma)$ for $j' = \max\{j; T^{n,j} \leq S^{n,i-1}\}$, $\bar{b}^2_{(j)} = \bar{b}^2_{(j)}(\sigma) = b^2(T^{n,j-1}, Y^1_{S^{n,i'}}, Y^2_{T^{n,j-1}}, \sigma)$ for $i' = \max\{i; S^{n,i} \leq T^{n,j-1}\}$ and

$$S(\sigma) = \begin{pmatrix} \operatorname{diag}\big(\{|\bar{b}^1_{(i)}|^2\}_i\big) & \{\bar{b}^1_{(i)} \cdot \bar{b}^2_{(j)} G_{ij}\}_{i,j} \\ \{\bar{b}^1_{(i)} \cdot \bar{b}^2_{(j)} G_{ij}\}_{j,i} & \operatorname{diag}\big(\{|\bar{b}^2_{(j)}|^2\}_j\big) \end{pmatrix}. \tag{2.6}$$

An intuitive meaning of H_n is as follows. If $\mu \equiv 0$, $b(t, x, \sigma)$ does not depend on x and Π is deterministic, then Z follows a zero-mean normal distribution. Moreover, the covariance matrix of Z is approximated as

$$E\left[\frac{Y^1_{S^{n,i}}-Y^1_{S^{n,i-1}}}{\sqrt{|I^i|}}\frac{Y^1_{S^{n,i'}}-Y^1_{S^{n,i'-1}}}{\sqrt{|I^{i'}|}}\right]\sim\left|\bar{b}^1_{(i)}\right|^2(\sigma_*)\delta_{i,i'},$$

$$E\left[\frac{Y^2_{T^{n,j}}-Y^2_{T^{n,j-1}}}{\sqrt{|J^j|}}\frac{Y^2_{T^{n,j'}}-Y^2_{T^{n,j'-1}}}{\sqrt{|J^{j'}|}}\right]\sim\left|\bar{b}^2_{(j)}\right|^2(\sigma_*)\delta_{j,j'},$$

$$E\left[\frac{Y^1_{S^{n,i}}-Y^1_{S^{n,i-1}}}{\sqrt{|I^i|}}\frac{Y^2_{T^{n,j}}-Y^2_{T^{n,j-1}}}{\sqrt{|J^j|}}\right]\sim\bar{b}^1_{(i)}(\sigma_*)\cdot\bar{b}^2_{(j)}(\sigma_*)G_{ij}.$$

Hence, $S(\sigma)$ is approximation of the covariance matrix of Z. Therefore, we can say $H_n(\sigma)$ is an approximate log-likelihood function. These arguments are valid only for this special case of μ, b and Π. However, Ogihara and Yoshida [19] define H_n as above for general cases of μ, b and Π and studied the quasi-maximum likelihood estimator and the Bayes-type estimator constructed by H_n.

Let $\pi:\Lambda\to(0,\infty)$ be a bounded continuous function. The quasi-maximum likelihood estimator $\hat{\sigma}_n$ and the Bayes-type estimator $\tilde{\sigma}_n$ for the prior density π are defined by $\hat{\sigma}_n=\operatorname{argmax}_{\sigma\in\mathrm{clos}(\Lambda)}H_n(\sigma)$ and

$$\tilde{\sigma}_n=\left(\int_\Lambda\exp\bigl(H_n(\sigma)\bigr)\pi(\sigma)\,\mathrm{d}\sigma\right)^{-1}\int_\Lambda\sigma\exp\bigl(H_n(\sigma)\bigr)\pi(\sigma)\,\mathrm{d}\sigma.$$

Let $\sigma^n_u=\sigma_*+b_n^{-1/2}u$ for $u\in\mathbb{R}^d$.

Theorem 2.2. *Let $\delta\in(0,1/2)$. Assume that $0<\inf_\sigma\pi(\sigma)\le\sup_\sigma\pi(\sigma)<\infty$ and that for any $q>0$, there exist $\delta'\ge1$ and $q'\in\mathbb{N}$ satisfying $2q'>q$ such that $[C1]$, $[C2\text{-}(2q'),\delta]$, $[C3\text{-}(2q'),\delta]$, $[C4\text{-}(2q'),\delta']$, $[C5]$ hold. Then*

$$\lim_{\alpha\to\infty}\liminf_{n\to\infty}\sup_{|u|\le\alpha}E_{\sigma^n_u}\bigl[l\bigl(\bigl|b_n^{1/2}\bigl(\hat{\sigma}_n-\sigma^n_u\bigr)\bigr|\bigr)\bigr]=E\bigl[l\bigl(\bigl|\Gamma^{-1/2}\mathcal{N}\bigr|\bigr)\bigr],$$

$$\lim_{\alpha\to\infty}\liminf_{n\to\infty}\sup_{|u|\le\alpha}E_{\sigma^n_u}\bigl[l\bigl(\bigl|b_n^{1/2}\bigl(\tilde{\sigma}_n-\sigma^n_u\bigr)\bigr|\bigr)\bigr]=E\bigl[l\bigl(\bigl|\Gamma^{-1/2}\mathcal{N}\bigr|\bigr)\bigr]$$

for any continuous function $l:[0,\infty)\to[0,\infty)$ that is nondecreasing, $l(0)=0$ and of at most polynomial growth.

Remark 2.2. Theorems 2.2 and 2.1 and the minimax theorem by Jeganathan [15] imply that estimators $\hat{\sigma}_n$ and $\tilde{\sigma}_n$ are asymptotically efficient under $[A1]$, $[A2]$ and the conditions in Theorem 2.2.

Outline of the proof of Theorem 2.2. Let $\mathbf{G}_u=b_n^{1/2}(\hat{\sigma}_n\circ(\Pi,Y_\Pi^{(\sigma^n_u)})-\sigma^n_u)$. Then Theorem 2 in Ogihara and Yoshida [19] yields

$$\lim_{n\to\infty}E\bigl[l\bigl(|\mathbf{G}_0|\bigr)\bigr]=E\bigl[l\bigl(\bigl|\Gamma^{-1/2}\mathcal{N}\bigr|\bigr)\bigr].\tag{2.7}$$

Moreover, for any $\varepsilon, \delta > 0$, there exists $n_1 \in \mathbb{N}$ such that $\sup_{|u|\le\alpha} P[|\mathbf{G}_u - \mathbf{G}_0| > \delta] < \varepsilon$ for $n \ge n_1$, by a similar argument to the proof of 1. of Theorem 2 in Ogihara and Yoshida [19] and relations $E[\sup_t |Y_t^{(\sigma_u^n)} - Y_t^{(\sigma_*)}|^q] \le C_q b_n^{-q/2}|u|^q$ for any $q \ge 2$.

Furthermore, we obtain $\sup_{|u|\le\alpha} E[|\mathbf{G}_u|^q] < \infty$ for any $\alpha > 0$, $q > 0$ and sufficiently large n, by a similar argument to the proof of Proposition 5 in Ogihara and Yoshida [19].

Then for any $\varepsilon > 0$, there exist M', n' and δ such that

$$\begin{aligned}\sup_{|u|\le\alpha}\bigl|E\bigl[l\bigl(|\mathbf{G}_u|\bigr)\bigr] - E\bigl[l\bigl(|\mathbf{G}_0|\bigr)\bigr]\bigr| &\le \sup_{|u|\le\alpha}\bigl|E\bigl[l\bigl(|\mathbf{G}_u|\bigr) - l\bigl(|\mathbf{G}_0|\bigr), |\mathbf{G}_u| \vee |\mathbf{G}_0| \le M'\bigr]\bigr| + \varepsilon \\ &\le \sup_{|x|\le M'} l(x) \sup_{|u|\le\alpha} P\bigl[|\mathbf{G}_u - \mathbf{G}_0| \ge \delta\bigr] + 2\varepsilon < 3\varepsilon\end{aligned}$$

for $n \ge n'$, by continuity of l.

Hence, we obtain

$$\lim_{\alpha\to\infty}\liminf_{n\to\infty}\sup_{|u|\le\alpha} E\bigl[l\bigl(|\mathbf{G}_u|\bigr)\bigr] = \lim_{n\to\infty} E\bigl[l\bigl(|\mathbf{G}_0|\bigr)\bigr] = E\bigl[l\bigl(\bigl|\Gamma^{-1/2}\mathcal{N}\bigr|\bigr)\bigr]$$

by (2.7). We can similarly obtain the result for the Bayes-type estimator $\tilde{\sigma}_n$. □

The following corollary is obtained by the argument in Section 6 in Ogihara and Yoshida [19].

Corollary 2.1. *Assume that* $0 < \inf_\sigma \pi(\sigma) \le \sup_\sigma \pi(\sigma) < \infty$ *and that* $[C1]$, $[B1]$ *and* $[H']$ *hold. Then the results in Theorem* 2.2 *hold true.*

3. Preliminary results

In the rest of this paper, we will prove Theorem 2.1. For this purpose, we will prove asymptotic equivalence between the log-likelihood ratio $\log(dP_{\sigma_u^n}/dP_{\sigma_*})(Y_\Pi)$ of the processes $Y^{(\sigma)}$ and the quasi-log-likelihood ratio $H_n(\sigma_u^n) - H_n(\sigma_*)$. Then we obtain Theorem 2.1 since $H_n(\sigma_u^n) - H_n(\sigma_*)$ has a LAMN-type property.

This section is devoted to some auxiliary results. We use Malliavin calculus techniques and prove estimates for transition density functions and their derivatives in Section 3.2. Section 3.3 is devoted to prove some tightness results of log-likelihood ratios. These results play essential roles in the proof of Theorem 2.1 in Section 4.

3.1. Some fundamental results

In this subsection, we define Euler–Maruyama-type processes and related notation. We also introduce a LAMN-type property of $H_n(\sigma_u^n) - H_n(\sigma_*)$.

First, we prepare several fundamental lemmas. The first one is about localization. To obtain Theorem 2.1, it is sufficient to consider the following stronger condition $[A1']$ instead of $[A1]$.

$[A1']$ Condition $[A1]$ holds true, $|Y_0| \le M$ a.s. for some $M > 0$, and b, μ and their derivatives are bounded on $[0, T] \times \mathbb{R}^2 \times \Lambda$. Moreover, there exist positive constants $\eta_{\min}$ and $\eta_{\max}$ such that

$$\eta_{\min}\mathcal{E}_2 \le \bigl(rb(t_1, x_1, \sigma) + (1-r)b(t_2, x_2, \sigma)\bigr)\bigl(rb(t_1, x_1, \sigma) + (1-r)b(t_2, x_2, \sigma)\bigr)^\star \le \eta_{\max}\mathcal{E}_2$$

for any $r \in [0, 1]$, $t_1, t_2 \in [0, T]$, $x_1, x_2 \in \mathbb{R}^2$ and $\sigma \in \Lambda$.

$[L]$ There exists a d-dimensional standard normal random variable $\mathcal{N}$ on an extension of $(\Omega, \mathcal{F}, P)$ such that $\mathcal{N}$ is independent of $\mathcal{F}$, $-b_n^{-1}\partial_\sigma^2 H_n(\sigma_*) \circ (\Pi, Y_\Pi) \to^p \Gamma$,

$$\log \frac{\mathrm{d}P_{\sigma_* + b_n^{-1/2}u,n}}{\mathrm{d}P_{\sigma_*,n}} - \left(u^\star b_n^{-1/2}\partial_\sigma H_n(\sigma_*) + \frac{1}{2}u^\star b_n^{-1}\partial_\sigma^2 H_n(\sigma_*)u\right) \to 0$$

in $P_{\sigma_*,n}$ probability, and

$$b_n^{-1/2}\partial_\sigma H_n(\sigma_*) \circ (\Pi, Y_\Pi) \to^{s\text{-}\mathcal{L}} \Gamma^{1/2}\mathcal{N}$$

for Γ defined in (2.3).

Let $\mathcal{H} = \{\omega \in \Omega; -\partial_\sigma^2 H_n(\sigma_*)(\omega)$ is positive definite$\}$,

$$\Gamma_n = -b_n^{-1}\partial_\sigma^2 H_n(\sigma_*)1_{\mathcal{H}} + \mathcal{E}_d 1_{\mathcal{H}^c}, \qquad \mathcal{N}_n = \bigl(-\partial_\sigma^2 H_n(\sigma_*)\bigr)^{-1/2}\partial_\sigma H_n(\sigma_*)1_{\mathcal{H}}. \tag{3.1}$$

Lemma 3.1. *Assume that* $[L]$ *holds true under* $[A1']$, $[A2]$ *and* $[A3]$. *Then Theorem* 2.1 *holds true with* Γ_n *and* $\mathcal{N}_n$ *in* (3.1).

Proof. Similar to the proof of Lemma 4.1. in Gobet [10] and we omit the details. □

The second lemma is Lemma 11 in Ogihara and Yoshida [19].

Lemma 3.2. *Let* $\{\mathsf{X}_n\}_{n\in\mathbb{N}}$ *be a sequence of integrable random variables on some probability space* $(\Omega', \mathcal{F}', P')$ *and* $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ *be sub* σ*-fields of* $\mathcal{F}'$. *Assume* $E'[\mathsf{X}_n|\mathcal{G}_n] \to^p 0$ *as* $n \to \infty$. *Then* $\mathsf{X}_n \to^p 0$ *as* $n \to \infty$.

Moreover, the following lemma is proved similarly to Lemma 3.2.

Lemma 3.3. *Let* Θ *be a set*, $\{\mathsf{X}_{n,\lambda}\}_{n\in\mathbb{N},\lambda\in\Theta}$ *be a family of integrable random variables on some probability space* $(\Omega', \mathcal{F}', P')$ *and* $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ *be sub* σ*-fields of* $\mathcal{F}'$. *Assume that for any* $\varepsilon > 0$, *there exists* $M > 0$ *such that* $\sup_{n,\lambda} P'[E'[|\mathsf{X}_{n,\lambda}||\mathcal{G}_n] > M] < \varepsilon$. *Then for any* $\varepsilon > 0$, *there exists* $M > 0$ *such that* $\sup_{n,\lambda} P'[|\mathsf{X}_{n,\lambda}| > M] < \varepsilon$.

Let $0 \le s < t \le T$, $r \in [0,1]$ and $\sigma \in \Lambda$. Under $[A1]$, a stochastic differential equation

$$\begin{cases} \mathcal{X}_v^{r,\sigma} = \{(1-r)\mu(v+s, \mathcal{X}_v^{r,\sigma}, \sigma) + r\mu(s, z_0, \sigma)\}\,\mathrm{d}v \\ \qquad\quad + \{(1-r)b(v+s, \mathcal{X}_v^{r,\sigma}, \sigma) + rb(s, z_0, \sigma)\}\,\mathrm{d}W_{v+s}, \qquad v \in [0, t-s], \\ \mathcal{X}_0^{r,\sigma} = z_0 \end{cases}$$

has a unique strong solution $\{\mathcal{X}_v^{r,\sigma}\}_{0 \le v \le t-s}$. Let $p(z_1; z_0, r, s, t, \sigma)$ be the probability density function of $\mathcal{X}_{t-s}^{r,\sigma}$.

The following lemma is classical estimate. See Theorem 1 in Aronson [5] or Proposition 5.1 in Gobet [10].

Lemma 3.4. *Assume* $[A1']$. *Then there exist positive constants* $\mu_1 < \mu_2$ *and* $C > 1$ *such that*

$$\frac{1}{C}\frac{\mu_2}{2\pi(t-s)}\exp\left(-\frac{\mu_2|z_1-z_0|^2}{2(t-s)}\right) \le p(z_1; z_0, r, s, t, \sigma) \le C\frac{\mu_1}{2\pi(t-s)}\exp\left(-\frac{\mu_1|z_1-z_0|^2}{2(t-s)}\right),$$

for $0 \le s < t \le T$, $r \in [0,1]$, $z_0, z_1 \in \mathbb{R}^2$ *and* $\sigma \in \Lambda$.

We will define some further notation. Let n_u be the minimum positive integer satisfying $\{\sigma_{vu}^n\}_{n \ge n_u, 0 \le v \le 1} \subset \Lambda$ for $u \in \mathbb{R}^d$. For $\bar{u} = ((s^i)_i, (t^j)_j) \in \mathcal{U}$, let $\check{u} = \{\check{u}^k(\bar{u})\}_{k=0}^{L_0(\bar{u})}$ be a strictly increasing sequence of the elements of $\bar{u}$ such that $\check{u}$ is equal to $\bar{u}$ as a set. Let $\Delta\check{u}^k = \check{u}^k - \check{u}^{k-1}$, $k_1(i) = k_1(i; \bar{u})$ be k satisfying $s^i = \check{u}^k$ and $k_2(j) = k_2(j; \bar{u})$ be k satisfying $t^j = \check{u}^k$,

$$i(k) = i(k; \bar{u}) = \max\{i;\ \text{there exists } j \text{ such that } s^i \le t^j \le \check{u}^{k-1}\},$$
$$j(k) = j(k; \bar{u}) = \max\{j;\ \text{there exists } i \text{ such that } t^j \le s^i \le \check{u}^{k-1}\}.$$

We define random times $\check{U}^k = \check{u}^k(\Pi)$ and $\check{U} = \{\check{U}^k\}_k$.

For $\bar{u} = ((s^i)_{i=0}^{L^1}, (t^j)_{j=0}^{L^2}) \in \mathcal{U}$ and $z = ((x_k)_{k=0}^{L_0(\bar{u})}, (y_k)_{k=0}^{L_0(\bar{u})}) \in \mathbb{R}^{2L_0(\bar{u})+2}$, we denote $\bar{z} = ((x_{k_1(i)})_{i=1}^{L^1}, (y_{k_2(j)})_{j=1}^{L^2})$ and $\hat{z} = ((x_k)_{k \notin \{k_1(i); 0 \le i \le L^1\}}, (y_k)_{k \notin \{k_2(j); 0 \le j \le L^2\}})$.

Now, let us define stochastic processes that *connect* the process $Y^{(\sigma)}$ and an Euler–Maruyama process. Let $u \in \mathbb{R}^d$, $\bar{u} \in \mathcal{U}$, $r \in [0,1]$ and $n \ge n_u$. Under $[A1']$, there exists a unique two-dimensional stochastic process $Y^{r,u} = \{Y_t^{r,u}\}_{0 \le t \le T} = \{(Y_t^{r,u,1}(\bar{u}), Y_t^{r,u,2}(\bar{u}))\}_{0 \le t \le T}$ satisfying

$$\begin{aligned} Y_t^{r,u} = Y_0 &+ \sum_{k=1}^{L_0} \int_{t \wedge \check{u}^{k-1}}^{t \wedge \check{u}^k} \{(1-r)\mu(s, Y_s^{r,u}, \sigma_u^n) + r\mu(\check{u}^{k-1}, Y_{\check{u}^{k-1}}^{r,u}, \sigma_u^n)\}\,\mathrm{d}s \\ &+ \sum_{k=1}^{L_0} \int_{t \wedge \check{u}^{k-1}}^{t \wedge \check{u}^k} \{(1-r)b(s, Y_s^{r,u}, \sigma_u^n) + rb(\check{u}^{k-1}, Y_{\check{u}^{k-1}}^{r,u}, \sigma_u^n)\}\,\mathrm{d}W_s, \qquad t \in [0,T]. \end{aligned}$$

Then we have $Y^{0,0} \equiv Y$.

Moreover, we define $\check{p}^r_{k,u}(z_0,z_1)=p(z_1;z_0,r,\check{u}^{k-1},\check{u}^k,\sigma^n_u)$, $\check{p}^{r,(1)}_{k,u}(z_0,z_1)=\partial_\sigma p(z_1;z_0,r,\check{u}^{k-1},\check{u}^k,\sigma^n_u)$,

$$\mathbb{P}^r_u(z,\bar{u})=\prod_{k=1}^{L_0}\check{p}^r_{k,u}\big((x_{k-1},y_{k-1}),(x_k,y_k)\big),\qquad \bar{\mathbb{P}}^r_u(z_0,\bar{z},\bar{u})=\int\mathbb{P}^r_u(z;\bar{u})\,\mathrm{d}\hat{z}$$

and $P^r_u=\mathbb{P}^r_u(z,\bar{u})P_{Y_0}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\,\mathrm{d}\hat{z}$ for $z=((x_k)_{k=0}^{L_0},(y_k)_{k=0}^{L_0})\in\mathbb{R}^{2L_0+2}$.

Then synchronous observations $Y^{r,u}_{\check{u}}$ follow the distribution P^r_u. Moreover, we have

$$\begin{aligned}P_{\sigma^n_u,n}&=P_{(\Pi,Y^{0,u}_\Pi)}=P_{Y^{0,u}_{\bar{u}}}(\mathrm{d}z_0\,\mathrm{d}\bar{z}|\Pi=\bar{u})P_\Pi(\mathrm{d}\bar{u})=P_{Y^{0,u}_{\bar{u}}}(\mathrm{d}z_0\,\mathrm{d}\bar{z})P_\Pi(\mathrm{d}\bar{u})\\&=\bar{\mathbb{P}}^0_u(z_0,\bar{z},\bar{u})P_{Y_0}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\,P_\Pi(\mathrm{d}\bar{u}).\end{aligned}$$

Therefore, we obtain

$$\log\frac{\mathrm{d}P_{\sigma^n_u,n}}{\mathrm{d}P_{\sigma_*,n}}(z_0,\bar{z},\bar{u})=\log\frac{\bar{\mathbb{P}}^0_u}{\bar{\mathbb{P}}^0_0}(z_0,\bar{z},\bar{u}).\tag{3.2}$$

So it is sufficient to investigate the asymptotic behavior of $\log(\bar{\mathbb{P}}^0_u/\bar{\mathbb{P}}^0_0)$.

For each function with respect to $(z,\bar{u})$ or $(z_0,\bar{z},\bar{u})$, we often omit the variable $\bar{u}$.

The following theorem gives a LAMN-type property of H_n (Proposition 3 and Proposition 10 in Ogihara and Yoshida [19]).

Theorem 3.1. *Assume* $[A1']$, $[A2]$ *and* $[A3]$. *Then there exists a random variable* $\mathcal{N}$ *on an extension of* $(\Omega,\mathcal{F},P)$ *such that* $\mathcal{N}$ *is independent of* $\mathcal{F}$, $-b_n^{-1}\partial_\sigma^2 H_n(\sigma_*)\circ(\Pi,Y_\Pi)\to^p\Gamma$,

$$\big\{H_n\big(\sigma^n_u\big)-H_n(\sigma_*)-\big(u^\star b_n^{-1/2}\partial_\sigma H_n(\sigma_*)+u^\star b_n^{-1}\partial_\sigma^2 H_n(\sigma_*)u/2\big)\big\}\circ(\Pi,Y_\Pi)\to^p 0,$$

$b_n^{-1/2}\partial_\sigma H_n(\sigma_*)\circ(\Pi,Y_\Pi)\to^{s\text{-}\mathcal{L}}\Gamma^{1/2}\mathcal{N}$ *as* $n\to\infty$, *where* Γ *is defined by* (2.3).

Remark 3.1. Though we need an assumption "$\partial_\sigma^k b$ $(0\le k\le 4)$ can be extended to a continuous function on $[0,T]\times\mathbb{R}^2\times\bar{\Lambda}$" to apply the results in Ogihara and Yoshida [19], the assumption can be removed by considering a relatively compact open subset of Λ containing σ_*.

By virtue of Lemma 3.1, Theorem 3.1 and (3.2), to obtain Theorem 2.1, it is sufficient to show asymptotic equivalence of $\log(\bar{\mathbb{P}}^0_u/\bar{\mathbb{P}}^0_0)(Y_\Pi)$ and $(H_n(\sigma^n_u)-H_n(\sigma_*))\circ(\Pi,Y_\Pi)$ under $[A1']$, $[A2]$ and $[A3]$. We will prove it in the rest of this paper.

