Bernoulli **21**(1), 2015, 489–504
DOI: 10.3150/13-BEJ576

Precise tail asymptotics of fixed points of the smoothing transform with general weights

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We consider solutions of the stochastic equation $R =_d \sum_{i=1}^N A_i R_i + B$, where $N > 1$ is a fixed constant, A_i are independent, identically distributed random variables and R_i are independent copies of R, which are independent both from A_i's and B. The hypotheses ensuring existence of solutions are well known. Moreover under a number of assumptions the main being $\mathbb{E}|A_1|^\alpha = 1/N$ and $\mathbb{E}|A_1|^\alpha \log |A_1| > 0$, the limit $\lim_{t\to\infty} t^\alpha \mathbb{P}[|R| > t] = K$ exists. In the present paper, we prove positivity of K.

Keywords: large deviations; linear stochastic equation; regular variation; smoothing transform

1. Introduction

Let $N > 1$ be an integer, $A_1, \dots, A_N, B$ real valued random variables such that A_i are independent and identically distributed (i.i.d.). On the set $P(\mathbb{R})$ of probability measures on the real line the smoothing transform is defined as follows

$$\mu \mapsto \mathcal{L}\left(\sum_{i=1}^N A_i R_i + B\right),$$

where $R_1, \dots, R_N$, are i.i.d. random variables with common distribution μ, independent of $(B, A_1, \dots, A_N)$ and $\mathcal{L}(R)$ denotes the law of the random variable R. A fixed point of the smoothing transform is given by any $\mu \in P(\mathbb{R})$ such that, if R has distribution μ, the equation

$$R =_d \sum_{i=1}^N A_i R_i + B, \tag{1.1}$$

holds true. We are going to distinguish between the case of $B = 0$ a.s. (the homogeneous smoothing transform) and the other one called the nonhomogeneous smoothing transform.

The homogeneous equation (1.1) is used for example, to study interacting particle systems [9] or the branching random walk [1,12]. In recent years, from practical reasons, the inhomogeneous equation has gained importance. It appears for example, in the stochastic analysis of the Pagerank algorithm (which is the heart of the Google engine) [13,14,18] as well as in the analysis of a large class of divide and conquer algorithms including the Quicksort algorithm [16,17]. Both the homogeneous and the inhomogeneous equation were recently used to describe equilibrium

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distribution of a class of kinetic models and used for example, to study the distribution of particle velocity in Maxwell gas (see, e.g., [6]).

Properties of the fixed points of equation (1.1) are governed by the function

$$m(s) = \mathbb{E}\left[\sum_{i=1}^{N} |A_i|^s\right] = N\mathbb{E}\big[|A_1|^s\big].$$

Suppose that $s_1 = \sup\{s\colon m(s) < \infty\}$ is strictly positive. Clearly m is convex and differentiable on $(0, s_1)$. We assume that there are $0 < \gamma < \alpha < s_1$ such that

$$m(\gamma) = m(\alpha) = 1.$$

Then

$$0 < m'(\alpha) = \mathbb{E}\left[\sum_{i=1}^{N} |A_i|^\alpha \log|A_i|\right]$$

and the latter quantity is finite. The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that*

- $\log|A_1|$ *is nonlattice*;
- $\mathbb{P}[A_1 > 0] > 0$ *and* $\mathbb{P}[A_1 < 0] > 0$;
- $s_1 > 0$;
- *there are* $0 < \gamma < \alpha < s_1$ *such that* $m(\gamma) = m(\alpha) = 1$;
- *there is* $\varepsilon > 0$ *such that* $\mathbb{E}|B|^{\gamma+\varepsilon} < \infty$.

Suppose that R *is a nontrivial solution to* (1.1) *such that* $\mathbb{E}|R|^{\gamma+\varepsilon} < \infty$. *Then*

$$\liminf_{t\to\infty} t^\alpha \mathbb{P}[R > t] > 0 \quad \textit{and} \quad \liminf_{t\to\infty} t^\alpha \mathbb{P}[R < -t] > 0.$$

Remark 1.2. Under the assumptions of Theorem 1.1 the random variable R is real valued and it attains both positive and negative values. If $\mathbb{P}[A_1 > 0] = \mathbb{P}[B > 0] = 1$ then R is a positive random variable and exactly the same proof shows that

$$\liminf_{t\to\infty} t^\alpha \mathbb{P}[R > t] > 0.$$

Existence of such a solution implies $\gamma < 2$ for the nonhomogeneous case and $1 \le \gamma < 2$ for the homogeneous one (see [3]). Then the solution is basically unique (given the mean of it exists) and, if $\mathbb{E}|B|^\alpha < \infty$ then for every $s < \alpha$

$$\mathbb{E}|R|^s < \infty. \tag{1.2}$$

In view of the result of Jelenkovic and Olvera-Cravioto (Theorem 4.6 in [15]), Theorem 1.1 implies.

Corollary 1.3. *Suppose that the assumptions of Theorem 1.1 are satisfied and additionally let* $\mathbb{E}|B|^\alpha < \infty$. *Then*

$$\lim_{t\to\infty} t^\alpha \mathbb{P}[R > t] = \lim_{t\to\infty} t^\alpha \mathbb{P}[R < -t] = K > 0. \tag{1.3}$$

The existence of the limit in (1.3) for such R, in a more general case of random N, was proved by Jelenkovic and Olvera-Cravioto [15], Theorem 4.6, but from the expression for K, given by their renewal theorem, it is not possible to conclude its strict positivity except of the very particular case when $A_1, \dots, A_N, B$ are positive and $\alpha \ge 1$. There are other solutions to (1.1) than those mentioned in the above corollary. For the full description of them see [2,4,5]. Clearly, Theorem 1.1 matters only for solutions satisfying (1.2).

