

A complete Riemann zeta distribution and the Riemann hypothesis

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Let $\sigma, t \in \mathbb{R}$, $s = \sigma + it$, $\Gamma(s)$ be the Gamma function, $\zeta(s)$ be the Riemann zeta function and $\xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the complete Riemann zeta function. We show that $\Xi_\sigma(t) := \xi(\sigma - it)/\xi(\sigma)$ is a characteristic function for any $\sigma \in \mathbb{R}$ by giving the probability density function. Next we prove that the Riemann hypothesis is true if and only if each $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function, which is defined in this paper, for each $1/2 < \sigma < 1$. Moreover, we show that $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function when $\sigma = 1$. Finally we prove that the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible for any $\sigma > 1$.

Keywords: characteristic function; Lévy–Khintchine representation; Riemann hypothesis; zeta distribution

1. Introduction and main results

1.1. Riemann zeta function and distribution

The famous Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$, for $\sigma > 1$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the letter p is a prime number, and the product of \prod_p is taken over all primes. The Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ and the Euler product $\prod_p (1 - p^{-s})^{-1}$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. The Riemann zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. Denote the Gamma function by $\Gamma(s)$. We have the following functional equation of the complete Riemann zeta function $\xi(s)$ (see, for example, Titchmarsh [15], (2.1.13))

$$\xi(s) = \xi(1-s), \quad \xi(s) := s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \quad (1.1)$$

In view of the Euler product, it is seen easily that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. It follows from the functional equation (1.1) and basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2m$, $m \in \mathbb{N}$. In 1859, Riemann

stated that it seems likely that all nontrivial zeros lie on the so-called critical line $\sigma = 1/2$. This is the famous, yet unproved Riemann hypothesis. In 1896, Hadamard and de la Vallée-Poussin independently proved that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}$ (see Titchmarsh [15], page 45). Hence, we can also see that no zeros of $\zeta(s)$ lie on the line $\Re(s) = 0$ by (1.1). Therefore, the Riemann hypothesis is rewritten equivalently as

$$\text{Riemann hypothesis } \zeta(s) \neq 0 \quad \text{for } 1/2 < \sigma < 1.$$

Put $Z_\sigma(t) := \zeta(\sigma - it)/\zeta(\sigma)$, $t \in \mathbb{R}$, then $Z_\sigma(t)$ is known to be a characteristic function when $\sigma > 1$ (see Khintchine [5] or Gnedenko and Kolmogorov [3], page 75). A distribution μ_σ on \mathbb{R} is said to be a Riemann zeta distribution with parameter σ if it has $Z_\sigma(t)$ as its characteristic function. Recently, the Riemann zeta distribution is investigated by Lin and Hu [7], and Gut [4]. On the other hand, in Aoyama and Nakamura [1], Remark 1.13, it is showed that $Z_\sigma(t)$ is not a characteristic function for any $1/2 \leq \sigma \leq 1$. Afterwards, Nakamura [9] showed that $F_\sigma(t)$, where $F_\sigma(t) := f_\sigma(t)/f_\sigma(0)$ and $f_\sigma(t) := \zeta(\sigma - it)/(\sigma - it)$, is a characteristic function for any $0 < \sigma \neq 1$.

Note that there are some other papers connected to Riemann zeta function in probabilistic view. Biane Pitman and Yor [2] reviewed known results about $\xi(s)$ which are related to one-dimensional Brownian motion and to higher dimensional Bessel processes. Lagarias and Rains [6] treated $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and its generalizations and gave results connected to infinite divisibility.

1.2. Infinitely divisible and quasi-infinitely divisible distributions

A probability measure μ on \mathbb{R} is infinitely divisible if, for any positive integer n , there is a probability measure μ_n on \mathbb{R} such that $\mu = \mu_n^{n*}$, where μ_n^{n*} is the n -fold convolution of μ_n . For instance, normal, degenerate, Poisson and compound Poisson distributions are infinitely divisible.

Let $\widehat{\mu}(t)$ be the characteristic function of a probability measure μ on \mathbb{R} and $ID(\mathbb{R})$ be the class of all infinitely divisible distributions on \mathbb{R} . The following Lévy–Khintchine representation is well known (see Sato [14], Section 2). Put $D_b := \{x \in \mathbb{R} : -b \leq x \leq b\}$, where $b > 0$. If $\mu \in ID(\mathbb{R})$, then one has

$$\widehat{\mu}(t) = \exp\left[-\frac{a}{2}t^2 + i\lambda t + \int_{\mathbb{R}}(e^{itx} - 1 - itx1_{D_b}(x))\nu(dx)\right], \quad t \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$, $\lambda \in \mathbb{R}$ and ν is a measure on \mathbb{R} satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty$. Moreover, the representation of $\widehat{\mu}$ in (1.2) by a, ν , and λ is unique. If the Lévy measure ν in (1.2) satisfies $\int_{|x|<1} |x|\nu(dx) < \infty$, then (1.2) can be written by

$$\widehat{\mu}(t) = \exp\left[-\frac{a}{2}t^2 + i\lambda_0 t + \int_{\mathbb{R}}(e^{itx} - 1)\nu(dx)\right], \quad \lambda_0 \in \mathbb{R}. \tag{1.3}$$

For example, the Lévy measure of $Z_\sigma(t) := \zeta(\sigma - it)/\zeta(\sigma)$ can be given as in the following (see Gnedenko and Kolmogorov [3], page 75). Let δ_x be the delta measure at x . Then we have

$$\log Z_\sigma(t) = \int_0^\infty (e^{itx} - 1)N_\sigma(dx), \quad N_\sigma(dx) := \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx). \tag{1.4}$$

On the other hand, there are non-infinitely divisible distributions whose characteristic functions are the quotients of two infinitely divisible characteristic functions. That class is called class of *quasi-infinitely divisible distributions* and is defined as follows.