3.2. Malliavin calculus techniques and estimates for transition densities

We will prepare results of estimates for transition density functions used later. To this end, we introduce some techniques from Malliavin calculus. We refer the reader to Chapter II in Nualart [17] and Gobet [10] for detailed expositions of this subsection.

We fix $u \in \mathbb{R}^d$, $\bar{u} \in \mathcal{U}$, $1 \le k \le L_0(\bar{u})$ and $n \ge n_u$ here. For $0 \le r \le 1$ and $x \in \mathbb{R}^2$, consider a unique two-dimensional process $\{\mathbf{Y}_t^{r,u,k,x}\}_{t\in[0,\Delta\check{u}^k]} = \{(\mathbf{Y}_t^{r,u,k,x,1}, \mathbf{Y}_t^{r,u,k,x,2})\}_{t\in[0,\Delta\check{u}^k]}$ satisfying

$$\mathbf{Y}_t^{r,u,k,x,i} = x + \int_0^t \{(1-r)\mu_s^{(0),r,i} + r\mu^i(\check{u}^{k-1}, x, \sigma_u^n)\}\,\mathrm{d}s + \sum_{j=1}^2 \int_0^t \{(1-r)b_s^{(0),r,i,j} + rb^{ij}(\check{u}^{k-1}, x, \sigma_u^n)\}\,\mathrm{d}W_s^j$$

for $t \in [0, \Delta\check{u}^k]$, where $\mu_{t,p_1,\dots,p_q}^{(q),r,i} = (\partial_x^q \mu^i(t + \check{u}^{k-1}, \mathbf{Y}_t^{r,u,k,x}, \sigma_u^n))_{p_1,\dots,p_q}$, $b_{t,p_1,\dots,p_q}^{(q),r,i,j} = (\partial_x^q b^{ij}(t + \check{u}^{k-1}, \mathbf{Y}_t^{r,u,k,x}, \sigma_u^n))_{p_1,\dots,p_q}$ for $q \in \mathbb{Z}_+$. We simply denote $\mathbf{Y}_t^r = \mathbf{Y}_t^{r,u,k,x}$ and $\mathbf{Y}_t^{r,i} = \mathbf{Y}_t^{r,u,k,x,i}$.

Under $[A1']$, Theorem 39 in Chapter V of Protter [21] ensures that $\partial_r \mathbf{Y}_t^r = (\partial_r \mathbf{Y}_t^{r,1}, \partial_r \mathbf{Y}_t^{r,2})$ exists for any $t \in [0, \Delta\check{u}^k]$ a.s. and satisfies

$$\partial_r \mathbf{Y}_t^{r,i} = \int_0^t \left[\sum_p (1-r)\mu_{s,p}^{(1),r,i} \partial_r \mathbf{Y}_s^{r,p} + \mu^i(\check{u}^{k-1}, x, \sigma_u^n) - \mu_s^{(0),r,i}\right]\mathrm{d}s + \sum_j \int_0^t \left[\sum_p (1-r) b_{s,p}^{(1),r,i,j} \partial_r \mathbf{Y}_s^{r,p} + b^{ij}(\check{u}^{k-1}, x, \sigma_u^n) - b_s^{(0),r,i,j}\right]\mathrm{d}W_s^j.$$

Define an isonormal Gaussian process W by $W(\xi) = \int_0^{\Delta\check{u}^k} (\mathrm{d}\xi_t/\mathrm{d}t)\cdot \mathrm{d}W_{t+\check{u}^{k-1}}$ for an $\mathbb{R}^2$-valued absolutely continuous function $\xi = \{\xi_t\}_{0\le t\le \Delta\check{u}^k}$ satisfying $\int_0^{\Delta\check{u}^k} |\mathrm{d}\xi_t/\mathrm{d}t|^2\,\mathrm{d}t < \infty$. We also consider the Malliavin derivative operator D and the divergence operator δ.

Let $\{\mathsf{V}_t^r\}_{t\in[0,\Delta\check{u}^k]} = \{\mathsf{V}_t^{r,i,j}\}_{t\in[0,\Delta\check{u}^k],i,j}$ be a stochastic process satisfying

$$\mathsf{V}_t^{r,i,j} = \delta_{ij} + \sum_p \int_0^t (1-r)\mu_{s,p}^{(1),r,i} \mathsf{V}_s^{r,p,j}\,\mathrm{d}s + \sum_{p,q} \int_0^t (1-r) b_{s,p}^{(1),r,i,q} \mathsf{V}_s^{r,p,j}\,\mathrm{d}W_s^q,$$

then the argument in Sections 2.2 and 2.3 of Nualart [17] yields

$$D_t^j \mathbf{Y}_{\Delta\check{u}^k}^{r,i} = \sum_{p,q} \mathsf{V}_{\Delta\check{u}^k}^{r,i,q} \left((\mathsf{V}_t^r)^{-1}\right)_{qp} \left[(1-r)b_t^{(0),r,p,j} + rb^{pj}(\check{u}^{k-1}, x, \sigma_u^n)\right]$$

for $t \in [0, \Delta\check{u}^k]$, where $((\mathsf{V}_t^r)^{-1})_{qp}$ represents the element of $(\mathsf{V}_t^r)^{-1}$. Moreover, we obtain

$$\sup_{t\in[0,\Delta\check{u}^k]} \left(E\left[\left|D_t \mathbf{Y}_{\Delta\check{u}^k}^r\right|^M\right] \vee E\left[\left|D_t \partial_r \mathbf{Y}_{\Delta\check{u}^k}^r\right|^M\right]\right) < \infty$$

for $M > 0$.

Lemma 3.5. *Let $u \in \mathbb{R}^d$ and $q \geq 1$. Assume $[A1']$. Then there exists a positive constant C_q such that*

$$\sup_k \frac{1}{(\Delta\check{u}^k)^{q/2}} \sup_{0\leq r\leq 1, z_{k-1}} \int \left|\frac{\partial_r \check{p}^r_{k,u}}{\check{p}^r_{k,u}}\right|^q \check{p}^r_{k,u}(z_{k-1}, z_k)\,\mathrm{d}z_k \leq C_q$$

for any $n \geq n_u$ and $\bar{u} \in \mathcal{U}$.

Proof. Let $\mathcal{B}^r_t = \{\mathcal{B}^{r,i}_{t,j}\}_{i,j} = ((1-r)b^{(0),r}_t + rb(\check{u}^{k-1}, x, \sigma^n_u))^{-1} V^r_t (V^r_{\Delta\check{u}^k})^{-1}$ for $t \in [0, \Delta\check{u}^k]$, where $b^{(0),r}_t = \{b^{(0),r,i,j}_t\}_{ij}$. Then by a similar argument to the proof of Proposition 4.1. in Gobet [10], we obtain

$$\frac{\partial_r \check{p}^r_{k,u}}{\check{p}^r_{k,u}}(z_{k-1}, z_k) = \frac{1}{\Delta\check{u}^k} E\big[\delta\big((\mathcal{B}^r)^\star \partial_r \mathbf{Y}^{r,u,k,z_{k-1}}_{\Delta\check{u}^k}\big) | \mathbf{Y}^{r,u,k,z_{k-1}}_{\Delta\check{u}^k} = z_k\big]. \tag{3.3}$$

Moreover, by Proposition 1.3.3. in Nualart [17], we obtain

$$\delta\big((\mathcal{B}^r)^\star \partial_r \mathbf{Y}^r_{\Delta\check{u}^k}\big) = \sum_{i=1}^2 \left\{\partial_r \mathbf{Y}^{r,i}_{\Delta\check{u}^k} \delta\big(\mathcal{B}^{r,i}\big) - \int_0^{\Delta\check{u}^k} D_t\big(\partial_r \mathbf{Y}^{r,i}_{\Delta\check{u}^k}\big) \cdot \mathcal{B}^{r,i}_t \,\mathrm{d}t\right\}. \tag{3.4}$$

Furthermore, we have

$$\begin{aligned}
&E\left[\left|\sum_{i=1}^2 \int_0^{\Delta\check{u}^k} D_t\big(\partial_r \mathbf{Y}^{r,i}_{\Delta\check{u}^k}\big) \cdot \mathcal{B}^{r,i}_t \,\mathrm{d}t\right|^q\right] \\
&\quad \leq C\big(\Delta\check{u}^k\big)^{q-1} \sum_{i=1}^2 \int_0^{\Delta\check{u}^k} E\big[\big|D_t \partial_r \mathbf{Y}^{r,i}_{\Delta\check{u}^k}\big|^{2q}\big]^{1/2} E\big[\big|\mathcal{B}^{r,i}_t\big|^{2q}\big]^{1/2} \,\mathrm{d}t \qquad (3.5)\\
&\quad = \mathrm{O}\big(\big(\Delta\check{u}^k\big)^{3q/2}\big),
\end{aligned}$$

and

$$E\left[\left|\sum_{i=1}^2 \partial_r \mathbf{Y}^{r,i}_{\Delta\check{u}^k} \delta\big(\mathcal{B}^{r,i}\big)\right|^q\right] = \mathrm{O}\big(\big(\Delta\check{u}^k\big)^q\big) \times \sum_{i=1}^2 E\big[\big|\delta\big(\mathcal{B}^{r,i}\big)\big|^{2q}\big]^{1/2}, \tag{3.6}$$

where we use the fact that any moments of $(\mathsf{V}^r_{\Delta\check{u}^k})^{-1}$ are bounded (see Section 2.3.1. in Nualart [17]).

By Propositions 1.3.8. and 1.5.7. in Nualart [17] and Clark–Ocone representation formula (Corollary A.2. in Nualart and Pardoux [18]), we have

$$\begin{aligned}
E\big[\big|\delta\big(\mathcal{B}^{r,i}\big)\big|^{2q}\big] &= E\left[\left|\int_0^{\Delta\check{u}^k} E\big[D_t \delta\big(\mathcal{B}^{r,i}\big) | \mathcal{F}_{t+\check{u}^{k-1}}\big] \cdot \mathrm{d}W_{t+\check{u}^{k-1}}\right|^{2q}\right] \\
&\leq \big(\Delta\check{u}^k\big)^{q-1} \int_0^{\Delta\check{u}^k} E\big[\big|D_t \delta\big(\mathcal{B}^{r,i}\big)\big|^{2q}\big] \,\mathrm{d}t \qquad (3.7)
\end{aligned}$$

$$= (\Delta\check{u}^k)^{q-1} \int_0^{\Delta\check{u}^k} E\big[\big|\mathcal{B}_t^{r,i} + \delta(D_t\mathcal{B}^{r,i})\big|^{2q}\big]\,\mathrm{d}t$$
$$= \mathrm{O}\big((\Delta\check{u}^k)^q\big).$$

By (3.4)–(3.7), we obtain $E[|\delta((\mathcal{B}^r)^\star\partial_r \mathbf{Y}^r_{\Delta\check{u}^k})|^q] = \mathrm{O}((\Delta\check{u}^k)^{3q/2})$. Therefore, we have

$$\int \left|\frac{\partial_r \check{p}^r_{k,u}}{\check{p}^r_{k,u}}\right|^q \check{p}^r_{k,u}(z_{k-1}, z_k)\,\mathrm{d}z_k \le C_q(\Delta\check{u}^k)^{q/2}$$

by (3.3). □

The following lemma is proved similarly.

Lemma 3.6. *Let $u \in \mathbb{R}^d$ and $q \ge 1$. Assume $[A1']$. Then*

$$\frac{\partial_\sigma p}{p}(z_k; z_{k-1}, r, \check{u}^{k-1}, \check{u}^k, \sigma_u^n) = \frac{1}{\Delta\check{u}^k} E\big[\delta\big((\mathcal{B}^r)^\star \partial_\sigma \mathbf{Y}^r_{\Delta\check{u}^k}\big)|\mathbf{Y}^r_{\Delta\check{u}^k} = z_k\big],$$

$$\frac{\partial^2_{(r,\sigma)} p}{p}(z_k; z_{k-1}, r, \check{u}^{k-1}, \check{u}^k, \sigma_u^n)$$
$$= E\Bigg[\frac{1}{\Delta\check{u}^k}\delta\big((\mathcal{B}^r)^\star \partial^2_{(r,\sigma)}\mathbf{Y}^r_{\Delta\check{u}^k}\big) + \frac{1}{(\Delta\check{u}^k)^2}\sum_{i=1}^2 \delta\big(\mathcal{B}^{r,i}\delta\big((\mathcal{B}^r)^\star \partial_{(r,\sigma)}\mathbf{Y}^r_{\Delta\check{u}^k}\partial_{(r,\sigma)}\mathbf{Y}^{r,i}_{\Delta\check{u}^k}\big)\big)\Big|\mathbf{Y}^r_{\Delta\check{u}^k} = z_k\Bigg],$$

and there exists a constant $C_q > 0$ such that

$$\sup_{k, 0\le r'\le 1, 0\le v\le 1, z_{k-1}} (\Delta\check{u}^k)^{-lq/2} \int \big|\partial_\sigma^j \partial_r^l \log p\big|^q (z_k; z_{k-1}, r', \check{u}^{k-1}, \check{u}^k, \sigma^n_{vu}) \times \check{p}^{r'}_{k,vu}(z_{k-1}, z_k)\,\mathrm{d}z_k \le C_q$$

for $r \in [0,1]$, $\bar{u} \in \mathcal{U}$, $n \ge n_u$, $1 \le j + l \le 2$.

3.3. Tightness results of some log-likelihood ratios

In this section, we will prove some tightness results, which are necessary later. First, we prove tightness of $\{\sup_{0\le r\le 1}|\log(\bar{\mathbb{P}}^r_u/\bar{\mathbb{P}}^0_0)|(Y_\Pi)\}_n$. To this end, we prove results about the log-likelihood ratio $\log(\mathbb{P}^r_u/\mathbb{P}^0_0)(Y_\Pi)$. Then we prove a key proposition (Proposition 3.1) which enables us to obtain tightness of a density ratio in a nonsynchronous scheme from properties of a density ratio in a synchronous scheme.

$[A3']$ The sequence $\{b_n^{-1}(\ell_{1,n}+\ell_{2,n})\}_n$ is tight.

Since $\operatorname{tr}(\mathcal{E}^1(T))+\operatorname{tr}(\mathcal{E}^2(T))=\ell_{1,n}+\ell_{2,n}$, $[A3]$ implies $[A3']$.

We prepare some results for the log-likelihood ratio $\log(\mathbb{P}_u^r/\mathbb{P}_0^0)$.

Lemma 3.7. *Let $u\in\mathbb{R}^d$. Assume $[A1']$ and $[A3']$. Then for any $\varepsilon>0$, there exists $M>0$ such that*

$$\sup_{n\geq n_u}\left\{P\left[\left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\bigl(Y_{\check{U}}^{0,u}\bigr)>M\right]\vee P\left[\left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\bigl(Y_{\check{U}}^{0,0}\bigr)>M\right]\right\}<\varepsilon.$$

Proof. By Lemmas 3.6 and 3.4, we obtain

$$\begin{aligned}
&E\left[\left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\bigl(Y_{\check{U}}^{0,u}\bigr)\middle|\Pi\right]\\
&\quad\leq E\left[\int_0^1\left|\frac{\partial_t(\mathbb{P}_{tu}^0)}{\mathbb{P}_{tu}^0}\right|\bigl(Y_{\check{U}}^{0,u}\bigr)\,\mathrm{d}t\middle|\Pi\right]\\
&\quad\leq E\left[b_n^{-1/2}|u|\left|\sum_k\frac{\check{p}_{k,u}^{0,(1)}}{\check{p}_{k,u}^0}\bigl(Y_{\check{U}^{k-1}}^{0,u},Y_{\check{U}^k}^{0,u}\bigr)\right|\right.\\
&\qquad\left.+b_n^{-1}|u|^2\int_0^1\int_{t_1}^1\left|\sum_k\partial_\sigma\left(\frac{\partial_\sigma p}{p}\right)\bigl(Y_{\check{U}^k}^{0,u};Y_{\check{U}^{k-1}}^{0,u},0,\check{U}^{k-1},\check{U}^k,\sigma_{t_2u}^n\bigr)\right|\mathrm{d}t_2\,\mathrm{d}t_1\middle|\Pi\right]\\
&\quad\leq Cb_n^{-1/2}(\ell_{1,n}+\ell_{2,n})^{1/2}+Cb_n^{-1}(\ell_{1,n}+\ell_{2,n}).
\end{aligned}$$

Hence, by Lemma 3.3 and the assumptions, for any $\varepsilon>0$ there exists $M>0$ such that $\sup_{n\geq n_u}P[|\log(\mathbb{P}_u^0/\mathbb{P}_0^0)|(Y_{\check{U}}^{0,u})>M]<\varepsilon$. Similarly, we obtain $\sup_{n\geq n_u}P[|\log(\mathbb{P}_u^0/\mathbb{P}_0^0)|(Y_{\check{U}}^{0,0})>M]<\varepsilon$. □

We define

$$A_M^n(\bar{u})=\left\{(x,y)\in\mathbb{R}^{L_0(\bar{u})+1}\times\mathbb{R}^{L_0(\bar{u})+1};\sup_{0\leq r\leq 1}\bigl|\log\bigl(\mathbb{P}_u^r/\mathbb{P}_0^0\bigr)\bigr|(x,y)\leq M\right\}$$

for $\bar{u}\in\mathcal{U}$ and $M>0$.

Lemma 3.8. *Let $u\in\mathbb{R}^d$. Assume $[A1']$ and $[A3']$. Then for any $\varepsilon>0$, there exists $M>0$ such that*

$$\sup_{n\geq n_u,r}\left\{P\bigl[Y_{\check{U}}^{0,0}\in\bigl(A_M^n\bigr)^c(\Pi)\bigr]\vee P\bigl[Y_{\check{U}}^{r,u}\in\bigl(A_M^n\bigr)^c(\Pi)\bigr]\vee E\left[\left|\frac{\partial_r\mathbb{P}_u^r}{\mathbb{P}_u^r}\right|1_{(A_M^n)^c}\bigl(Y_{\check{U}}^{r,u}\bigr)\right]\right\}<\varepsilon.$$

Proof. Fix $\varepsilon > 0$. Then for $r \in [0,1]$, we obtain

$$E\left[\sup_{0\le r'\le 1}\left|\log\frac{\mathbb{P}_u^{r'}}{\mathbb{P}_u^0}\right|\left(Y_{\check{U}}^{r,u}\right)\Big|\Pi\right] \le E\left[\int_0^1\left|\frac{\partial_r\mathbb{P}_u^{r'}}{\mathbb{P}_u^{r'}}\right|\left(Y_{\check{U}}^{r,u}\right)\mathrm{d}r'\Big|\Pi\right] = \int_0^1 E\left[\left|\frac{\partial_r\mathbb{P}_u^{r'}}{\mathbb{P}_u^{r'}}\right|\left(Y_{\check{U}}^{r,u}\right)\Big|\Pi\right]\mathrm{d}r'.$$

On the other hand, by Lemmas 3.6 and 3.4, we have

$$\begin{aligned} E\left[\left|\frac{\partial_r\mathbb{P}_u^{r'}}{\mathbb{P}_u^{r'}}\right|\left(Y_{\check{U}}^{r,u}\right)\Big|\Pi\right] &= E\left[\left|\frac{\partial_r\mathbb{P}_u^{r}}{\mathbb{P}_u^{r}} + \int_r^{r'}\partial_r\left(\frac{\partial_r\mathbb{P}_u^{r_1}}{\mathbb{P}_u^{r_1}}\right)\mathrm{d}r_1\right|\left(Y_{\check{U}}^{r,u}\right)\Big|\Pi\right] \\ &\le E\left[\sum_{k=1}^{L_0}\left(\frac{\partial_r\check{p}_{k,u}^r}{\check{p}_{k,u}^r}\right)^2\left(Y_{\check{U}^{k-1}}^{r,u}, Y_{\check{U}^k}^{r,u}\right)\Big|\Pi\right]^{1/2} \\ &\quad + \sup_{r'} E\left[\sum_{k=1}^{L_0}\left|\partial_r\left(\frac{\partial_r\check{p}_{k,u}^{r'}}{\check{p}_{k,u}^{r'}}\right)\right|\left(Y_{\check{U}^{k-1}}^{r,u}, Y_{\check{U}^k}^{r,u}\right)\Big|\Pi\right] \\ &\le C\sqrt{T} + CT. \end{aligned}$$

Hence, by Lemma 3.3, for any $\varepsilon > 0$ there exists $M_1 > 0$ such that

$$\sup_{0\le r\le 1} P\left[\sup_{0\le r'\le 1}\left|\log\frac{\mathbb{P}_u^{r'}}{\mathbb{P}_u^0}\right|\left(Y_{\check{U}}^{r,u}\right) > \frac{M_1}{2}\right] < \frac{\varepsilon}{2}.$$

Therefore, Lemma 3.7 yields

$$\begin{aligned} &\sup_{n\ge n_u, r} P\left[Y_{\check{U}}^{r,u} \in \left(A_M^n\right)^c(\Pi)\right] \\ &\quad \le \sup_{n\ge n_u,r} P\left[\left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\left(Y_{\check{U}}^{r,u}\right) > \frac{M}{2} \text{ and } \sup_{0\le r'\le 1}\left|\log\frac{\mathbb{P}_u^{r'}}{\mathbb{P}_u^0}\right|\left(Y_{\check{U}}^{r,u}\right) \le \frac{M_1}{2}\right] + \frac{\varepsilon}{2} \\ &\quad \le \sup_{n\ge n_u,r} E\left[\frac{\mathbb{P}_u^r}{\mathbb{P}_u^0}\left(Y_{\check{U}}^{0,u}\right), \left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\left(Y_{\check{U}}^{0,u}\right) > \frac{M}{2} \text{ and } \sup_{0\le r'\le 1}\left|\log\frac{\mathbb{P}_u^{r'}}{\mathbb{P}_u^0}\right|\left(Y_{\check{U}}^{0,u}\right) \le \frac{M_1}{2}\right] + \frac{\varepsilon}{2} \\ &\quad \le \mathrm{e}^{M_1/2}\sup_{n\ge n_u} P\left[\left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\left(Y_{\check{U}}^{0,u}\right) > \frac{M}{2}\right] + \frac{\varepsilon}{2} < \varepsilon \end{aligned} \tag{3.8}$$

for sufficiently large $M > 0$.

Moreover, by Lemma 3.5, we obtain

$$\sup_{n\ge n_u,r} E\left[\left|\partial_r\mathbb{P}_u^r/\mathbb{P}_u^r\right|^2\left(Y_{\check{U}}^{r,u}\right)\right] = \sup_{n\ge n_u,r} E\left[\sum_k\left(\partial_r\check{p}_{k,u}^r/\check{p}_{k,u}^r\right)^2\left(Y_{\check{U}^{k-1}}^{r,u}, Y_{\check{U}^k}^{r,u}\right)\right] < \infty.$$

Hence, by (3.8), we have $\sup_{n\ge n_u,r} E[|\partial_r\mathbb{P}_u^r/\mathbb{P}_u^r|1_{(A_M^n)^c}(Y_{\check{U}}^{r,u})] < \varepsilon$ for sufficiently large $M > 0$.