Some partial results concerning positivity of K are contained in [7] and [3]. The paper [7] deals with matrices but Theorem 2.12 and Proposition 2.13 there can be specified to our case. Under additional assumption that $\mathbb{E}|B|^{s_0} < \infty$ for some $\alpha < s_0 < s_1$ they say that either $K > 0$ or $\mathbb{E}|R|^s < \infty$ for all $s < s_0$. If R is not constant, the latter is not possible when there is $\beta \le s_0$ such that $\mathbb{E}|A_1|^\beta = 1$. Indeed, then R becomes the solution of

$$R = AR + Q$$

with $Q = \sum_{i=2}^N A_i R_i + B$ and the conclusion of Goldie's theorem [11] would be violated. It is interesting that for the asymptotics in (1.3) in the case of N being constant the implicit renewal theorem of Jelenkovic and Olvera-Cravioto is not needed. The usual one on R is sufficient [7], Theorem 2.8. For positivity of K in the general case of random N see [3], Theorem 9.

Clearly, Theorem 1.1 improves considerably the results of [7] specialised to the one dimensional case. Also, the technique is purely probabilistic while in [7] holomorphicity of $\mathbb{E}|R|^z$ and the Landau theorem is used.

Let μ_A be the law of A_i. In Section 2, we show some necessary properties of the random walks with the transition probability μ_A. A version of the Bahadur, Rao theorem ([8], Theorem 3.7.4) is needed and its proof is included in the Appendix. Section 3 is devoted to the proof of Theorem 1.1.

2. Random walk generated by the measure μ_A

In this section, we will study properties of the random walk $\{|\widetilde{A}_1 \cdots \widetilde{A}_n|\}_{n\in\mathbb{N}}$, where $\widetilde{A}_i$ are independent and distributed according to the measure μ_A (it is convenient for our purpose to use the multiplicative notation). Since $\mathbb{E}\log|\widetilde{A}_1| < 0$, by the strong law of large numbers, this random walk converges to 0 a.s. Nevertheless, our aim is to describe a sufficiently large set on which trajectories of the process exceed an arbitrary large, but fixed number t. Given n, one can prove that the probability of the event $\{|\widetilde{A}_1 \cdots \widetilde{A}_n| > t\}$ is largest when n is comparable with n_0 defined by

$$n_0 = \left\lfloor \frac{\log t}{N\rho} \right\rfloor, \tag{2.1}$$

where $\rho = \mathbb{E}[|\widetilde{A}_1|^\alpha \log|\widetilde{A}_1|]$. Notice that n_0 depends on t. However, since we are interested only in estimates from below we need less and for our purpose it is sufficient to consider sets

$$V_n = \{|\widetilde{A}_1 \cdots \widetilde{A}_n| \geq t \text{ and } |\widetilde{A}_1 \cdots \widetilde{A}_s| \leq \mathrm{e}^{-(n-s)\delta} t C_0 \text{ for every } s \leq n-1\}, \tag{2.2}$$

where C_0 is a large constant and δ is a small constant (both will be defined later).

Our main result of this section is the following theorem.

Theorem 2.1. *Assume* $\mathbb{E}[|\widetilde{A}_1|^{\alpha+\delta}] < \infty$, $\mathbb{E}[|\widetilde{A}_1|^\alpha] = \frac{1}{N}$ *and* $0 < \rho < \infty$. *There are constants* C_0, C_1, C_2 *such that for sufficiently large* t *and for* $n_0 - \sqrt{n_0} \leq n \leq n_0$

$$\frac{C_1}{\sqrt{n} t^\alpha N^n} < \mathbb{P}[V_n] \leq \frac{C_2}{\sqrt{n} t^\alpha N^n}.$$

In order to prove the theorem above we will need precise estimates of $\mathbb{P}[|\widetilde{A}_1 \cdots \widetilde{A}_s| > t]$. We will use the following extension of the Bahadur, Rao theorem ([8], Theorem 3.7.4, see also Example 3.7.10).

Proposition 2.2. *Assume* $\mathbb{E}[|\widetilde{A}_1|^{\alpha+\delta}] < \infty$, $\mathbb{E}[|\widetilde{A}_1|^\alpha] = \frac{1}{N}$ *and* $0 < \rho < \infty$. *There is* C *such that for every* $d \geq 0$ *and every* $n \in \mathbb{N}$

$$\mathbb{P}\{|\widetilde{A}_1 \cdots \widetilde{A}_n| > \mathrm{e}^d \mathrm{e}^{\rho n N}\} \leq \frac{C}{\sqrt{2\pi}\alpha\lambda\sqrt{n}\mathrm{e}^{\rho\alpha n N} N^n \mathrm{e}^{\alpha d}}, \tag{2.3}$$

where $\lambda = \sqrt{\Lambda''(\alpha)}$ *for* $\Lambda(s) = \log \mathbb{E}[|\widetilde{A}_1|^s]$.

Moreover, let $\theta \geq 0$ *and*

$$0 \leq \frac{d}{\sqrt{n}} \leq \theta \tag{2.4}$$

for sufficiently large n. *Then there is* $C = C(\theta)$ *such that for large* n:

$$\sqrt{2\pi}\alpha\lambda\sqrt{n}\mathrm{e}^{\rho\alpha n N} N^n \mathrm{e}^{\alpha d} \mathrm{e}^{d^2/(2\lambda^2 n)} \cdot \mathbb{P}\{|\widetilde{A}_1 \cdots \widetilde{A}_n| > \mathrm{e}^d \mathrm{e}^{\rho n N}\} = 1 + C(\theta)\mathrm{o}(1), \tag{2.5}$$

where as usual $\lim_{n\to\infty} \mathrm{o}(1) = 0$ *uniformly for* d *satisfying* (2.4).

The proof is a slight modification of the proof of Theorem 3.7.4 in [8]. For reader's convenience we give all the details but we postpone the proof to the Appendix.