Quasi-infinitely divisible distribution. A distribution μ on \mathbb{R} is called *quasi-infinitely divisible* if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure ν is a signed measure on \mathbb{R} with total variation measure $|\nu|$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (|x|^2 \wedge 1)|\nu|(dx) < \infty$.

We have to mention that the triplet (a, ν, λ) in this case is also unique if each component exists and that infinitely divisible distributions on \mathbb{R} are quasi-infinitely divisible if and only if $a \geq 0$ and the negative part of ν in the Jordan decomposition equals zero. The measure ν is called *quasi-Lévy measure* and has appeared in some books and papers, for example, Gnedenko and Kolmogorov [3], page 81, Lindner and Sato [8], Niedbalska-Rajba [10], and others (see also Sato [13], Section 2.4).

1.3. Main results

In the present paper, we give a complete Riemann zeta distribution by the normalized complete Riemann zeta function

$$\Xi_\sigma(t) := \frac{\xi(\sigma - it)}{\xi(\sigma)},$$

$$\xi(\sigma - it) := (\sigma - it)(\sigma - 1 - it)\pi^{(it-\sigma)/2}\Gamma\left(\frac{\sigma - it}{2}\right)\zeta(\sigma - it),$$

for any $\sigma \in \mathbb{R}$. It should be mentioned that $\Xi_\sigma(t)$ is symmetric about the vertical axis $\sigma = 1/2$ by the functional equation (1.1). Therefore, we only have to consider the case $\sigma \geq 1/2$. In order to state the main results, we introduce the following pretended-infinitely divisible distribution.

Pretended-infinitely divisible distribution. A distribution μ on \mathbb{R} is called *pretended-infinitely divisible* if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure ν is a signed measure on \mathbb{R} with $\nu(\{0\}) = 0$.

Namely, pretended-infinitely divisible distributions are infinitely divisible or quasi-infinitely divisible distributions without the condition $\int_{\mathbb{R}} (|x|^2 \wedge 1)|\nu|(dx) < \infty$.

The main results in this paper are following four theorems.

Theorem 1.1. *The function $\Xi_\sigma(t)$ is a characteristic function for any $\sigma \in \mathbb{R}$. Moreover, the probability density function $P_\sigma(y)$ is given as follows:*

$$P_\sigma(y) := \begin{cases} \frac{2}{\xi(\sigma)} \sum_{n=1}^{\infty} f(ne^{-y})e^{-\sigma y}, & y \leq 0, \\ \frac{2}{\xi(\sigma)} \sum_{n=1}^{\infty} f(ne^y)e^{(1-\sigma)y}, & y > 0, \end{cases} \tag{1.5}$$

where $f(x) := 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2}$.

Let \mathcal{Z} and \mathcal{Z}_+ be the set of zeros of the Riemann zeta function which lie in the critical strip $\{s \in \mathbb{C}: 0 < \Re(s) < 1\}$, and the region $\{s \in \mathbb{C}: 0 < \Re(s) < 1, \Im(s) > 0\}$, respectively. If the Riemann hypothesis is true, then each $\rho \in \mathcal{Z}_+$ can be expressed by $\rho = 1/2 + i\gamma$, where $\gamma > 0$.

Theorem 1.2. *The characteristic function $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function for any $1/2 < \sigma < 1$ if and only if the Riemann hypothesis is true. Furthermore, we have*

$$\begin{aligned} \Xi_\sigma(t) &= \exp\left[\int_0^\infty (e^{itx} - 1)v_\sigma(dx)\right], \\ v_\sigma(dx) &:= - \sum_{1/2+i\gamma \in \mathcal{Z}_+} \frac{2 \cos(\gamma x)}{x e^{(\sigma-1/2)x}}(dx), \end{aligned} \tag{1.6}$$

under the Riemann hypothesis.

Let \mathcal{Z}_+^R be the set of zeros of $\zeta(s)$ which lie on the half line $\{s \in \mathbb{C}: \Re(s) = 1/2, \Im(s) > 0\}$ and \mathcal{Z}_+^N be the set of zeros of $\zeta(s)$ which lie in the region $\{s \in \mathbb{C}: 1/2 < \Re(s) < 1, \Im(s) > 0\}$. Note that $\mathcal{Z}_+^N = \emptyset$ if and only if the Riemann hypothesis is true. One has $\mathcal{Z} = \{\rho, 1 - \rho: \rho \in \mathcal{Z}_+^R\} \cup \{\rho, 1 - \rho, \bar{\rho}, 1 - \bar{\rho}: \rho \in \mathcal{Z}_+^N\}$ from $\xi(s) = \xi(1 - s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$.