On the other hand, there exists $M_2 > 0$ such that $\sup_{n \ge n_u} P[|\log(\mathbb{P}_u^0/\mathbb{P}_0^0)|(Y_{\check{U}}^{0,0}) > M_2] < \varepsilon/2$ by Lemma 3.7. Therefore, we have

$$P\big[Y_{\check{U}}^{0,0} \in \big(A_M^n\big)^c(\Pi)\big] \le P\left[\sup_{0 \le r \le 1}\left|\log\frac{\mathbb{P}_u^r}{\mathbb{P}_0^0}\right|\big(Y_{\check{U}}^{0,0}\big) > M, \left|\log\frac{\mathbb{P}_u^0}{\mathbb{P}_0^0}\right|\big(Y_{\check{U}}^{0,0}\big) \le M_2\right] + \frac{\varepsilon}{2}$$
$$\le \mathrm{e}^{M_2} P\left[\sup_{0 \le r \le 1}\left|\log\frac{\mathbb{P}_u^r}{\mathbb{P}_0^0}\right|\big(Y_{\check{U}}^{0,u}\big) > M\right] + \frac{\varepsilon}{2} < \varepsilon$$

for sufficiently large $M > 0$ by (3.8). □

Let $\mathsf{Z}^n = \{\mathsf{Z}_t^n\}_{0 \le t \le T}$ and $\mathsf{Z}^{n,r} = \{\mathsf{Z}_t^{n,r}(\bar{u})\}_{0 \le t \le T}$ be two-dimensional continuous $\mathbf{F}$-adapted processes satisfying that $\mathsf{Z}_0^{n,r} = \mathsf{Z}_0^n$ for $n \in \mathbb{N}$, $\bar{u} \in \mathcal{U}$ and $0 \le r \le 1$, $(t, \bar{u}, \omega) \mapsto Z_t^{n,r}(\bar{u})(\omega)$ is measurable, and $(Z_t^n, Z_t^{n,r}(\bar{u}))_{t,r,\bar{u}}$ are independent of $\sigma((\Pi_n)_n)$. Let the distributions of $\mathsf{Z}_{\check{u}}^n$ and $\mathsf{Z}_{\check{u}}^{n,r}(\bar{u})$ be given by $F_n(z_0, \bar{z}, \hat{z}) P_{Z_0^n}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\,\mathrm{d}\hat{z}$ and $F_n^r(z_0, \bar{z}, \hat{z}) P_{Z_0^n}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\,\mathrm{d}\hat{z}$, respectively, for some positive-valued Borel functions F_n and F_n^r. Let $K_M^n = K_M^n(\bar{u})$ be a Borel set in $\mathbb{R}^{2(L_0+1)}$, $\bar{K}_M^n = \bar{K}_M^n(z_0, \bar{z}, \bar{u}) = \{\hat{z}; (z_0, \bar{z}, \hat{z}) \in K_M^n\}$, $\bar{F}_n(z_0, \bar{z}) = \int F_n(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z}$, $\bar{F}_{n,M}(z_0, \bar{z}) = \int_{\bar{K}_M^n(z_0,\bar{z})} F_n(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z}$ and $\bar{F}_n^r$ and $\bar{F}_{n,M}^r$ be defined similarly.

The following proposition is a key result to deduce properties of density ratios in the nonsynchronous scheme.

Proposition 3.1. *Suppose F_n^r can be continuously differentiable with respect to r and $\int \partial_r F_n^r 1_{(K_M^n)^c}\,\mathrm{d}\hat{z}$ exists and is continuous with respect to r for each $n, z_0, \bar{z}$ and $\hat{z}$.*

1. *Suppose for any $\varepsilon > 0$, there exists $M_1 > 0$ such that*

$$\sup_n P\left[\sup_r\left|\log\frac{F_n^r}{F_n}\right|\big(\mathsf{Z}_{\check{U}}^n\big) > M\right] < \varepsilon \quad \textit{and}$$
$$\sup_{n,r}\left\{P\big[\mathsf{Z}_{\check{U}}^{n,0} \in \big(K_M^n\big)^c(\Pi)\big] \vee E\left[\left|\frac{\partial_r F_n^r}{F_n^r}\right| 1_{(K_M^n)^c}\big(\mathsf{Z}_{\check{U}}^{n,r}\big)\right]\right\} < \varepsilon$$

for $M \ge M_1$. Then for any $\varepsilon, \eta > 0$, there exists $M_2 > 0$ such that

$$\sup_n P\Big[\sup_r\big|\log\big(\bar{F}_n^r/\bar{F}_{n,M}^r\big)\big|\big(\mathsf{Z}_\Pi^n\big) \ge \eta\Big] < \varepsilon$$

for $M \ge M_2$.

2.
$$\sup_n P\left[\sup_r\left|\log\frac{\bar{F}_n^r}{\bar{F}_{n,M}^r}\right|\big(Z_\Pi^n\big) \ge \eta\Big|\Pi\right]$$
$$\le \frac{\mathrm{e}^{M'}}{1 - \mathrm{e}^{-\eta}}\left\{P\big[Z_{\check{U}}^{n,0} \in \big(K_M^n\big)^c|\Pi\big] + \sup_r E\left[\left|\frac{\partial_r F_n^r}{F_n^r}\right| 1_{(K_M^n)^c}\big(Z_{\check{U}}^{n,r}\big)\Big|\Pi\right]\right\}$$
$$+ P\left[\sup_r\left|\log\frac{F_n^r}{F_n}\right|\big(Z_{\check{U}}^n\big) > M'\Big|\Pi\right]$$

for any $\eta, M, M' > 0$.

Proof. We first prove 1. By the assumptions, for any $\varepsilon, \eta > 0$, there exist $M_1, M_2 > 0$ such that

$$\sup_n P\left[\sup_r \left|\log \frac{F_n^r}{F_n}\right|(\mathsf{Z}_{\check{U}}^n) > M_1\right] < \frac{\varepsilon}{3} \quad \text{and}$$

$$\sup_{n,r}\left\{P\left[\mathsf{Z}_{\check{U}}^{n,0} \in (K_M^n)^c(\Pi)\right] \vee E\left[\left|\frac{\partial_r F_n^r}{F_n^r}\right| 1_{(K_M^n)^c}(\mathsf{Z}_{\check{U}}^{n,r})\right]\right\} < \frac{\varepsilon\eta'}{3\mathrm{e}^{M_1}}$$

for $M \geq M_2$, where $\eta' = 1 - \mathrm{e}^{-\eta}$. Hence, we obtain

$$\begin{aligned}
&P\left[\sup_r \left|\log \frac{\bar{F}_n^r}{\bar{F}_{n,M}^r}\right|(\mathsf{Z}_\Pi^n) \geq \eta\right] \\
&\leq P\left[\sup_r \left|1 - \frac{\bar{F}_{n,M}^r}{\bar{F}_n^r}\right|(\mathsf{Z}_\Pi^n) \geq \eta', \mathsf{Z}_{\check{U}}^n \in L_{M_1}^n\right] + \frac{\varepsilon}{3} \\
&\leq \frac{1}{\eta'} E\left[\int \sup_r \left|1 - \frac{\bar{F}_{n,M}^r}{\bar{F}_n^r}\right| 1_{L_{M_1}^n}(z_0, \bar{z}, \hat{z}) F_n(z_0, \bar{z}, \hat{z}) P_{Z_0^n}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\,\mathrm{d}\hat{z}\Big|_{\bar{u}=\Pi}\right] + \frac{\varepsilon}{3} \\
&\leq \frac{1}{\eta'} E\left[\int \sup_r \frac{\int F_n 1_{L_{M_1}^n}(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z}}{\bar{F}_n^r} \sup_r \left|\bar{F}_n^r - \bar{F}_{n,M}^r\right| P_{Z_0^n}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\Big|_{\bar{u}=\Pi}\right] + \frac{\varepsilon}{3},
\end{aligned}$$

where $L_M^n = \{(z_0, \bar{z}, \hat{z}); \sup_r |\log(F_n^r/F_n)|(z_0, \bar{z}, \hat{z}) \leq M\}$.

Since

$$\sup_r \frac{\int F_n 1_{L_{M_1}^n}(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z}}{\bar{F}_n^r} = \sup_r \frac{1}{\bar{F}_n^r} \int \frac{F_n}{F_n^r} F_n^r 1_{L_{M_1}^n}(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z} \leq \mathrm{e}^{M_1},$$

we obtain

$$\begin{aligned}
&P\left[\sup_r \left|\log(\bar{F}_n^r/\bar{F}_{n,M}^r)\right|(\mathsf{Z}_\Pi^n) \geq \eta\right] \\
&\leq \frac{\mathrm{e}^{M_1}}{\eta'} E\left[\int \left\{\left|\bar{F}_n^0 - \bar{F}_{n,M}^0\right| + \int_0^1 \left|\partial_r(\bar{F}_n^r - \bar{F}_{n,M}^r)\right|\mathrm{d}r\right\} P_{Z_0^n}(\mathrm{d}z_0)\,\mathrm{d}\bar{z}\Big|_{\bar{u}=\Pi}\right] + \frac{\varepsilon}{3} \\
&\leq \frac{\mathrm{e}^{M_1}}{\eta'}\left\{P\left[\mathsf{Z}_{\check{U}}^{n,0} \in (K_M^n)^c(\Pi)\right] + \sup_r E\left[\left|\frac{\partial_r F_n^r}{F_n^r}\right| 1_{(K_M^n)^c}(\mathsf{Z}_{\check{U}}^{n,r})\right]\right\} + \frac{\varepsilon}{3} < \varepsilon
\end{aligned}$$

for $M \geq M_2$. Hence, we obtain 1.

The result in 2. is proved by a similar argument as above. □

Let

$$\bar{A}_M^n(z_0, \bar{z}) = \left\{\hat{z}; \sup_r \left|\log(\mathbb{P}_u^r/\mathbb{P}_0^0)(z_0, \bar{z}, \hat{z})\right| \leq M\right\}, \qquad \bar{\mathbb{P}}_{M,u}^r(z_0, \bar{z}) = \int_{\bar{A}_M^n(z_0,\bar{z})} \mathbb{P}_u^r(z_0, \bar{z}, \hat{z})\,\mathrm{d}\hat{z}$$

for $M > 0$.

Lemma 3.9. *Let $u \in \mathbb{R}^d$. Assume* $[A1']$ *and* $[A3']$. *Then for any* $\varepsilon, \eta > 0$, *there exists* $M' > 0$ *such that*

$$\sup_{n \geq n_u} P\left[\sup_r \left|\log \frac{\bar{\mathbb{P}}_u^r}{\bar{\mathbb{P}}_{M,u}^r}\right|(Y_\Pi) \geq \eta\right] < \varepsilon, \qquad \sup_{n \geq n_u} P\left[\left|\log \frac{\bar{\mathbb{P}}_0^0}{\bar{\mathbb{P}}_{M,0}^0}\right|(Y_\Pi) \geq \eta\right] < \varepsilon$$

for $M \geq M'$.

Proof. The results are obtained by using Proposition 3.1 and Lemma 3.8. The first inequality is obtained by setting $\mathsf{Z}^n = Y$, $\mathsf{Z}^{n,r} = Y^{r,u}$ and $K_M^n = A_M^n$ in Proposition 3.1. For the second inequality, set $\mathsf{Z}^n = \mathsf{Z}^{n,r} = Y$ and $K_M^n = A_M^n$. □

Proposition 3.2. *Let $u \in \mathbb{R}^d$. Assume* $[A1']$ *and* $[A3']$. *Then* $\{\sup_{0 \leq r \leq 1} |\log(\bar{\mathbb{P}}_u^r / \bar{\mathbb{P}}_0^0)|(Y_\Pi)\}_{n \geq n_u}$ *is tight.*

Proof. We easily obtain the result by Lemma 3.9 and an estimate

$$\sup_r \left|\log \frac{\bar{\mathbb{P}}_{M,u}^r}{\bar{\mathbb{P}}_{M,0}^0}\right| \leq \sup_r \left|\log \frac{1}{\bar{\mathbb{P}}_{M,0}^0} \int_{\bar{A}_M^n} \frac{\mathbb{P}_u^r}{\mathbb{P}_0^0} \mathbb{P}_0^0 \, \mathrm{d}\hat{z}\right| \leq M$$

for sufficiently large $M > 0$. □

The following lemma is similarly proved and used later.

Lemma 3.10. *Let $u \in \mathbb{R}^d$. Assume* $[A1']$ *and* $[A3']$. *Then* $\{\sup_{0 \leq v \leq 1} |\log(\mathbb{P}_{vu}^0 / \mathbb{P}_0^0)|(Y_{\check{U}})\}_{n \geq n_u}$ *and* $\{\sup_{0 \leq v \leq 1} |\log(\bar{\mathbb{P}}_{vu}^0 / \bar{\mathbb{P}}_0^0)|(Y_\Pi)\}_{n \geq n_u}$ *are tight.*

4. The proof of LAMN property

In this section, we will complete the proof of the LAMN property of $\{P_{\sigma,n}\}_{\sigma,n}$.

It is essential in the proof to replace $\bar{\mathbb{P}}_u^0$ in (3.2) by the function $\int \exp(\sum_k \tilde{f}_k^u) \, \mathrm{d}\hat{z}$ below so that coefficient b is predictable and does not depend on $\hat{z}$. For this purpose, we use Itô's rule and martingale properties and estimate the difference. However, the proof is technically complicated because the function $\log \bar{\mathbb{P}}_u^0$ contains a $\mathrm{d}\hat{z}$-integral of an exponential function. This integral is far more difficult to handle than a simple function of increments of the process, which appears in synchronous sampling models of Gobet [10]. We estimate the difference step by step in Lemmas 4.3 and 4.8. The function $\log \int \exp(\sum_k \tilde{f}_k^u) \, \mathrm{d}\hat{z}$ can be rewritten in a simple function of increments of the process as seen in Lemma 4.6. Then the proof is completed by proving asymptotic equivalence of the replaced likelihood ratio and the quasi-likelihood ratio $H_n(\sigma) - H_n(\sigma_*)$.

In the following, we assume that $[A2]$ holds true. Let $\mu_{(k)}(z,\sigma) = \mu(\check{u}^{k-1}, z_{k-1}, \sigma)$, $b_{(k)}(z,\sigma) = b(\check{u}^{k-1}, z_{k-1}, \sigma)$, $\tilde{b}_{(k)}(z,\sigma) = b(\check{u}^{k-1}, x_{k_1(i(k))}, y_{k_2(j(k))}, \sigma)$,

$$\begin{aligned} f_{(k)}(z,\sigma) &= -\tfrac{1}{2}\bigl(z_k - z_{k-1} - \Delta\check{u}^k \mu_{(k)}(z,\sigma)\bigr)^\star \bigl(\Delta\check{u}^k b_{(k)} b_{(k)}^\star(z,\sigma)\bigr)^{-1} \\ &\quad \times \bigl(z_k - z_{k-1} - \Delta\check{u}^k \mu_{(k)}(z,\sigma)\bigr) \\ &\quad - \tfrac{1}{2}\log\det\bigl(\Delta\check{u}^k b_{(k)} b_{(k)}^\star(z,\sigma)\bigr) - \log(2\pi), \\ \tilde{f}_{(k)}(z,\sigma) &= -\tfrac{1}{2}(z_k - z_{k-1})^\star \bigl(\Delta\check{u}^k \tilde{b}_{(k)} \tilde{b}_{(k)}^\star(z,\sigma)\bigr)^{-1}(z_k - z_{k-1}) \\ &\quad - \tfrac{1}{2}\log\det\bigl(\Delta\check{u}^k \tilde{b}_{(k)} \tilde{b}_{(k)}^\star(z,\sigma)\bigr) - \log(2\pi) \end{aligned}$$

for $z = (z_k)_{k=0}^{L_0(\bar{u})} = ((x_k)_{k=0}^{L_0(\bar{u})}, (y_k)_{k=0}^{L_0(\bar{u})}) \in \mathbb{R}^{2L_0(\bar{u})+2}$, and let $\mu_k^u(z) = \mu_{(k)}(z,\sigma_u^n)$, $b_k^u(z) = b_{(k)}(z,\sigma_u^n)$, $\tilde{b}_k^u(z) = \tilde{b}_{(k)}(z,\sigma_u^n)$, $f_k^u(z) = f_{(k)}(z,\sigma_u^n)$, $\tilde{f}_k^u(z) = \tilde{f}_{(k)}(z,\sigma_u^n)$, $f_k^{u,(1)}(z) = \partial_\sigma f_{(k)}(z,\sigma_u^n)$, $\tilde{f}_k^{u,(1)}(z) = \partial_\sigma \tilde{f}_{(k)}(z,\sigma_u^n)$.

Then we obtain $\bar{\mathbb{P}}_u^1 = \int \exp(\sum_k f_k^u(z))\,\mathrm{d}\hat{z}$.

Moreover, let κ be a positive constant satisfying

$$\delta_2 \vee (\delta_1 + \delta_3) < \kappa < \left(\frac{1}{4} - \frac{(3\delta_1 + 2\delta_3) \vee (\delta_1 + \delta_2)}{2}\right) \wedge \left(\frac{1}{6} - \frac{\delta_1}{2}\right),$$

$h = h_n = [b_n^\kappa]$ and

$$\acute{f}_{k'}^{k,u}(z) = \begin{cases} \tilde{f}_{k'}^u(z), & |k - k'| \le h, \\ \log \check{p}_{k',u}^0(z) & \text{otherwise}, \end{cases}$$

where $\{\delta_j\}_{j=1}^3$ appears in $[A2]$. Then we obtain

$$\begin{aligned} \log\frac{\bar{\mathbb{P}}_u^0}{\bar{\mathbb{P}}_0^0}(Y_\Pi) &= \int_0^1 \frac{\partial_v(\bar{\mathbb{P}}_{vu}^0)}{\bar{\mathbb{P}}_{vu}^0}\,\mathrm{d}v(Y_\Pi) \\ &= b_n^{-1/2} u \int_0^1 \frac{\int \sum_k (\check{p}_{k,vu}^{0,(1)}/\check{p}_{k,vu}^0)\exp(\sum_{k'}\log\check{p}_{k',vu}^0)(z)\,\mathrm{d}\hat{z}}{\int \exp(\sum_{k'}\log\check{p}_{k',vu}^0)(z)\,\mathrm{d}\hat{z}}\,\mathrm{d}v(Y_\Pi). \end{aligned} \tag{4.1}$$

If we have asymptotic equivalence of $\log(\bar{\mathbb{P}}_u^0/\bar{\mathbb{P}}_0^0)(Y_\Pi)$ and

$$\begin{aligned} &\log\biggl(\int \exp\biggl(\sum_k \tilde{f}_k^u\biggr)\mathrm{d}\hat{z} \bigg/ \int \exp\biggl(\sum_k \tilde{f}_k^0\biggr)\mathrm{d}\hat{z}\biggr)(Y_\Pi) \\ &\quad = b_n^{-1/2} u \int_0^1 \frac{\int \sum_k \tilde{f}_k^{vu,(1)} \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\,\mathrm{d}\hat{z}}{\int \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\,\mathrm{d}\hat{z}}\,\mathrm{d}v(Y_\Pi), \end{aligned} \tag{4.2}$$

then Lemma 4.6 gives a simple asymptotic representation of $\log(\bar{\mathbb{P}}_u^0/\bar{\mathbb{P}}_0^0)(Y_\Pi)$ using the increments of processes. However, it is difficult to estimate directly the difference of these two quantities since $\exp(\sum_{k'} \log \check{p}^0_{k',vu}) - \exp(\sum_{k'} \tilde{f}^{vu}_{k'})$ is not asymptotically negligible. So we first prove asymptotic equivalence of $\log(\bar{\mathbb{P}}^0_u/\bar{\mathbb{P}}^0_0)(Y_\Pi)$ and

$$b_n^{-1/2} u \int_0^1 \frac{\int \sum_k \tilde{f}^{vu,(1)}_k \exp(\sum_{k'} \acute{f}^{k,vu}_{k'})\, d\hat{z}}{\int \exp(\sum_{k'} \acute{f}^{k,vu}_{k'})\, d\hat{z}}\, dv(Y_\Pi) \tag{4.3}$$

in Lemmas 4.3 and 4.8. Then we prove asymptotic equivalence of (4.2) and (4.3) in Lemma 4.7, using a simpler expression of (4.3) obtained by calculating $d\hat{z}$-integral partially by the virtue of Lemma 4.6.

We start with preparation of several lemmas. The first one is proved similarly to Lemma 5 in Ogihara and Yoshida [19], so we omit details.

Lemma 4.1. *Assume* [A2] *and* [A3′]. *Then*

$$b_n^{-1/2+\delta} \sum_{p_1,p_2=0}^{\infty} \frac{\sum_{l_1,l_2} |\theta_{p_1,l_1} \cap \theta_{p_2,l_2}|}{(p_1+1)^5 (p_2+1)^5} \to^p 0$$

as $n \to \infty$ *for any* δ *satisfying* $0 < \delta < 1/2 - (3\delta_1 + 2\delta_3) \vee (\delta_1 + \delta_2)$.

Lemma 4.2. *Let* $u \in \mathbb{R}^d$. *Assume* [A1′], [A2] *and* [A3′]. *Then for any* $\varepsilon, \eta > 0$, *there exists* $M' > 0$ *such that*

$$\sup_{n \geq n_u} P\left[\sup_{0 \leq v \leq 1} \left|\log \frac{\bar{\mathbb{P}}^0_{vu}}{\tilde{\bar{\mathbb{P}}}^0_{M,vu}}\right|(Y_\Pi) \geq \eta\right] < \varepsilon$$

for $M \geq M'$, *where*

$$B^n_M = \left\{z \in \mathbb{R}^{2L_0+2}; \sup_{k',v'} \left|\tilde{f}^{v'u}_{k'} - \log \check{p}^0_{k',v'u}(z_{k'-1}, z_{k'})\right| \leq M b_n^{-1/3-\kappa}\right\},$$

$\bar{B}^n_M(z_0, \bar{z}) = \{\hat{z}; (z_0, \bar{z}, \hat{z}) \in B^n_M\}$ *and* $\tilde{\mathbb{P}}^0_{M,u}(z_0, \bar{z}) = \int_{\bar{B}^n_M(z_0,\bar{z})} \mathbb{P}^0_u(z)\, d\hat{z}$.

Proof. We will apply 2. of Proposition 3.1. By using the Burkholder–Davis–Gundy inequality and Lemma 3.6, we have

$$\begin{aligned}
&E\left[\sup_v \left|\log \frac{\mathbb{P}^0_{vu}}{\mathbb{P}^0_0}\right|(Y_{\check{U}})\middle|\Pi\right] \\
&= E\left[\sup_v \left|\int_0^v \frac{\partial_s(\mathbb{P}^0_{su})}{\mathbb{P}^0_{su}}\, ds\right|(Y_{\check{U}})\middle|\Pi\right] \\
&\leq b_n^{-1/2} |u| \int_0^1 E\left[\left|\sum_k \frac{\check{p}^{0,(1)}_{k,vu}}{\check{p}^0_{k,vu}}(Y_{\check{U}^{k-1}}, Y_{\check{U}^k})\right|\middle|\Pi\right] dv \qquad (4.4)
\end{aligned}$$

$$
\begin{aligned}
&\le b_n^{-1/2}|u|E\left[\left|\sum_k \frac{\check{p}_{k,0}^{0,(1)}}{\check{p}_{k,0}^{0}}(Y_{\check{U}^{k-1}},Y_{\check{U}^k})\right|\middle|\Pi\right] \\
&\quad + b_n^{-1/2}|u|\int_0^1\int_0^v E\left[\left|\sum_k \partial_v\left(\frac{\check{p}_{k,v_2u}^{0,(1)}}{\check{p}_{k,v_2u}^{0}}\right)(Y_{\check{U}^{k-1}},Y_{\check{U}^k})\right|\middle|\Pi\right]\mathrm{d}v_2\,\mathrm{d}v \\
&\le Cb_n^{-1/2}|u|(\ell_{1,n}+\ell_{2,n})^{1/2}+Cb_n^{-1}|u|^2(\ell_{1,n}+\ell_{2,n}).
\end{aligned}
$$

On the other hand, for any $\varepsilon > 0$, Lemmas 3.4 and 3.6 yield

$$
\begin{aligned}
&\sup_v P\left[Y_{\check{U}}^{0,vu}\in\left(B_M^n\right)^c|\Pi\right] \\
&\quad\le \frac{b_n^{q/3+q\kappa}}{M^q}\sup_v E\left[\sup_{k',v'}\left|\tilde{f}_{k'}^{v'u}-f_{k'}^{v'u}+\int_0^1\frac{\partial_r\check{p}_{k',v'u}^r}{\check{p}_{k',v'u}^r}\,\mathrm{d}r\right|^q\left(Y_{\check{U}}^{0,vu}\right)\middle|\Pi\right] \\
&\quad\le C_q\frac{b_n^{q/3+q\kappa}}{M^q}\Bigg(r_n^{q/2}(\ell_{1,n}+\ell_{2,n}) \qquad (4.5)\\
&\qquad\qquad + \sup_{r,v}E\left[\sum_{k'}\left(\left|\frac{\partial_r\check{p}_{k',0}^r}{\check{p}_{k',0}^r}\right|^q+\int_0^1\left|\partial_v\left(\frac{\partial_r\check{p}_{k',v'u}^r}{\check{p}_{k',v'u}^r}\right)\right|^q\mathrm{d}v'\right)\left(Y_{\check{U}}^{0,vu}\right)\middle|\Pi\right]\Bigg) \\
&\quad\le C_q b_n^{q/3+q\kappa}M^{-q}r_n^{q/2}(\ell_{1,n}+\ell_{2,n})
\end{aligned}
$$

for any $q > 0$ and $M > 0$.