We will also use the following. Since $\mathbb{E}[|\widetilde{A}_1|^\beta] < \frac{1}{N}$ for $\beta < \alpha$ and sufficiently close to α, one can find $\beta < \alpha$ and $\gamma > 0$ such that

$$\mathbb{E}[|\widetilde{A}_1|^\beta] = \frac{1}{N^{1+\gamma}}. \tag{2.6}$$

Proof of Theorem 2.1. Denote

$$U_n = \{|\widetilde{A}_1 \cdots \widetilde{A}_n| > t\},$$
$$W_{s,n} = \{|\widetilde{A}_1 \cdots \widetilde{A}_s| > \mathrm{e}^{-\delta(n-s)} C_0 t\}.$$

We have

$$\begin{aligned}\mathbb{P}[V_n] &= \mathbb{P}\Big[U_n \cap \bigcap_{s<n} W_{s,n}^c\Big] \\ &= \mathbb{P}[U_n] - \mathbb{P}\Big[U_n \cap \Big(\bigcap_{s<n} W_{s,n}^c\Big)^c\Big] \\ &= \mathbb{P}[U_n] - \mathbb{P}\Big[\bigcup_{s<n}(U_n \cap W_{s,n})\Big].\end{aligned}$$

By Proposition 2.2 ($s = n, d = N\rho(n_0 - n), \theta = N\rho + 1$)

$$\begin{aligned}\mathbb{P}[U_n] &= \mathbb{P}\big[|\widetilde{A}_1 \cdots \widetilde{A}_n| > t\big] = \mathbb{P}\big[|\widetilde{A}_1 \cdots \widetilde{A}_n| > \mathrm{e}^{N\rho n}\mathrm{e}^{N\rho(n_0-n)}\big] \\ &\geq \frac{C_1 \mathrm{e}^{-N\rho\alpha(n_0-n)}}{\sqrt{n}\mathrm{e}^{N\rho\alpha n} N^n} = \frac{C_1}{\sqrt{n}\mathrm{e}^{N\rho\alpha n_0} N^n} = \frac{C_1}{\sqrt{n} t^\alpha N^n}\end{aligned}$$

for sufficiently large t and $C_1 = \frac{1+C(N\rho+1)o(1)}{\sqrt{2\pi}\alpha\lambda}\exp(-\frac{(N\rho+1)^2}{2\lambda^2})$. Exactly in the same way (2.5) gives estimates from above with $C_2 = \frac{1+C(N\rho+1)o(1)}{\sqrt{2\pi}\alpha\lambda}$. Therefore to prove the theorem, it is sufficient to justify that

$$\mathbb{P}\Big[\bigcup_{s<n}(U_n \cap W_{s,n})\Big] \leq \frac{\varepsilon}{\sqrt{n} t^\alpha N^n}. \tag{2.7}$$

We fix t, n_0 and n such that $n_0 - \sqrt{n_0} \leq n \leq n_0$. First we estimate $\mathbb{P}[U_n \cap W_{s,n}]$ for $s < n - D\log n$, where the constant D will be defined later. By the Chebyshev inequality and (2.6), we have

$$\begin{aligned}\mathbb{P}[U_n \cap W_{s,n}] &= \sum_{m=0}^{\infty} \mathbb{P}\big[\mathrm{e}^m \mathrm{e}^{-\delta(n-s)} C_0 t < |\widetilde{A}_1 \cdots \widetilde{A}_s| \leq \mathrm{e}^{m+1}\mathrm{e}^{-\delta(n-s)} C_0 t \text{ and } |\widetilde{A}_1 \cdots \widetilde{A}_n| > t\big] \\ &\leq \sum_{m=0}^{\infty} \mathbb{P}\big[|\widetilde{A}_1 \cdots \widetilde{A}_s| > \mathrm{e}^m \mathrm{e}^{-\delta(n-s)} C_0 t\big]\mathbb{P}\big[|\widetilde{A}_{s+1} \cdots \widetilde{A}_n| > \mathrm{e}^{-(m+1)}\mathrm{e}^{\delta(n-s)} C_0^{-1}\big] \\ &\leq \sum_{m=0}^{\infty} \frac{\mathrm{e}^{\delta\alpha(n-s)}}{\mathrm{e}^{m\alpha} C_0^\alpha t^\alpha}\big(\mathbb{E}|\widetilde{A}_1|^\alpha\big)^s \cdot \frac{\mathrm{e}^{\beta(m+1)} C_0^\beta}{\mathrm{e}^{\delta\beta(n-s)}}\big(\mathbb{E}|\widetilde{A}_1|^\beta\big)^{n-s} \\ &\leq \frac{\mathrm{e}^{\delta\alpha(n-s)}}{C_0^{\alpha-\beta} t^\alpha} \cdot \frac{1}{N^s} \cdot \frac{1}{\mathrm{e}^{\delta\beta(n-s)} N^{n-s} N^{\gamma(n-s)}} \cdot \sum_{m=0}^{\infty} \frac{\mathrm{e}^\beta}{\mathrm{e}^{m(\alpha-\beta)}} \\ &\leq \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n \mathrm{e}^{(\gamma\log N + \delta(\beta-\alpha))(n-s)}} = \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n \mathrm{e}^{\gamma_1(n-s)}},\end{aligned}$$

where $\gamma_1 := \gamma \log N + (\beta - \alpha)\delta$ and choosing appropriately small δ we can assume that $\gamma_1 > 0$. Hence, for $s < n - D\log n$

$$\mathbb{P}[U_n \cap W_{s,n}] \le \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n e^{\gamma_1(n-s)}}. \tag{2.8}$$