Theorem 1.3. *When $\sigma \geq 1$, we have*

$$\begin{aligned} \Xi_\sigma(t) &= \exp\left[\int_0^\infty (e^{itx} - 1)v_\sigma(dx)\right], \\ v_\sigma(dx) &:= - \sum_{1/2+i\gamma \in \mathcal{Z}_+^R} \frac{2 \cos(\gamma x)}{x e^{(\sigma-1/2)x}}(dx) - \sum_{\beta+i\gamma \in \mathcal{Z}_+^N} \left(\frac{2 \cos(\gamma x)}{x e^{(\sigma-\beta)x}} + \frac{2 \cos(\gamma x)}{x e^{(\sigma-1+\beta)x}}\right)(dx). \end{aligned} \tag{1.7}$$

Especially, $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function when $\sigma = 1$.

Theorem 1.4. *When $\sigma > 1$, we have*

$$\Xi_\sigma(t) = \exp\left[it\lambda_\sigma + \int_0^\infty (e^{itx} - 1 - itx 1_{D_{1/2}}(x))v_\sigma(dx)\right],$$

$$\begin{aligned} \lambda_\sigma &:= \frac{e^{-\sigma/2} - 1}{\sigma} + \frac{e^{(1-\sigma)/2} - 1}{\sigma - 1} + \frac{\log \pi}{2} \\ &\quad + \frac{1}{2} \int_0^1 \left(\frac{e^{-\sigma x/2}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - \frac{1}{2} \int_1^\infty e^{-x} \frac{dx}{x}, \\ \nu_\sigma(dx) &:= \frac{1(dx)}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1 + e^x}{xe^{\sigma x}}(dx) + \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx). \end{aligned}$$

Therefore, the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible when $\sigma > 1$.

We call the distribution defined by the characteristic function $\Xi_\sigma(t)$ the *completed Riemann zeta distribution*. It is well known that $\zeta(s)$ has zeros on $\Re(s) = 1/2$ (see Titchmarsh [15], Section 10). By the definition of pretended-infinitely divisible distribution and the fact that $\exp(z) \neq 0$ for any $z \in \mathbb{C}$, the characteristic function does not have zeros. Thus, $\Xi_\sigma(t)$ is not even a pretended-infinitely divisible characteristic function when $\sigma = 1/2$.

2. Proofs

2.1. Proof of Theorem 1.1

We quote the following fact from Patterson [12] (see also Biane Pitman and Yor [2], Section 2).

Lemma 2.1 (see Patterson [12], Section 2.10). *Let $f(x) := 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2}$. Then we have*

$$\xi(s) = 2 \int_1^\infty \sum_{n=1}^\infty f(nx) (x^{s-1/2} + x^{1/2-s}) x^{-1/2} dx. \tag{2.1}$$

Note that the last integral is absolutely convergent for all values of s .

Proof of Theorem 1.1. By (2.1) and the change of variables $x = e^{-y}$ and $x = e^y$, we have

$$\begin{aligned} \xi(\sigma - it) &= 2 \int_1^\infty \sum_{n=1}^\infty f(nx) x^{\sigma-it-1} dx + 2 \int_1^\infty \sum_{n=1}^\infty f(nx) x^{it-\sigma} dx \\ &= 2 \int_0^{-\infty} \sum_{n=1}^\infty f(ne^{-y}) e^{(1+it-\sigma)y} (-e^{-y}) dy + 2 \int_0^\infty \sum_{n=1}^\infty f(ne^y) e^{(it-\sigma)y} (e^y) dy \\ &= 2 \int_{-\infty}^0 e^{ity} \sum_{n=1}^\infty f(ne^{-y}) e^{-\sigma y} dy + 2 \int_0^\infty e^{ity} \sum_{n=1}^\infty f(ne^y) e^{(1-\sigma)y} dy. \end{aligned}$$

Obviously, we have $f(x) = 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2} > 0$ for any $x \geq 1$. Hence, one has $f(ne^{-y}) > 0$ for any $y \leq 0$ and $n \in \mathbb{N}$, and $f(ne^y) > 0$ for any $y > 0$ and $n \in \mathbb{N}$. Thus it holds that

$$\sum_{n=1}^{\infty} f(ne^{-y})e^{-\sigma y} > 0, \quad y \leq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} f(ne^y)e^{(1-\sigma)y} > 0, \quad y > 0.$$

On the other hand, we have

$$\xi(\sigma) = 2 \int_{-\infty}^0 \sum_{n=1}^{\infty} f(ne^{-y})e^{-\sigma y} dy + 2 \int_0^{\infty} \sum_{n=1}^{\infty} f(ne^y)e^{(1-\sigma)y} dy > 0$$

from (2.1) and the argument above. Hence, $P_\sigma(y)$ defined by (1.5) is nonnegative. Therefore, we have $\Xi_\sigma(t) = \int_{\mathbb{R}} e^{ity} P_\sigma(y) dy$, where $P_\sigma(y)$ is the probability density function. \square

