Bernoulli **21**(3), 2015, 1341–1360
DOI: 10.3150/13-BEJ588

Mimicking self-similar processes

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We construct a family of self-similar Markov martingales with given marginal distributions. This construction uses the self-similarity and Markov property of a reference process to produce a family of Markov processes that possess the same marginal distributions as the original process. The resulting processes are also self-similar with the same exponent as the original process. They can be chosen to be martingales under certain conditions. In this paper, we present two approaches to this construction, the transition-randomising approach and the time-change approach. We then compute the infinitesimal generators and obtain some path properties of the resulting processes. We also give some examples, including continuous Gaussian martingales as a generalization of Brownian motion, martingales of the squared Bessel process, stable Lévy processes as well as an example of an artificial process having the marginals of $t^\kappa V$ for some symmetric random variable V. At the end, we see how we can mimic certain Brownian martingales which are non-Markovian.

Keywords: Lévy processes; martingales with given marginals; self-similar

1. Introduction

Constructing martingales with given marginal distributions has been an active area of research over the last decade (e.g. [1,2,9,10,14,15]). (Here and in the entire paper, marginal distributions (also marginals) refer to the 1-dimensional distributions.) A condition for the existence of such martingales is given by Kellerer [12] (see Hirsch and Roynette [11] for a new and improved proof).

Three constructions of Markov martingales with pre-specified marginal distributions were given by Madan and Yor [14], namely the Skorokhod embedding method, the time-changed Brownian motion and the continuous martingale approach pioneered by Dupire [4]. Recently, Hirsch *et al*. [10] gave six different methods for constructing martingales whose marginal distributions match those of a given family of probability measures. They also tackle the tedious task of finding sufficient conditions to ensure that the chosen family is indeed increasing in the convex order, or as they coined it, a peacock.

In this paper, we deal with a different, albeit related, scenario. We do not start with a family of probability distributions, rather we start with a given martingale (the existence of which is assumed) and produce a large family (as opposed to just a handful) of new martingales having the same marginal distributions as the original process. We say that these martingales "mimic" the original process.

This same task was undertaken in [9] for the Brownian motion. It gave rise to the papers [1,2] and [15] (who coined the term Faked Brownian Motion). Albin [1] and Oleszkiewicz [15] answered the question of the existence of a continuous martingale with Brownian marginals. However, their constructions yield non-Markov processes. Baker *et al*. [2] then generalised Albin's construction and produced a sequence of (non-Markov) martingales with Brownian marginals. In this paper, we extend the construction of [9] to a much larger class of processes, namely self-similar Markov martingales.

Before formulating a solution to this problem we give a brief account on the origin and relevance of the mimicking question to finance, and more specifically to option pricing; that is the pricing of a contract that gives the holder the right to buy (or sell) the instrument (a stock) at a future time T for a specified price K. The theoretical valuation of an option is performed in such a way as not to allow arbitrage opportunities – arbitrage occurs when riskless trading results in profit. The first fundamental theorem (e.g., [18], page 231) states that the absence of arbitrage in a market with stock price S_t, $0 \le t \le T$, is essentially equivalent to the existence of an equivalent probability measure under which the stock price is a martingale. (Here without loss of generality, we let the riskless interest rate be zero.) The second fundamental theorem (e.g., [18], page 232) implies that the arbitrage-free price of an option is given by $\mathbb{E}[(S_T - K)^+]$, where the expectation is taken under the equivalent martingale measure. In the classical model of Black and Scholes the stock price S_t is given, under the martingale measure, by the exponential Brownian motion $S_t = S_0 \exp(\sigma B_t - t\sigma^2/2)$. The parameter σ is known as the volatility, and is assumed to be a constant. The resulting expectation produces the well-known Black–Scholes formula for option prices. However, empirical evidence shows that in order to match the Black–Scholes formula to market prices of options one needs to vary σ. As a consequence it is natural to ask whether there exist alternative models of stock prices that result in a prescribed option pricing formula, such as Black–Scholes. Finally, it is easy to see that the collective knowledge of $\{\mathbb{E}[(S_T - K)^+], K \ge 0\}$ determines the distribution of S_T or the marginal distribution, for example [8]. Therefore, if one wants to keep the option prices given by the original formula but without the limitations of the original process (such as constant volatility) one has to look for martingales (to have the model arbitrage free) with given marginals (to keep the same option prices). This question received much attention in the last ten years, see the pioneering work of Madan and Yor [14].

Throughout this paper, we assume that all processes are càdlàg and progressively measurable. We will use the notation $\stackrel{d}{=}$ to mean equal in distribution for random variables or equal in finite-dimensional distributions for processes, and this will be clear in the context. For a given random measure $M(\mathrm{d}x)$, the measure $M(c\,\mathrm{d}x)$ for $c > 0$ is defined by $\int g(x)M(c\,\mathrm{d}x) = \int g(x/c)M(\mathrm{d}x)$. We will also write $\mathbb{E}[M(\mathrm{d}x)]$ to mean the measure defined by $\int g(x) \times \mathbb{E}[M(\mathrm{d}x)] = \mathbb{E}[\int g(x)M(\mathrm{d}x)]$ for any positive function g.

We start with a martingale Z which is also a Markov process, that is, for any bounded measurable function g, $\mathbb{E}[g(Z_t)|\mathcal{F}_s] = \mathbb{E}[g(Z_t)|Z_s]$ for $s \le t$, where $(\mathcal{F}_t)_{t\ge0}$ denotes the natural filtration of $(Z_t)_{t\ge0}$. We aim to construct new processes that will have the same marginal distributions as Z while retaining the martingale and Markov properties. We assume further that Z is self-similar, that is, there exists a (strictly) positive function $q(c)$ such that $(Z_{ct})_{t\ge0} \stackrel{d}{=} (q(c)Z_t)_{t\ge0}$ for all $c > 0$. For Z to be non-trivial and stochastically continuous at $t = 0$, we must have (see, e.g., [5]) that $q(c) = c^\kappa$ for some $\kappa > 0$, i.e.,

$$(Z_{ct})_{t\ge0} \stackrel{d}{=} \left(c^\kappa Z_t\right)_{t\ge0} \qquad \forall c > 0$$

(which implies that $Z_0 = 0$). We then say that Z is self-similar with exponent κ, or simply, κ-self-similar.

Denote the transition function of Z by $P(s,t,x,\mathrm{d}y) := \mathbb{P}(Z_t \in \mathrm{d}y | Z_s = x)$ for $s \le t$. Then the self-similarity of Z translates to the following scaling property on P:

$$P\big(cs, ct, c^\kappa x, c^\kappa\,\mathrm{d}y\big) = P(s,t,x,\mathrm{d}y) \qquad \forall c > 0, s \le t \text{ and } x.$$

If L is the infinitesimal generator associated with P, this scaling property is equivalent to

$$c\pi_{c^\kappa} L_{cs} \pi_{c^{-\kappa}} = L_s, \tag{1.1}$$

where π_c is the operator defined by $\pi_c f(x) = f(cx)$. From this, we see that

$$L_s = s^{-1} \pi_{s^{-\kappa}} L_1 \pi_{s^\kappa}. \tag{1.2}$$

In the following, we present a mimicking scheme to the process Z by randomising the transition functions. We will see that this is equivalent to time-changing the process together with an appropriate scaling. We then obtain some properties of the resulting processes and give some examples.

2. Mimicking scheme

Let Z be a Markov process with transition function P, which is self-similar with exponent $\kappa > 0$. Note that if Z is a martingale, then it has a càdlàg version; if Z is not a martingale, there are conditions for a Markov process to be càdlàg. In this section, we construct new Markov processes that possess the same marginal distributions as Z and show that the resulting processes are martingales under certain conditions.