For $s > n - D\log n$, we estimate

$$\begin{aligned}\mathbb{P}[U_n \cap W_{s,n}] &= \sum_{m=0}^{\infty} \mathbb{P}\big[e^m e^{-\delta(n-s)} C_0 t < |\widetilde{A}_1 \cdots \widetilde{A}_s| \le e^{m+1} e^{-\delta(n-s)} C_0 t \text{ and } |a_1 \cdots a_n| > t\big] \\ &\le \sum_{m=0}^{\infty} \mathbb{P}\big[|\widetilde{A}_1 \cdots \widetilde{A}_s| > e^m e^{-\delta(n-s)} C_0 t\big] \mathbb{P}\big[|\widetilde{A}_{s+1} \cdots \widetilde{A}_n| > e^{-(m+1)} e^{\delta(n-s)} C_0^{-1}\big].\end{aligned}$$

We denote the first factor of the sum by I_m. To estimate it, we will use Proposition 2.2. Namely let

$$\begin{aligned} k &= n - s, \qquad k_0 = n_0 - s, \\ d_1 &= -\delta k + m + \log C_0 + N\rho k_0, \\ d_2 &= d_1 + 1, \end{aligned}$$

then (recall $\log t = (s + k_0)N\rho$)

$$e^m e^{-\delta(n-s)} C_0 t = e^{d_1} e^{N\rho s}.$$

So, by Proposition 2.2:

$$\mathbb{P}\big[|\widetilde{A}_1 \cdots \widetilde{A}_s| > e^{d_1 + N\rho s}\big] \le \frac{C}{\sqrt{s}} N^{-s} e^{-N\rho\alpha s - \alpha d_1} \le \frac{C e^{\delta\alpha k}}{C_0^\alpha e^{\alpha m} t^\alpha N^s \sqrt{s}}.$$

The second factor we estimate exactly in the same way as previously and we obtain

$$\begin{aligned}\mathbb{P}[U_n \cap W_{s,n}] &= \sum_{m=0}^{\infty} \frac{C e^{\delta\alpha(n-s)}}{C_0^\alpha e^{\alpha m} t^\alpha N^s \sqrt{s}} \cdot \frac{e^{\beta(m+1)} C_0^\beta}{e^{\delta\beta(n-s)} N^{(1+\gamma)(n-s)}} \\ &\le \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n \sqrt{n} e^{\gamma_1(n-s)}}.\end{aligned} \tag{2.9}$$

Next, in view of (2.8) and (2.9)

$$\begin{aligned}\mathbb{P}\Big[\bigcup_{s<n} (U_n \cap W_{s,n})\Big] &\le \sum_{s<n-D\log n} \mathbb{P}[U_n \cap W_{s,n}] + \sum_{n-D\log n \le s < n} \mathbb{P}[U_n \cap W_{s,n}] \\ &\le \sum_{s<n-D\log n} \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n e^{\gamma_1(n-s)}} + \sum_{n-D\log n \le s<n} \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n \sqrt{n} e^{\gamma_1(n-s)}}\end{aligned}$$

$$\le \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n}\left(\frac{n}{n^{\gamma_1 D}} + \frac{1}{\sqrt{n}} \sum_{s<D\log n} \frac{1}{e^{\gamma_1 s}}\right)$$

$$\le \frac{C}{C_0^{\alpha-\beta} t^\alpha N^n}\left(\frac{1}{n^{\gamma_1 D-1}} + \frac{1}{\sqrt{n}}\right)$$

$$\le \frac{\varepsilon}{\sqrt{n} t^\alpha N^n}$$

assuming that $\frac{C}{C_0^{\alpha-\beta}} < \varepsilon$ and $\gamma_1 D \ge \frac{3}{2}$. Hence, (2.7) and the proof is finished. □

3. Proof of Theorem 1.1

We start with the following lemma.

Lemma 3.1. *If* $\mathbb{E}[|A_1|^\beta \log|A_1|] > 0$ *for some* $\beta > 0$, $\mathbb{P}[A_1 > 0] > 0$ *and* $\mathbb{P}[A_1 < 0] > 0$, *then any nontrivial solution of* (1.1) *is unbounded at* $+\infty$ *and* $-\infty$.

Proof. Suppose that R is a bounded solution of (1.1) and $R \neq C$ a.s. for any C. Assume first that R is bounded a.s. from below and from above. Let $[r, s]$ be the smallest interval containing the support of R for some finite numbers r and s. Of course $r \neq s$. Denote $\widetilde{B} = \sum_{i=2}^N A_i R_i + B$, then

$$R =_d A_1 R_1 + \widetilde{B}. \tag{3.1}$$

Since $\mathbb{E}[|A_1|^\beta \log|A_1|] > 0$, the probability of the set $U = \{(A_1, \widetilde{B})\colon |A_1| > 1\}$ is strictly positive. Then by (3.1) we must have

$$A_1 r + \widetilde{B} \ge r \quad \text{and} \quad A_1 s + \widetilde{B} \le s \qquad \text{a.s.}$$

But if we take a random pair $(A_1, \widetilde{B}) \in U$, then

$$\left|(A_1 r + \widetilde{B}) - (A_1 s + \widetilde{B})\right| = |A_1||r-s| > |r-s|.$$

Thus, we are led to a contradiction and at least one constant r or s must be infinite. Without loss of generality, we can assume that $s = +\infty$. In view of our assumptions, we can choose a large constant M and a small constant ε such that the probability of the set $V = \{(A_1, \widetilde{B})\colon A_1 < -\varepsilon, \widetilde{B} < M\}$ is strictly positive. Now, take any $x > (r-M)/(-\varepsilon)$ belonging to the support of R. Then for any $(A_1, \widetilde{B}) \in V$ we have

$$A_1 x + \widetilde{B} < -\varepsilon x + M < r.$$

Thus, by (3.1), r cannot be a lower bound of the support of R and must be equal $-\infty$. □