Remark 2.2. It should be emphasised that $\Xi_\sigma(t)$ is a characteristic function for any $\sigma \in \mathbb{R}$. On the other hand, $F_\sigma(t) := f_\sigma(t)/f_\sigma(0)$, where $f_\sigma(t) := \zeta(\sigma - it)/(\sigma - it)$, is not a characteristic function for $\sigma = 0, 1$ and $\sigma < -1/2$. This is proved as follows. When $\sigma = 1$, it is well known that $\zeta(1 + it) \neq 0, t \neq 0$, and $\zeta(s)$ has an only one pole at $s = 1$. Hence, we have

$$F_1(t) = \frac{1}{\zeta(1)} \frac{\zeta(1 + it)}{1 + it} = 0 \quad \text{for any } t \neq 0,$$

which contradicts the uniform continuity of characteristic function $\widehat{\mu}(t)$ and $\widehat{\mu}(0) = 1$. A similar argument can be done when $\sigma = 0$ since $\zeta(s)/s$ has a simple pole at $s = 0$. By (1.1) and Stirling’s formula, one has

$$|\zeta(s)| = \pi^{\sigma-1/2} (|t/2| + 2)^{-\sigma+1/2} (1 + O((|t| + 2)^{-1})) |\zeta(1 - s)|$$

for $\sigma < 0$. On the other hand, for any $\varepsilon > 0$ there are arbitrarily large t which satisfy $|\zeta(\sigma + it)| > (1 - \varepsilon)\zeta(\sigma)$ when $\sigma > 1$ (see Titchmarsh [15], Theorem 8.4). Thus, we can find t which satisfies $|\zeta(s)| > \pi^{\sigma-1/2} |t/2|^{-\sigma+1/2} \zeta(1 - \sigma)/2$. Hence, there exists $t \in \mathbb{R}$ such that $|F_\sigma(t)| > 1$ when $\sigma < -1/2$ by the factor $|t/2|^{-\sigma+1/2}$.

The absolute value of a characteristic function is not greater than 1 (see for instance Sato [14], Proposition 2.5). Hence, we have the following inequality by Theorem 1.1.

Corollary 2.3 (see Patterson [12], Section 2.11). For any $t \in \mathbb{R}$ and $1/2 \leq \sigma$, we have

$$\left| (\sigma + it)(\sigma - 1 + it)\pi^{-(\sigma+it)/2} \Gamma\left(\frac{\sigma + it}{2}\right) \zeta(\sigma + it) \right| \leq \sigma(\sigma - 1)\pi^{-\sigma/2} \Gamma\left(\frac{\sigma}{2}\right) \zeta(\sigma).$$

2.2. Proof of Theorem 1.2

Recall that \mathcal{Z} is the set of zeros of the Riemann zeta function which lie in the critical strip $\{s \in \mathbb{C}: 0 < \Re(s) < 1\}$ (see Section 1.3). Observe that by the functional equation and $\overline{\zeta(s)} = \zeta(\overline{s})$

if $\rho \in \mathcal{Z}$ then $\bar{\rho}, 1 - \rho, 1 - \bar{\rho} \in \mathcal{Z}$. There are no real elements of \mathcal{Z} since $\xi(\sigma) < 0$ and $0 < \Gamma(\sigma/2)$ when $0 < \sigma < 1$ (see Section 1.1 and the proof of Theorem 1.1). Now we quote the following fact from Patterson [12].

Lemma 2.4 (see Patterson [12], page 34). *Let $\mathcal{Z}_+ := \{\rho \in \mathcal{Z} : \Im(\rho) > 0\}$. Then $\sum_{\rho \in \mathcal{Z}_+} |\rho|^{-a}$ converges for all $a > 1$ and it holds that*

$$\xi(s) = s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \prod_{\rho \in \mathcal{Z}_+} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1 - \rho}\right) \tag{2.2}$$

the product being absolutely convergent for all $s \in \mathbb{C}$.

Proof of Theorem 1.2. If $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function for any $1/2 < \sigma < 1$, then $\zeta(s) \neq 0$ for any $1/2 < \sigma < 1$ by $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, $\Gamma(s) \neq 0$ for any $1/2 < \sigma < 1$ and the representation (1.2).

Next suppose that the Riemann hypothesis is true. Then we have $\rho = 1/2 + i\gamma$ and $1 - \rho = 1/2 - i\gamma$, where $\gamma > 0$ for $\rho \in \mathcal{Z}_+$. Note that the exponential distribution with parameter $a > 0$ is defined by $\mu(B) := a \int_{B \cap (0, \infty)} e^{-ax} dx$, where $B \in \mathfrak{B}(\mathbb{R})$. The characteristic function is given by $\widehat{\mu}(t) = a/(a - it)$ (see, for example, Sato [14], page 13). Moreover, it is well known that

$$\frac{a}{a - iz} = \exp\left[\int_0^\infty (e^{izx} - 1)x^{-1}e^{-ax} dx\right], \quad a > 0, z \in \mathbb{R} \tag{2.3}$$