By (4.4), (4.5) and 2. of Proposition 3.1, we obtain

$$
\begin{aligned}
&P\left[\sup_v\left|\log\frac{\bar{\mathbb{P}}_{vu}^0}{\tilde{\mathbb{P}}_{M,vu}^0}\right|(Y_\Pi)\ge\eta\middle|\Pi\right] \\
&\quad\le \frac{C_q\mathrm{e}^{M'}}{1-\mathrm{e}^{-\eta}}\left\{\left(\frac{1}{M^q}+\frac{M_2}{M^q}\right)b_n^{q/3+q\kappa}r_n^{q/2}(\ell_{1,n}+\ell_{2,n})+\frac{1}{M_2}\sup_v E\left[\left|\frac{\partial_v\mathbb{P}_{vu}^0}{\mathbb{P}_{vu}^0}\right|^2\middle|\Pi\right]\right\} \\
&\qquad + \frac{C_q}{M'}\left(1+b_n^{-1}|u|^2(\ell_{1,n}+\ell_{2,n})\right)
\end{aligned}
$$

for any $M, M', M_2 > 0$.

Hence, we have

$$
\sup_n P\left[\sup_v\left|\log\frac{\bar{\mathbb{P}}_{vu}^0}{\tilde{\mathbb{P}}_{M,vu}^0}\right|(Y_\Pi)\ge\eta\right]=\sup_n E\left[P\left[\sup_v\left|\log\frac{\bar{\mathbb{P}}_{vu}^0}{\tilde{\mathbb{P}}_{M,vu}^0}\right|(Y_\Pi)\ge\eta\middle|\Pi\right]\wedge 1\right]<\varepsilon
$$

for sufficiently large $M > 0$. □

Similarly to (4.1), we obtain

$$\log\frac{\tilde{\mathbb{P}}^0_{M,u}}{\tilde{\mathbb{P}}^0_{M,0}}(Y_\Pi)=\int_0^1\frac{\partial_v(\tilde{\mathbb{P}}^0_{M,vu})}{\tilde{\mathbb{P}}^0_{M,vu}}\,\mathrm{d}v(Y_\Pi)$$
$$=b_n^{-1/2}u\int_0^1\frac{\int_{\bar{B}^n_M}\sum_k(\check{p}^{0,(1)}_{k,vu}/\check{p}^0_{k,vu})\exp(\sum_{k'}\log\check{p}^0_{k',vu})(z)\,\mathrm{d}\hat{z}}{\tilde{\mathbb{P}}^0_{M,vu}}\,\mathrm{d}v(Y_\Pi). \tag{4.6}$$

Let $\bar{\mathbb{P}}^{2,u}_k(g)(z_0,\bar{z})=\int g(z)\exp(\sum_{k'}\acute{f}^{k,u}_{k'}(z))\,\mathrm{d}\hat{z}$ for an integrable function g. For $1\le k\le L_0(\bar{u})$ and $p\in\mathbb{Z}_+$, let $\tilde{\theta}(p,k;\bar{u})$ be $\theta(p,l;\bar{u})$, where an integer l satisfies $1\le l\le L^1$ and $[\check{u}^{k-1},\check{u}^k)\subset[s^{i-1},s^i)$. Let $\tilde{\theta}_{p,k}=\tilde{\theta}(p,k;\Pi)$.

The following lemma is the first step to replace $\bar{\mathbb{P}}^0_u$ by $\int\exp(\sum_k\tilde{f}^u_k)\,\mathrm{d}\hat{z}$.

Lemma 4.3. *Let $u\in\mathbb{R}^d$. Assume $[A1']$, $[A2]$ and $[A3']$. Then for any $\varepsilon,\eta>0$, there exist $M'>0$ and $\{N_M\}_{M\ge M'}\subset\mathbb{N}$ such that*

$$P\left[\left|\log\frac{\bar{\mathbb{P}}^0_u}{\bar{\mathbb{P}}^0_0}(Y_\Pi)-b_n^{-1/2}u\int_0^1\sum_k\frac{\bar{\mathbb{P}}^{2,vu}_k(\tilde{f}^{vu,(1)}_k1_{B^n_M})}{\tilde{\mathbb{P}}^0_{M,vu}}\,\mathrm{d}v(Y_\Pi)\right|\ge\eta\right]<\varepsilon$$

for $M\ge M'$ and $n\ge N_M$.

Proof. Fix $\varepsilon,\eta\in(0,1)$. By Lemmas 3.10 and 4.2, there exists $M'>0$ such that $\sup_{n\ge n_u}P[Y_\Pi\in(\mathcal{K}^1_M)^c(\Pi)]<\varepsilon/2$ for $M\ge M'$, where

$$\mathcal{K}^1_M(\bar{u})=\left\{(z_0,\bar{z});\ \sup_{0\le v\le 1}\left|\log\big(\bar{\mathbb{P}}^0_{vu}/\bar{\mathbb{P}}^0_0\big)\right|(z_0,\bar{z})\le M\text{ and }\sup_{0\le v\le 1}\left|\log\big(\bar{\mathbb{P}}^0_{vu}/\tilde{\mathbb{P}}^0_{M,vu}\big)\right|(z_0,\bar{z})\le 1\right\}.$$

Therefore by (4.6), Lemmas 3.2 and 4.2, it is sufficient to show that

$$\Phi_n = E\Bigg[\Bigg|b_n^{-1/2}u\int_0^1\sum_k\Bigg(\int_{\bar{B}^n_M}\Bigg\{\big(\check{p}^{0,(1)}_{k,vu}/\check{p}^0_{k,vu}\big)\exp\Bigg(\sum_{k'}\log\check{p}^0_{k',vu}\Bigg)(z) - \tilde{f}^{vu,(1)}_k\exp\Bigg(\sum_{k'}\acute{f}^{k,vu}_{k'}\Bigg)\Bigg\}\,\mathrm{d}\hat{z}\Bigg)\Big/\big(\tilde{\mathbb{P}}^0_{M,vu}\big)\,\mathrm{d}v\Bigg| \times 1_{\mathcal{K}^1_M(\Pi)}(Y_\Pi)\Bigg|\Pi\Bigg] \to^p 0$$

as $n\to\infty$ for any $M>0$.

By the definition of $\mathcal{K}_M^1$ and the relation $|\exp(x)-1-x|\leq Cx^2$ for $|x|\leq 3M$, we obtain

$$\begin{aligned}
\Phi_n &\leq \mathrm{e}^{M+1}|u| \\
&\quad\times \sup_v E\Bigg[\Bigg|b_n^{-1/2}\sum_k\Bigg\{\frac{\check{p}_{k,vu}^{0,(1)}}{\check{p}_{k,vu}^0}-\tilde{f}_k^{vu,(1)}\exp\Bigg(\sum_{k';|k'-k|\leq h}\big(\tilde{f}_{k'}^{vu}-\log\check{p}_{k',vu}^0\big)\Bigg)\Bigg\}\Bigg| \\
&\qquad\qquad\times 1_{B_M^n}\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\leq C\sup_v E\Bigg[\Bigg|b_n^{-1/2}\sum_k\Bigg\{\frac{\check{p}_{k,vu}^{0,(1)}}{\check{p}_{k,vu}^0}-\tilde{f}_k^{vu,(1)}\Bigg(1+\sum_{k';|k'-k|\leq h}\big(\tilde{f}_{k'}^{vu}-\log\check{p}_{k',vu}^0\big)\Bigg)\Bigg\}\Bigg|\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\quad+\mathrm{o}_p(1) \\
&\leq C\sup_v E\Bigg[\Bigg|b_n^{-1/2}\sum_k\Bigg\{\frac{\check{p}_{k,vu}^{0,(1)}}{\check{p}_{k,vu}^0}-\frac{\check{p}_{k,vu}^{1,(1)}}{\check{p}_{k,vu}^1}\Bigg\}\Bigg|\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\quad+C\sup_v E\Bigg[\Bigg|b_n^{-1/2}\sum_k\tilde{f}_k^{vu,(1)}\sum_{k';|k'-k|\leq h}\big(\log\check{p}_{k',vu}^1-\log\check{p}_{k',vu}^0\big)\Bigg|\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\quad+C\sup_v E\Bigg[\Bigg|b_n^{-1/2}\sum_k\Bigg\{f_k^{vu,(1)}-\tilde{f}_k^{vu,(1)}\Bigg(1+\sum_{k';|k'-k|\leq h}\big(\tilde{f}_{k'}^{vu}-f_{k'}^{vu}\big)\Bigg)\Bigg\}\Bigg|\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\quad+\mathrm{o}_p(1) \\
&=\Phi_{n,1}+\Phi_{n,2}+\Phi_{n,3}+\mathrm{o}_p(1).
\end{aligned}\tag{4.7}$$

The quantity $\Phi_{n,1}$ is estimated as

$$\begin{aligned}
\Phi_{n,1} &\leq C\sup_{r,v}E\Bigg[\Bigg|b_n^{-1/2}\sum_k\partial_r\Bigg(\frac{\check{p}_{k,vu}^{r,(1)}}{\check{p}_{k,vu}^r}\Bigg)\Bigg|\big(Y_{\check{U}}^{0,vu}\big)\Bigg|\Pi\Bigg] \\
&\leq Cb_n^{-1/2}\sum_k\sup_{r,v}E\Bigg[\Bigg|E\Bigg[\partial_r\Bigg(\frac{\check{p}_{k,vu}^{r,(1)}}{\check{p}_{k,vu}^r}\Bigg)\big(Y_{\check{u}^{k-1}}^{0,vu},Y_{\check{u}^k}^{0,vu}\big)\Big|\mathcal{F}_{\check{u}^{k-1}}\Bigg]\Bigg|\Bigg]\Bigg|_{\bar{u}=\Pi} \\
&\quad+C\sup_{r,v}E\Bigg[\Bigg|b_n^{-1/2}\sum_k\Bigg(\partial_r\Bigg(\frac{\check{p}_{k,vu}^{r,(1)}}{\check{p}_{k,vu}^r}\Bigg)\big(Y_{\check{u}^{k-1}}^{0,vu},Y_{\check{u}^k}^{0,vu}\big) \\
&\qquad\qquad-E\Bigg[\partial_r\Bigg(\frac{\check{p}_{k,vu}^{r,(1)}}{\check{p}_{k,vu}^r}\Bigg)\big(Y_{\check{u}^{k-1}}^{0,vu},Y_{\check{u}^k}^{0,vu}\big)\Big|\mathcal{F}_{\check{u}^{k-1}}\Bigg]\Bigg)\Bigg|\Bigg]\Bigg|_{\bar{u}=\Pi} \\
&\leq Cb_n^{-1/2}\sum_k\sup_{r,v}E\Bigg[\Bigg|\int\partial_r\Bigg(\frac{\check{p}_{k,vu}^{r,(1)}}{\check{p}_{k,vu}^r}\Bigg)\check{p}_{k,vu}^0(z_{k-1},z_k)\,\mathrm{d}z_k\Big|_{z_{k-1}=Y_{\check{u}^{k-1}}^{0,vu}}\Bigg|\Bigg]\Bigg|_{\bar{u}=\Pi}+\mathrm{o}_p(1).
\end{aligned}$$

Then we have $\Phi_{n,1} = \mathrm{o}_p(1)$ since

$$\begin{aligned}
&\left|\int \partial_r \left(\frac{\check{p}^{r,(1)}_{k,vu}}{\check{p}^{r}_{k,vu}}\right)\check{p}^{0}_{k,vu}(z_{k-1}, z_k)\,\mathrm{d}z_k\right| \\
&\quad = \left|\int \partial_r \left(\frac{\check{p}^{r,(1)}_{k,vu}}{\check{p}^{r}_{k,vu}}\right)\left(\check{p}^{r}_{k,vu} - \int_0^r \partial_r \check{p}^{r'}_{k,vu}\,\mathrm{d}r'\right)(z_{k-1}, z_k)\,\mathrm{d}z_k\right| \\
&\quad \le \left|\int \partial_r \left(\frac{\check{p}^{r,(1)}_{k,vu}}{\check{p}^{r}_{k,vu}}\right)\check{p}^{r}_{k,vu}(z_{k-1}, z_k)\,\mathrm{d}z_k\right| \\
&\qquad + \left\{\int \left(\partial_r \left(\frac{\check{p}^{r,(1)}_{k,vu}}{\check{p}^{r}_{k,vu}}\right)\right)^2 \check{p}^{r}_{k,vu}\,\mathrm{d}z_k\right\}^{1/2} \sup_{r'} \left(\int \left(\frac{\partial_r \check{p}^{r'}_{k,vu}}{\check{p}^{r}_{k,vu}}\right)^2 \check{p}^{r}_{k,vu}\,\mathrm{d}z_k\right)^{1/2},
\end{aligned}$$

$E[|\check{p}^{r,(1)}_{k,vu}/\check{p}^{r}_{k,vu} - \check{p}^{1,(1)}_{k,vu}/\check{p}^{1}_{k,vu}|^p(Y^{r,vu}_{\check{u}})]^{1/p} = \mathrm{O}((\Delta\check{u}^k)^{1/2})$ and

$$\begin{aligned}
&\int \frac{\partial_r \check{p}^{r}_{k,vu}}{\check{p}^{r}_{k,vu}} \frac{\check{p}^{1,(1)}_{k,vu}}{\check{p}^{1}_{k,vu}} \check{p}^{r}_{k,vu}(z_{k-1}, z_k)\,\mathrm{d}z_k \\
&\quad = -\frac{1}{2} E_{z_{k-1}}\Bigg[\frac{\delta(\mathcal{B}^r \partial_r \mathbf{Y}^{r,u,k,z_{k-1}}_{\Delta\check{u}^k})}{\Delta\check{u}^k} \\
&\qquad\qquad \times \partial_\sigma\left(\left(\Delta\bar{\mathbf{Y}}^k\right)^\star \frac{(bb^\star)^{-1}(\check{u}^{k-1}, z_{k-1}, \sigma^n_{vu})}{\Delta\check{u}^k}\Delta\bar{\mathbf{Y}}^k \right.\\
&\qquad\qquad\qquad \left. + \log\det\left(bb^\star\right)\left(\check{u}^{k-1}, z_{k-1}, \sigma^n_{vu}\right)\right)\Bigg] \\
&\quad = \mathrm{O}\left(\Delta\check{u}^k\right),
\end{aligned}$$

where $\Delta\bar{\mathbf{Y}}^k = (\mathbf{Y}^r_{\Delta\check{u}^k} - z_{k-1} - \Delta\check{u}^k\mu(\check{u}^{k-1}, z_{k-1}, \sigma^n_{vu}))$.

Similarly, $\Phi_{n,2}$ is estimated as

$$\begin{aligned}
\Phi_{n,2} &\le C\sup_{r,v} E\left[\left|b_n^{-1/2}\sum_k \tilde{f}^{vu,(1)}_k \sum_{|k-k'|\le h} \frac{\partial_r \check{p}^{r}_{k',vu}}{\check{p}^{r}_{k',vu}}\right|\left(Y^{0,vu}_{\check{U}}\right)\Big|\Pi\right] \\
&\le C\sup_{r,v} E\left[\left|b_n^{-1/2}\sum_k \left(\tilde{f}^{vu,(1)}_k - E\left[\tilde{f}^{vu,(1)}_k|\mathcal{F}_{\check{u}^{k-1}}\right]\right)\sum_{|k-k'|\le h} \frac{\partial_r \check{p}^{r}_{k',vu}}{\check{p}^{r}_{k',vu}}\right|\left(Y^{0,vu}_{\check{u}}\right)\right]\Bigg|_{\bar{u}=\Pi} \\
&\quad + \mathrm{o}_p\left(b_n^{-1/2} b_n r_n b_n^{\kappa}\right) \\
&\le C\sup_{r,v} E\left[\left|b_n^{-1/2}\sum_k \left(\tilde{f}^{vu,(1)}_k - E\left[\tilde{f}^{vu,(1)}_k|\mathcal{F}_{\check{u}^{k-1}}\right]\right)\sum_{k';k-h\le k'\le k} \frac{\partial_r \check{p}^{r}_{k',vu}}{\check{p}^{r}_{k',vu}}\right|\left(Y^{0,vu}_{\check{u}}\right)\right]\Bigg|_{\bar{u}=\Pi}
\end{aligned}$$

$$+ C\sup_{r,v} E\bigg[\bigg|b_n^{-1/2}\sum_{k'}\bigg(\sum_{k;k'-h\le k<k'}\big(\tilde{f}_k^{vu,(1)} - E\big[\tilde{f}_k^{vu,(1)}|\mathcal{F}_{\breve{u}^{k-1}}\big]\big)\bigg) \times \frac{\partial_r \check{p}^r_{k',vu}}{\check{p}^r_{k',vu}}\bigg|\big(Y^{0,vu}_{\breve{u}}\big)\bigg]\bigg|_{\bar{u}=\Pi} + o_p(1) = o_p(1),$$

by the Burkholder–Davis–Gundy inequality.

Finally, we will prove $\Phi_{n,3} = o_p(1)$. Let $t_k^0 = S^{n,i(k;\Pi)} \wedge T^{n,j(k;\Pi)}$, $\hat{b}_k^v = b(t_k^0, Y^{0,vu}_{t_k^0}, \sigma_{vu}^n)$ and

$$\mathcal{A}_k^1 = -\Delta W_k^\star\big(\hat{b}_k^v\big)^\star\big(\partial_\sigma\big(\big(b_{(k)}b_{(k)}^\star\big)^{-1}\big) - \partial_\sigma\big(\big(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star\big)^{-1}\big)\big)\big(Y^{0,vu}_{\check{U}}, \sigma_{vu}^n\big)\hat{b}_k^v\Delta W_k/\big(2\Delta\check{U}^k\big) - \frac{1}{2}\partial_\sigma \log\frac{\det(b_{(k)}b_{(k)}^\star)}{\det(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)}\big(Y^{0,vu}_{\check{U}}, \sigma_{vu}^n\big) + \partial_\sigma\big(\mu_{(k)}^\star\big(b_{(k)}b_{(k)}^\star\big)^{-1}\big)\big(Y^{0,vu}_{\check{U}}, \sigma_{vu}^n\big)\hat{b}_k^v\Delta W_k,$$

$$\mathcal{A}_k^2 = -\Delta W_k^\star\big(\hat{b}_k^v\big)^\star\partial_\sigma\big(\big(\hat{b}_k^v\big(\hat{b}_k^v\big)^\star\big)^{-1}\big)\hat{b}_k^v\Delta W_k/\big(2\Delta\check{U}^k\big) - \frac{1}{2}\partial_\sigma \log\det\big(\hat{b}_k^v\big(\hat{b}_k^v\big)^\star\big),$$

$$\mathcal{A}_{k,k'}^3 = \bigg\{-\frac{1}{2}\operatorname{tr}\big(\big(\hat{b}_{k'}^v\big)^\star\big(\big(\tilde{b}_{k'}^{vu}\big(\tilde{b}_{k'}^{vu}\big)^\star\big)^{-1} - \big(b_{k'}^{vu}\big(b_{k'}^{vu}\big)^\star\big)^{-1}\big)\big(Y^{0,vu}_{\check{U}}\big)\hat{b}_{k'}^v\big) - \frac{1}{2}\log\frac{\det(\tilde{b}_{k'}^{vu}(\tilde{b}_{k'}^{vu})^\star)}{\det(b_{k'}^{vu}(b_{k'}^{vu})^\star)}\big(Y^{0,vu}_{\check{U}}\big)\bigg\}1_{\{|k-k'|\le h\}},$$

$$\mathcal{A}_{k,k'}^4 = -\int_{\breve{u}^{k'-1}}^{\breve{u}^{k'}}(W_t - W_{\breve{u}^{k'-1}})^\star\big(\hat{b}_{k'}^v\big)^\star\frac{(\tilde{b}_{k'}^{vu}(\tilde{b}_{k'}^{vu})^\star)^{-1} - (b_{k'}^{vu}(b_{k'}^{vu})^\star)^{-1}}{\Delta\breve{u}^{k'}}\hat{b}_{k'}^v\,\mathrm{d}W_t\bigg|_{\bar{u}=\Pi}1_{\{|k-k'|\le h\}}.$$

Then $\mathcal{A}_k^1, \mathcal{A}_k^2, \mathcal{A}_{k,k'}^3, \mathcal{A}_{k,k'}^4$ satisfy $E[|f_k^{vu,(1)} - \tilde{f}_k^{vu,(1)} - \mathcal{A}_k^1||\Pi] \le C|\tilde{\theta}_{1,k}|$, $E[|\tilde{f}_k^{vu,(1)} - \mathcal{A}_k^2|^2|\Pi]^{1/2} \le C|\tilde{\theta}_{1,k}|^{1/2}$ and $E[|(\tilde{f}_{k'}^{vu} - f_{k'}^{vu} - \mu_{(k')}^\star(b_{(k')}b_{(k')}^\star)^{-1}b_{(k')}\Delta W_{k'})1_{\{|k-k'|\le h\}} - \mathcal{A}_{k,k'}^3 - \mathcal{A}_{k,k'}^4|^2|\Pi]^{1/2} \le C|\tilde{\theta}_{1,k'}|1_{\{|k-k'|\le h\}}$.

Hence, we obtain

$$\begin{aligned}\Phi_{n,3} &\le C\sup_v E\bigg[\bigg|b_n^{-1/2}\sum_k\bigg(\mathcal{A}_k^1 - \mathcal{A}_k^2\sum_{k';|k'-k|\le h}\big(\mathcal{A}_{k,k'}^3 + \mathcal{A}_{k,k'}^4\big)\bigg)\bigg|\bigg|\Pi\bigg] \\ &\quad + O_p\big(b_n^{-1/2}r_nb_n^\kappa(\ell_{1,n} + \ell_{2,n})\big) + o_p(1) \qquad (4.8)\\ &= C\sup_v E\bigg[\bigg|b_n^{-1/2}\sum_k\bigg(\mathcal{A}_k^1 - \mathcal{A}_k^2\sum_{k';|k'-k|\le h}\big(\mathcal{A}_{k,k'}^3 + \mathcal{A}_{k,k'}^4\big)\bigg)\bigg|\bigg|\Pi\bigg] + o_p(1),\end{aligned}$$

where we use the Cauchy–Schwarz inequality, tightness of $\{b_n^{-1}(\ell_{1,n} + \ell_{2,n})\}_n$ and $r_nb_n^{1/2+\kappa} = o_p(1)$ by the definition of κ.