Let $\mathcal{T}$ be an N-ary rooted tree, that is, the tree with a distinguished vertex o called root, such that every vertex has N daughters and one mother (except the root). The tree $\mathcal{T}$ can be identified with the set of finite words over the alphabet $\{1, 2, \ldots, N\}$:

$$\mathcal{T} = \bigcup_{k=0}^{\infty} \{1, 2, \ldots, N\}^k,$$

where the empty word $\varnothing$ is the root and given $i_1 i_2 \cdots i_n \in \mathcal{T}$ its daughters are the words of the form $i_1 i_2 \cdots i_n j$ for $j = 1, \ldots, N$. We denote a typical vertex of the tree by $\gamma = i_1 i_2 \cdots i_n$ and we identify it with the shortest path connecting γ with o. We write $|\gamma| = n$ for the length of γ and $\gamma_{|k} = i_1 \cdots i_k$ for the curtailment of γ after k steps. Conventionally, $|\varnothing| = 0$ and $\gamma_{|0} = \varnothing$. If $\gamma_1 = i_1^1 i_2^1 \cdots i_{n_1}^1 \in \mathcal{T}$ and $\gamma_2 = i_1^2 i_2^2 \cdots i_{n_2}^2 \in \mathcal{T}$ then we write $\gamma_1 \gamma_2 = i_1^1 i_2^1 \cdots i_{n_1}^1 i_1^2 i_2^2 \cdots i_{n_2}^2$ for the element of $\mathcal{T}$ obtained by juxtaposition. In particular, $\gamma \varnothing = \varnothing \gamma = \gamma$. We partially order $\mathcal{T}$ by writing $\gamma_1 \le \gamma_2$ if there exists $\gamma_0 \in \mathcal{T}$ such that $\gamma_2 = \gamma_1 \gamma_0$. For two vertices γ_1 and γ_2, we denote by $\gamma_0 = \gamma_1 \wedge \gamma_2$ the longest common subsequence of γ_1 and γ_2 that is, the maximal γ_0 such that both $\gamma_0 \le \gamma_1$ and $\gamma_0 \le \gamma_2$.

To every vertex $\gamma \in \mathcal{T}$ we associate random variables $(A_{\gamma 1}, \ldots, A_{\gamma N}, B_\gamma, R_{\gamma 1}, \ldots, R_{\gamma N})$ which are independent copies of $(A_1, \ldots, A_N, B, R_1, \ldots, R_N)$ defined in (1.1). It is more convenient to think that $A_{\gamma i}$ and $R_{\gamma i}$ are indeed attached not to the vertex γ but to the edge connecting γ with γi. We write $\Pi_\gamma = A_{\gamma_{|1}} A_{\gamma_{|2}} \cdots A_\gamma$, then Π_γ is just the product of random variables $A_{\gamma_{|k}}$ which are associated with consecutive edges connecting the root o with γ.

We fix $\gamma = i_1 \cdots i_n$ and we apply n times the stochastic equation (1.1) in such a way that in kth step we apply recursively this equation to $R_{\gamma_{|k}}$:

$$\begin{aligned}
R &=_d \sum_{i=1}^{N} A_i R_i + B_0 \\
&=_d A_{i_1} \left(\sum_{j=1}^{N} A_{i_1 j} R_{i_1 j} + B_{i_1} \right) + \sum_{i \ne i_1} A_i R_i + B_0 \\
&=_d A_{i_1} A_{i_1 i_2} R_{i_1 i_2} + A_{i_1} \left(\sum_{j \ne i_2}^{N} A_{i_1 j} R_{i_1 j} + B_{i_1} \right) + \sum_{i \ne i_1} A_i R_i + B_0 \\
&=_d \Pi_{\gamma_{|2}} R_{\gamma_{|2}} + \sum_{j \ne i_2}^{N} \Pi_{(\gamma_{|1} j)} R_{(\gamma_{|1} j)} + A_{i_1} B_{i_1} + \sum_{i \ne i_1} A_i R_i + B_0 \qquad (3.2) \\
&=_d \Pi_{\gamma_{|2}} \left(\sum_{i=1}^{N} A_{(\gamma_{|2} i)} R_{(\gamma_{|2} i)} + B_{\gamma_{|2}} \right) + \sum_{i \ne i_2}^{N} \Pi_{(\gamma_{|1} i)} R_{(\gamma_{|1} i)} + \sum_{i \ne i_1} A_i R_i + A_{i_1} B_{i_1} + B_0 \\
&=_d \cdots \\
&=_d \Pi_\gamma R_\gamma + \sum_{k<n} \sum_{i \ne i_k} \Pi_{(\gamma_{|k} i)} R_{(\gamma_{|k} i)} + \sum_{k<n} \Pi_{\gamma_{|k}} B_{\gamma_{|k}}.
\end{aligned}$$

We define

$$V_\gamma = \big\{|\Pi_\gamma| \ge t \text{ and } |\Pi_{\gamma_{|s}}| \le e^{-(|\gamma|-s)\delta} C_0 t \text{ for every } s < |\gamma|\big\}.$$

Notice that if we denote $\widetilde{A}_k = A_{\gamma_{|k}}$, then the set V_γ coincides with the set $V_{|\gamma|}$ defined in (2.2). Thus, by Theorem 2.1 we can choose large C_0 such that if $n = |\gamma|$ and $n_0 - \sqrt{n_0} < n < n_0$, then

$$\mathbb{P}[V_\gamma] \ge \frac{C}{\sqrt{n} t^\alpha N^n}.$$

For a sufficiently large constant d (defined later) and $D = \frac{Nd^2+d}{1-e^{-\delta/2}}$, we define sets

$$\begin{aligned}
W_\gamma &= \big\{|R_{(\gamma_{|s} i)}| < d e^{(|\gamma|-s)\delta/4}, |A_{(\gamma_{|s} i)}| < d e^{(|\gamma|-s)\delta/4}, |B_{\gamma_{|s}}| < d e^{(|\gamma|-s)\delta/2}, \\
&\qquad s = 0, \dots, |\gamma|-1; i \ne i_{s+1}\big\}; \\
W_\gamma^+ &= W_\gamma \cap \{R_\gamma > 2D\}; \\
W_\gamma^- &= W_\gamma \cap \{R_\gamma < -2D\}; \\
V_\gamma^+ &= V_\gamma \cap \{\Pi_\gamma > 0\}; \\
V_\gamma^- &= V_\gamma \cap \{\Pi_\gamma < 0\}.
\end{aligned}$$