(see, for instance, Sato [14], page 45). The formula above holds if a is replaced by α with $\Re(\alpha) > 0$. This is proved as follows. Put $\alpha = a + ib$, $a > 0$ and $b \in \mathbb{R}$. Then one has

$$\begin{aligned} \frac{\alpha}{\alpha - iz} &= \frac{a + ib}{a} \frac{a}{a + ib - iz} \\ &= \exp\left[\int_0^\infty (e^{i(z-b)x} - 1)x^{-1}e^{-ax} dx - \int_0^\infty (e^{-ibx} - 1)x^{-1}e^{-ax} dx\right] \\ &= \exp\left[\int_0^\infty (e^{izx} - 1)x^{-1}e^{-ax} dx\right], \quad \Re(\alpha) > 0, \end{aligned} \tag{2.4}$$

by (2.3). Thus, it holds that

$$\begin{aligned} \left(1 - \frac{\sigma - it}{\rho}\right)\left(1 - \frac{\sigma}{\rho}\right)^{-1} &= \frac{1/2 - \sigma + i(\gamma + t)}{1/2 + i\gamma} \frac{1/2 + i\gamma}{1/2 - \sigma + i\gamma} = \frac{\sigma - 1/2 - i\gamma - it}{\sigma - 1/2 - i\gamma} \\ &= \exp\left[-\int_0^\infty (e^{itx} - 1)e^{(1/2 - \sigma + i\gamma)x} \frac{dx}{x}\right], \end{aligned}$$

where $\sigma > 1/2$. It should be noted that we have $\sigma - it \neq \rho, 1 - \rho$ when $\sigma > 1/2$ under the Riemann hypothesis. Therefore, one has

$$\begin{aligned} \varphi_\rho(t) &:= \left(1 - \frac{\sigma - it}{\rho}\right) \left(1 - \frac{\sigma}{\rho}\right)^{-1} \left(1 - \frac{\sigma - it}{1 - \rho}\right) \left(1 - \frac{\sigma}{1 - \rho}\right)^{-1} \\ &= \frac{\sigma - 1/2 - i\gamma - it}{\sigma - 1/2 - i\gamma} \frac{\sigma - 1/2 + i\gamma - it}{\sigma - 1/2 + i\gamma} \\ &= \exp\left[-2 \int_0^\infty (e^{itx} - 1) \frac{\cos(\gamma x)}{x e^{(\sigma - 1/2)x}} dx\right]. \end{aligned} \tag{2.5}$$

We remark that $x^{-1} \cos(\gamma x) e^{(1/2 - \sigma)x} (dx)$ is not a measure but a signed measure since one has $-1 \leq \cos(\gamma x) \leq 1$ when $\gamma \in \mathbb{R}$. By (2.2) and the definition of $\Xi_\sigma(t)$, we have

$$\begin{aligned} \Xi_\sigma(t) &= \prod_{\gamma \in \mathcal{Z}_+} \frac{\sigma - 1/2 - i\gamma + it}{\sigma - 1/2 - i\gamma} \frac{\sigma - 1/2 + i\gamma + it}{\sigma - 1/2 + i\gamma} \\ &= \exp\left[-2 \sum_{1/2 + i\gamma \in \mathcal{Z}_+} \int_0^\infty (e^{itx} - 1) \frac{\cos(\gamma x)}{x e^{(\sigma - 1/2)x}} dx\right]. \end{aligned}$$

This equality implies (1.6). □

Remark 2.5. It should be mentioned that $\varphi_{1/2+i\gamma}(t)$ defined by (2.5) is not a characteristic function for any $\sigma > 1/2$. It is proved by as follows. Obviously, one has

$$|\varphi_{1/2+i\gamma}(t)|^2 = \frac{(\sigma - 1/2)^2 + \gamma^2 - t^2 + (2\sigma - 1)it}{(\sigma - 1/2)^2 + \gamma^2}.$$

If we take $t^2 = 2((\sigma - 1/2)^2 + \gamma^2)$, then $|\varphi_{1/2+i\gamma}(t)|^2 > 1$.

2.3. Proof of Theorem 1.3

Recall that \mathcal{Z} , \mathcal{Z}_+^R and \mathcal{Z}_+^N is the set of zeros of $\zeta(s)$ which lie in $\{s \in \mathbb{C}: 0 < \Re(s) < 1\}$, $\{s \in \mathbb{C}: \Re(s) = 1/2, \Im(s) > 0\}$ and $\{s \in \mathbb{C}: 1/2 < \Re(s) < 1, \Im(s) > 0\}$, respectively. Then one has $\mathcal{Z} = \{\rho, 1 - \rho: \rho \in \mathcal{Z}_+^R\} \cup \{\rho, 1 - \rho, \bar{\rho}, 1 - \bar{\rho}: \rho \in \mathcal{Z}_+^N\}$. We have the following by Lemma 2.4.

Lemma 2.6. *The sums $\sum_{\rho \in \mathcal{Z}_+^R} |\rho|^{-a}$ and $\sum_{\rho \in \mathcal{Z}_+^N} |\rho|^{-a}$ converge for all $a > 1$ and it holds that*

$$\begin{aligned} \xi(s) &= \prod_{1/2+i\gamma \in \mathcal{Z}_+^R} \left(1 - \frac{s}{1/2+i\gamma}\right) \left(1 - \frac{s}{1/2-i\gamma}\right) \\ &\quad \times \prod_{\rho \in \mathcal{Z}_+^N} \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{1-\rho}\right) \left(1 - \frac{s}{\bar{\rho}}\right) \left(1 - \frac{s}{1-\bar{\rho}}\right), \end{aligned} \tag{2.6}$$

the products being absolutely convergent for all $s \in \mathbb{C}$.