Lemma 2.1. *Let $0 < s \le t \le u$. For any $a, b \in [0,1]$ and a measurable set B,*

$$\int P(0,s,0,\mathrm{d}x) P\big(at, t, (at/s)^\kappa x, B\big) = P(0,t,0,B)$$

and

$$\int P\big(at, t, (at/s)^\kappa x, \mathrm{d}y\big) P\big(bu, u, (bu/t)^\kappa y, B\big) = P\big(abu, u, (abu/s)^\kappa x, B\big).$$

Proof. Here we prove only the second equality, the first one is proved similarly. Suppose that $a, b \in (0,1]$. By the scaling property, we have

$$\begin{aligned}
&\int P\big(at, t, (at/s)^\kappa x, \mathrm{d}y\big) P\big(bu, u, (bu/t)^\kappa y, B\big) \\
&\quad = \int P\big((bu/t)at, (bu/t)t, (bu/t)^\kappa (at/s)^\kappa x, (bu/t)^\kappa\,\mathrm{d}y\big) P\big(bu, u, (bu/t)^\kappa y, B\big) \\
&\quad = \int P\big(abu, bu, (abu/s)^\kappa x, \mathrm{d}y\big) P(bu, u, y, B) = P\big(abu, u, (abu/s)^\kappa x, B\big).
\end{aligned}$$

Notice that when a or b is 0, the two equalities in the lemma trivially hold true. $\square$

Proposition 2.1. *Let* $(G_{s,t})_{s\le t}$ *be a family of probability distributions on the set* $(0,1]$, *where* $G_{s,s}=\delta_1$, *the Dirac measure at* 1. *Suppose that for any bounded measurable function* h *and* $s\le t\le u$, *we have*

$$\int\int h(ab)G_{s,t}(\mathrm{d}a)G_{t,u}(\mathrm{d}b)=\int h(r)G_{s,u}(\mathrm{d}r). \tag{2.1}$$

Then $\widetilde{P}$ *defined as follows is a transition function,*

$$\widetilde{P}(0,t,0,\mathrm{d}y)=P(0,t,0,\mathrm{d}y),$$
$$\widetilde{P}(s,t,x,\mathrm{d}y)=\int P\bigl(rt,t,(t/s)^\kappa r^\kappa x,\mathrm{d}y\bigr)G_{s,t}(\mathrm{d}r), \qquad s\le t.$$

Proof. Clearly, for each (s,t,x), $\widetilde{P}(s,t,x,\mathrm{d}y)$ is a probability measure and for each (s,t,B), $\widetilde{P}(s,t,x,B)$ is measurable in x. Note also that $\widetilde{P}(s,s,x,B)=\delta_x(B)$. Next, using Lemma 2.1, we obtain, for $0<s\le t\le u$, $\int\widetilde{P}(0,t,0,\mathrm{d}y)\widetilde{P}(t,u,y,B)=\widetilde{P}(0,u,0,B)$ and

$$\begin{aligned}\int\widetilde{P}(s,t,x,\mathrm{d}y)\widetilde{P}(t,u,y,B)&=\int\int P\bigl(abu,u,(u/s)^\kappa(ab)^\kappa x,B\bigr)G_{s,t}(\mathrm{d}a)G_{t,u}(\mathrm{d}b)\\&=\int P\bigl(ru,u,(u/s)^\kappa r^\kappa x,B\bigr)G_{s,u}(\mathrm{d}r)=\widetilde{P}(s,u,x,B);\end{aligned}$$

in other words, $\widetilde{P}$ satisfies the Chapman–Kolmogorov equations. □

Proposition 2.2. *If* $G_{s,t}$ *depends on* s *and* t *only through* t/s (*i.e.* $G_{s,t}=G_{t/s}$), *then the scaling property of* P *carries over to* $\widetilde{P}$:

$$\widetilde{P}\bigl(cs,ct,c^\kappa x,c^\kappa\,\mathrm{d}y\bigr)=\widetilde{P}(s,t,x,\mathrm{d}y) \qquad \forall c>0,\ s\le t \text{ and } x.$$

Proof. This follows immediately from the definition of $\widetilde{P}$ and the scaling of P. □

Let, for $s\le t$, $R_{s,t}$ be a random variable having distribution $G_{s,t}$. Property (2.1) is equivalent to the property that if, for $s\le t\le u$, $R_{s,t}$ and $R_{t,u}$ are independent random variables, then $R_{s,t}R_{t,u}\overset{d}{=}R_{s,u}$. Further, if we let $V_{a,b}=-\ln R_{\mathrm{e}^a,\mathrm{e}^b}$ and write $K_{a,b}$ for the distributions of $V_{a,b}$ (with $K_{a,a}=\delta_0$), then Property (2.1) translates to the convolution identity

$$K_{a,b}*K_{b,c}=K_{a,c}, \qquad a\le b\le c.$$

As we seek to retain the scaling property of the original process, we assume that $G_{s,t}=G_{t/s}$ and immediately reduce Property (2.1) to

$$K_a*K_b=K_{a+b}, \qquad a,b\ge 0.$$

The family $(K_a)_{a\geq 0}$ defines a subordinator (process with positive, independent and stationary increments) and by Lévy–Khintchine it has Laplace transforms of the form

$$\int \mathrm{e}^{-\lambda v} K_a(\mathrm{d}v) = \exp\bigl(-a\psi(\lambda)\bigr), \qquad \lambda \geq 0.$$

The function ψ, known as the Laplace exponent, takes the form

$$\psi(\lambda) = \beta\lambda + \int_0^\infty \bigl(1 - \mathrm{e}^{-\lambda x}\bigr)\nu(\mathrm{d}x), \tag{2.2}$$

with drift $\beta \geq 0$ and Lévy measure ν satisfying $\nu(\{0\}) = 0$ and $\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty$. Conversely, to each ψ (i.e., to each pair (β, ν) as above) corresponds a convolution semigroup $(K_a)_{a\geq 0}$ and in turn a family $(G_u)_{u\geq 1}$ which satisfies Property (2.1). For details see, for example, [3], Section 1.2. This ensures the existence of $(G_u)_{u\geq 1}$ and a process with transition function $\widetilde{P}$.

Theorem 2.1. *Let Z be a κ-self-similar Markov process. To each ψ, Laplace exponent of a subordinator, corresponds a κ-self-similar Markov process X, starting from 0 and having the marginals of Z. Furthermore, if Z is a martingale and $\psi(\kappa) = \kappa$, then X is also a martingale.*

Writing in terms of $R_{t/s}$, $s \leq t$, the new transition function

$$\widetilde{P}(s, t, x, \mathrm{d}y) = \mathbb{E}\bigl[P\bigl(R_{t/s}t, t, (t/s)^\kappa R_{t/s}^\kappa x, \mathrm{d}y\bigr)\bigr], \qquad s \leq t$$

can be seen as a randomisation of $P(s, t, x, \mathrm{d}y)$. Furthermore, the condition on $(G_{t/s})_{s\leq t}$ for X to be a martingale can be written as $\mathbb{E}[R_{t/s}^\kappa] = (s/t)^\kappa$.

Proof. By the Kolmogorov extension theorem, there exists a Markov process X with transition function $\widetilde{P}(s, t, x, \mathrm{d}y)$. As for the martingale property, we first observe that

$$\psi(\lambda) = -\frac{1}{a}\ln \int \mathrm{e}^{-\lambda v} K_a(\mathrm{d}v) = -\frac{1}{a}\ln \int r^\lambda G_{\mathrm{e}^a}(\mathrm{d}r),$$

so that $\psi(\kappa) = \kappa$ translates to $u^\kappa \int r^\kappa G_u(\mathrm{d}r) = 1$ for $u \geq 1$. Then we have

$$\int y\widetilde{P}(s, t, x, \mathrm{d}y) = \int (t/s)^\kappa r^\kappa x G_{t/s}(\mathrm{d}r) = x$$

using the definition of $\widetilde{P}$ and the martingale property of Z. □

The process X can also be obtained using subordination (with a suitable scaling in the state space and an appropriate time-change). The idea of subordination was suggested by Bertoin[1] in the context of Brownian marginals. However, subordination alone (albeit with a suitable scaling in the state space) is not sufficient. A further logarithmic change of time is needed. This naturally creates an issue at 0. To deal with this, we follow Oleszkiewicz [15].

[1] Private communication.

Proposition 2.3. *Let $(\zeta_t)_{t\geq 0}$ be a subordinator independent of Z. Let, for $a \in \mathbb{R}$,*

$$X_t^{(a)} = t^\kappa \mathrm{e}^{-\kappa \zeta_{a+\ln t}} Z_{\mathrm{e}^{\zeta_{a+\ln t}}}, \qquad t \geq \mathrm{e}^{-a}.$$

Then the process $(X_t^{(a)})_{t\geq \mathrm{e}^{-a}}$ has the same marginal distributions as $(Z_t)_{t\geq \mathrm{e}^{-a}}$ and there exists a process $(X_t)_{t\geq 0}$ such that for any $a \in \mathbb{R}$, $(X_t)_{t\geq \mathrm{e}^{-a}} \stackrel{d}{=} (X_t^{(a)})_{t\geq \mathrm{e}^{-a}}$. The process $(X_t)_{t\geq 0}$ is a κ-self-similar Markov process with transition function

$$Q(0,t,0,\mathrm{d}y) = P(0,t,0,\mathrm{d}y),$$
$$Q(s,t,x,\mathrm{d}y) = \mathbb{E}\big[P\big(\mathrm{e}^{-\zeta_{\ln(t/s)}}t, t, (t/s)^\kappa \mathrm{e}^{-\kappa\zeta_{\ln(t/s)}}x, \mathrm{d}y\big)\big], \qquad s \leq t.$$

Moreover, X is a martingale provided that Z is a martingale and $\mathbb{E}[\mathrm{e}^{-\kappa\zeta_{\ln(t/s)}}] = (s/t)^\kappa$.

Proof. Since $Z_t(\omega)$ is measurable in ω and right-continuous in t, it is measurable as a function of (t,ω). Hence, for each $t \geq \mathrm{e}^{-a}$, $X_t^{(a)}(\omega)$ is a random variable.