Finally we define

$$\widetilde{V}_\gamma = \big(V_\gamma^+ \cap W_\gamma^+\big) \cup \big(V_\gamma^- \cap W_\gamma^-\big).$$

Lemma 3.2. *Assume $\gamma \in \mathcal{T}$. Then on the set $\widetilde{V}_\gamma$ we have*

$$R > At.$$

Proof. Let $n = |\gamma|$, then by (3.2) on $\widetilde{V}_\gamma$ we have

$$\begin{aligned}
R &\ge \Pi_\gamma R_\gamma - \left| \sum_{k<n} \sum_{i \ne i_k} \Pi_{(\gamma_{|k} i)} R_{(\gamma_{|k} i)} + \sum_{k<n} \Pi_{\gamma_{|k}} B_{\gamma_{|k}} \right| \\
&\ge 2Dt - \sum_{k<n} \big(Nd^2 + d\big) e^{-(n-k)\delta/2} C_0 t \\
&\ge Dt.
\end{aligned}$$

□

We are going to prove that for some $\eta > 0$

$$\mathbb{P}\Bigg[\bigcup_{\{\gamma \in \mathcal{T}: n_0 - \sqrt{n_0} < |\gamma| < n_0\}} \widetilde{V}_\gamma\Bigg] \ge \eta t^{-\alpha}, \tag{3.3}$$

which immediately implies that

$$\liminf_{t\to\infty}\mathbb{P}\{R>t\}t^{\alpha}>0.$$

Lemma 3.3. *Let X_i be a sequence of i.i.d. random variables such that $\mathbb{E}|X_1|^{\varepsilon}<\infty$ for some $\varepsilon>0$. Let $\delta_0>0$. Then there exist constants d_0 and $p_0>0$ such that for every n*

$$\mathbb{P}\bigl[|X_i|<d_0\mathrm{e}^{(n-i)\delta_0}, i=1,2,\dots,n-1\bigr]\ge p_0.$$

Proof. By the Chebyshev inequality, we have

$$\mathbb{P}\bigl[|X_i|\ge d_0\mathrm{e}^{(n-i)\delta_0}\bigr]\le\frac{\mathbb{E}|X_i|^{\varepsilon}}{d_0^{\varepsilon}}\mathrm{e}^{-(n-i)\delta_0\varepsilon}.$$

Take d_0 such that $d_0^{\varepsilon}>3\mathbb{E}|X_i|^{\varepsilon}$. Then, since $1-\frac{x}{3}>\mathrm{e}^{-x}$ for $x\in[0,1]$ we have

$$\mathbb{P}\bigl[|X_i|<d_0\mathrm{e}^{(n-i)\delta_0}\bigr]\ge 1-\tfrac{1}{3}\mathrm{e}^{-(n-i)\delta_0\varepsilon}\ge\exp\bigl(-\bigl(\mathrm{e}^{-\delta_0\varepsilon}\bigr)^{n-i}\bigr).$$

Therefore,

$$\begin{aligned}&\mathbb{P}\bigl[|X_i|<d_0\mathrm{e}^{(n-i)\delta_0}, i=1,2,\dots,n-1\bigr]\\&\quad=\prod_{i=1}^{n-1}\mathbb{P}\bigl[|X_i|<d_0\mathrm{e}^{(n-i)\delta_0}\bigr]\ge\prod_{i=1}^{n-1}\mathrm{e}^{-(\mathrm{e}^{-\delta_0\varepsilon})^{n-i}}\\&\quad=\exp\left(-\sum_{i=1}^{n-1}\bigl(\mathrm{e}^{-\delta_0\varepsilon}\bigr)^{i}\right)\ge\exp\bigl(-\bigl(1-\mathrm{e}^{-\delta_0\varepsilon}\bigr)^{-1}\bigr)=:p_0.\end{aligned}$$

□

Since B and R have absolute moments of order bigger then γ we obtain the following corollary.

Corollary 3.4. *There are constants d and $p>0$ such that for every $\gamma\in\mathcal{T}$*

$$\mathbb{P}\bigl[W_{\gamma}^{+}\bigr]\ge p\quad\textit{and}\quad\mathbb{P}\bigl[W_{\gamma}^{-}\bigr]\ge p.$$

In view of the last result to obtain (3.3), it is sufficient to prove

$$\mathbb{P}\Biggl[\bigcup_{\{\gamma\in\mathcal{T}:n_0-\sqrt{n_0}<|\gamma|<n_0\}}V_{\gamma}\Biggr]\ge\eta_1 t^{-\alpha},$$

for some $\eta_1>0$.

In fact, we will estimate from below much smaller sum over a sparse subset of $\mathcal{T}$. The details are as follows.