Proof of Theorem 1.3. Put $\bar{s} = \sigma - it$. Then we have

$$\begin{aligned} & \left(1 - \frac{\bar{s}}{\rho}\right) \left(1 - \frac{\sigma}{\rho}\right)^{-1} \left(1 - \frac{\bar{s}}{1-\rho}\right) \left(1 - \frac{\sigma}{1-\rho}\right)^{-1} \\ &= \frac{\sigma - \beta - i\gamma - it}{\sigma - \beta - i\gamma} \frac{\sigma - 1 + \beta - i\gamma - it}{\sigma - 1 + \beta - i\gamma} \\ &= \exp \left[- \int_0^\infty (e^{itx} - 1) e^{(\beta - \sigma + i\gamma)x} \frac{dx}{x} - \int_0^\infty (e^{itx} - 1) e^{(1 - \beta - \sigma + i\gamma)x} \frac{dx}{x} \right] \end{aligned}$$

from (2.4). By replacing ρ by $\bar{\rho}$, we obtain

$$\begin{aligned} & \left(1 - \frac{\bar{s}}{\bar{\rho}}\right) \left(1 - \frac{\sigma}{\bar{\rho}}\right)^{-1} \left(1 - \frac{\bar{s}}{1-\bar{\rho}}\right) \left(1 - \frac{\sigma}{1-\bar{\rho}}\right)^{-1} \\ &= \exp \left[- \int_0^\infty (e^{itx} - 1) e^{(\beta - \sigma - i\gamma)x} \frac{dx}{x} - \int_0^\infty (e^{itx} - 1) e^{(1 - \beta - \sigma - i\gamma)x} \frac{dx}{x} \right]. \end{aligned}$$

We have to mention that one has $\beta - \sigma < 0$ and $1 - \beta - \sigma < 0$ since $\zeta(s) \neq 0$ for $\sigma \geq 1$ (see Remark 2.7 below). Hence, one has

$$\begin{aligned} & \frac{(1 - \bar{s}/\rho)(1 - \bar{s}/\bar{\rho})(1 - \bar{s}/(1-\rho))(1 - \bar{s}/(1-\bar{\rho}))}{(1 - \sigma/\rho)(1 - \sigma/\bar{\rho})(1 - \sigma/(1-\rho))(1 - \sigma/(1-\bar{\rho}))} \\ &= \exp \left[-2 \int_0^\infty (e^{itx} - 1) \cos(\gamma x) (e^{(\beta - \sigma)x} + e^{(1 - \beta - \sigma)x}) \frac{dx}{x} \right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Xi_\sigma(t) &= \exp \left[-2 \sum_{1/2+i\gamma \in \mathcal{Z}_+^R} \int_0^\infty (e^{itx} - 1) \cos(\gamma x) e^{(1/2 - \sigma)x} \frac{dx}{x} \right. \\ &\quad \left. - 2 \sum_{\beta+i\gamma \in \mathcal{Z}_+^N} \int_0^\infty (e^{itx} - 1) \cos(\gamma x) (e^{(\beta - \sigma)x} + e^{(1 - \beta - \sigma)x}) \frac{dx}{x} \right] \end{aligned}$$

by (2.6) and the definition of $\Xi_\sigma(t)$. □

Remark 2.7. By modifying the proof above, we can see that one has (1.7) for any $\sigma \geq \sigma_0 > 1/2$ if $\zeta(s)$ does not vanish for $\sigma \geq \sigma_0$.

2.4. Proof of Theorem 1.4

In order to prove Theorem 1.4, we first prove the following lemma which is an analogue of Nikeghbali and Yor [11], Lemma 2.9.

Lemma 2.8. Let $G_\sigma(t) = \Gamma(\sigma - it)/\Gamma(\sigma)$ for $0 < \sigma$. Then $G_\sigma(t)$ is an infinitely divisible characteristic function for any $\sigma > 0$. Moreover, one has

$$\begin{aligned} \log G_\sigma(t) &= it\lambda_\sigma^\# + \int_0^\infty (e^{itx} - 1 - itx1_{[0,1]}(x))\nu_\sigma^\#(dx), \\ \lambda_\sigma^\# &= C(\sigma) := \int_0^1 \left(\frac{e^{-\sigma x}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - \int_1^\infty e^{-x} \frac{dx}{x}, \\ \nu_\sigma^\#(dx) &:= \frac{1(dx)}{xe^{\sigma x}(1 - e^{-x})}. \end{aligned}$$

Proof. By the integral representation of $\Gamma(s)$ and the change of variables $x = e^{-y}$, we have

$$\begin{aligned} G_\sigma(t) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-x} x^{\sigma-1-it} dx = \frac{-1}{\Gamma(\sigma)} \int_\infty^{-\infty} e^{-e^{-y}} e^{y(1-\sigma+it)y} e^{-y} dy \\ &= \frac{1}{\Gamma(\sigma)} \int_{-\infty}^\infty e^{ity} \exp(-\sigma y - e^{-y}) dy, \quad \sigma > 0. \end{aligned}$$

Therefore, the probability density function is given by $\exp(-\sigma y - e^{-y})/\Gamma(\sigma)$.