For $c > 0$, let $\widehat{Z}_s = \mathrm{e}^{-\kappa\zeta_c} Z_{s\mathrm{e}^{\zeta_c}}$ and $\widehat{\zeta}_s = \zeta_{c+s} - \zeta_c$, so that $(\widehat{Z}_s)_{s\geq 0} \stackrel{d}{=} (Z_s)_{s\geq 0}$ and $(\widehat{\zeta}_s)_{s\geq 0} \stackrel{d}{=} (\zeta_s)_{s\geq 0}$. Then we have, for $t \geq \mathrm{e}^{-b} \geq \mathrm{e}^{-a}$ and with $c = a-b$,

$$\big(X_t^{(a)}\big)_{t\geq \mathrm{e}^{-b}} = \big(t^\kappa \mathrm{e}^{-\kappa\widehat{\zeta}_{b+\ln t}} \widehat{Z}_{\mathrm{e}^{\widehat{\zeta}_{b+\ln t}}}\big)_{t\geq \mathrm{e}^{-b}} \stackrel{d}{=} \big(X_t^{(b)}\big)_{t\geq \mathrm{e}^{-b}}.$$

For $\mathrm{e}^{-a} \leq t_1 < \cdots < t_n$, let $\mu_{t_1,\dots,t_n}$ be the law of $(X_{t_1}^{(a)},\dots,X_{t_n}^{(a)})$. Then the measures $(\mu_{t_1,\dots,t_n})_{n,0<t_1<\cdots<t_n}$ are consistent and by the Kolmogorov extension theorem, there exists a process $(X_t)_{t>0}$ with finite-dimensional distributions $(\mu_{t_1,\dots,t_n})_{n,0<t_1<\cdots<t_n}$.

A similar argument shows that $(X_{ct_1}^{(a)},\dots,X_{ct_n}^{(a)}) \stackrel{d}{=} c^\kappa(X_{t_1}^{(a)},\dots,X_{t_n}^{(a)})$, from which we deduce that $(X_t)_{t>0}$ is κ-self-similar. As such, X extends by continuity to $t \geq 0$ by letting $X_0 = 0$. The equality of marginal distributions of $X^{(a)}$ and Z follows from the scaling property of Z as $X_t^{(a)} = t^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln t}} Z_{\mathrm{e}^{\zeta_{a+\ln t}}} \stackrel{d}{=} t^\kappa Z_1 \stackrel{d}{=} Z_t$ for any fixed $t \geq \mathrm{e}^{-a}$.

Using successively Lemmas A.3, A.4, A.2 and A.1 (see the Appendix), we see that $X^{(a)}$ is Markovian. By the scaling property of P and the stationarity of subordinator ζ, we obtain, for $\mathrm{e}^{-a} \leq s \leq t$, the transition function of $X^{(a)}$ as

$$Q^{(a)}(s,t,x,\mathrm{d}y) = \mathbb{P}\big(X_t^{(a)} \in \mathrm{d}y | X_s^{(a)} = x\big) = \mathbb{E}\big[P\big(\mathrm{e}^{-\zeta_{\ln(t/s)}}t, t, (t/s)^\kappa \mathrm{e}^{-\kappa\zeta_{\ln(t/s)}}x, \mathrm{d}y\big)\big].$$

Notice that the transition function $Q^{(a)}$ does not depend on a and it is the same as $\widetilde{P}$ defined earlier with $G_{t/s}$ being the distribution of $\exp(-\zeta_{\ln(t/s)})$. The rest of the assertions of the proposition then follows immediately from Theorem 2.1. (Alternatively, we can carry out the proof independently, without referring to $\widetilde{P}$, following Oleszkiewicz [15].) □

The process constructed in Proposition 2.3 is identical (in law) to the process obtained in Theorem 2.1 with $G_{t/s}$ being the distribution of $\exp(-\zeta_{\ln(t/s)})$, or $R_{t/s} = \exp(-\zeta_{\ln(t/s)})$.

Remark 2.1. For X to be a martingale, β in (2.2) must be at most 1, and is 1 if and only if $\zeta_t = t$ ($\nu = 0$ and $\psi(\lambda) = \lambda$ for any λ) and $X = Z$.

Remark 2.2. In Proposition 2.3, we cannot replace $\zeta_{a+\ln t}$ with a two-sided subordinator $\zeta_t = \zeta_t^1 \mathbf{1}_{t\geq 0} - \zeta_{-t}^2 \mathbf{1}_{t<0}$ for $t \in \mathbb{R}$, where $(\zeta_t^1)_{t\geq 0}$ and $(\zeta_t^2)_{t\geq 0}$ are independent subordinators. This is because by doing that, we will not have independent increments. In particular, since $\zeta_0 = 0$, then for $t < 0$, the increment $\zeta_0 - \zeta_t = -\zeta_t \in \mathcal{G}_t$, where $\mathcal{G}_t$ denotes the filtration generated by ζ.

3. Properties

In this section, we obtain the infinitesimal generators of the process X and display some of their path properties. We will work within the martingale framework, that is, unless otherwise stated, we will assume that our initial process Z is a martingale and we will use a subordinator ζ with drift β, Lévy measure ν and Lapace exponent satisfying $\psi(\kappa) = \kappa$.

Theorem 3.1. *Suppose that Z has infinitesimal generator L_t. Then the infinitesimal generator of the process X is given by*

$$\begin{aligned} A_0 f(x) &= L_0 f(x), \\ A_t f(x) &= \beta L_t f(x) + (1-\beta)\frac{\kappa}{t} x f'(x) \\ &\quad + \frac{1}{t}\int_{(0,\infty)} \int_{-\infty}^{\infty} \big(f(y) - f(x)\big) P\big(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-u\kappa}, \mathrm{d}y\big)\nu(\mathrm{d}u), \qquad t > 0, \end{aligned}$$

for f differentiable and in the domain of L.

Proof. First, from Lemma A.3, $\widehat{Z}_t := \mathrm{e}^{-t\kappa} Z_{\mathrm{e}^t}$ is time-homogeneous with generator $\widehat{L} f(x) = L_1 f(x) - \kappa x f'(x)$ and transition semigroup $\widehat{P}_t f(x) = \int f(y) P(\mathrm{e}^{-t}, 1, x\mathrm{e}^{-t\kappa}, \mathrm{d}y)$. Next, applying Lemma A.4, the generator of the process $\bar{Z}_t := \widehat{Z}_{\zeta_t} = \mathrm{e}^{-\kappa\zeta_t} Z_{\mathrm{e}^{\zeta_t}}$ is

$$\bar{L} f(x) = \beta L_1 f(x) - \kappa\beta x f'(x) + \int_{(0,\infty)} \int \big(f(y) - f(x)\big) P\big(\mathrm{e}^{-u}, 1, x\mathrm{e}^{-u\kappa}, \mathrm{d}y\big)\nu(\mathrm{d}u).$$

Then, let $\widetilde{Z}_t = \mathrm{e}^{\kappa(t-a)} \bar{Z}_t = \mathrm{e}^{\kappa(t-a)} \mathrm{e}^{-\kappa\zeta_t} Z_{\mathrm{e}^{\zeta_t}}$ and using Lemma A.2, the generator of $\widetilde{Z}$ is

$$\begin{aligned} \widetilde{L}_t f(x) &= \beta \pi_{\mathrm{e}^{-\kappa(t-a)}} L_1 \pi_{\mathrm{e}^{\kappa(t-a)}} f(x) + (1-\beta)\kappa x f'(x) \\ &\quad + \int_{(0,\infty)} \int \big(f\big(\mathrm{e}^{\kappa(t-a)} y\big) - f(x)\big) P\big(\mathrm{e}^{-u}, 1, \mathrm{e}^{-\kappa(t-a)} x\mathrm{e}^{-u\kappa}, \mathrm{d}y\big)\nu(\mathrm{d}u), \end{aligned}$$

since $\pi_{\mathrm{e}^{-\kappa(t-a)}} \Lambda \pi_{\mathrm{e}^{\kappa(t-a)}} f(x) = \Lambda f(x)$ for $\Lambda f(x) = x f'(x)$. Finally, we time-change the process $\widetilde{Z}_t$ with $a + \ln t$ to get $X_t^{(a)}$. Thus, by Lemma A.1, the generator of $X_t^{(a)}$ is

$$\begin{aligned} A_t^{(a)} f(x) &= \beta L_t f(x) + (1-\beta)\frac{\kappa}{t} x f'(x) \\ &\quad + \frac{1}{t}\int_{(0,\infty)} \int \big(f(y) - f(x)\big) P\big(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-u\kappa}, \mathrm{d}y\big)\nu(\mathrm{d}u) \end{aligned}$$

due to a change of variable, the scaling property of P and identity (1.2). The generator of X_t is established by noting that $A_t^{(a)}$ does not depend on a.