Moreover, by using the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned}
&\sup_v E\left[\left|b_n^{-1/2}\sum_k \mathcal{A}_k^1\right|\middle|\Pi\right] \\
&\quad = \sup_v E\left[\left|\frac{b_n^{-1/2}}{2}\sum_k\Bigl(\operatorname{tr}\bigl((\hat{b}_k^v)^\star\bigl(\partial_\sigma\bigl(\bigl(b_{(k)}b_{(k)}^\star\bigr)^{-1}\bigr)-\partial_\sigma\bigl(\bigl(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star\bigr)^{-1}\bigr)\bigr)\bigl(Y_{\check{U}}^{0,vu},\sigma_{vu}^n\bigr)\hat{b}_k^v\bigr)\right.\right. \\
&\qquad\qquad \left.\left. +\,\partial_\sigma\log\frac{\det(b_{(k)}b_{(k)}^\star)}{\det(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)}\bigl(Y_{\check{U}}^{0,vu},\sigma_{vu}^n\bigr)\Bigr)\right|\middle|\Pi\right]+\mathrm{o}_p(1).
\end{aligned}$$

Furthermore, by applying Itô's formula to $(\partial_\sigma((b_{(k)}b_{(k)}^\star)^{-1})-\partial_\sigma((\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)^{-1}))$ and $\partial_\sigma\log(\det(b_{(k)}b_{(k)}^\star)/\det(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star))$, 1. of Lemma A.1 in the Appendix and Lemma 4.1, we have

$$\begin{aligned}
\sup_v E\left[\left|b_n^{-1/2}\sum_k \mathcal{A}_k^1\right|\middle|\Pi\right] &\le Cb_n^{-1/2}\Bigl(\sum_{l_1,l_2}\sum_k \Delta\check{U}^k 1_{\{[\check{U}^{k-1},\check{U}^k)\subset\tilde{\theta}_{1,l_1}\cap\tilde{\theta}_{1,l_2}\}}\Bigr)^{1/2} \\
&\quad + Cb_n^{-1/2}\sum_k|\tilde{\theta}_{1,k}|+\mathrm{o}_p(1) \qquad (4.9)\\
&\le Cb_n^{-1/2}\Bigl(\sum_{l_1,l_2}|\tilde{\theta}_{1,l_1}\cap\tilde{\theta}_{1,l_2}|\Bigr)^{1/2}+\mathrm{o}_p(1)=\mathrm{o}_p(1).
\end{aligned}$$

We can see $\mathcal{A}_{k,k'}^3$ can be decomposed as $\mathcal{A}_{k,k'}^3=\sum_{\tilde{k}}\hat{\mathcal{A}}_{k,k',\tilde{k}}^3+\mathrm{O}_p(|\tilde{\theta}_{1,k'}|)$, where $\{\sum_{\tilde{k}\le l}\hat{\mathcal{A}}_{k,k',\tilde{k}}^3|_{\Pi=\bar{u}}\}_{l=0}^{L_0(\bar{u})}$ is a martingale for any $\bar{u}\in\mathcal{U}$, $E[|\sum_{\tilde{k};\tilde{k}<l}\hat{\mathcal{A}}_{k,k',\tilde{k}}^3|^4|\Pi]^{1/4}\le C|\tilde{\theta}_{1,k'}|^{1/2}$ for any l and $E[|\hat{\mathcal{A}}_{k,k',\tilde{k}}^3|^4|\Pi]^{1/4}\le C|\Delta\check{U}^{\tilde{k}}|^{1/2}1_{\{[\check{U}^{\tilde{k}-1},\check{U}^{\tilde{k}})\subset\tilde{\theta}_{1,k'}\}}$. Hence by Lemmas 4.1 and A.1, we obtain

$$\begin{aligned}
&\sup_v E\left[\left|b_n^{-1/2}\sum_k\mathcal{A}_k^2\sum_{k';|k'-k|\le h}\mathcal{A}_{k,k'}^3\right|\middle|\Pi\right] \\
&\quad\le \sup_v E\left[\left|b_n^{-1/2}\sum_k\mathcal{A}_k^2\sum_{k';|k'-k|\le h}\hat{\mathcal{A}}_{k,k',k}^3\right|\middle|\Pi\right]+\mathrm{O}_p\Bigl(b_n^{-1/2}\sum_k\sum_{k';|k'-k|\le h}|\tilde{\theta}_{1,k'}|\Bigr) \\
&\qquad + Cb_n^{-1/2}b_n^\kappa\Bigl\{\sum_{l_1,l_2;|l_1-l_2|\le 2h}|\tilde{\theta}_{1,l_1}|^{1/2}|\tilde{\theta}_{1,l_2}|^{1/2}+\sum_{l_1,l_2}\sum_k\Delta U^k 1_{\{[U^{k-1},U^k)\subset\tilde{\theta}_{1,l_1}\cap\tilde{\theta}_{1,l_2}\}}\Bigr\}^{1/2} \qquad (4.10)\\
&\quad\le \sup_v E\left[\left|b_n^{-1/2}\sum_k\mathcal{A}_k^2\sum_{k';|k'-k|\le h}\hat{\mathcal{A}}_{k,k',k}^3\right|\middle|\Pi\right]+\mathrm{O}_p\bigl(b_n^{-1/2}b_n^\kappa r_n(\ell_{1,n}+\ell_{2,n})\bigr)
\end{aligned}$$

$$+ Cb_n^{-1/2} b_n^{\kappa} \left\{ r_n^{1/2} b_n^{\kappa/2} (\ell_{1,n} + \ell_{2,n})^{1/2} + \left(\sum_{l_1,l_2} |\tilde{\theta}_{1,l_1} \cap \tilde{\theta}_{1,l_2}| \right)^{1/2} \right\}$$

$$\le \sup_v E\left[\left| b_n^{-1/2} \sum_k \mathcal{A}_k^2 \sum_{k';|k'-k|\le h} \hat{\mathcal{A}}^3_{k,k',k} \right| \Big| \Pi \right] + \mathrm{o}_p(1).$$

Moreover, by using Lemma A.1 with relations $E[|\mathcal{A}_k^2|^4|\Pi]^{1/4} \le C$ and $E[|\mathcal{A}^4_{k,k'}|^4|\Pi]^{1/4} \le C|\tilde{\theta}_{1,k'}|^{1/2}$, we obtain

$$\begin{aligned}
&\sup_v E\left[\left| b_n^{-1/2} \sum_k \mathcal{A}_k^2 \sum_{k';0<|k'-k|\le h} \mathcal{A}^4_{k,k'} \right| \Big| \Pi \right] \\
&\quad \le Cb_n^{-1/2} b_n^{\kappa} \left(\sum_{l_2,k_2';|l_2-k_2'|\le 2h} |\tilde{\theta}_{1,l_2}|^{1/2} |\tilde{\theta}_{1,k_2'}|^{1/2} \right)^{1/2} \\
&\quad \le Cb_n^{-1/2+\kappa} \left((\ell_{1,n} + \ell_{2,n}) \cdot \left(4b_n^{\kappa} + 1\right) r_n \right)^{1/2} = \mathrm{o}_p(1).
\end{aligned} \tag{4.11}$$

By (4.8)–(4.11), we obtain

$$\begin{aligned}
\Phi_{n,3} \le C \Bigg\{ &\sup_v E\left[\left| b_n^{-1/2} \sum_k \mathcal{A}_k^2 \sum_{k';|k'-k|\le h} \hat{\mathcal{A}}^3_{k,k',k} \right| \Big| \Pi \right] \\
&+ \sup_v E\left[\left| b_n^{-1/2} \sum_k \mathcal{A}_k^2 \mathcal{A}^4_{k,k} \right| \Big| \Pi \right] \Bigg\} + \mathrm{o}_p(1).
\end{aligned}$$

By using Itô's formula, the Burkholder–Davis–Gundy inequality and Lemma A.1 1. similarly, we obtain $\Phi_{n,3} = \mathrm{o}_p(1)$. □

We proceed to the second step. We will prove

$$\log \frac{\bar{\mathbb{P}}_u^0}{\bar{\mathbb{P}}_0^0}(Y_\Pi) - b_n^{-1/2} u \int_0^1 \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)} \, \mathrm{d}v(Y_\Pi) \to^p 0$$

as $n \to \infty$ under $[A1']$, $[A2]$ and $[A3']$. To this end, we need to estimate

$$\begin{aligned}
&b_n^{-1/2} u \int_0^1 \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)} 1_{B_M^n})}{\tilde{\mathbb{P}}^0_{M,vu}} \, \mathrm{d}v(Y_\Pi) - b_n^{-1/2} u \int_0^1 \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)} 1_{B_M^n})}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})} \, \mathrm{d}v(Y_\Pi) \\
&\quad = b_n^{-1/2} u \int_0^1 \frac{1}{\tilde{\mathbb{P}}^0_{M,vu}} \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)} 1_{B_M^n})}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})} \\
&\qquad \times \left(\exp\left(\sum_{|k-k'|\le h} \left(\tilde{f}_{k'}^{vu} - \log \check{p}^0_{k',vu} \right) \right) - 1 \right) 1_{B_M^n} \mathbb{P}^0_{vu} \, \mathrm{d}\hat{z} \, \mathrm{d}v(Y_\Pi).
\end{aligned}$$

If $\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)} 1_{B_M^n})/\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})$ has *good* properties, we can apply techniques from Itô calculus similarly to the proof of Lemma 4.3. So we will investigate properties of this quantity.

We prepare some additional lemmas.

Lemma 4.4. *Let $u \in \mathbb{R}^d$. Assume [A1′], [A2] and [A3′]. Then $\{\sup_{k,v} |\log(\bar{\mathbb{P}}_k^{2,vu}(1)/\bar{\mathbb{P}}_0^0)|(Y_\Pi)\}_n$ is tight.*

Proof. Let $\mathbb{P}_k^{2,vu} = \exp(\sum_{k'} \acute{f}_{k'}^k)$, $\mathcal{K}_2'(\bar{u}) = \{z; \sup_{k,v} |\tilde{f}_k^{vu} - \log \check{p}_{k,vu}^0|(z) \le b_n^{-2\kappa}\}$, and $Y_{\check{U}}^{2,vu,k}$ be random variables with the Π-conditional distribution $\mathbb{P}_k^{2,vu}(z)\,\mathrm{d}\hat{z}\,\mathrm{d}\bar{z}\,P_{Y_0}(\mathrm{d}z_0)$. For any $q > 0$, let $q' \ge 2(q+1)/(1-\delta_1-4\kappa)$. Then we obtain

$$\begin{aligned}
&P\bigl[Y_{\check{U}} \in \bigl(\mathcal{K}_2'\bigr)^c(\Pi)|\Pi\bigr] \\
&\quad\le b_n^{2\kappa q'} \sum_k E\Bigl[\sup_v \bigl|\tilde{f}_k^{vu} - \log \check{p}_{k,vu}^0\bigr|^{q'}(Y_{\check{U}})|\Pi\Bigr] \\
&\quad\le C b_n^{2\kappa q'} \sum_k E\Bigl[\sup_v \bigl|\tilde{f}_k^{vu} - f_k^{vu}\bigr|^{q'}(Y_{\check{U}})|\Pi\Bigr] \\
&\qquad + C b_n^{2\kappa q'} \sup_r \sum_k E\biggl[\sup_v \biggl|\frac{\partial_r \check{p}_{k,vu}^r}{\check{p}_{k,vu}^r}\biggr|^{q'}(Y_{\check{U}})\bigg|\Pi\biggr] \\
&\quad\le C b_n^{2\kappa q'} r_n^{q'/2}(\ell_{1,n} + \ell_{2,n}) = \mathrm{O}_p\bigl(b_n^{-q}\bigr).
\end{aligned} \tag{4.12}$$

Similarly, we have $\sup_v P[Y_{\check{U}}^{0,vu} \in (\mathcal{K}_2')^c(\Pi)|\Pi] = \mathrm{O}_p(b_n^{-q})$ and $\sup_{k,v} P[Y_{\check{U}}^{2,vu,k} \in (\mathcal{K}_2')^c(\Pi)|\Pi] = \mathrm{O}_p(b_n^{-q})$ for any $q > 0$.

Hence, we have

$$\begin{aligned}
&P\biggl[\sup_{k,v} \biggl|\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{(\mathcal{K}_2')^c})}{\bar{\mathbb{P}}_{vu}^0}\biggr|(Y_\Pi) > \frac{b_n^{-1}}{2}, \sup_v \biggl|\log \frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_0^0}\biggr|(Y_\Pi) \le M' \bigg|\Pi\biggr] \\
&\quad\le 2\mathrm{e}^{M'} b_n \int \sup_{k,v} \mathbb{P}_k^{2,vu} 1_{(\mathcal{K}_2')^c}\,\mathrm{d}\hat{z}\,\mathrm{d}\bar{z}\,P_{Y_0}(\mathrm{d}z_0)\bigg|_{\bar{u}=\Pi} \\
&\quad\le 2\mathrm{e}^{M'} b_n \sum_k \int \biggl(\mathbb{P}_k^{2,0} + \int_0^1 \bigl|\partial_v \mathbb{P}_k^{2,vu}\bigr|\,\mathrm{d}v\biggr) 1_{(\mathcal{K}_2')^c}\,\mathrm{d}\hat{z}\,\mathrm{d}\bar{z}\,P_{Y_0}(\mathrm{d}z_0)\bigg|_{\bar{u}=\Pi} \\
&\quad\le 2\mathrm{e}^{M'} b_n \sum_k P\bigl[Y_{\check{U}}^{2,0,k} \in \bigl(\mathcal{K}_2'\bigr)^c|\Pi\bigr] + 2\mathrm{e}^{M'} b_n \sum_k \sup_v E\biggl[\biggl|\frac{\partial_v \mathbb{P}_k^{2,vu}}{\mathbb{P}_k^{2,vu}}\biggr| 1_{(\mathcal{K}_2')^c}\bigl(Y_{\check{U}}^{2,vu,k}\bigr)\bigg|\Pi\biggr] \\
&\quad= \mathrm{O}_p\bigl(b_n^{-q}\bigr)
\end{aligned}$$

for any $q > 0$ and $M' > 0$.

Therefore, for any $\varepsilon > 0$, there exists $N > 0$ such that

$$P\left[\sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{(\mathcal{K}_2')^c})}{\bar{\mathbb{P}}_{vu}^0}\right|(Y_\Pi) > \frac{b_n^{-1}}{2}\right] < \varepsilon \tag{4.13}$$

for $n \geq N$ by Lemma 3.10.

Moreover, for any $\delta > 0$ and $M' > 0$, we have

$$\begin{aligned}
&P\left[\sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{\mathcal{K}_2'})}{\bar{\mathbb{P}}_{vu}^0} - 1\right|(Y_\Pi) > \delta, \sup_v\left|\log\frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_0^0}\right|(Y_\Pi) \leq M'\middle|\Pi\right] \\
&\leq \frac{e^{M'}}{\delta}\int \sup_{k,v}\left|\mathbb{P}_k^{2,vu}1_{\mathcal{K}_2'} - \mathbb{P}_{vu}^0\right| d\hat{z}\, d\bar{z}\, P_{Y_0}(dz_0)\Big|_{\bar{u}=\Pi} \\
&\leq \frac{e^{M'}}{\delta}\int\left\{\sup_k\left|\mathbb{P}_k^{2,0}1_{\mathcal{K}_2'} - \mathbb{P}_0^0\right| + \int_0^1 \sup_k\left|\partial_v\mathbb{P}_k^{2,vu}1_{\mathcal{K}_2'} - \partial_v\mathbb{P}_{vu}^0\right| dv\right\} d\hat{z}\, d\bar{z}\, P_{Y_0}(dz_0)\Big|_{\bar{u}=\Pi} \\
&\leq \frac{e^{M'}}{\delta}\Bigg\{P\left[Y_{\check{U}} \in (\mathcal{K}_2')^c(\Pi)|\Pi\right] + O_p\left(b_n^{-\kappa}\right) + \sup_v E\left[\left|\frac{\partial_v\mathbb{P}_{vu}^0}{\mathbb{P}_{vu}^0}\right| 1_{(\mathcal{K}_2')^c}\left(Y_{\check{U}}^{0,vu}\right)\middle|\Pi\right] \\
&\qquad + \sup_v\int\sup_k\left|\sum_{k''}\partial_v \acute{f}_{k''}^{k,vu}\exp\left(\sum_{|k'-k|\leq h}\left(\tilde{f}_{k'}^{vu} - \log\check{p}_{k',vu}^0\right)\right) - \sum_{k'}\frac{\partial_v\check{p}_{k',vu}^0}{\check{p}_{k',vu}^0}\right| \\
&\qquad \times 1_{\mathcal{K}_2'}\mathbb{P}_{vu}^0\, d\hat{z}\, d\bar{z}\, P_{Y_0}(dz_0)\Big|_{\bar{u}=\Pi}\Bigg\} \\
&= o_p(1),
\end{aligned}$$

by (4.12). Hence,

$$P\left[\sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{\mathcal{K}_2'})}{\bar{\mathbb{P}}_{vu}^0} - 1\right|(Y_\Pi) > \delta\right] < \varepsilon \tag{4.14}$$

for sufficiently large n.

Lemmas 3.4 and 3.10, (4.13) and (4.14) complete the proof. □

Let $\mathcal{K}_{2,M}'' = \{(z_0, \bar{z}); \inf_{k,v}\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})(z_0, \bar{z}) > 0\}$ and

$$\mathcal{K}_M^2(\bar{u}) = \mathcal{K}_M^1(\bar{u}) \cap \left\{(z_0, \bar{z}); \sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)}1_{B_M^n})}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})} - \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}\right|(z_0, \bar{z}) \leq b_n^{-1}\right\} \cap \mathcal{K}_{2,M}''$$

for $u \in \mathbb{R}^d$, $\bar{u} \in \mathcal{U}$, $1 \leq k \leq L_0(\bar{u})$ and $M > 0$.

Lemma 4.5. *Let $u \in \mathbb{R}^d$. Assume [A1′], [A2] and [A3′]. Then for any $\varepsilon > 0$, there exists $M' > 0$ and $\{N_M\}_{M \geq M'} \subset \mathbb{N}$ such that $\sup_{n \geq N_M} P[Y_\Pi \in (\mathcal{K}_M^2)^c(\Pi)] < \varepsilon$ for $M \geq M'$.*

Proof. By (4.13), for any $\varepsilon > 0$, there exists $N_1' \in \mathbb{N}$ such that $P[\sup_{k,v}(\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n}1_{(\mathcal{K}_2')^c})/\bar{\mathbb{P}}_{vu}^0)(Y_\Pi) > b_n^{-1}/2] < \varepsilon$ for $n \geq N_1'$ and $M > 0$. Moreover, by (4.5), we have

$$P\left[\sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n}1_{\mathcal{K}_2'})}{\bar{\mathbb{P}}_{vu}^0} - 1\right|(Y_\Pi) > \delta\right] < \varepsilon$$

for any $\delta, \varepsilon > 0$ and sufficiently large n and M, similarly to the derivation of (4.14).

Therefore, there exist $N_2' \in \mathbb{N}$ and $M_2 > 0$ such that

$$P\Big[\inf_{k,v}\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})(Y_\Pi) > 0\Big] > 1 - \varepsilon$$

and

$$P\Big[\sup_{k,v}\big|\log\big(\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})/\bar{\mathbb{P}}_{vu}^0\big)\big|(Y_\Pi) > \delta\Big] < \varepsilon \tag{4.15}$$

for $M > M_2$ and $n \geq N_2'$. Moreover, we have $\sup_{k,v}|\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})/\bar{\mathbb{P}}_{vu}^0|(Y_\Pi) = \mathrm{O}_p(b_n^2)$.

Since

$$\begin{aligned}
&\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)}1_{B_M^n})}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})} - \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)} \\
&\quad = -\frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})}\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)}1_{(B_M^n)^c})}{\bar{\mathbb{P}}_{vu}^0} \\
&\qquad + \frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_k^{2,vu}(1_{B_M^n})}\frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_k^{2,vu}(1)}\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_{vu}^0}\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{(B_M^n)^c})}{\bar{\mathbb{P}}_{vu}^0},
\end{aligned}$$

there exist M', $M_1 > 0$ and $\{N_M\}_{M\geq M'} \subset \mathbb{N}$ such that

$$\begin{aligned}
&P\big[Y_\Pi \in \big(\mathcal{K}_M^2\big)^c(\Pi)\big] \\
&\quad \leq P\big[Y_\Pi \in \big(\mathcal{K}_M^1\big)^c(\Pi) \cup \big(\mathcal{K}_{2,M}''\big)^c\big] \\
&\qquad + P\left[\sup_{k,v}\left|2\frac{\bar{\mathbb{P}}_k^{2,vu}(|\tilde{f}_k^{vu,(1)}|1_{(B_M^n)^c})}{\bar{\mathbb{P}}_{vu}^0} + 4b_n^3\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{(B_M^n)^c})}{\bar{\mathbb{P}}_{vu}^0}\right|(Y_\Pi) > b_n^{-1}\right] + \varepsilon \\
&\quad \leq P\left[\sup_{k,v}\left|2\frac{\bar{\mathbb{P}}_k^{2,vu}(|\tilde{f}_k^{vu,(1)}|1_{(B_M^n)^c}1_{\mathcal{K}_2'})}{\bar{\mathbb{P}}_{vu}^0} + 4b_n^3\frac{\bar{\mathbb{P}}_k^{2,vu}(1_{(B_M^n)^c}1_{\mathcal{K}_2'})}{\bar{\mathbb{P}}_{vu}^0}\right|(Y_\Pi) > \frac{b_n^{-1}}{2},\right. \\
&\qquad \left.\sup_v\left|\log\frac{\bar{\mathbb{P}}_{vu}^0}{\bar{\mathbb{P}}_0^0}\right|(Y_\Pi) \leq M_1\right] + 5\varepsilon \\
&\quad \leq E\left[\left(4b_n e^{M_1}\int\sup_{k,v}\big\{\big(2\big|\tilde{f}_k^{vu,(1)}\big| + 4b_n^3\big)\mathbb{P}_{vu}^0\big\}1_{(B_M^n)^c}1_{\mathcal{K}_2'}\,\mathrm{d}\hat{z}\,\mathrm{d}\bar{z}\,P_{Y_0}(\mathrm{d}z_0)\bigg|_{\bar{u}=\Pi}\right) \wedge 1\right] + 5\varepsilon
\end{aligned}$$

$$\leq CE\bigg[\bigg(\bigg(b_n^4 P\big[Y_{\check{U}} \in (B_M^n)^c|\Pi\big] + b_n^{-2}(\ell_{1,n} + \ell_{2,n})$$

$$+ \sup_v E\bigg[\bigg|\frac{\partial_v \mathbb{P}_{vu}^0}{\mathbb{P}_{vu}^0}\bigg|\bigg(b_n^4 + b_n \sum_k \sup_v \big|\tilde{f}_k^{vu,(1)}\big|\bigg) 1_{(B_M^n)^c}\big(Y_{\check{U}}^{0,vu}\big)\bigg|\Pi\bigg]\bigg) \wedge 1\bigg] + 5\varepsilon$$

$$< 6\varepsilon$$

for $M \geq M'$ and $n \geq N_M$, by (4.15) and similar arguments to (4.12) and (4.13). □

In the following, we see that the integral $\int \exp(\sum_{\nu \leq k' \leq \chi} \tilde{f}_{k'}^{vu})\,\mathrm{d}\hat{z}$ has a simple representation of a function of increments for $1 \leq \nu \leq \chi \leq L_0(\bar{u})$. To see this, we will define some notation related to the observation times and increments of processes in the interval $(\check{u}^{\nu-1}, \check{u}^{\chi}]$.