Next, we quote Malmstén’s formula (see, for example, Whittaker and Watson [16], page 249)

$$\log \Gamma(s) = \int_0^\infty \left(\frac{e^{-sx} - e^{-x}}{1 - e^{-x}} + (s - 1)e^{-x} \right) \frac{dx}{x}, \quad \sigma > 0.$$

Hence, it holds that

$$\begin{aligned} \log G_\sigma(t) &= \int_0^\infty \left(\frac{e^{-(\sigma-it)x} - e^{-\sigma x}}{1 - e^{-x}} - ite^{-x} \right) \frac{dx}{x} \\ &= \int_0^1 \left(\frac{e^{itx} - 1 - itx}{e^{\sigma x}(1 - e^{-x})} - ite^{-x} + \frac{itxe^{-\sigma x}}{1 - e^{-x}} \right) \frac{dx}{x} + \int_1^\infty \left(\frac{e^{itx} - 1}{e^{\sigma x}(1 - e^{-x})} - ite^{-x} \right) \frac{dx}{x} \\ &= \int_0^\infty \frac{e^{itx} - 1 - itx1_{[0,1]}(x)}{xe^{\sigma x}(1 - e^{-x})} dx + it \int_0^1 \left(\frac{e^{-\sigma x}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - it \int_1^\infty e^{-x} \frac{dx}{x}. \end{aligned}$$

Therefore, we obtain Lemma 2.8. □

For the reader’s convenience, we give a proof of (1.4). By the Euler product of $\zeta(s)$ and the Taylor expansion of $\log(1 - x)$, $|x| < 1$, one has

$$\begin{aligned} \log \frac{\zeta(\sigma - it)}{\zeta(\sigma)} &= \sum_p \log \frac{1 - p^{-\sigma}}{1 - p^{-\sigma+it}} = \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} (p^{rit} - 1) \\ &= \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} (e^{rit \log p} - 1) = \int_{-\infty}^\infty (e^{itx} - 1) \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{r\sigma} \delta_{r \log p}(dx). \end{aligned}$$

This equality implies (1.4).

Proof of Theorem 1.4. We have

$$\Xi_\sigma(t) = \pi^{it/2} G_{\sigma/2}(t/2) \frac{\sigma - it}{\sigma} \frac{\sigma - 1 - it}{\sigma - 1} \frac{\zeta(\sigma - it)}{\zeta(\sigma)}$$

by the definition of $\Xi_\sigma(t)$. It holds that

$$\begin{aligned} \log G_{\sigma/2}(t/2) &= \frac{it}{2} C(\sigma/2) + \int_0^\infty \frac{e^{i(t/2)x} - 1 - i(t/2)x 1_{[0,1]}(x)}{xe^{\sigma x/2}(1 - e^{-x})} dx \\ &= it \frac{C(\sigma/2)}{2} + \int_0^\infty \frac{e^{itx} - 1 - itx 1_{[0,1/2]}(x)}{xe^{\sigma x}(1 - e^{-2x})} dx \end{aligned}$$

from Lemma 2.8. Obviously, one has $1/2 < r \log p$ for any integer r and prime number p since $\log 2 = 0.6931471806\dots$. Hence by using (1.4), we have

$$\log \frac{\zeta(\sigma - it)}{\zeta(\sigma)} = \int_{-\infty}^\infty (e^{itx} - 1 - itx 1_{[0,1/2]}(x)) \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} \delta_{r \log p}(dx).$$

When $\sigma > 1$, one has

$$\begin{aligned} \frac{\sigma - it}{\sigma} &= \exp \left[- \int_0^\infty (e^{itx} - 1 - itx 1_{[0,1/2]} + itx 1_{[0,1/2]}) e^{-\sigma x} \frac{dx}{x} \right] \\ &= \exp \left[- \int_0^\infty \frac{e^{itx} - 1 - itx 1_{[0,1/2]}(x)}{xe^{\sigma x}} dx - it \frac{1 - e^{-\sigma/2}}{\sigma} \right] \end{aligned}$$

by (2.3). Thus, it holds that

$$\begin{aligned} &\frac{\sigma - it}{\sigma} \frac{\sigma - 1 - it}{\sigma - 1} \\ &= \exp \left[- \int_0^\infty (e^{itx} - 1 - itx 1_{[0,1/2]}(x)) \frac{1 + e^x}{xe^{\sigma x}} dx - it \left(\frac{1 - e^{-\sigma/2}}{\sigma} + \frac{1 - e^{-(\sigma-1)/2}}{\sigma - 1} \right) \right]. \end{aligned}$$

If x is sufficiently large, then we have

$$\frac{1}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1 + e^x}{xe^{\sigma x}} < 0.$$

Thus ν_σ in Theorem 1.4 is not a measure but a signed measure.