Since Z is self-similar, $\pi_c f$ is in the domain of L for all $c > 0$ whenever f is in the domain of L. Therefore, f is in the domain of A, if f is also differentiable. □

Note that when $\beta = 1$ and $\nu \equiv 0$, we recover the process Z and $A_t = L_t$.

For a measurable function f, if there exists a measurable function g such that for each t, $\int_0^t |g(X_s)|\,\mathrm{d}s < \infty$ almost surely and $f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s$ is a martingale, then f is said to belong to the domain of the extended infinitesimal generator of X and the extended infinitesimal generator $A_s f(X_s) = g(X_s)$. If $f(x) = x^2$ belongs to the domain of the extended infinitesimal generator of X, then X has predictable quadratic variation

$$\langle X, X\rangle_t = \int_0^t A_s f(X_s)\,\mathrm{d}s. \tag{3.1}$$

See examples in Section 4 for the computation $\langle X, X\rangle$ in some specific cases.

Proposition 3.1. *Suppose that Z is continuous in probability. Then the process X is also continuous in probability, that is, for every t,*

$$\forall c > 0, \qquad \lim_{s\to t} \mathbb{P}\bigl(|X_t - X_s| > c\bigr) = 0.$$

Proof. We have, for $s, t \geq \mathrm{e}^{-a}$,

$$\begin{aligned}\mathbb{P}\bigl(|X_t - X_s| > c\bigr) \leq \mathbb{P}\biggl(\Bigl|\bigl(t^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln t}} - s^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln s}}\bigr) Z_{\mathrm{e}^{\zeta_{a+\ln t}}}\Bigr| > \frac{c}{2}\biggr) \\ + \mathbb{P}\biggl(\Bigl|s^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln s}} (Z_{\mathrm{e}^{\zeta_{a+\ln t}}} - Z_{\mathrm{e}^{\zeta_{a+\ln s}}})\Bigr| > \frac{c}{2}\biggr).\end{aligned} \tag{3.2}$$

However, the first term

$$\begin{aligned}&\mathbb{P}\biggl(\Bigl|\bigl(t^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln t}} - s^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln s}}\bigr) Z_{\mathrm{e}^{\zeta_{a+\ln t}}}\Bigr| > \frac{c}{2}\biggr) \\ &\quad\leq \mathbb{P}\biggl(\Bigl|\bigl(t^\kappa - s^\kappa\bigr)\mathrm{e}^{-\kappa\zeta_{a+\ln t}} Z_{\mathrm{e}^{\zeta_{a+\ln t}}}\Bigr| > \frac{c}{4}\biggr) \\ &\qquad + \mathbb{P}\biggl(\Bigl|s^\kappa\bigl(\mathrm{e}^{-\kappa\zeta_{a+\ln t}} - \mathrm{e}^{-\kappa\zeta_{a+\ln s}}\bigr) Z_{\mathrm{e}^{\zeta_{a+\ln t}}}\Bigr| > \frac{c}{4}\biggr),\end{aligned}$$

which converges to 0 as $s \to t$, since ζ is continuous in probability as a subordinator.

To deal with the last term in (3.2) we first observe that a process that is continuous in probability does not jump at fixed points so that $\mathbb{P}(\Delta \mathrm{e}^{\zeta_{a+\ln t}} = 0) = 1$. Further, since Z is also continuous in probability, $\lim_{s\to t}\mathbb{P}(|Z_{y_t}-Z_{y_s}|>\varepsilon)=0$ as soon as $\lim_{s\to t} y_s = y_t$. Therefore, for $s\le t+1$,

$$\begin{aligned}
&\lim_{s\to t}\mathbb{P}\biggl(\bigl|s^\kappa \mathrm{e}^{-\kappa\zeta_{a+\ln s}}(Z_{\mathrm{e}^{\zeta_{a+\ln t}}}-Z_{\mathrm{e}^{\zeta_{a+\ln s}}})\bigr|>\frac{c}{2}\biggr)\\
&\quad\le \lim_{s\to t}\mathbb{P}\biggl(|Z_{\mathrm{e}^{\zeta_{a+\ln t}}}-Z_{\mathrm{e}^{\zeta_{a+\ln s}}}|>\frac{c}{2}(t+1)^{-\kappa}\biggr)\\
&\quad=\mathbb{E}\biggl[\lim_{s\to t}\mathbb{P}\biggl(|Z_{\mathrm{e}^{\zeta_{a+\ln t}}}-Z_{\mathrm{e}^{\zeta_{a+\ln s}}}|>\frac{c}{2}(t+1)^{-\kappa}\bigl|\hat{\mathcal{G}}_t\biggr)\mathbf{1}_{\{\Delta \mathrm{e}^{\zeta_{a+\ln t}}=0\}}\biggr]=0,
\end{aligned}$$

where $\hat{\mathcal{G}}_t=\sigma(\zeta_s, s\le a+\ln t)$. $\square$

Proposition 3.2. *If Z is continuous in probability with finite second moments and ζ has no drift $(\beta=0)$, then X is a purely discontinuous martingale.*

Proof. Let $U_t=\mathrm{e}^{\zeta_{a+\ln t}}$ so that $X_t=t^\kappa U_t^{-\kappa}Z_{U_t}$ for $t\ge \mathrm{e}^{-a}$. First, we observe that with probability one, Z does not jump at U_{t-} if U jumps at t. Indeed, if Γ is a countable set of points in $[0,\infty)$, then, as Z is continuous in probability, $\mathbb{P}(\exists t\in\Gamma \text{ s.t. } \Delta Z_t\ne 0)=0$. Let $\Lambda=\{t>\mathrm{e}^{-a}\colon\ \Delta U_t>0\}$ and $\Gamma_U=U^-(\Lambda)$ where $U_t^-=U_{t-}$, then

$$\mathbb{P}\bigl(\exists t>\mathrm{e}^{-a} \text{ s.t. } \Delta U_t>0, \Delta Z_{(U_{t-})}\ne 0\bigr)=\mathbb{E}\bigl[\mathbb{P}(\exists s\in\Gamma_U \text{ s.t. } \Delta Z_s\ne 0|\mathcal{G}_\infty)\bigr]=0,$$

where $\mathcal{G}$ denotes the filtration of U. Taking a to infinity, we obtain the desired result.

To show that X is purely discontinuous, that is, $\langle X^c,X^c\rangle_t=0$, we compute the sum of the square of jumps of X. In general, we have

$$\begin{aligned}
\mathbb{E}\biggl[\sum_{\mathrm{e}^{-a}<s\le t}(\Delta X_s)^2\biggr]&=\mathbb{E}\biggl[\sum_{\mathrm{e}^{-a}<s\le t}(\Delta X_s)^2\mathbf{1}_{\Delta U_s>0,\Delta Z_{(U_{s-})}\ne 0}\biggr]\\
&\quad+\mathbb{E}\biggl[\sum_{\mathrm{e}^{-a}<s\le t}(\Delta X_s)^2\mathbf{1}_{\Delta U_s>0,\Delta Z_{(U_{s-})}=0}\biggr]\\
&\quad+\mathbb{E}\biggl[\sum_{\mathrm{e}^{-a}<s\le t}(\Delta X_s)^2\mathbf{1}_{\Delta U_s=0}\biggr].
\end{aligned}$$

As Z is continuous in probability, the first term is zero due to the observation at the start of the proof. We write $l(a,t)=\mathbb{E}[\sum_{\mathrm{e}^{-a}<s\le t}(\Delta X_s)^2\mathbf{1}_{\Delta U_s=0}]$ for the third term.

For the second term, we have, on the set $\{\Delta U_s>0, \Delta Z_{(U_{s-})}=0\}$,

$$\begin{aligned}
(\Delta X_s)^2&=s^{2\kappa}\bigl(U_s^{-\kappa}(Z_{U_s}-Z_{(U_{s-})})+Z_{(U_{s-})}\bigl(U_s^{-\kappa}-U_{s-}^{-\kappa}\bigr)\bigr)^2\\
&=s^{2\kappa}\bigl(U_s^{-2\kappa}(Z_{U_s}-Z_{(U_{s-})})^2+Z_{(U_{s-})}^2\bigl(U_s^{-\kappa}-U_{s-}^{-\kappa}\bigr)^2\\
&\quad+2U_s^{-\kappa}\bigl(U_s^{-\kappa}-U_{s-}^{-\kappa}\bigr)Z_{(U_{s-})}(Z_{U_s}-Z_{(U_{s-})})\bigr).
\end{aligned}$$