For $\bar{u} = ((s^i)^i, (t^j)^j) \in \mathcal{U}$ and $1 \leq \nu \leq \chi \leq L_0(\bar{u})$, let

$$\mathcal{I}_{\nu,\chi} = \big(\{k_1(i)\}_i \cap [\nu, \chi - 1]\big) \cup \{\nu - 1, \chi\},$$
$$\mathcal{J}_{\nu,\chi} = \big(\{k_2(j)\}_j \cap [\nu, \chi - 1]\big) \cap \{\nu - 1, \chi\},$$
$$\mathcal{I}_{\nu,\chi}^- = \mathcal{I}_{\nu,\chi} \setminus \{\chi\}, \qquad \mathcal{J}_{\nu,\chi}^- = \mathcal{J}_{\nu,\chi} \setminus \{\chi\}.$$

Moreover, for $z = (x_k, y_k)_{k=0}^{L_0(\bar{u})}$, define

$$\hat{z}_{\nu,\chi} = \big((x_k)_{\nu \leq k \leq \chi-1, k \notin \{k_1(i)\}_i}, (y_k)_{\nu \leq k \leq \chi-1, k \notin \{k_2(j)\}_j}\big),$$
$$x(\mathcal{I}_{\nu,\chi}) = \{x_{i_k} - x_{i_{k-1}}\}_{k=1}^{L_{\nu,\chi}^1}, \qquad y(\mathcal{J}_{\nu,\chi}) = \{y_{j_k} - y_{j_{k-1}}\}_{k=1}^{L_{\nu,\chi}^2},$$

where $\mathcal{I}_{\nu,\chi} = \{i_k\}_{k=0}^{L_{\nu,\chi}^1}$, $\mathcal{J}_{\nu,\chi} = \{j_k\}_{k=0}^{L_{\nu,\chi}^2}$ and $\nu - 1 = i_0 < \cdots < i_{L_{\nu,\chi}^1} = \chi$, $\nu - 1 = j_0 < \cdots < j_{L_{\nu,\chi}^2} = \chi$. For $p = 1, 2$, $k \in \mathcal{I}_{\nu,\chi}$ and $l \in \mathcal{J}_{\nu,\chi}$, let $\check{b}_k^p = \check{b}_{k,vu}^{n,p} = b^p(\check{u}^{k-1}, x_{i(k)}, y_{j(k)}, \sigma_{vu}^n)$, $\tilde{I}_{\nu,\chi}^k = [s^{i-1}, s^i) \cap [\check{u}^{\nu-1}, \check{u}^{\chi})$, $\tilde{J}_{\nu,\chi}^k = [t^{j-1}, t^j) \cap [\check{u}^{\nu-1}, \check{u}^{\chi})$, where i, j satisfy $[\check{u}^{k-1}, \check{u}^k) \subset [s^{i-1}, s^i)$ and $[\check{u}^{l-1}, \check{u}^l) \subset [t^{j-1}, t^j)$. Let $L_{\nu,\chi} = L_{\nu,\chi}^1 + L_{\nu,\chi}^2$, $\tilde{K}_{k'} = [\check{u}^{k'-1}, \check{u}^{k'})$ and

$$S_{\nu,\chi} = \begin{pmatrix} \operatorname{diag}\Big(\Big\{\sum_{k'} |\check{b}_{k'}^1|^2 |\tilde{I}_{\nu,\chi}^{i_k} \cap \tilde{K}_{k'}|\Big\}_{1 \leq k \leq L_{\nu,\chi}^1}\Big) & \Big\{\sum_{k'} \check{b}_{k'}^1 \cdot \check{b}_{k'}^2 |\tilde{I}_{\nu,\chi}^{i_k} \cap \tilde{J}_{\nu,\chi}^{j_l} \cap \tilde{K}_{k'}|\Big\}_{1 \leq k \leq L_{\nu,\chi}^1, 1 \leq l \leq L_{\nu,\chi}^2} \\ \Big\{\sum_{k'} \check{b}_{k'}^1 \cdot \check{b}_{k'}^2 |\tilde{I}_{\nu,\chi}^{i_k} \cap \tilde{J}_{\nu,\chi}^{j_l} \cap \tilde{K}_{k'}|\Big\}_{1 \leq l \leq L_{\nu,\chi}^2, 1 \leq k \leq L_{\nu,\chi}^1} & \operatorname{diag}\Big(\Big\{\sum_{k'} |\check{b}_{k'}^2|^2 |\tilde{J}_{\nu,\chi}^{j_l} \cap \tilde{K}_{k'}|\Big\}_{1 \leq l \leq L_{\nu,\chi}^2}\Big) \end{pmatrix}.$$

Let $\varphi(x; V)$ be the density function of $N(0, V)$ for a symmetric, positive definite matrix V. The following lemma enables us to calculate integrals of exponential functions of $\tilde{f}_k^{vu}$.

Lemma 4.6. *Let* $u \in \mathbb{R}^d$, $\bar{u} \in \mathcal{U}$, $n \in \mathbb{N}$ *and* $1 \le \nu \le \chi \le L_0(\bar{u})$. *Assume* [*A*1]. *Then* $\det S_{\nu,\chi} > 0$ *and*

$$\int \exp\Big(\sum_{\nu \le k' \le \chi} \tilde{f}^{\nu u}_{k'}(z)\Big)\,\mathrm{d}\hat{z}_{\nu,\chi} = \varphi\big(\big(x(\mathcal{I}_{\nu,\chi})^\star, y(\mathcal{J}_{\nu,\chi})^\star\big)^\star; S_{\nu,\chi}\big). \tag{4.16}$$

Proof. We see $\det S_{\nu,\chi} > 0$ by a similar argument to the proof of Proposition 1 in Ogihara and Yoshida [19], so we omit the details.

We prove (4.16) by induction on χ. The results obviously hold true for $\chi = \nu$.

Let $\chi > \nu$ and assume the results hold for $\chi - 1$. We give the proof only for the case $\check{u}^{\chi-1} \notin (s^i)_i$ and $\check{u}^{\chi-1} \in (t^j)_j$. The other cases are proved similarly.

By the induction assumption, we obtain

$$\begin{aligned}&\int \exp\Big(\sum_{\nu \le k' \le \chi} \tilde{f}^{\nu u}_{k'}\Big)\,\mathrm{d}\hat{z}_{\nu,\chi}\\ &\quad= \int \varphi\big(\big(x(\mathcal{I}_{\nu,\chi-1})^\star, y(\mathcal{J}_{\nu,\chi-1})^\star\big)^\star; S_{\nu,\chi-1}\big)\varphi\big(z_\chi - z_{\chi-1}; \tilde{b}^{\nu u}_\chi\big(\tilde{b}^{\nu u}_\chi\big)^\star\Delta\check{u}^\chi\big)\,\mathrm{d}x_{\chi-1}.\end{aligned}$$

Let Z_1 and Z_2 be random variables independent of each other, satisfying $Z_1 \sim N(0, S_{\nu,\chi-1})$ and $Z_2 \sim N(0, \tilde{b}^{\nu u}_\chi(\tilde{b}^{\nu u}_\chi)^\star\Delta\check{u}^\chi)$. Moreover, let $\mathcal{D}$ be an $(L_{\nu,\chi-1}+1) \times (L_{\nu,\chi-1}+2)$ matrix with $\mathcal{D}_{pq} = \delta_{p,q}$ for $1 \le p, q \le L_{\nu,\chi-1}$, $\mathcal{D}_{pq} = 1$ for $(p,q) = (L^1_{\nu,\chi-1}, L_{\nu,\chi-1}+1)$ or $(L_{\nu,\chi-1}+1, L_{\nu,\chi-1}+2)$, and $\mathcal{D}_{pq} = 0$ in other cases. Then the covariance matrix of $\mathcal{D}(Z_1^\star, Z_2^\star)^\star$ is

$$\mathcal{D}\begin{pmatrix} S_{\nu,\chi-1} & 0 \\ 0 & \tilde{b}^{\nu u}_\chi\big(\tilde{b}^{\nu u}_\chi\big)^\star\Delta\check{u}^\chi\end{pmatrix}\mathcal{D}^\star = S_{\nu,\chi}.$$

Hence, we obtain the result by considering relations between densities of Z_1, Z_2 and $\mathcal{D}(Z_1^\star, Z_2^\star)^\star$. □

Remark 4.1. We emphasize that we can prove the above lemma because $\tilde{b}^{\nu u}_\chi$ does not depend on $\hat{z}_{\nu,\chi}$.

We now give another representation of $\bar{\mathbb{P}}^{2,\nu u}_k(\tilde{f}^{\nu u,(1)}_k)/\bar{\mathbb{P}}^{2,\nu u}_k(1)(z_0, \bar{z})$ consisting of a quadratic form of increments. This representation is useful to apply Itô's rule and martingale properties.

Let $\Theta(n,k,1;\bar{u}) = \{i; 1 \le i \le L^1, s^{i-1} > \check{u}^{(k-h-1)\vee 0}, s^i < \check{u}^{(k+h)\wedge L_0}\}$, $\Theta(n,k,2;\bar{u}) = \{j; 1 \le j \le L^2, t^{j-1} > \check{u}^{(k-h-1)\vee 0}, t^j < \check{u}^{(k+h)\wedge L_0}\}$, $\mathbf{M} = \sharp(\Theta(n,k,1;\bar{u})) + \sharp(\Theta(n,k,2;\bar{u}))$ and

$$\mathcal{Z}_k = \left(\left(\frac{x_{k_1(i)} - x_{k_1(i-1)}}{\sqrt{s^i - s^{i-1}}}\right)^\star_{i \in \Theta(n,k,1;\bar{u})}, \left(\frac{y_{k_2(j)} - y_{k_2(j-1)}}{\sqrt{t^j - t^{j-1}}}\right)^\star_{j \in \Theta(n,k,2;\bar{u})}\right)^\star$$

for $\bar{u} = ((s^i)_{i=0}^{L^1}, (t^j)_{j=0}^{L^2}) \in \mathcal{U}$ and $z = ((x_k)_{k=0}^{L_0(\bar{u})}, (y_k)_{k=0}^{L_0(\bar{u})}) \in \mathbb{R}^{2(L_0(\bar{u})+1)}$.

Lemma 4.7. *Let $u \in \mathbb{R}^d$. Assume $[A1'], [A2]$ and $[A3']$. Then there exist an $\mathbb{R}^d \otimes \mathbb{R}^{\mathbf{M}} \otimes \mathbb{R}^{\mathbf{M}}$-valued function $\mathcal{Q}_1^{k,v}(z_0, \bar{z}, \bar{u})$, $\mathbb{R}^d$-valued functions $\{\mathcal{Q}_p^{k,v}(z_0, \bar{z}, \bar{u})\}_{p=2}^4$ ($v \in [0,1]$, $\bar{u} = ((s^i)_i, (t^j)_j) \in \mathcal{U}$, $1 \le k \le L_0(\bar{u})$) and a constant $C > 0$ such that $\mathcal{Q}_1^{k,v}(Y_{\bar{u}}^{r,v'u}, \bar{u})$ and $\mathcal{Q}_2^{k,v}(Y_{\bar{u}}^{r,v'u}, \bar{u})$ are $\mathcal{F}_{\inf \tilde{\theta}((k-h-1)\vee 1, 2; \bar{u})}$-measurable,*

$$\sup_{n,v,\bar{u},k,z_0,\bar{z}} \left(\left\| \mathcal{Q}_1^{k,v}(z_0, \bar{z}; \bar{u})\right\| \vee \left|\mathcal{Q}_2^{k,v}(z_0, \bar{z}; \bar{u})\right|\right) \le C,$$

$$\sup_{r,k,v,v'} E\left[\left|\mathcal{Q}_3^{k,v}\left(Y_\Pi^{r,v'u}; \Pi\right)\right|^q \middle| \Pi\right]^{1/q} \le C r_n^{1/2} b_n^{3\kappa/2} \qquad a.s.,$$

$\sup_{k,v} |\mathcal{Q}_4^{k,v}(Y_\Pi; \Pi)| = \mathrm{o}_p(b_n^{-q})$ and

$$\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}(z_0, \bar{z}) = \mathcal{Z}_k^\star \mathcal{Q}_1^{k,v}(z_0, \bar{z}, \bar{u}) \mathcal{Z}_k 1_{\{\mathcal{Z}_k \ne \varnothing\}} + \sum_{p=2}^4 \mathcal{Q}_p^{k,v}(z_0, \bar{z}, \bar{u}) \tag{4.17}$$

for $v, v', r \in [0,1]$, $n \ge n_u$, $q > 0$, $\bar{u} = ((s^i)_i, (t^j)_j) \in \mathcal{U}$ and $1 \le k \le L_0(\bar{u})$. Moreover,

$$\sup_{k,v} \left| \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)} - \frac{\int \tilde{f}_k^{vu,(1)} \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\, d\hat{z}}{\int \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\, d\hat{z}} \right| (Y_\Pi) = \mathrm{o}_p\left(b_n^{-q}\right) \tag{4.18}$$

for any $q > 0$.

Proof. We only consider the case that $L_{k-h,k-1}^1 \wedge L_{k-h,k-1}^2 \wedge L_{k+1,k+h}^1 \wedge L_{k+1,k+h}^2 > 1$, $k \ge h+1$, $k+h \le L_0(\bar{u})$, $\check{u}^{k-1}, \check{u}^k \notin (s^i)_i$ and $\check{u}^{k-1}, \check{u}^k \in (t^j)_j$. Other cases are proved in a similar way.

The proof is rather complicated. It is divided in several steps.

Step 1. In this step, we will have an expression of $\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})/\bar{\mathbb{P}}_k^{2,vu}(1)$ similar to (4.17) by using elementary formulas (Lemma 4.6 and Lemma A.2 in the Appendix) of Gaussian distributions.

Let $L_k = L_{k-h,k-1}$, $L_k^j = L_{k-h,k-1}^j$ for $j = 1, 2$, $\mathcal{I}_k = \mathcal{I}_{k-h,k-1}$, $\mathcal{J}_k = \mathcal{J}_{k-h,k-1}$ and $S_k = S_{k-h,k-1}$. Then Lemma 4.6 yields

$$\begin{aligned} &\int \tilde{f}_k^{vu,(1)} \exp\left(\sum_{k'; k' \le k} \acute{f}_{k'}^{k,vu}\right)(z)\, d\hat{z}_{1,k} \\ &= \int \left(-\frac{1}{2}(z_k - z_{k-1})^\star \frac{\partial_\sigma((\tilde{b}_{(k)} \tilde{b}_{(k)}^\star)^{-1})(z, \sigma_{vu}^n)}{\Delta \check{u}^k}(z_k - z_{k-1}) \right. \\ &\qquad \left. - \frac{1}{2} \partial_\sigma \log\det\left(\tilde{b}_{(k)} \tilde{b}_{(k)}^\star\right)\left(z, \sigma_{vu}^n\right)\right) \end{aligned} \tag{4.19}$$

$$\times \varphi\big(z_k - z_{k-1}; \Delta\check{u}^k \tilde{b}_k^{vu}\big(\tilde{b}_k^{vu}\big)^\star\big)\varphi\big(\big(x(\mathcal{I}_k)^\star, y(\mathcal{J}_k)^\star\big)^\star; S_k\big)\,\mathrm{d}x_{k-1}$$
$$\times \exp\bigg(\sum_{k' \le k-h-1} f_{k'}^{vu}\bigg)\,\mathrm{d}\hat{z}_{1,k-h}.$$

Let $\mathcal{M}_1 = ((S_k^{-1})_{ij})_{i,j \ne L_k^1}$, $\mathcal{M}_2 = ((S_k^{-1})_{i,L_k^1})_{i \ne L_k^1}$, $\mathcal{M}_3 = (S_k^{-1})_{L_k^1, L_k^1}$, $\tilde{\mathcal{Z}}_2 = x(\mathcal{I}_k)_{L_k^1}$, $\tilde{\mathcal{Z}}_1 = (x(\mathcal{I}_k)^\star, y(\mathcal{J}_k)^\star)^\star \setminus \tilde{\mathcal{Z}}_2$, $v_1 = ((\tilde{b}_k^{vu}(\tilde{b}_k^{vu})^\star \Delta\check{u}^k)^{-1})_{11}$ and

$$\mathcal{M}_4 = \begin{pmatrix} \mathcal{M}_3(v_1 + \mathcal{M}_3)^{-1} & -\big(\big(\tilde{b}_k^{vu}\big(\tilde{b}_k^{vu}\big)^\star \Delta\check{u}^k\big)^{-1}\big)_{12}(v_1 + \mathcal{M}_3)^{-1} \\ 0 & 1 \end{pmatrix},$$

where $x(\mathcal{I}_k) = \{x(\mathcal{I}_k)_i\}_{i=1}^{L_{k-h,k-1}^1}$ and $y(\mathcal{J}_k) = \{y(\mathcal{J}_k)_j\}_{j=1}^{L_{k-h,k-1}^2}$. Then we obtain

$$\begin{aligned}
&\big(x(\mathcal{I}_k)^\star, y(\mathcal{J}_k)^\star\big)S_k^{-1}\big(x(\mathcal{I}_k)^\star, y(\mathcal{J}_k)^\star\big)^\star \\
&\quad = \tilde{\mathcal{Z}}_1^\star \mathcal{M}_1 \tilde{\mathcal{Z}}_1 + 2\tilde{\mathcal{Z}}_1^\star \mathcal{M}_2 \tilde{\mathcal{Z}}_2 + \tilde{\mathcal{Z}}_2^\star \mathcal{M}_3 \tilde{\mathcal{Z}}_2 \\
&\quad = \big(\tilde{\mathcal{Z}}_2 + \mathcal{M}_3^{-1}\mathcal{M}_2^\star \tilde{\mathcal{Z}}_1\big)^\star \mathcal{M}_3 \big(\tilde{\mathcal{Z}}_2 + \mathcal{M}_3^{-1}\mathcal{M}_2^\star \tilde{\mathcal{Z}}_1\big) \\
&\qquad + \tilde{\mathcal{Z}}_1^\star \mathcal{M}_1 \tilde{\mathcal{Z}}_1 - \tilde{\mathcal{Z}}_1^\star \mathcal{M}_2 \mathcal{M}_3^{-1} \mathcal{M}_2^\star \tilde{\mathcal{Z}}_1.
\end{aligned} \tag{4.20}$$

By (4.19), (4.20) and Lemma A.2 in the Appendix, we have

$$\begin{aligned}
&\int \tilde{f}_k^{vu,(1)} \exp\bigg(\sum_{k' \le k} \acute{f}_{k'}^{k,vu}\bigg)(z)\,\mathrm{d}\hat{z}_{1,k} \\
&\quad = \int \bigg\{-\tilde{\mathcal{Z}}_3^\star \mathcal{M}_4^\star \frac{\partial_\sigma((\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)^{-1})}{2\Delta\check{u}^k}\mathcal{M}_4 \tilde{\mathcal{Z}}_3 + \Upsilon_6\bigg\} \exp\bigg(\sum_{k' \le k} \acute{f}_{k'}^{k,vu}\bigg)(z)\,\mathrm{d}\hat{z}_{1,k},
\end{aligned}$$

where $k_1' = \max \mathcal{I}_{k-h,k-1}^-$, $\tilde{\mathcal{Z}}_3 = (x_k - x_{k_1'} + \mathcal{M}_3^{-1}\mathcal{M}_2^\star \tilde{\mathcal{Z}}_1, y_k - y_{k-1})^\star$ and $\Upsilon_6 = -\partial_\sigma((\tilde{b}_{(k)} \times \tilde{b}_{(k)}^\star)^{-1})_{11}(\Delta\check{u}^k)^{-1}(v_1 + \mathcal{M}_3)^{-1}/2 - \partial_\sigma \log\det(\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)/2$.

Moreover, a similar argument yields

$$\begin{aligned}
&\int \tilde{f}_k^{vu,(1)} \exp\bigg(\sum_{k'} \acute{f}_{k'}^{k,vu}\bigg)(z)\,\mathrm{d}\hat{z} \\
&\quad = \int \bigg\{-\tilde{\mathcal{Z}}_5^\star \mathcal{M}_4^\star \frac{\partial_\sigma((\tilde{b}_{(k)}\tilde{b}_{(k)}^\star)^{-1})}{2\Delta\check{u}^k}\mathcal{M}_4 \tilde{\mathcal{Z}}_5 + \tilde{\mathcal{Q}}_2^{k,v}(z_0, \bar{z}; \bar{u})\bigg\} \exp\bigg(\sum_{k'} \acute{f}_{k'}^{k,vu}\bigg)(z)\,\mathrm{d}\hat{z},
\end{aligned}$$

where $\mathcal{P} = \{L_{k-h,k}^1, L_{k-h,k}\}$, $\mathcal{M}_2' = ((S_{k-h,k}^{-1})_{ij})_{i \notin \mathcal{P}, j \in \mathcal{P}}$, $\mathcal{M}_3' = ((S_{k-h,k}^{-1})_{ij})_{i,j \in \mathcal{P}}$, $\tilde{\mathcal{Z}}_1' = (x(\mathcal{I}_{k-h,k})^\star, y(\mathcal{J}_{k-h,k})^\star)^\star \setminus (x(\mathcal{I}_{k-h,k})_{L_{k-h,k}^1}, y(\mathcal{J}_{k-h,k})_{L_{k-h,k}^2})^\star$, $\mathcal{M}_5 = ((S_{k+1,k+h}^{-1})_{i1})_{i \ne 1}$, $\mathcal{M}_6 =$

$(S^{-1}_{k+1,k+h})_{11}$, $\tilde{\mathcal{Z}}_4 = (x(\mathcal{I}_{k+1,k+h})^\star, y(\mathcal{J}_{k+1,k+h})^\star)^\star \setminus x(\mathcal{I}_{k+1,k+h})_1$, $k'_3 = \min(\mathcal{I}_{k+1,k+h} \setminus \{k\})$,

$$\begin{aligned}\tilde{\mathcal{Z}}_5 = &\begin{pmatrix} \mathcal{M}_6\big((\mathcal{M}'_3)_{11} + \mathcal{M}_6\big)^{-1} & -(\mathcal{M}'_3)_{12}\big((\mathcal{M}'_3)_{11} + \mathcal{M}_6\big)^{-1} \\ 0 & 1 \end{pmatrix} \\ &\times \left\{ \begin{pmatrix} x_{k'_3} - x_{k'_1} + \mathcal{M}_6^{-1}\mathcal{M}_5^\star \tilde{\mathcal{Z}}_4 \\ y_k - y_{k-1} \end{pmatrix} + (\mathcal{M}'_3)^{-1}(\mathcal{M}'_2)^\star \tilde{\mathcal{Z}}'_1 \right\} \\ &+ \big(\mathcal{M}_3^{-1}\mathcal{M}_2^\star \tilde{\mathcal{Z}}_1, 0\big)^\star - (\mathcal{M}'_3)^{-1}(\mathcal{M}'_2)^\star \tilde{\mathcal{Z}}'_1,\end{aligned}$$

and

$$\tilde{\mathcal{Q}}_2^{k,v}(z_0, \bar{z}; \bar{u}) = \Upsilon_6 - \big(\mathcal{M}_4^\star \partial_\sigma\big((\tilde{b}_{(k)}\tilde{b}^\star_{(k)})^{-1}\big)\mathcal{M}_4\big)_{11}\big((\mathcal{M}_3)_{11} + \mathcal{M}_6\big)^{-1} / \big(2\Delta\check{u}^k\big).$$

Let $\hat{\mathcal{Z}}_5$ be a vector obtained by substituting 0 for $x(\mathcal{I}_k)_1, y(\mathcal{J}_k)_1, x(\mathcal{I}_{k-h,k})_1, y(\mathcal{J}_{k-h,k})_1$, $x(\mathcal{I}_{k+1,k+h})_{L^1_{k+1,k+h}}$ and $y(\mathcal{J}_{k+1,k+h})_{L^2_{k+1,k+h}}$ in $\tilde{\mathcal{Z}}_5$. Then since $\mathcal{M}_4$, $\hat{\mathcal{Z}}_5$ and $\tilde{\mathcal{Q}}_2^{k,v}$ do not depend on $\hat{z}$, we obtain

$$\begin{aligned}\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}(z_0, \bar{z}) = &-\hat{\mathcal{Z}}_5^\star \mathcal{M}_4^\star \frac{\partial_\sigma((\tilde{b}_{(k)}\tilde{b}^\star_{(k)})^{-1})}{2\Delta\check{u}^k}\mathcal{M}_4\hat{\mathcal{Z}}_5 \\ &+ \tilde{\mathcal{Q}}_2^{k,v}(z_0, \bar{z}; \bar{u}) + \tilde{\mathcal{Q}}_4^{k,v}(z_0, \bar{z}; \bar{u}),\end{aligned} \tag{4.21}$$

where $\Upsilon_7 = (\hat{\mathcal{Z}}_5 + \tilde{\mathcal{Z}}_5)^\star \mathcal{M}_4^\star \partial_\sigma((\tilde{b}_{(k)}\tilde{b}^\star_{(k)})^{-1})\mathcal{M}_4(\hat{\mathcal{Z}}_5 - \tilde{\mathcal{Z}}_5)/(2\Delta\check{u}^k)$ and

$$\tilde{\mathcal{Q}}_4^{k,v}(z_0, \bar{z}; \bar{u}) = \int \Upsilon_7 \exp\Big(\sum_{k'} \acute{f}_{k'}^{k,vu}\Big)(z)\,\mathrm{d}\hat{z} \Big/ \int \exp\Big(\sum_{k'} \acute{f}_{k'}^{k,vu}\Big)(z)\,\mathrm{d}\hat{z}.$$

Step 2. We will prove $\sup_{v,k} |\tilde{\mathcal{Q}}_4^{k,v}(Y_\Pi; \Pi)| = o_p(b_n^{-q})$ for any $q > 0$ in this step. We follow the approach in Section 2 of Ogihara and Yoshida [19].