We fix a large integer C_1 (determined later) and an arbitrary element $\overline{\gamma}$ of $\mathcal{T}$ such that $|\overline{\gamma}| = C_1$ (e.g., $\overline{\gamma}$ can be chosen as the word consisting of n one's). We define a sparse subset of vertices of $\mathcal{T}$:

$$\overline{\mathcal{T}} = \{\gamma \in \mathcal{T}\colon \bigl(|\gamma| \mod C_1\bigr) = 0, \gamma = \gamma_{||\gamma|-C_1}\overline{\gamma}, n_0 - \sqrt{n_0} < |\gamma| < n_0\},$$

that is, $\overline{\mathcal{T}}$ is the set of vertices of $\mathcal{T}$ located on the level kC_1 (for some integer k) such that $n_0 - \sqrt{n_0} < kC_1 < n_0$ and such that the last n letters of γ form the word $\overline{\gamma}$. Notice that for every γ such that $|\gamma| = kC_1$ the set

$$\left\{\gamma\gamma_1, \gamma_1 \in \bigcup_{i=1}^{C_1}\{1, \dots, N\}^i\right\}$$

contains exactly one element of $\mathcal{T}$. Thus there are exactly N^{kC_1} elements of $\overline{\mathcal{T}}$ of length $(k+1)C_1$. Moreover, the crucial property of the set $\overline{\mathcal{T}}$, that will be strongly used below, is that the distance between two different elements of $\overline{\mathcal{T}}$ is at least C_1 (by "distance" we mean the usual distance on graphs, that is, the minimal number of edges connecting two vertices).

Theorem 3.5. *There is $\eta > 0$ such that*

$$\mathbb{P}\Bigl(\bigcup_{\gamma \in \overline{\mathcal{T}}} V_\gamma\Bigr) \geq \frac{C\eta}{N^{C_1}C_1t^\alpha}.$$

Proof. By the inclusion–exclusion principle, we have

$$\mathbb{P}\Bigl(\bigcup_{\gamma \in \overline{\mathcal{T}}} V_\gamma\Bigr) \geq \sum_{\gamma \in \overline{\mathcal{T}}} \mathbb{P}(V_\gamma) - \sum_{\gamma \in \overline{\mathcal{T}}}\sum_{U_\gamma} \mathbb{P}(V_\gamma \cap V_{\gamma'}), \tag{3.4}$$

where $U_\gamma = \{\gamma' \in \overline{\mathcal{T}} \setminus \{\gamma\}\colon |\gamma'| \leq |\gamma|\}$.

Therefore, we have to estimate

$$\sum_{\gamma \in \overline{\mathcal{T}}} \mathbb{P}(V_\gamma) \quad \text{and} \quad \sum_{\gamma \in \overline{\mathcal{T}}}\sum_{U_\gamma} \mathbb{P}(V_\gamma \cap V_{\gamma'}).$$

Let K be the set of levels on which there are some elements of $\overline{\mathcal{T}}$, that is,

$$K = \{kC_1\colon n_0 - \sqrt{n_0} < kC_1 < n_0\}.$$

Let $L = |K|$ be the number of elements of the set K and let n_j be the jth element of K.

Observe that for given $n \in K$ there are exactly N^{n-C_1} elements of $\overline{\mathcal{T}}$ located on the level n and for every such γ, by Theorem 2.1, we have $\mathbb{P}(V_\gamma) \geq \frac{C}{\sqrt{n}N^n t^\alpha}$. Hence,

$$\sum_{\gamma \in \overline{\mathcal{T}}} \mathbb{P}(V_\gamma) \geq \sum_{j=1}^{L} \frac{C}{\sqrt{n_j}N^{n_j}t^\alpha}N^{n_j - C_1} \geq \frac{C}{N^{C_1}C_1t^\alpha}. \tag{3.5}$$

Now, let us estimate the sum of intersections. We fix $\gamma \in \overline{\mathcal{T}}$ and $\gamma' \in U_\gamma$. Let $\gamma_0 = \gamma \wedge \gamma'$ and let s be the length of γ_0. We have

$$\begin{aligned}\mathbb{P}[V_\gamma \cap V_{\gamma'}] &\le \mathbb{P}\bigl[V_\gamma \cap \bigl\{|\Pi_{\gamma_0}| < \mathrm{e}^{-\delta(|\gamma|-s)} C_0 t, |\Pi_{\gamma'}| > t\bigr\}\bigr] \\ &\le \mathbb{P}[V_\gamma]\mathbb{P}\bigl[|A_{\gamma'_{|s+1}} A_{\gamma'_{|s+2}} \cdots A_{\gamma'}| > \mathrm{e}^{\delta(|\gamma|-s)} C_0^{-1}\bigr] \\ &\le \mathbb{P}[V_\gamma] \cdot \frac{C_0^\alpha}{\mathrm{e}^{\alpha\delta(|\gamma|-s)} N^{|\gamma'|-s}},\end{aligned} \tag{3.6}$$

where for the last inequality we have used the Chebyshev inequality. We fix $\gamma \in \overline{\mathcal{T}}$ and we consider $\gamma' \in U_\gamma$. Notice that if γ and γ' connect on the level s, that is, $\gamma_{|s} = \gamma \wedge \gamma'$, then s must be smaller than $|\gamma| - C_1$. Given s let us estimate the number of elements $\gamma' \in U_\gamma$ such that $\gamma_{|s} = \gamma \wedge \gamma'$. All these elements must be located on levels $|\gamma|, |\gamma| - C_1, \ldots, |\gamma| - kC_1$, where k is the largest number such that $|\gamma| - kC_1 \ge \max\{s, n_0 - \sqrt{n_0}\}$, that is,

$$k \le \frac{1}{C_1} \min\bigl\{|\gamma| - s, |\gamma| - n_0 + \sqrt{n_0}\bigr\} \le \frac{1}{C_1}\bigl(|\gamma| - s\bigr).$$