Let $\theta_t = \mathbb{E}[Z_t^2]$. Since Z is κ-self-similar, $\theta_t = t^{2\kappa}\theta_1$. As Z is a martingale, $\mathbb{E}[Z_s(Z_t - Z_s)] = 0$ and $\mathbb{E}[(Z_t - Z_s)^2] = \mathbb{E}[Z_t^2] - \mathbb{E}[Z_s^2]$. Thus, we obtain

$$\begin{aligned}
&\mathbb{E}\big[(\Delta X_s)^2 | \mathcal{G}_\infty \cap \{\Delta U_s > 0, \Delta Z_{(U_{s-})} = 0\}\big] \\
&\quad = s^{2\kappa}\big(U_s^{-2\kappa}(\theta_{U_s} - \theta_{(U_{s-})}) + \theta_{(U_{s-})}\big(U_s^{-\kappa} - U_{s-}^{-\kappa}\big)^2\big) \\
&\quad = 2s^{2\kappa}\theta_1\big(1 - U_s^{-\kappa}U_{s-}^{\kappa}\big).
\end{aligned}$$

Since $\{\Delta Z_{(U_{s-})} = 0\}$ has probability one on the set $\{\Delta U_s > 0\}$, it follows that

$$\begin{aligned}
\mathbb{E}\bigg[\sum_{e^{-a}<s\le t}(\Delta X_s)^2 \mathbf{1}_{\Delta U_s>0, \Delta Z_{(U_{s-})}=0}\bigg] &= \mathbb{E}\bigg[\sum_{e^{-a}<s\le t} 2s^{2\kappa}\theta_1\big(1 - U_s^{-\kappa}U_{s-}^{\kappa}\big)\mathbf{1}_{\Delta U_s>0}\bigg] \\
&= \mathbb{E}\bigg[\sum_{0<r\le a+\ln t} 2e^{2\kappa(r-a)}\theta_1\big(1 - e^{-\kappa\Delta\zeta_r}\big)\mathbf{1}_{\Delta\zeta_r>0}\bigg].
\end{aligned}$$

Writing in terms of ν and ψ, and using (2.2) with $\psi(\kappa) = \kappa$,

$$\begin{aligned}
\mathbb{E}\bigg[\sum_{e^{-a}<s\le t}(\Delta X_s)^2 \mathbf{1}_{\Delta U_s>0, \Delta Z_{(U_{s-})}=0}\bigg] &= \theta_1 \int_0^{a+\ln t} 2e^{2\kappa(r-a)}\,dr \int_{(0,\infty)}\big(1 - e^{-z\kappa}\big)\nu(dz) \\
&= \theta_1\big(t^{2\kappa} - e^{-2a\kappa}\big)(1-\beta).
\end{aligned}$$

Adding those three terms and taking limit as $a \to \infty$, we have

$$\mathbb{E}\bigg[\sum_{0<s\le t}(\Delta X_s)^2\bigg] = \theta_1 t^{2\kappa}(1-\beta) + \lim_{a\to\infty} l(a,t).$$

Since X is square integrable on any finite interval if Z has finite second moments, it has quadratic variation with expectation $\mathbb{E}[[X,X]_t] = \mathbb{E}[X_t^2] = \mathbb{E}[Z_t^2] = t^{2\kappa}\theta_1$. Therefore,

$$\begin{aligned}
\mathbb{E}\big[\langle X^c, X^c\rangle_t\big] &= \mathbb{E}\big[[X,X]_t\big] - \mathbb{E}\bigg[\sum_{0<s\le t, \Delta X_s\neq 0}(\Delta X_s)^2\bigg] \\
&= \theta_1 t^{2\kappa}\beta - \lim_{a\to\infty} l(a,t).
\end{aligned}$$

Note that both $\mathbb{E}[\langle X^c, X^c\rangle_t]$ and $l(a,t)$ are non-negative. Thus, when $\beta = 0$, we must have $\lim_{a\to\infty} l(a,t) = 0$, which gives $\mathbb{E}[\langle X^c, X^c\rangle_t] = 0$ and hence $\langle X^c, X^c\rangle_t = 0$. □

4. Examples

Given any self-similar Markov martingale Z with transition function P and infinitesimal generator L, we can mimic Z as per Section 2. We construct a new Markov martingale X that has the same marginal distributions as Z and possesses the same self-similarity Z enjoys from each ζ.

We assume that the subordinator ζ has drift β and Lévy measure ν with Laplace exponent ψ satisfying $\psi(\kappa)=\kappa$, or $\mathbb{E}[e^{-\kappa\zeta_{\ln(t/s)}}]=(s/t)^\kappa$.

Examples of subordinators include Poisson process, compound Poisson process with positive jumps, gamma process and stable subordinators. For example, we can take ζ as a Poisson process with rate $\kappa/(1-\exp(-\kappa))$ to satisfy $\psi(\kappa)=\kappa$. In the following, we provide some examples of mimicking with the infinitesimal generators and the predictable quadratic variations computed explicitly to have a better understanding of the processes.

We finish this section with a discussion on modifying our construction to mimic some Brownian related martingales and its limitation.

4.1. Gaussian continuous martingales

For any $k\geq 0$, define the process $(Z_t)_{t\geq 0}$ by $Z_t=\int_0^t s^k\,\mathrm{d}B_s$, where $(B_t)_{t\geq 0}$ is a Brownian motion. Note that

$$(Z_t)_{t\geq 0}\stackrel{d}{=}(B_{(1/(2k+1))t^{2k+1}})_{t\geq 0}.$$

This is a Gaussian process with zero mean and covariance function $\operatorname{Cov}(Z_t,Z_{t+u})=\frac{1}{2k+1}t^{2k+1}$. It is a Markov process and also a martingale. Moreover, it is $(k+\frac{1}{2})$-self-similar ($\kappa=k+\frac{1}{2}$) since for all $c>0$,

$$(Z_{ct})_{t\geq 0}\stackrel{d}{=}(B_{(1/(2k+1))c^{2k+1}t^{2k+1}})_{t\geq 0}\stackrel{d}{=}\bigl(c^{k+1/2}B_{(1/(2k+1))t^{2k+1}}\bigr)_{t\geq 0}\stackrel{d}{=}\bigl(c^{k+1/2}Z_t\bigr)_{t\geq 0}.$$

A key aspect of the construction in [9] is the following representation of the mimic X_t when $k=0$:

$$X_t=\sqrt{t/s}\bigl(\sqrt{R_{s,t}}X_s+\sqrt{s}\sqrt{1-R_{s,t}}\xi_{s,t}\bigr),\qquad t\geq s,$$

where $R_{s,t}$ has distribution $G_{t/s}$, $\xi_{s,t}$ is standard normal and, $R_{s,t}$, $\xi_{s,t}$ and X_s are independent. This representation extends to the case of other Gaussian continuous martingales. In fact, it also extends to the case of stable processes – see Proposition 4.4.

Proposition 4.1. *With $\kappa=k+\frac{1}{2}$, the mimic $(X_t)_{t\geq 0}$ has the representation*

$$X_t=(t/s)^\kappa\bigl(R_{s,t}^\kappa X_s+s^\kappa(1-R_{s,t})^\kappa\xi_{s,t}\bigr),\qquad t\geq s>0,$$

where $R_{s,t}\stackrel{d}{=}e^{-\zeta_{\ln(t/s)}}$, $\xi_{s,t}\stackrel{d}{=}Z_1$ and, $R_{s,t}$, $\xi_{s,t}$ and X_s are independent.

Proof. Since Z and ζ have independent and stationary increments, for $e^{-a}\leq s<t$,

$$\begin{aligned}X_t^{(a)}&\stackrel{d}{=}t^\kappa e^{-\kappa\zeta_{a+\ln t}}Z_{e^{\zeta_{a+\ln s}}}+t^\kappa e^{-\kappa\zeta_{a+\ln t}}\bigl(e^{\zeta_{a+\ln t}}-e^{\zeta_{a+\ln s}}\bigr)^\kappa\xi_{s,t}\\&\stackrel{d}{=}(t/s)^\kappa e^{-\kappa\widehat{\zeta}_{\ln(t/s)}}X_s^{(a)}+t^\kappa\bigl(1-e^{-\widehat{\zeta}_{\ln(t/s)}}\bigr)^\kappa\xi_{s,t},\end{aligned}$$

where $\widehat{\zeta}$ is an independent copy of ζ, and $\xi_{s,t}$ is a random variable distributed as Z_1. Note that this representation holds also for $t=s$. □

Knowing that Z has generator $L_t f(x) = \frac{1}{2} t^{2k} f''(x)$ for $t \geq 0$, we can compute the generator of X following Theorem 3.1 and obtain, for $t > 0$,

$$\begin{aligned}A_t f(x) &= \frac{1}{2}\beta t^{2k} f''(x) + (1-\beta)\frac{2k+1}{2t} x f'(x) \\ &\quad + \frac{1}{t}\int_{(0,\infty)} \int_{-\infty}^{\infty} \big(f(y) - f(x)\big) P\big(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-(k+1/2)u}, \mathrm{d}y\big)\nu(\mathrm{d}u).\end{aligned}$$

Taking $f(x) = x^2$, then Equation (3.1), along with routine calculations and Equation (2.2), gives the following result.