Let

$$D'_k = \mathrm{diag}\bigg(\Big(\sum_{k'} |\check{b}^1_{k'}|^2 |\tilde{I}^{i_p}_{k-h,k-1} \cap \tilde{K}_{k'}|\Big)_{p=1}^{L^1_k}, \Big(\sum_{k'} |\check{b}^2_{k'}|^2 |\tilde{J}^{j_q}_{k-h,k-1} \cap \tilde{K}_{k'}|\Big)_{q=1}^{L^2_k}\bigg),$$

$$G_k = \left\{ \frac{|\tilde{I}^{i_p}_{k-h,k-1} \cap \tilde{J}^{j_q}_{k-h,k-1}|}{|\tilde{I}^{i_p}_{k-h,k-1}|^{1/2} |\tilde{J}^{j_q}_{k-h,k-1}|^{1/2}} \right\}_{1 \le p \le L^1_k, 1 \le q \le L^2_k},$$

$$\tilde{G}_k = \left\{ \frac{\sum_{k'} \check{b}^1_{k'} \cdot \check{b}^2_{k'} |\tilde{I}^{i_p}_{k-h,k-1} \cap \tilde{J}^{j_q}_{k-h,k-1} \cap \tilde{K}_{k'}|}{((D'_k)_{p,p})^{1/2}((D'_k)_{q+L^1_k, q+L^1_k})^{1/2}} \right\}_{1 \le p \le L^1_k, 1 \le q \le L^2_k}.$$

Then we obtain

$$S_k^{-1} = \big(D'_k\big)^{-1/2} \begin{pmatrix} (\mathcal{E} - \tilde{G}_k\tilde{G}_k^\star)^{-1} & -(\mathcal{E} - \tilde{G}_k\tilde{G}_k^\star)^{-1}\tilde{G}_k \\ -\tilde{G}_k^\star(\mathcal{E} - \tilde{G}_k\tilde{G}_k^\star)^{-1} & (\mathcal{E} - \tilde{G}_k^\star\tilde{G}_k)^{-1} \end{pmatrix} \big(D'_k\big)^{-1/2}, \tag{4.22}$$

by a standard formula for block matrices.

Moreover, the argument in Lemma 2 of Ogihara and Yoshida [19] yields

$$\|\tilde{G}_k\| \vee \|\tilde{G}_k^\star\| \le \tilde{\rho}(\bar{u})\|G_k\| \vee \|G_k^\star\| \le \tilde{\rho}(\bar{u}), \tag{4.23}$$

where

$$\begin{aligned}\tilde{\rho}(\bar{u}) &= \tilde{\rho}(z_0, \bar{z}, \bar{u}) \\ &= \sup\Bigg\{\frac{|\check{b}^1_{k_1} \cdot \check{b}^2_{k_2}|}{|\check{b}^1_{k_3}||\check{b}^2_{k_4}|}; v \in [0,1], 1 \le k \le L_0(\bar{u}) \text{ and there exist } l_1, l_2 \text{ such that} \\ &\qquad \tilde{I}^{l_1}_{k-h,k+h} \cap \tilde{J}^{l_2}_{k-h,k+h} \ne \varnothing, \tilde{K}_{k_1}, \tilde{K}_{k_3} \subset \tilde{I}^{l_1}_{k-h,k+h} \text{ and } \tilde{K}_{k_2}, \tilde{K}_{k_4} \subset \tilde{J}^{l_2}_{k-h,k+h}\Bigg\}.\end{aligned}$$

Let $\tilde{l}_1(k;\bar{u}) = \min\{l \in \mathbb{Z}_+; ((G_k G_k^\star)^l G_k)_{1,L_k^2} > 0\}$, then (4.22), (4.23) and relations $(\mathcal{E} - \tilde{G}_k\tilde{G}_k^\star)^{-1} = \sum_{l=0}^\infty (\tilde{G}_k\tilde{G}_k^\star)^l$ and $(\mathcal{E} - \tilde{G}_k^\star\tilde{G}_k)^{-1} = \sum_{l=0}^\infty (\tilde{G}_k^\star\tilde{G}_k)^l$ yield

$$\begin{aligned}&\big|(D'_k)^{1/2}_{11}(\mathcal{M}_2)_1(D'_k)^{1/2}_{L_k^1,L_k^1}\big| \vee \big|(D'_k)^{1/2}_{L_k^1+1,L_k^1+1}(\mathcal{M}_2)_{L_k^1}(D'_k)^{1/2}_{L_k^1,L_k^1}\big| \\ &\quad \le C\tilde{\rho}(\bar{u})^{2\tilde{l}_1-1}/\big(1-\tilde{\rho}(\bar{u})^2\big)\end{aligned} \tag{4.24}$$

if $\tilde{\rho}(\bar{u}) < 1$.

On the other hand, we have

$$\mathcal{M}_3^{-1} = (S_k)_{L_k^1,L_k^1} - \big((S_k)_{i,L_k^1}\big)^\star_{i\ne L_k^1}\big(((S_k)_{ij})_{i,j\ne L_k^1}\big)^{-1}\big((S_k)_{i,L_k^1}\big)_{i\ne L_k^1}$$

by a standard formula for block matrices, and hence

$$\big|(D'_k)^{-1/2}_{L_k^1,L_k^1}\mathcal{M}_3^{-1}(D'_k)^{-1/2}_{L_k^1,L_k^1}\big| \le C\big(1-\tilde{\rho}(\bar{u})^2\big)^{-1} \tag{4.25}$$

if $\tilde{\rho}(\bar{u}) < 1$.

Moreover we have

$$v_1^{-1} + \mathcal{M}_3^{-1} \ge \big(1-\tilde{\rho}^2\big)\big(\big(\tilde{b}_k^{vu}\big(\tilde{b}_k^{vu}\big)^\star \Delta\check{u}^k\big)_{11} + (S_k)_{L_k^1,L_k^1}\big) \ge \big(1-\tilde{\rho}^2\big)(S_{k-h,k})_{L_k^1,L_k^1},$$

and consequently we obtain $(v_1 + \mathcal{M}_3)^{-1} \le C(1-\tilde{\rho}^2)^{-1}\Delta\check{u}^k(\check{u}^{k-1} - \check{u}^{k'_1})/(\check{u}^k - \check{u}^{k'_1})$. Similarly we have $((\mathcal{M}'_3)_{11} + \mathcal{M}_6)^{-1} \le C(1-\tilde{\rho}^2)^{-1}(\check{u}^k - \check{u}^{k'_1})(\check{u}^{k'_3} - \check{u}^k)/(\check{u}^{k'_3} - \check{u}^{k'_1})$.

Therefore, we obtain

$$\sup_{k,v}\big|\tilde{\mathcal{Q}}_4^{k,v}(Y_\Pi;\Pi)\big| \le C\sup_k \frac{\tilde{\rho}(Y_\Pi;\Pi)^{2(\tilde{l}_1(k;\Pi)\wedge\tilde{l}_2(k;\Pi))-1}}{(1-\tilde{\rho}(Y_\Pi;\Pi)^2)^6} \times O_p\big((\ell_{1,n}+\ell_{2,n})^3\big) \tag{4.26}$$

on $\{\tilde{\rho}(Y_\Pi;\Pi)<1\}$ by Lemma 4.4, (4.24), (4.25) and similar estimates for $\{\mathcal{M}_l\}_{2\le l\le 6}$ and $\{\mathcal{M}'_l\}_{l=2,3}$, where

$$G'_k=\left\{\frac{|\tilde{I}^{i_p}_{k+1,k+h}\cap\tilde{J}^{j_q}_{k+1,k+h}|}{|\tilde{I}^{i_p}_{k+1,k+h}|^{1/2}|\tilde{J}^{j_q}_{k+1,k+h}|^{1/2}}\right\}_{1\le p\le L^1_{k+1,k+h},1\le q\le L^2_{k+1,k+h}}$$

and $\tilde{l}_2(k;\bar{u})=\min\{l\in\mathbb{Z}_+;((G'_k(G'_k)^\star)^lG'_k)_{1,L^2_{k+1,k+h}}>0\}$.

By the definitions of $\tilde{l}_1$, we obtain

$$\big(2\tilde{l}_1(k;\bar{u})+2\big)\max_{i,j}\big(\big|s^i-s^{i-1}\big|\vee\big|t^j-t^{j-1}\big|\big)\ge\big|\check{u}^{k-1}-\check{u}^{k-h-1}\big| \tag{4.27}$$

for any k. Moreover, since the numbers of elements of $(s^i)_i\cap[\check{u}^{k-h-1},\check{u}^{k-1}]$ or $(t^j)_j\cap[\check{u}^{k-h-1},\check{u}^{k-1}]$ is equal to or greater than $(h+1)/2$, $[A2]$ and $[A3']$ yields

$$\liminf_{n\to\infty}P\Big[\inf_k\big|U^{k-1}-U^{k-h-1}\big|>b_n^{-1-\delta_3}\big((h+1)/2-1\big)\Big]\ge\liminf_{n\to\infty}P[\mathbf{A}_n]=1, \tag{4.28}$$

where

$$\mathbf{A}_n=\bigcap_{|j_2-j_1|\ge b_n^{\delta_2}}\Bigg[\left\{\frac{|S^{n,j_2}-S^{n,j_1}|}{|j_2-j_1|}>b_n^{-1-\delta_3}\text{ if }j_1\vee j_2\le\ell_{1,n}\right\}\cap\left\{\frac{|T^{n,j_2}-T^{n,j_1}|}{|j_2-j_1|}>b_n^{-1-\delta_3}\text{ if }j_1\vee j_2\le\ell_{2,n}\right\}\Bigg].$$

By (4.27) and (4.28), we have

$$\lim_{n\to\infty}P\Big[\inf_k\tilde{l}_1(k;\Pi)>b_n^{\kappa-\delta_1-\delta_3}/5\Big]=1. \tag{4.29}$$

Similarly, we obtain

$$\lim_{n\to\infty}P\Big[\inf_k\tilde{l}_2(k;\Pi)>b_n^{\kappa-\delta_1-\delta_3}/5\Big]=1. \tag{4.30}$$

Let $\bar{\rho}=\sup_{t,x,y,\sigma}|b^1|^{-1}|b^2|^{-1}|b^1\cdot b^2|(t,x,y,\sigma)$. Then by virtue of $[A1']$ and the relation $\det(bb^\star)=|b^1|^2|b^2|^2-(b^1\cdot b^2)^2$, we obtain $\bar{\rho}<1$. Moreover, we obtain

$$\lim_{n\to\infty}P\big[\tilde{\rho}(Y_\Pi;\Pi)>1-(1-\bar{\rho})/2\big]=0, \tag{4.31}$$

since $r_n\to^p 0$ and $b(t,x,y,\sigma)$ is continuous with respect to (t,x,y,σ).

By (4.26), (4.29), (4.30) and (4.31), we have $\sup_{v,k}|\tilde{\mathcal{Q}}_4^{k,v}(Y_\Pi;\Pi)|=o_p(b_n^{-q})$ for any $q>0$. Furthermore, we can write

$$\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}(z_0,\bar{z})=\mathcal{Z}_k^\star\tilde{\mathcal{Q}}_1^{k,v}(z_0,\bar{z};\bar{u})\mathcal{Z}_k+\tilde{\mathcal{Q}}_2^{k,v}(z_0,\bar{z};\bar{u})+\tilde{\mathcal{Q}}_4^{k,v}(z_0,\bar{z};\bar{u}),$$

where $\sup_{v,k}\|\tilde{\mathcal{Q}}_1^{k,v}(z_0,\bar{z};\bar{u})\|\le C(1-\tilde{\rho}^2(\bar{u}))^{-6}$ and $\sup_{k,v}|\tilde{\mathcal{Q}}_2^{k,v}(z_0,\bar{z};\bar{u})|\le C(1-\tilde{\rho}(\bar{u})^2)^{-3}$.

Step 3. We now complete the proof.

Let $\mathcal{Q}_p^{k,v}$ be obtained by substituting the same values in $\tilde{\mathcal{Q}}_p^{k,v}$ as

$$\mathcal{Q}_p^{k,v}(z_0,\bar{z};\bar{u}) = \tilde{\mathcal{Q}}_p^{k,v}\big(z_0,\big((x_{\hat{k}_1})_{i=1}^{L^1},(y_{\hat{k}_2})_{j=1}^{L^2}\big);\bar{u}\big)$$

for $p=1,2$, where $\hat{k}_1,\hat{k}_2$ are the maximum integers satisfying $\check{u}^{\hat{k}_1}=s^i$, $\check{u}^{\hat{k}_2}=t^j$, $s^i\vee t^j\le \inf(\tilde{\theta}(k-h-1,2;\bar{u}))$ for some i and j. Then we have

$$\sup_{n,v,\bar{u},k,z_0,\bar{z}}\big(\big\|\mathcal{Q}_1^{k,v}(z_0,\bar{z};\bar{u})\big\|\vee\big|\mathcal{Q}_2^{k,v}(z_0,\bar{z};\bar{u})\big|\big)\le C\big(1-\bar{\rho}^2\big)^{-6}\le C.$$

Therefore, by setting

$$\mathcal{Q}_3^{k,v} = \big(\mathcal{Z}_k^\star\big(\tilde{\mathcal{Q}}_1^{k,v}-\mathcal{Q}_1^{k,v}\big)\mathcal{Z}_k+\tilde{\mathcal{Q}}_2^{k,v}-\mathcal{Q}_2^{k,v}\big)1_{\{\tilde{\rho}(\bar{u})\le 1-(1-\bar{\rho})/2\}},$$
$$\mathcal{Q}_4^{k,v} = \tilde{\mathcal{Q}}_4^{k,v}+\big(\mathcal{Z}_k^\star\big(\tilde{\mathcal{Q}}_1^{k,v}-\mathcal{Q}_1^{k,v}\big)\mathcal{Z}_k+\tilde{\mathcal{Q}}_2^{k,v}-\mathcal{Q}_2^{k,v}\big)1_{\{\tilde{\rho}(\bar{u})> 1-(1-\bar{\rho})/2\}},$$

we obtain $\sup_{k,v}|\mathcal{Q}_4^{k,v}(Y_\Pi;\Pi)| = \mathrm{o}_p(b_n^{-q})$, $\sup_{n,r,k,v,v'}(r_n^{-1/2}b_n^{-3\kappa/2}E[|\mathcal{Q}_3^{k,v}(Y_\Pi^{r,v'u};\Pi)|^q|\Pi]^{1/q})\le C$ a.s. by (4.31).

Furthermore, a similar argument for

$$\int \tilde{f}_k^{vu,(1)}\exp\Big(\sum_{k'}\tilde{f}_{k'}^{vu}\Big)\,\mathrm{d}\hat{z}\Big/\int\exp\Big(\sum_{k'}\tilde{f}_{k'}^{vu}\Big)\,\mathrm{d}\hat{z}$$

yields

$$\sup_{k,v}\left|\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}-\frac{\int\tilde{f}_k^{vu,(1)}\exp(\sum_{k'}\tilde{f}_{k'}^{vu})\,\mathrm{d}\hat{z}}{\int\exp(\sum_{k'}\tilde{f}_{k'}^{vu})\,\mathrm{d}\hat{z}}\right|(Y_\Pi)$$
$$\le \mathrm{O}_p\big((\ell_{1,n}+\ell_{2,n})^3\cdot\tilde{\rho}(\Pi)^{2\inf_k(\tilde{l}_1\wedge\tilde{l}_2(k;\Pi))-1}\big)+\mathrm{o}_p\big(b_n^{-q}\big)=\mathrm{o}_p\big(b_n^{-q}\big)$$

for any $q>0$. □

The following lemma enables us to replace $\bar{\mathbb{P}}_u^0$ and $\bar{\mathbb{P}}_0^0$ in $\log(\bar{\mathbb{P}}_u^0/\bar{\mathbb{P}}_0^0)$ by the function $\int\exp(\sum_k\tilde{f}_k^u)\,\mathrm{d}\hat{z}$ and $\int\exp(\sum_k\tilde{f}_k^0)\,\mathrm{d}\hat{z}$, respectively.

Lemma 4.8. *Let $u\in\mathbb{R}^d$. Assume* [A1′], [A2] *and* [A3′]. *Then*

$$\log\frac{\bar{\mathbb{P}}_u^0}{\bar{\mathbb{P}}_0^0}(Y_\Pi)-b_n^{-1/2}u\int_0^1\sum_k\frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)}\,\mathrm{d}v(Y_\Pi)\to^p 0$$

as $n\to\infty$.

Proof. Let

$$\mathcal{A}^5_{k,v}(z_0,\bar{z}) = \mathcal{Z}^\star_k \mathcal{Q}^{k,v}_1(z_0,\bar{z},\bar{u})\mathcal{Z}_k 1_{\{\mathcal{Z}_k\neq\varnothing\}} + \mathcal{Q}^{k,v}_2(z_0,\bar{z},\bar{u}) + \mathcal{Q}^{k,v}_3(z_0,\bar{z},\bar{u})$$

and

$$\mathcal{K}^3_M(\bar{u}) = \mathcal{K}^2_M(\bar{u}) \cap \Big\{(z_0,\bar{z}); \sup_{k,v}\big|\mathcal{Q}^{k,v}_4(z_0,\bar{z};\bar{u})\big| \le b_n^{-1}\Big\}$$

for $M>0$ and $\bar{u}\in\mathcal{U}$.

By Lemmas 4.3, 4.5 and 4.7 and the definition of $\mathcal{K}^2_M$, it is sufficient to show that

$$\Phi'_n = \sup_v E\left[\left|b_n^{-1/2}\sum_k\left(\frac{\bar{\mathbb{P}}^{2,vu}_k(\tilde{f}^{vu,(1)}_k 1_{B^n_M})}{\tilde{\mathbb{P}}^0_{M,vu}} - \frac{\bar{\mathbb{P}}^{2,vu}_k(\tilde{f}^{vu,(1)}_k 1_{B^n_M})}{\bar{\mathbb{P}}^{2,vu}_k(1_{B^n_M})}\right)1_{\mathcal{K}^3_M(\Pi)}\right|(Y_\Pi)\Big|\Pi\right] \to^p 0$$

as $n\to\infty$ for any $M>0$.