Finally, we show $\int_{\mathbb{R}} (|x|^2 \wedge 1) |\nu_\sigma|(dx) < \infty$ when $\sigma > 1$. By using $(1 - e^{-2})x \leq 1 - e^{-2x}$ for $0 \leq x < 1$ and $1 - e^{-2} \leq 1 - e^{-2x}$ for $x \geq 1$, we have

$$\int_0^\infty \frac{(1 - e^{-2})(|x|^2 \wedge 1)}{xe^{\sigma x}(1 - e^{-2x})} dx \leq \int_0^1 \frac{dx}{e^{\sigma x}} + \int_1^\infty \frac{dx}{xe^{\sigma x}} < \int_0^\infty \frac{dx}{e^{\sigma x}} < \infty.$$

Obviously, it holds that

$$\int_0^\infty \frac{(1 + e^x)(|x|^2 \wedge 1)}{xe^{\sigma x}} dx < 2 \int_0^\infty \frac{(|x|^2 \wedge 1) dx}{xe^{(\sigma-1)x}} < 2 \int_0^\infty \frac{dx}{e^{(\sigma-1)x}} < \infty.$$

From $\sum_p p^{-\sigma} < \sum_{n=2}^\infty n^{-\sigma} = \zeta(\sigma) - 1$, one has

$$\begin{aligned} & \int_0^\infty \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} \delta_{r \log p}(dx) \\ &= \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} < \sum_p \sum_{r=1}^\infty p^{-r\sigma} < \sum_{n=1}^\infty n^{-\sigma} + \sum_p \sum_{r=2}^\infty p^{-r\sigma} \\ &= \zeta(\sigma) + \sum_p \frac{p^{-2\sigma}}{1 - p^{-\sigma}} < \zeta(\sigma) + \sum_{n=2}^\infty \frac{n^{-2\sigma}}{1 - 2^{-\sigma}} \\ &< \zeta(\sigma) + (1 - 2^{-\sigma})^{-1} \zeta(2\sigma) < \infty. \end{aligned}$$

Therefore the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible. □

Remark 2.9. Suppose $\sigma \neq 1$ and put

$$\Xi_\sigma^*(t) := \frac{\sigma - 1}{\sigma - 1 - it} \Xi_\sigma(t).$$

Then $\Xi_\sigma^*(t)$ is a characteristic function for any $\sigma \neq 1$ by the fact that the product of a finite number of characteristic functions is also a characteristic function. By modifying the proof above, we have

$$\begin{aligned} \Xi_\sigma^*(t) &= \exp \left[it\lambda_\sigma^* + \int_0^\infty (e^{itx} - 1 - itx 1_{[0,1/2]}(x)) \nu_\sigma^*(dx) \right], \\ \lambda_\sigma^* &:= \frac{1 - e^{-\sigma/2}}{\sigma} + \frac{\log \pi}{2} + \frac{1}{2} \int_0^1 \left(\frac{e^{-\sigma x/2}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - \frac{1}{2} \int_1^\infty e^{-x} \frac{dx}{x}, \\ \nu_\sigma^*(dx) &:= \frac{1(dx)}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1(dx)}{xe^{\sigma x}} + \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx) \end{aligned}$$

for $\sigma > 1$. Therefore the characteristic function $\Xi_\sigma^*(t)$ is infinitely divisible for any $\sigma > 1$ since one has

$$\frac{1}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1}{xe^{\sigma x}} > 0, \quad x > 0.$$

Moreover, we can see that every characteristic function $\Xi_\sigma^*(t)$ is a pretended-infinitely divisible characteristic function for each $1/2 < \sigma < 1$ if and only if the Riemann hypothesis is true by

an argument similar to that in the proof of Theorem 1.2. In addition, it holds that

$$\begin{aligned}\Xi_{\sigma}^*(t) &= \exp\left[\int_{-\infty}^{\infty} (e^{itx} - 1)v_{\sigma}^*(dx)\right], \\ v_{\sigma}^*(dx) &:= \frac{1_{(-\infty,0)}(dx)}{-xe^{(\sigma-1)x}} - \sum_{1/2+i\gamma \in \mathcal{Z}_+} \frac{2\cos(\gamma x)}{xe^{(\sigma-1/2)x}} 1_{(0,\infty)}(dx),\end{aligned}$$

for $1/2 < \sigma < 1$, under the Riemann hypothesis. This is proved by (1.6) and

$$\begin{aligned}\frac{\sigma - 1}{\sigma - 1 - it} &= \frac{1 - \sigma}{1 - \sigma + it} = \exp\left[\int_0^{\infty} (e^{-itx} - 1)e^{(\sigma-1)x} \frac{dx}{x}\right] \\ &= \exp\left[\int_0^{-\infty} \frac{(e^{itx} - 1) dx}{xe^{(\sigma-1)x}}\right] = \exp\left[-\int_{-\infty}^0 \frac{(e^{itx} - 1) dx}{xe^{(\sigma-1)x}}\right]\end{aligned}$$

when $1/2 < \sigma < 1$.

It is well known that convolving a density with a normal density to make distributions more well-behaved. In this case the exponential distribution is the one that makes things nicer since when $\sigma > 1$, the complete Riemann zeta distribution defined by $\Xi_{\sigma}(t)$ and the distribution defined by $\Xi_{\sigma}^*(t)$ are quasi-infinitely divisible and infinitely divisible, respectively.

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