Proposition 4.2. *The predictable quadratic variation of X is*

$$\langle X, X\rangle_t = \frac{1}{(2k+1)^2} t^{2k+1} \psi(2k+1) + \big(2k+1-\psi(2k+1)\big)\int_0^t \frac{1}{s} X_s^2 \,\mathrm{d}s.$$

Since $\langle Z, Z\rangle_t = \int_0^t s^{2k}\,\mathrm{d}s$, we can also write

$$\langle X, X\rangle_t = \frac{1}{2k+1}\psi(2k+1)\langle Z, Z\rangle_t + \big(2k+1-\psi(2k+1)\big)\int_0^t \frac{1}{s} X_s^2 \,\mathrm{d}s.$$

Remark 4.1. When $k = 0$, Z is a Brownian motion and our results agree with [9].

4.2. Martingale of squared Bessel process

A process $(S_t)_{t\geq 0}$ is a squared Bessel process of dimension δ, for some $\delta \geq 0$, if it satisfies $\mathrm{d}S_t = 2\sqrt{S_t}\,\mathrm{d}B_t + \delta\,\mathrm{d}t$, where B denotes a Brownian motion. The squared Bessel process S started at 0 is a continuous Markov process satisfying the self-similarity with $\kappa = 1$.

Let $Z_t = S_t - \delta t$. Then $(Z_t)_{t\geq 0}$ is a 1-self-similar Markov process and satisfies the SDE

$$\mathrm{d}Z_t = 2\sqrt{Z_t + \delta t}\,\mathrm{d}B_t.$$

Note that S is stochastically dominated by the square of the norm of an n-dimensional Brownian motion, where $n \geq \delta$, thus $\mathbb{E}[S_t] \leq nt$ and $\mathbb{E}[\langle Z, Z\rangle_t] \leq 2nt^2$. It follows that Z is a true martingale (see, e.g., [13], Theorem 7.35).

The infinitesimal generator of Z is $L_t f(x) = 2(x + \delta t) f''(x)$, $t \geq 0$, thus, that of X is

$$\begin{aligned}A_0 f(x) &= L_0 f(x) = 2x f''(x), \\ A_t f(x) &= 2\beta(x+\delta t) f''(x) + \frac{1}{t}(1-\beta) x f'(x) \\ &\quad + \frac{1}{t}\int_{(0,\infty)} \int_{-\infty}^{\infty} \big(f(y) - f(x)\big) P\big(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-u}, \mathrm{d}y\big)\nu(\mathrm{d}u), \qquad t > 0.\end{aligned}$$

Proposition 4.3. *The predictable quadratic variation of X is*

$$\langle X, X\rangle_t = \delta t^2 \psi(2) + 4\big(\psi(2) - 1\big)\int_0^t X_s \,\mathrm{d}s + \big(2 - \psi(2)\big)\int_0^t \frac{1}{s} X_s^2 \,\mathrm{d}s, \qquad t \geq 0,$$

where ψ is the Laplace exponent of ζ.

Proof. From $\mathrm{d}Z_t = 2\sqrt{Z_t + \delta t}\,\mathrm{d}B_t$, we have for $u > 0$,

$$Z_t^2 = Z_{t\mathrm{e}^{-u}}^2 + 4\int_{t\mathrm{e}^{-u}}^t Z_s\sqrt{Z_s + \delta s}\,\mathrm{d}B_s + 4\int_{t\mathrm{e}^{-u}}^t Z_s \,\mathrm{d}s + 2\delta t^2\big(1 - \mathrm{e}^{-2u}\big)$$

and taking conditional expectation ($\int_0^t Z_s\sqrt{Z_s + \delta s}\,\mathrm{d}B_s$ is a true martingale – see above),

$$\mathbb{E}\big[Z_t^2 | Z_{t\mathrm{e}^{-u}}\big] = Z_{t\mathrm{e}^{-u}}^2 + 4t\big(1 - \mathrm{e}^{-u}\big)Z_{t\mathrm{e}^{-u}} + 2\delta t^2\big(1 - \mathrm{e}^{-2u}\big).$$

Thus, using Equation (2.2) we obtain, with $f(x) = x^2$,

$$A_t f(x) = 2\delta t \psi(2) + 4x\big(\psi(2) - \psi(1)\big) + \frac{1}{t}x^2\big(2 - \psi(2)\big).$$

However $\psi(1) = 1$. The result then follows from Equation (3.1). Note that if $\beta = 1$, $\psi(\lambda) = \lambda$ for any λ and we recover $\langle X, X\rangle_t = 2\delta t^2 + 4\int_0^t X_s \,\mathrm{d}s$. □

4.3. Stable processes with $1 < \alpha < 2$

Suppose $(Z_t)_{t\geq 0}$ is an α-stable process with $1 < \alpha < 2$. Then Z is a Markov process and

$$(Z_{c^\alpha t})_{t\geq 0} \stackrel{d}{=} (cZ_t)_{t\geq 0}, \qquad \forall c > 0,$$

that is, Z is κ-self-similar with $\kappa = \frac{1}{\alpha}$. It is a Lévy process with Lévy triplet $(0, \nu_Z, \gamma)$, where

$$\nu_Z(\mathrm{d}z) = (A\mathbf{1}_{z>0} + B\mathbf{1}_{z<0})|z|^{-(\alpha+1)}\,\mathrm{d}z$$

for some positive constants A and B. Assume that Z is a martingale, in which case the Lévy triplet must satisfy

$$\gamma + \int_{|z|\geq 1} z\nu_Z(\mathrm{d}z) = 0.$$

Proposition 4.4. *The mimic $(X_t)_{t\geq 0}$ has the representation*

$$X_t = (t/s)^\kappa\big(R_{s,t}^\kappa X_s + s^\kappa(1 - R_{s,t})^\kappa \xi_{s,t}\big), \qquad t \geq s,$$

where $R_{s,t} \stackrel{d}{=} \mathrm{e}^{-\zeta_{\ln(t/s)}}$, $\xi_{s,t} \stackrel{d}{=} Z_1$ and, $R_{s,t}$, $\xi_{s,t}$ and X_s are independent.

Proof. See Proposition 4.1. □

The stable process Z has infinitesimal generator

$$Lf(x) = \gamma f'(x) + \int_{\mathbb{R}\setminus\{0\}} \bigl(f(x+y) - f(x) - yf'(x)\mathbf{1}_D(y)\bigr)\nu_Z(\mathrm{d}y),$$

where $D = \{x\colon\ |x| \le 1\}$. To distinguish the characteristics of ζ from that of Z, we add the subscript ζ to the drift and Lévy measure of the subordinator ζ. Then Theorem 3.1 gives the infinitesimal generator of X for $t > 0$ as

$$\begin{aligned} A_t f(x) &= \beta_\zeta \gamma f'(x) + (1-\beta_\zeta)\frac{x\kappa}{t} f'(x) \\ &\quad + \beta_\zeta \int_{\mathbb{R}\setminus\{0\}} \bigl(f(x+y) - f(x) - yf'(x)\mathbf{1}_D(y)\bigr)\nu_Z(\mathrm{d}y) \\ &\quad + \frac{1}{t}\int_{(0,\infty)} \int_{-\infty}^{\infty} \bigl(f(w) - f(x)\bigr) P\bigl(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-u\kappa}, \mathrm{d}w\bigr)\nu_\zeta(\mathrm{d}u). \end{aligned}$$

4.4. Martingale with marginals $t^\kappa V$ with V symmetric

Let V be an integrable, symmetric random variable (i.e. $V \stackrel{d}{=} -V$) and $(B_t)_{t\ge0}$ be a Brownian motion independent of V. Following and extending [10], page 283, for any κ let $T_t = \inf\{u \ge 0\colon\ |B_u| = t^\kappa\}$ and $Z_t = B_{T_t} V$. Then $(Z_t)_{t\ge0}$ is a Markov martingale such that for each $t \ge 0$, $Z_t \stackrel{d}{=} t^\kappa V$. Moreover, Z is κ-self-similar. Indeed, using the Brownian motion $B_t^{(c)} := cB_{c^{-2}t}$, we have

$$T_t^{(c)} = \inf\bigl\{u \ge 0\colon\ \bigl|B_u^{(c)}\bigr| = t^\kappa\bigr\} = c^2 \inf\bigl\{s \ge 0\colon\ |B_s| = c^{-1}t^\kappa\bigr\} = c^2 T_{c^{-1/\kappa}t}.$$

It follows that $B^{(c)}_{T_t^{(c)}} = cB_{c^{-2}c^2 T_{c^{-1/\kappa}t}} = cB_{T_{c^{-1/\kappa}t}}$ and $(B_{T_{ct}})_{t\ge0} \stackrel{d}{=} (c^\kappa B_{T_t})_{t\ge0}$. Hence,

$$(Z_{ct})_{t\ge0} = (B_{T_{ct}} V)_{t\ge0} \stackrel{d}{=} \bigl(c^\kappa B_{T_t} V\bigr)_{t\ge0} = \bigl(c^\kappa Z_t\bigr)_{t\ge0}.$$