Fix $M>0$. Then Lemma 4.7 and the definition of $\mathcal{K}^3_M$ yield

$$\begin{aligned}
\Phi'_n &= \sup_v E\left[\left|b_n^{-1/2}\sum_k \frac{\bar{\mathbb{P}}^{2,vu}_k(\tilde{f}^{vu,(1)}_k 1_{B^n_M})}{\bar{\mathbb{P}}^{2,vu}_k(1_{B^n_M})}\left(\frac{\bar{\mathbb{P}}^{2,vu}_k(1_{B^n_M})}{\tilde{\mathbb{P}}^0_{M,vu}} - 1\right)1_{\mathcal{K}^3_M(\Pi)}\right|(Y_\Pi)\Big|\Pi\right] \\
&\le \sup_v E\left[\left|b_n^{-1/2}\sum_k \mathcal{A}^5_{k,v}(Y_\Pi)\left(\frac{\bar{\mathbb{P}}^{2,vu}_k(1_{B^n_M})}{\tilde{\mathbb{P}}^0_{M,vu}} - 1\right)1_{\mathcal{K}^3_M(\Pi)}\right|(Y_\Pi)\Big|\Pi\right] \\
&\quad + 2\sup_v E\left[b_n^{-3/2}\sum_k\left|\frac{\bar{\mathbb{P}}^{2,vu}_k(1_{B^n_M})}{\tilde{\mathbb{P}}^0_{M,vu}} - 1\right|1_{\mathcal{K}^3_M(\Pi)}(Y_\Pi)\Big|\Pi\right].
\end{aligned}$$

The second term of the right-hand side in the above inequality is equal to or smaller than

$$Cb_n^{-3/2}\sup_v\sum_k\int\left|\mathbb{P}^{2,vu}_k 1_{B^n_M} - \mathbb{P}^0_{vu}1_{B^n_M}\right|\mathrm{d}\hat{z}\,\mathrm{d}\bar{z}\,P_{Y_0}(\mathrm{d}z_0)\Big|_{\bar{u}=\Pi} = \mathrm{o}_p(1).$$

Hence, by a similar argument to the proof of Lemma 4.3, we obtain

$$\begin{aligned}
\Phi'_n &\le \mathrm{e}^{M+1} \\
&\quad\times\sup_v E\left[\left|b_n^{-1/2}\sum_k\mathcal{A}^5_{k,v}\big(Y^{0,vu}_\Pi\big)\left\{\exp\left(\sum_{k';|k'-k|\le h}\big(\tilde{f}^{vu}_{k'} - \log\check{p}^0_{k',vu}\big)\right)-1\right\}\right.\right. \\
&\qquad\left.\left.\times 1_{B^n_M}\right|\big(Y^{0,vu}_{\check{U}}\big)\Big|\Pi\right] \\
&\quad+\mathrm{o}_p(1) \\
&\le C\sup_v E\left[\left|b_n^{-1/2}\sum_k\mathcal{A}^5_{k,v}\big(Y^{0,vu}_\Pi\big)\sum_{k';|k'-k|\le h}\big(\tilde{f}^{vu}_{k'} - \log\check{p}^0_{k',vu}\big)1_{B^n_M}\right|\big(Y^{0,vu}_{\check{U}}\big)\Big|\Pi\right]
\end{aligned}$$

$$+ \mathrm{O}_p\big(b_n^{-1/2} b_n b_n^{\kappa} b_n^{2\kappa} b_n^{-2/3-2\kappa}\big) + \mathrm{o}_p(1)$$

$$\leq C \sup_v E\bigg[\bigg| b_n^{-1/2} \sum_k \mathcal{A}^5_{k,v}\big(Y^{0,vu}_\Pi\big) \sum_{k';|k'-k|\leq h} \big(\log \check{p}^1_{k',vu} - \log \check{p}^0_{k',vu}\big)\bigg|\big(Y^{0,vu}_{\check{U}}\big)\bigg|\Pi\bigg]$$

$$+ C \sup_v E\bigg[\bigg| b_n^{-1/2} \sum_k \mathcal{A}^5_{k,v}\big(Y^{0,vu}_\Pi\big) \sum_{k';|k'-k|\leq h} \bigg(\sum_{\tilde{k}} \hat{\mathcal{A}}^3_{k,k',\tilde{k}} + \mathcal{A}^4_{k,k'}\bigg)\bigg|\bigg|\Pi\bigg] + \mathrm{o}_p(1)$$

$$= \Phi'_{1,n} + \Phi'_{2,n} + \mathrm{o}_p(1).$$

Furthermore, let $\tilde{\mathcal{A}}^5_{k,v}(z_0,\bar{z}) = \mathcal{Z}_k^\star \mathcal{Q}_1^{k,v}(z_0,\bar{z},\bar{u})\mathcal{Z}_k 1_{\{\mathcal{Z}_k\neq\varnothing\}}$, then Lemma 4.7 and the Burkholder–Davis–Gundy inequality yield

$$\Phi'_{1,n} \leq \sup_{r,v} E\bigg[\bigg| b_n^{-1/2} \sum_k \tilde{\mathcal{A}}^5_{k,v}\big(Y^{0,vu}_\Pi\big) \sum_{k';|k'-k|\leq h} \frac{\partial_r \check{p}^r_{k',vu}}{\check{p}^r_{k',vu}}\big(Y^{0,vu}_{\check{U}}\big)\bigg|\bigg|\Pi\bigg] + \mathrm{o}_p(1),$$

and

$$\Phi'_{2,n} \leq C \sup_v E\bigg[\bigg| b_n^{-1/2} \sum_k \tilde{\mathcal{A}}^5_{k,v}\big(Y^{0,vu}_\Pi\big) \sum_{k';|k'-k|\leq h} \bigg(\sum_{\tilde{k}} \hat{\mathcal{A}}^3_{k,k',\tilde{k}} + \mathcal{A}^4_{k,k'}\bigg)\bigg|\bigg|\Pi\bigg] + \mathrm{o}_p(1).$$

Let $\acute{\mathcal{Z}}^j = \{\acute{\mathcal{Z}}^j_k\}_{k=1}^{L_0(\bar{u})}$ $(j=1,2)$,

$$\acute{\mathcal{Z}}^1_k = \frac{Y^{0,vu,1}_{\check{U}^k} - Y^{0,vu,1}_{\check{U}^{k-1}}}{|\tilde{\theta}_{0,k}|^{1/2}}, \qquad \acute{\mathcal{Z}}^2_k = \frac{Y^{0,vu,2}_{\check{U}^k} - Y^{0,vu,2}_{\check{U}^{k-1}}}{|\inf\{T^j; T^j \geq \check{U}^k\} - \sup\{T^j; T^j \leq \check{U}^{k-1}\}|^{1/2}},$$

and $\check{\mathcal{Q}}^{k,v,j_1,j_2} = \{(\check{\mathcal{Q}}^{k,v,j_1,j_2})_{l_1,l_2}\}_{l_1,l_2}$ be a certain symmetric matrix $(1 \leq j_1, j_2 \leq 2)$ satisfying $\mathcal{Z}_k^\star \mathcal{Q}_1^{k,v} \mathcal{Z}_k 1_{\{\mathcal{Z}_k\neq\varnothing\}} = \sum_{j_1,j_2=1}^2 (\acute{\mathcal{Z}}^{j_1})^\star \check{\mathcal{Q}}^{k,v,j_1,j_2} \acute{\mathcal{Z}}^{j_2}$. Moreover, let

$$\tilde{\mathcal{X}}^1_{l,k} = \sum_{j_1,j_2=1}^2 \big(\check{\mathcal{Q}}^{l,v,j_1,j_2}\big)_{k,k} \acute{\mathcal{Z}}^{j_1}_k \acute{\mathcal{Z}}^{j_2}_k + 2\sum_{l_2<k} \sum_{j_1,j_2=1}^2 \big(\check{\mathcal{Q}}^{l,v,j_1,j_2}\big)_{k,l_2} \acute{\mathcal{Z}}^{j_1}_k \acute{\mathcal{Z}}^{j_2}_{l_2},$$

$$\tilde{\mathcal{X}}^2_{l,k',\tilde{k}} = \hat{\mathcal{A}}^3_{l,k',\tilde{k}} + \mathcal{A}^4_{l,k'} 1_{\{k'=\tilde{k}\}}.$$

Then we have

$$\Phi'_{1,n} \leq C \sup_{r,v} E\bigg[\bigg| b_n^{-1/2} \sum_{\tilde{k}} \sum_{k;|\tilde{k}-k|\leq h} \bigg(\sum_{k';|k'-k|\leq h, k'\leq\tilde{k}} \frac{\partial_r \check{p}^r_{k',vu}}{\check{p}^r_{k',vu}}\bigg)\tilde{\mathcal{X}}^1_{k,\tilde{k}}$$

$$+ b_n^{-1/2} \sum_{k'} \bigg(\sum_{k;|k'-k|\leq h} \sum_{\tilde{k};|\tilde{k}-k|\leq h, \tilde{k}<k'} \tilde{\mathcal{X}}^1_{k,\tilde{k}}\bigg) \frac{\partial_r \check{p}^r_{k',vu}}{\check{p}^r_{k',vu}}\bigg|\big(Y^{0,vu}_{\check{U}}\big)\bigg|\Pi\bigg] + \mathrm{o}_p(1)$$

$$= \mathrm{o}_p(1),$$

by a similar argument to the estimate of $\Phi_{n,2}$ in the proof of Lemma 4.3.

Moveover, 4. of Lemma A.1 in the Appendix and estimates in the proof of Lemma 4.3 yield

$$\Phi'_{2,n} \leq C \sup_v E\left[\left|b_n^{-1/2} \sum_{l,k} \tilde{\mathcal{X}}^1_{l,k} \sum_{k';|k'-l|\leq h} \tilde{\mathcal{X}}^2_{l,k',k}\right| \Big| \Pi\right] + o_p(1).$$

Therefore, we can see $\Phi'_{2,n} \to^p 0$ by using the Burkholder–Davis–Gundy inequality and estimates in the proof of Lemma 4.3. □

Proof of Theorem 2.1. By virtue of (3.2), Theorem 3.1, Lemmas 3.1 and 4.8, it is sufficient to show that

$$b_n^{-1/2} u \int_0^1 \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)} \mathrm{d}v(Y_\Pi) - \big(H_n(\sigma_u) - H_n(\sigma_*)\big) \circ (\Pi, Y_\Pi) \to^p 0$$

as $n \to \infty$ under $[A1']$, $[A2]$ and $[A3']$.

Lemmas 4.7 and 4.6 yield

$$\begin{aligned}
&b_n^{-1/2} u \int_0^1 \sum_k \frac{\bar{\mathbb{P}}_k^{2,vu}(\tilde{f}_k^{vu,(1)})}{\bar{\mathbb{P}}_k^{2,vu}(1)} \mathrm{d}v(Y_\Pi) \\
&\quad = b_n^{-1/2} u \int_0^1 \sum_k \frac{\int \tilde{f}_k^{vu,(1)} \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\, \mathrm{d}\hat{z}}{\int \exp(\sum_{k'} \tilde{f}_{k'}^{vu})\, \mathrm{d}\hat{z}} \mathrm{d}v(Y_\Pi) + o_p(1) \\
&\quad = \int_0^1 \partial_v \left(-\frac{1}{2} \tilde{Z}^\star S_{1,L_0(\Pi)}^{-1} \tilde{Z} - \frac{1}{2} \log\det S_{1,L_0(\Pi)}\right) \mathrm{d}v + o_p(1),
\end{aligned}$$

where $\tilde{Z} = ((Y^1_{S^{n,i}} - Y^1_{S^{n,i-1}})_i^\star, (Y^2_{T^{n,j}} - Y^2_{T^{n,j-1}})_j^\star)^\star$.

Let $\tilde{D} = \mathrm{diag}((|I^i|)_i, (|J^j|)_j)$, then the difference between $\tilde{D}^{-1/2} S_{1,L_0(\Pi)} \tilde{D}^{-1/2}$ and $S(\sigma^n_{vu})$ in (2.6) is only the substituted values of b^1, b^2. Then we can see the right-hand side of the above equation is equal to

$$\int_0^1 \partial_v H_n\big(\sigma^n_{vu}\big)\, \mathrm{d}v \circ (\Pi, Y_\Pi) + o_p(1) = \big(H_n\big(\sigma^n_u\big) - H_n(\sigma_*)\big) \circ (\Pi, Y_\Pi) + o_p(1),$$

by $[A2]$, Lemma 4.1 and a similar argument to the proof of Lemma 13 in Ogihara and Yoshida [19]. We omit the details. □

Appendix

Lemma A.1. *Let $g, L \in \mathbb{N}$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space, $\tilde{\mathbf{F}} = \{\tilde{\mathcal{F}}_k\}_{k=0}^L$ be a filtration. Denote by $\tilde{E}$ the integral with respect to $\tilde{P}$.*

1. *Let* $\{\mathcal{X}_{k,k'}\}_{1\leq k,k'\leq L}$ *be random variables. Suppose* $\{\sum_{1\leq k'\leq l}\mathcal{X}_{k,k'}\}_{l=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale for* $1\leq k\leq L$. *Moreover, assume that there exists a sequence* $\{\mathcal{C}_{k,k'}\}_{1\leq k'\leq L}$ *of positive numbers such that* $\tilde{E}[|\mathcal{X}_{k,k'}|^2]^{1/2}\leq \mathcal{C}_{k,k'}$ *for* $1\leq k,k'\leq L$. *Then*

$$\tilde{E}\left[\left|\sum_{k=1}^L\sum_{k'=1}^L\mathcal{X}_{k,k'}\right|\right]\leq\left(\sum_{l_1,l_2=1}^L\sum_{k=1}^L\mathcal{C}_{l_1,k}\mathcal{C}_{l_2,k}\right)^{1/2}.$$

2. *Let* $\{\mathcal{X}_k^1\}_{k=1}^L,\{\mathcal{X}_{k,k'}^2\}_{1\leq k,k'\leq L}$ *be random variables. Suppose* $\{\sum_{1\leq k\leq l}\mathcal{X}_k^1\}_{l=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale,* $\{\sum_{1\leq k'\leq l}\mathcal{X}_{k,k'}^2\}_{l=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale for* $1\leq k\leq L$. *Moreover, assume that there exist a positive constant* $\mathcal{C}^1$ *and a sequence* $\{\mathcal{C}_k^2\}_{1\leq k\leq L}$ *of positive numbers such that* $\tilde{E}[|\mathcal{X}_k^1|^4]^{1/4}\leq\mathcal{C}^1$ *and* $\tilde{E}[|\mathcal{X}_{k,k'}^2|^4]^{1/4}\leq\mathcal{C}_{k'}^2$ *for* $1\leq k,k'\leq L$. *Then*

$$\tilde{E}\left[\left|\sum_{k=1}^L\mathcal{X}_k^1\sum_{k';0<|k'-k|\leq g}\mathcal{X}_{k,k'}^2\right|\right]\leq 2(2g+1)\mathcal{C}^1\left\{\sum_{l_1,l_2;|l_1-l_2|\leq 2g}\mathcal{C}_{l_1}^2\mathcal{C}_{l_2}^2\right\}^{1/2}.$$

3. *Let* $\{\mathcal{X}_k^1\}_{k=1}^L,\{\mathcal{X}_{k,k',\tilde{k}}^2\}_{1\leq k,k',\tilde{k}\leq L}$ *be random variables. Suppose* $\{\sum_{1\leq k\leq l}\mathcal{X}_k^1\}_{l=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale,* $\{\sum_{1\leq\tilde{k}\leq l}\mathcal{X}_{k,k',\tilde{k}}^2\}_{l=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale for* $1\leq k,k'\leq L$. *Moreover, assume that there exist a positive constant* $\mathcal{C}^1$ *and sequences* $\{\mathcal{C}_k^2\}_{1\leq k\leq L}$ *and* $\{\mathcal{C}_{k',\tilde{k}}^3\}_{1\leq k',\tilde{k}\leq L}$ *of positive numbers such that* $\tilde{E}[|\mathcal{X}_k^1|^4]^{1/4}\leq\mathcal{C}^1$, $\tilde{E}[|\sum_{\tilde{k};\tilde{k}<l}\mathcal{X}_{k,k',\tilde{k}}^2|^4]^{1/4}\leq\mathcal{C}_{k'}^2$ *and* $\tilde{E}[|\mathcal{X}_{k,k',\tilde{k}}^2|^4]^{1/4}\leq\mathcal{C}_{k',\tilde{k}}^3$ *for* $1\leq l,k,k',\tilde{k}\leq L$. *Then*

$$\tilde{E}\left[\left|\sum_{k=1}^L\mathcal{X}_k^1\sum_{k';|k'-k|\leq g}\sum_{\tilde{k};\tilde{k}\neq k}\mathcal{X}_{k,k',\tilde{k}}^2\right|\right]$$

$$\leq\sqrt{2}(2g+1)\mathcal{C}^1\left\{\sum_{l_1,l_2;|l_1-l_2|\leq 2g}\mathcal{C}_{l_1}^2\mathcal{C}_{l_2}^2+\sum_{l_1,l_2}\sum_k\mathcal{C}_{l_1,k}^3\mathcal{C}_{l_2,k}^3\right\}^{1/2}.$$

4. *Let* $\{\mathcal{X}_{l,k}^1\}_{1\leq k,l\leq L},\{\mathcal{X}_{l,k',\tilde{k}}^2\}_{1\leq l,k',\tilde{k}\leq L}$ *be random variables. Suppose* $\{\sum_{1\leq k\leq p}\mathcal{X}_{l,k}^1\}_{p=0}^L$ *and* $\{\sum_{1\leq\tilde{k}\leq p}\mathcal{X}_{l,k',\tilde{k}}^2\}_{p=0}^L$ *is* $\tilde{\mathbf{F}}$*-martingale for* $1\leq l,k'\leq L$. *Moreover, assume that there exist sequences* $\{\mathcal{C}_{l,k}^1\}_{1\leq l,k\leq L}$, $\{\mathcal{C}_{k'}^2\}_{1\leq k'\leq L}$ *and* $\{\mathcal{C}_{k',\tilde{k}}^3\}_{1\leq k',\tilde{k}\leq L}$ *of positive numbers such that* $\tilde{E}[|\mathcal{X}_{l,k}^1|^4]^{1/4}\leq\mathcal{C}_{l,k}^1$, $\tilde{E}[|\sum_{\tilde{k};\tilde{k}<k}\mathcal{X}_{l,k',\tilde{k}}^2|^4]^{1/4}\leq\mathcal{C}_{k'}^2$ *and* $\tilde{E}[|\mathcal{X}_{l,k',\tilde{k}}^2|^4]^{1/4}\leq\mathcal{C}_{k',\tilde{k}}^3$ *for* $1\leq l,k,k',\tilde{k}\leq L$. *Then*

$$\tilde{E}\left[\left|\sum_{k,l}\mathcal{X}_{l,k}^1\sum_{k';|k'-l|\leq g}\sum_{\tilde{k};\tilde{k}\neq k}\mathcal{X}_{l,k',\tilde{k}}^2\right|\right]$$

$$\leq\sqrt{2}\left\{\sum_{l_1}\left(\sum_l\mathcal{C}_{l,l_1}^1\sum_{k';|k'-l|\leq g}\mathcal{C}_{k'}^2\right)^2+\sum_{l_1}\left(\sum_l\sum_{l_2<l_1}\mathcal{C}_{l,l_2}^1\sum_{k';|k'-l|\leq g}\mathcal{C}_{k',l_1}^3\right)^2\right\}^{1/2}.$$

Proof. We first prove 4. By using the Cauchy–Schwarz inequality and Lemma 9 in Ogihara and Yoshida [19] repeatedly, we obtain

$$\begin{aligned}
&\tilde{E}\Bigg[\bigg|\sum_{k,l}\mathcal{X}^1_{l,k}\sum_{k';|k'-l|\le g}\sum_{\tilde{k};\tilde{k}\neq k}\mathcal{X}^2_{l,k',\tilde{k}}\bigg|\Bigg]^2 \\
&\le \tilde{E}\Bigg[\bigg|\sum_{l_1,l}\sum_{l_2<l_1}\bigg(\mathcal{X}^1_{l,l_1}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_2}+\mathcal{X}^1_{l,l_2}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_1}\bigg)\bigg|^2\Bigg] \\
&\le \sum_{l_1}\Bigg\{2\tilde{E}\Bigg[\bigg(\sum_{l}\mathcal{X}^1_{l,l_1}\sum_{l_2<l_1}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_2}\bigg)^2\Bigg] \\
&\qquad + 2\tilde{E}\Bigg[\bigg(\sum_{l}\sum_{l_2<l_1}\mathcal{X}^1_{l,l_2}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_1}\bigg)^2\Bigg]\Bigg\} \\
&\le 2\sum_{l_1}\Bigg(\sum_{l}\mathcal{C}^1_{l,l_1}\tilde{E}\Bigg[\bigg(\sum_{l_2<l_1}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_2}\bigg)^4\Bigg]^{1/4}\Bigg)^2 \\
&\quad + 2\sum_{l_1}\Bigg(\sum_{l}\sum_{l_2<l_1}\tilde{E}\Bigg[\bigg(\mathcal{X}^1_{l,l_2}\sum_{k';|k'-l|\le g}\mathcal{X}^2_{l,k',l_1}\bigg)^2\Bigg]^{1/2}\Bigg)^2 \\
&\le 2\sum_{l_1}\bigg(\sum_{l}\mathcal{C}^1_{l,l_1}\sum_{k';|k'-l|\le g}\mathcal{C}^2_{k'}\bigg)^2 + 2\sum_{l_1}\bigg(\sum_{l}\sum_{l_2<l_1}\mathcal{C}^1_{l,l_2}\sum_{k';|k'-l|\le g}\mathcal{C}^3_{k',l_1}\bigg)^2.
\end{aligned}$$

Then we obtain 4.

We obtain 3. by setting $\mathcal{X}^1_{l,k}=\mathcal{X}^1_k 1_{\{l=k\}}$ in 4. We can prove 2. by setting $\mathcal{X}^2_{k,k',\tilde{k}}=\mathcal{X}^2_{k,k'}1_{\{k'=\tilde{k}\}}$ in 3. Moreover, we can easily check 1. □

The following lemma is proved by elementary calculation. We omit proofs.

Lemma A.2. *Let* $A=\{A_{ij}\}_{i,j=1}^2$ *be a* 2×2 *symmetric matrix*, V_1 *be a* 2×2 *symmetric*, *positive definite matrix*, $\alpha,\beta\in\mathbb{R}$ *and* $v_2>0$. *Then*

$$\begin{aligned}
&\int_{\mathbb{R}}(x_1-y_1+\alpha,x_2-y_2+\beta)A(x_1-y_1+\alpha,x_2-y_2+\beta)^\star \\
&\quad\times\varphi\big((x_1-y_1,x_2-y_2)^\star;V_1\big)\varphi(y_1-w;v_2)\,\mathrm{d}y_1 \\
&\quad=\bigg\{(\mathcal{W},x_2-y_2+\beta)A(\mathcal{W},x_2-y_2+\beta)^\star+\frac{A_{11}}{(V_1^{-1})_{11}+v_2^{-1}}\bigg\} \\
&\qquad\times\int_{\mathbb{R}}\varphi\big((x_1-y_1,x_2-y_2)^\star;V_1\big)\varphi(y_1-w;v_2)\,\mathrm{d}y_1
\end{aligned}$$

for $x_1, x_2, y_2, w \in \mathbb{R}$, *where* $(V_1^{-1})_{ij}$ *denotes the element of* V_1^{-1} *and* $\mathcal{W} = (v_2^{-1}(x_1 - w) - (V_1^{-1})_{12}(x_2 - y_2))/((V_1^{-1})_{11} + v_2^{-1}) + \alpha$.

Proof of Lemma 2.2. Let $\delta \in (3/q, \delta_2 \wedge \delta_3)$ and

$$\mathsf{A} = \bigcap_{i=1}^{2} \bigcap_{k=1}^{[Tb_n]+1} \{\mathbf{N}^i_{(b_n^{-1}k)\wedge T} - \mathbf{N}^i_{b_n^{-1}(k-1)} \le b_n^\delta\}.$$

Then for sufficiently large n, we obtain

$$\begin{aligned} P[\mathsf{A}^c] &\le \sum_{k=1}^{[Tb_n]+1} \sum_{i=1}^{2} P\big[\mathbf{N}^i_{(b_n^{-1}k)\wedge T} - \mathbf{N}^i_{b_n^{-1}(k-1)} > b_n^\delta\big] \\ &\le b_n^{-q\delta} \sum_{k=1}^{[Tb_n]+1} \sum_{i=1}^{2} E\big[\big(\mathbf{N}^i_{(b_n^{-1}k)\wedge T} - \mathbf{N}^i_{b_n^{-1}(k-1)}\big)^q\big] \le C b_n^{1-q\delta}. \end{aligned}$$

On the other hand, for any $k \in \mathbb{Z}_+$,

$$\big|S^{n,j_2} - S^{n,j_1}\big| \le k b_n^{-1} \quad \Rightarrow \quad |j_2 - j_1| \le (k+1) b_n^\delta$$

on A. Hence, we have

$$|j_2 - j_1| > (k+1) b_n^\delta \quad \Rightarrow \quad \big|S^{n,j_2} - S^{n,j_1}\big| > k b_n^{-1} \qquad \text{on } \mathsf{A}.$$

For sufficiently large n, if $|j_2 - j_1| \ge b_n^{\delta_2}$ and $\omega \in \mathsf{A}$, there exists $k \in \mathbb{N}$ such that $(k+1)b_n^\delta < |j_2 - j_1| \le (k+2) b_n^\delta$. Then since $|S^{n,j_2} - S^{n,j_1}| > k b_n^{-1}$, we have

$$\frac{|S^{n,j_2} - S^{n,j_1}|}{|j_2 - j_1|} > \frac{k b_n^{-1}}{(k+2) b_n^\delta} \ge \frac{1}{3} b_n^{-1-\delta} \ge b_n^{-1-\delta_3}.$$

Therefore, we obtain

$$b_n^2 \sup_{j_1, j_2 \in \mathbb{N}, |j_2 - j_1| \ge b_n^{\delta_2}} P\left[\ell_{1,n} \ge j_1 \vee j_2 \text{ and } \frac{|S^{n,j_2} - S^{n,j_1}|}{|j_2 - j_1|} \le b_n^{-1-\delta_3}\right] \le b_n^2 P\big[\mathsf{A}^c\big] \le C b_n^{3-q\delta} \to 0$$

as $n \to \infty$. Similar estimates for $\{T^{n,j}\}$ hold true.

In particular, under $[B1]$, Proposition 8 in Ogihara and Yoshida [19] yields $\limsup_{n\to\infty} E[b_n^{q-1} r_n^q] < \infty$ for any $q > 0$. Then we have $[A2]$. □

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