Since Z is a Markov process with transition semigroup

$$\begin{aligned} P_{0,t} f(x) &= \int f\bigl(t^\kappa v\bigr)\,\mathrm{d}F(v), \qquad t > 0, \\ P_{s,t} f(x) &= \frac{1}{2}\bigl(1 + (s/t)^\kappa\bigr) f\bigl((t/s)^\kappa x\bigr) + \frac{1}{2}\bigl(1 - (s/t)^\kappa\bigr) f\bigl(-(t/s)^\kappa x\bigr), \qquad 0 < s \le t, \end{aligned}$$

where F is the cumulative distribution function of V, it has infinitesimal generator

$$\begin{aligned} L_0 f(0) &= 0, \\ L_t f(x) &= \frac{\kappa}{t} x f'(x) - \frac{\kappa}{2t} f(x) + \frac{\kappa}{2t} f(-x), \qquad t > 0. \end{aligned}$$

Using Theorem 3.1, we obtain the infinitesimal generator of the mimic X, for $t > 0$,

$$A_t f(x) = \frac{\kappa}{t}\left(xf'(x) + \frac{1}{2}\beta f(-x) - \frac{1}{2}\beta f(x)\right) + \frac{1}{t}\int_{(0,\infty)}\int_{-\infty}^{\infty}\big(f(y) - f(x)\big)P\big(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-\kappa u}, \mathrm{d}y\big)\nu(\mathrm{d}u).$$

Proposition 4.5. *The predictable quadratic variation of X is*

$$\langle X, X\rangle_t = 2\kappa\int_0^t \frac{1}{s}X_s^2\,\mathrm{d}s, \qquad t \geq 0.$$

Proof. Let $f(x) = x^2$. Since $\int_{-\infty}^{\infty} f(y)P(t\mathrm{e}^{-u}, t, x\mathrm{e}^{-\kappa u}, \mathrm{d}y) = P_{t\mathrm{e}^{-u},t} f(x\mathrm{e}^{-\kappa u}) = x^2$ and $A_t f(x) = \frac{2\kappa}{t}x^2$, the result follows immediately from Equation (3.1). □

Note that $\langle Z, Z\rangle_t = 2\kappa\int_0^t \frac{1}{s}Z_s^2\,\mathrm{d}s$. The predictable quadratic variations of X and Z are given by the same functional of the process.

4.5. Extension to mimicking Brownian martingales

Now we discuss how we can (and cannot) alter our martingale condition to mimic some Brownian related processes, including the martingales associated with the Hermite polynomials and the exponential martingale of Brownian motion.

Consider the Hermite polynomials h_n which are defined by

$$\sum_{n\geq 0}\frac{u^n}{n!}h_n(x) = \exp\big(ux - u^2/2\big) \qquad \forall u, x \in \mathbb{R},$$

equivalently, $h_n(x) = (-1)^n\mathrm{e}^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2/2})$. Let

$$H_n(x,t) = t^{n/2}h_n(x/\sqrt{t}) \qquad \forall x \in \mathbb{R}, t > 0$$

and $H_n(x, 0) = x^n$. Then $H_n(B_t, t)$, where B denotes a Brownian motion, is a local martingale for every n since $H_n(x,t)$ is space–time harmonic, that is, $\frac{\partial H_n}{\partial t} + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2} = 0$.

Take $n = 2$, the process $H_2(B_t, t) = B_t^2 - t$ is Markovian and 1-self-similar, thus can be mimicked using our mimicking scheme with any subordinator that satisfies $\psi(1) = 1$.

For $n \geq 3$, $H_n(B_t, t)$ is $\frac{n}{2}$-self-similar, but it is not Markovian (see [6]). So we are not able to mimic this process by a direct application of the method described above. However, a slight modification of our construction proves sufficient to achieve our aim.

Let X_t be a mimic of the Brownian motion as in Section 4.1 with $k = 0$, but without the requirement that X_t be a martingale. Then we have the following result.

Proposition 4.6. *For each n, the process $H_n(X_t, t)$ has the same marginal distributions as $H_n(B_t, t)$ and is a martingale if and only if $\psi(n/2) = n/2$, or $\mathbb{E}[\mathrm{e}^{-(n/2)\zeta_{\ln(t/s)}}] = (s/t)^{n/2}$.*

Proof. It is obvious that $H_n(X_t, t)$ and $H_n(B_t, t)$ have the same marginals. Here we prove the martingale condition. The transition function of X is

$$\widetilde{P}(s, t, x, \mathrm{d}y) = \int P\big(rt, t, (t/s)^{1/2} r^{1/2} x, \mathrm{d}y\big) G_{s,t}(\mathrm{d}r), \qquad s \le t,$$

where P is the transition function of B and $G_{s,t}$ is the distribution of $\mathrm{e}^{-\zeta_{\ln(t/s)}}$. Writing $\mathcal{F}_t$ as the natural filtrations of $H_n(X_t, t)$ and since $H_n(B_t, t)$ is a martingale, we have

$$\begin{aligned}\mathbb{E}\big[H_n(X_t, t)|\mathcal{F}_s\big] &= \mathbb{E}\Big[\int H_n(y, t)\widetilde{P}(s, t, X_s, \mathrm{d}y)\big|\mathcal{F}_s\Big]\\ &= \mathbb{E}\Big[\int H_n\big((t/s)^{1/2} r^{1/2} X_s, rt\big) G_{s,t}(\mathrm{d}r)\big|\mathcal{F}_s\Big].\end{aligned}$$

Using that $H_n(ax, t) = a^n H_n(x, \frac{t}{a^2})$ for any $a > 0$, we then obtain

$$\mathbb{E}\big[H_n(X_t, t)|\mathcal{F}_s\big] = (t/s)^{n/2} H_n(X_s, s)\mathbb{E}\big[\mathrm{e}^{-(n/2)\zeta_{\ln(t/s)}}\big].$$

Hence, $H_n(X_t, t)$ is a martingale if and only if $\mathbb{E}[\mathrm{e}^{-(n/2)\zeta_{\ln(t/s)}}] = (s/t)^{n/2}$. $\square$

Therefore, in order to mimic the process $H_n(B_t, t)$, we can mimic B_t, with the martingale requirement $\psi(\frac{1}{2}) = \frac{1}{2}$ changed to $\psi(\frac{n}{2}) = \frac{n}{2}$, and then apply the function $H_n(x, t)$ to the resulting process. It is of interest to ask whether the above trick extends to other space–time harmonic functions. In particular, could this enable us to mimic the geometric Brownian motion $\exp(B_t - t/2)$. Unfortunately, this is not the case. In fact, $H_n(x, t)$, $n \ge 1$, are the only analytic functions for which this trick works.

Proposition 4.7. *Suppose that $H(x, t) = \sum_{m,n=0}^{\infty} a_{m,n} x^m t^n$ and there exists $r > 1$ such that $\sum_{m,n} |a_{m,n}| r^{m+n} < \infty$, in other words, $H(x, t)$ is analytic on the set $(-r, r)^2$. Suppose further that H is space–time harmonic, so that $H(B_t, t)$ is a martingale. Suppose that X_t mimics B_t with the martingale requirement replaced with $\psi(k/2) = k/2$ for a positive integer k. Then $H(X_t, t)$ has the same marginals as $H(B_t, t)$ and is a martingale if and only if $H(x, t) = t^{k/2} h_k(x/\sqrt{t})$ where $h_k(x)$ is the kth Hermite polynomial.*

Proof. For $H(X_t, t)$ to be a martingale, we must have for any x and $s \le t$,

$$H(x, s) = \int H(y, t)\widetilde{P}(s, t, x, \mathrm{d}y) = \int H\big((t/s)^{1/2} r^{1/2} x, rt\big) G_{s,t}(\mathrm{d}r),$$

that is, $\mathbb{E}[H(xu^{1/2}R_u^{1/2}, suR_u)] = H(x,s)$ for any x, s and $u \geq 1$. Letting $u = \mathrm{e}^t$ and $V_t = \mathrm{e}^{t-\zeta_t}$, this is equivalent to $\mathbb{E}[H(xV_t^{1/2}, sV_t)] = H(x,s)$ for all x and s, or

$$\sum_{m,n=0}^{\infty} a_{m,n} x^m s^n \mathbb{E}\bigl[V_t^{n+m/2}\bigr] = \sum_{m,n=0}^{\infty} a_{m,n} x^m s^n.$$

Therefore, for all m, n and $t < \frac{1}{2}\ln r$, we must have $a_{m,n}\mathbb{E}[V_t^{n+m/2}] = a_{m,n}$. Thus, either $a_{m,n} = 0$ or $\mathbb{E}[V_t^{n+m/2}] = 1$.

Recall that $\mathbb{E}[V_t^{\lambda}] = \mathbb{E}[\exp(\lambda t - \lambda\zeta_t)] = \exp(-t(\psi(\lambda) - \lambda))$ and there is at most one λ satisfying $\psi(\lambda) = \lambda$. Now, choose ζ_t such that $\mathbb{E}[V_t^{k/2}] = 1$ for a $k \in \mathbb{N}^*$. Then, for all (m,n) such that $m + 2n \neq k$, $a_{m,n} = 0$. Therefore,

$$H(x,s) = \sum_{n=0}^{\lfloor k/2 \rfloor} a_{k-2n,n} x^{k-2n} s^n.$$

Furthermore,

$$H(cx,s) = c^k \sum_{n=0}^{\lfloor k/2 \rfloor} a_{k-2n,n} x^{k-2n} \bigl(s/c^2\bigr)^n = c^k H\bigl(x, s/c^2\bigr).$$

By Plucińska [16] and Fitzsimmons [7], $H(x,1)$ is the kth Hermite polynomial. □

Corollary 4.1. *Let X_t be any mimic of B_t in the sense of Section 2 but without the martingale requirement. Although the process $\exp(X_t - t/2)$ has the same marginal distributions as $\exp(B_t - t/2)$, it is not a martingale unless $\zeta_t = t$, in which case $X = B$.*

Appendix

Lemma A.1. *Let $(Y_t)_{t\geq 0}$ be a Markov process with infinitesimal generator A_t and c_t be a deterministic, differentiable, increasing function in t with derivative c_t'. Then the time-changed process $(\widetilde{Y}_t)_{t\geq 0} := (Y_{c_t})_{t\geq 0}$ is also a Markov process with infinitesimal generator $\widetilde{A}_t = c_t' A_{c_t}$. Furthermore, if f is in the domain of A, then f is in the domain of $\widetilde{A}$.*

Proof. Let $\mathcal{F}$ be the filtration of Y and $\widetilde{\mathcal{F}}$ be the filtration of $\widetilde{Y}$. For any bounded measurable function g, we have, for $s \leq t$,

$$\mathbb{E}\bigl[g(\widetilde{Y}_t)|\widetilde{\mathcal{F}}_s\bigr] = \mathbb{E}\bigl[g(Y_{c_t})|\widetilde{\mathcal{F}}_s\bigr] = \mathbb{E}\bigl[g(Y_{c_t})|\mathcal{F}_{c_s}\bigr] = \mathbb{E}\bigl[g(Y_{c_t})|Y_{c_s}\bigr] = \mathbb{E}\bigl[g(\widetilde{Y}_t)|\widetilde{Y}_s\bigr].$$

For t where the function c is strictly increasing, the infinitesimal generator of Y_{c_t} is

$$\widetilde{A}_t f(x) = \lim_{u\downarrow t} \frac{\mathbb{E}[f(Y_{c_u})|Y_{c_t} = x] - f(x)}{c_u - c_t} \frac{c_u - c_t}{u - t} = A_{c_t} f(x) c_t'.$$

If $c_u = c_t$ in a small neighbourhood of t, then $c'_t = 0$ and

$$\widetilde{A}_t f(x) = \lim_{u \downarrow t} \frac{\mathbb{E}[f(Y_{c_t})|Y_{c_t} = x] - f(x)}{u - t} = 0 = A_{c_t} f(x) c'_t. \qquad \square$$

Lemma A.2. *Let $(Y_t)_{t\geq 0}$ be a Markov process with infinitesimal generator A_t and c_t be a deterministic, differentiable function in t with derivative c'_t and $c_t \neq 0$ for any t. Then the process $(\widetilde{Y}_t)_{t\geq 0} := (c_t Y_t)_{t\geq 0}$ is also a Markov process and has generator*

$$\widetilde{A}_t f(x) = \pi_{1/c_t} A_t \pi_{c_t} f(x) + \frac{c'_t}{c_t} x f'(x),$$

where π_c is an operator defined by $\pi_c f(x) = f(cx)$. Furthermore, if $\pi_c f$ is in the domain of A for any c and f is differentiable, then f is in the domain of $\widetilde{A}$.

Proof. Let $\mathcal{F}$ be the filtration of Y and $\widetilde{\mathcal{F}}$ be the filtration of $\widetilde{Y}$. Let h_t be a function such that $h_t(x) = c_t x$. Since h_t is one-to-one, $\sigma(h_u(Y_u), u \leq s) = \sigma(Y_u, u \leq s)$. Therefore, for any bounded measurable function g, we have

$$\mathbb{E}\big[g(\widetilde{Y}_t)|\widetilde{\mathcal{F}}_s\big] = \mathbb{E}\big[g \circ h_t(Y_t)|\widetilde{\mathcal{F}}_s\big] = \mathbb{E}\big[g \circ h_t(Y_t)|\mathcal{F}_s\big] = \mathbb{E}\big[g \circ h_t(Y_t)|Y_s\big] = \mathbb{E}\big[g(\widetilde{Y}_t)|\widetilde{Y}_s\big].$$

The infinitesimal generator of $\widetilde{Y}$ is

$$\begin{aligned}\widetilde{A}_t f(x) &= \lim_{u \downarrow t} \left(\mathbb{E}\left[\pi_{c_u} f(Y_u) \Big| Y_t = \frac{1}{c_t} x\right] - \pi_{c_u} f\left(\frac{1}{c_t} x\right) + f\left(\frac{c_u}{c_t} x\right) - f(x) \right) \Big/ (u - t) \\ &= A_t \pi_{c_t} f\left(\frac{1}{c_t} x\right) + \frac{c'_t}{c_t} x f'(x). \qquad \square\end{aligned}$$

Lemma A.3. *Suppose $(Z_t)_{t\geq 0}$ is a κ-self-similar Markov process. Suppose $P(s,t,x,\mathrm{d}y)$ and L_t are, respectively, the transition function and infinitesimal generator of Z. Let $\widehat{Z}_t = \mathrm{e}^{-t\kappa} Z_{\mathrm{e}^t}$. Then $(\widehat{Z}_t)_{t\in\mathbb{R}}$ is a time-homogeneous Markov process with transition semigroup*

$$\widehat{P}_t f(x) = \int f(y) P\big(\mathrm{e}^{-t}, 1, x\mathrm{e}^{-t\kappa}, \mathrm{d}y\big)$$

and infinitesimal generator

$$\widehat{L} f(x) = L_1 f(x) - \kappa x f'(x).$$

Furthermore, if f is in the domain of L and differentiable, then it is in the domain of $\widehat{L}$.

Proof. By the scaling property of P, we have

$$\mathbb{P}(\widehat{Z}_t \in \mathrm{d}y|\widehat{Z}_s = x) = P\big(\mathrm{e}^s, \mathrm{e}^t, x\mathrm{e}^{s\kappa}, \mathrm{e}^{t\kappa}\,\mathrm{d}y\big) = P\big(\mathrm{e}^{-(t-s)}, 1, x\mathrm{e}^{-(t-s)\kappa}, \mathrm{d}y\big).$$

It follows that $\widehat{Z}$ is time-homogeneous and $\widehat{P}_t f(x) = \int f(y) P(\mathrm{e}^{-t}, 1, x\mathrm{e}^{-t\kappa}, \mathrm{d}y)$. The generator of $\widehat{Z}$ can be obtained by applying Lemma A.1 and Lemma A.2, and seeing that $\pi_{\mathrm{e}^{t\kappa}} \mathrm{e}^t L_{\mathrm{e}^t} \pi_{\mathrm{e}^{-t\kappa}} = L_1$ from Equation (1.1) with $c = \mathrm{e}^t$ and $s = 1$.

Note that for all $c > 0$, $\pi_c f$ is in the domain of L by the scaling property of Z. Thus, writing $\check{L}$ as the generator of $\check{Z}_t := Z_{e^t}$, $\pi_c f$ is in the domain of $\check{L}$ by Lemma A.1. □

Lemma A.4. *Suppose $(\chi_t)_{t\geq 0}$ is a time-homogeneous Markov process with semigroup P_t and generator L, and $(\eta_t)_{t\geq 0}$ is a subordinator independent of χ with drift β and Lévy measure ν. Set $Y_t = \chi_{\eta_t}$. Then the process $(Y_t)_{t\geq 0}$ is a time-homogeneous Markov process with generator A where*

$$Af(x) = \beta Lf(x) + \int_{(0,\infty)} \bigl(P_u f(x) - f(x)\bigr)\nu(\mathrm{d}u).$$

Furthermore, if f is in the domain of L, then it is in the domain of A.

If η has zero drift, then Y is a pure jump process.

Proof. See Sato [17], Theorem 32.1. □

Acknowledgements

The authors are grateful to two referees and an associate editor for their careful reading of an earlier version of the paper and a number of suggestions and improvements.

This research was supported by the Australian Research Council Grant DP0988483 and the first author was the recipient of a Victorian International Research Scholarship.

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Received September 2012 and revised August 2013