Moreover on the level $|\gamma| - jC_1$ ($j < k$), there are exactly $N^{|\gamma| - jC_1 - s - C_1}$ elements of U_γ. Thus for C_1 sufficiently large, by (3.6), we have

$$\begin{aligned}&\sum_{\gamma\in\overline{\mathcal{T}}} \sum_{\gamma'\in U_\gamma} \mathbb{P}[V_\gamma \cap V_{\gamma'}] \\ &\quad\le \sum_{\gamma\in\overline{\mathcal{T}}} \sum_{s\le|\gamma|-C_1} \sum_{\{\gamma'\in U_\gamma : \gamma_{|s} = \gamma\wedge\gamma'\}} \mathbb{P}[V_\gamma] \cdot \frac{C_0^\alpha}{\mathrm{e}^{\alpha\delta(|\gamma|-s)} N^{|\gamma'|-s}} \\ &\quad\le \sum_{\gamma\in\overline{\mathcal{T}}} \mathbb{P}[V_\gamma] \sum_{s\le|\gamma|-C_1} \sum_{0\le j\le(1/C_1)(|\gamma|-s)} \sum_{\{\gamma'\in U_\gamma : \gamma_{|s} = \gamma\wedge\gamma', |\gamma'| = |\gamma| - jC_1\}} \frac{C_0^\alpha}{\mathrm{e}^{\alpha\delta(|\gamma|-s)} N^{|\gamma'|-s}} \\ &\quad\le \sum_{\gamma\in\overline{\mathcal{T}}} \mathbb{P}[V_\gamma] \sum_{s\le|\gamma|-C_1} \sum_{0\le j\le(1/C_1)(|\gamma|-s)} \frac{C_0^\alpha}{\mathrm{e}^{\alpha\delta(|\gamma|-s)} N^{|\gamma|-jC_1-s}} \cdot N^{|\gamma|-jC_1-s-C_1} \\ &\quad\le \sum_{\gamma\in\overline{\mathcal{T}}} \mathbb{P}[V_\gamma] \sum_{s\le|\gamma|-C_1} \frac{C_0^\alpha(|\gamma| - s)}{C_1 N^{C_1} \mathrm{e}^{\alpha\delta(|\gamma|-s)}} \\ &\quad\le \sum_{\gamma\in\overline{\mathcal{T}}} \mathbb{P}[V_\gamma] \frac{C_0^\alpha}{C_1 N^{C_1} \mathrm{e}^{\alpha\delta C_1/2}} \le \frac{1}{2} \sum_{\gamma\in\overline{\mathcal{T}}} \mathbb{P}[V_\gamma].\end{aligned}$$

Finally, combining the above estimates with (3.4) and (3.5), we obtain

$$\mathbb{P}\biggl[\bigcup_{\gamma\in\overline{\mathcal{T}}} V_\gamma\biggr] \ge \frac{1}{2} \frac{C}{N^{C_1} C_1 t^\alpha}.$$

□

Appendix: Proof of Proposition 2.2

Proof. We proceed as in [8] and for reader's convenience we use the same notation. Define $X_i = \log|\widetilde{A}_i|$ and $\widehat{S}_n = \frac{1}{n}\sum_{i=1}^n X_i$. We introduce a new probability measure: $\widetilde{\mu}(dx) = N e^{\alpha x}\mu(dx)$, where μ is the law of X_i. Next, we normalize X_i and we define new random variables: $Y_i = \frac{X_i - N\rho}{\sqrt{\Lambda''(\alpha)}}$ and $W_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i$. Then $\mathbb{E}_{\widetilde{\mu}} Y_i = 0$ and

$$\widehat{S}_n - N\rho = \frac{\lambda}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i = \frac{\lambda}{\sqrt{n}} W_n,$$

where $\lambda = \sqrt{\Lambda''(\alpha)}$ and $\Lambda(s) = \log(\mathbb{E}[|\widetilde{A}_1|^s])$. Let F_n be the distribution of W_n with respect to the changed measure $\widetilde{\mu}$. Let $\psi_n = \alpha\lambda\sqrt{n}$. Then,

$$\begin{aligned}
\mathbb{P}\big\{|\widetilde{A}_1\cdots\widetilde{A}_n| > e^d e^{N\rho n}\big\} &= \mathbb{P}\{\widehat{S}_n > N\rho + d/n\} \\
&= \mathbb{P}\left\{W_n > \frac{d}{\lambda\sqrt{n}}\right\} = \mathbb{E}_{\widetilde{\mu}}\big[N^{-n}|\widetilde{A}_1\cdots\widetilde{A}_n|^{-\alpha}\mathbf{1}_{\{W_n > d/(\lambda\sqrt{n})\}}\big] \\
&= e^{-\alpha n\rho N} N^{-n}\mathbb{E}_{\widetilde{\mu}}\big[e^{-\psi_n W_n}\mathbf{1}_{\{W_n > d/(\lambda\sqrt{n})\}}\big] \\
&= e^{-\alpha n\rho N} N^{-n}\int_{d/(\lambda\sqrt{n})}^{\infty} e^{-\psi_n x}\, dF_n(x).
\end{aligned}$$

We will use here the Berry–Esseen expansion for nonlattice distributions of F_n (see [10], page 538):

$$\lim_{n\to\infty}\left(\sqrt{n}\sup_x\left|F_n(x) - \Phi(x) - \frac{m_3}{6\sqrt{n}}\big(1 - x^2\big)\phi(x)\right|\right) = 0, \tag{A.1}$$

where $m_3 = \mathbb{E}_{\widetilde{\mu}}[Y_1^3] < \infty$, $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the standard normal density, and $\Phi(x) = \int_{-\infty}^x \phi(y)\, dy$ is its distribution function.

First, we integrate by parts and then we use the above result

$$\begin{aligned}
J &= \alpha\lambda\sqrt{n} e^{N\rho\alpha n} N^n \mathbb{P}\big\{|\widetilde{A}_1\cdots\widetilde{A}_n| > e^d e^{N\rho n}\big\} \\
&= \int_{d/(\lambda\sqrt{n})}^{\infty} \psi_n e^{-\psi_n x}\, dF_n(x) \\
&= \psi_n e^{-\psi_n x} F_n(x)\Big|_{d/(\lambda\sqrt{n})}^{\infty} + \int_{d/(\lambda\sqrt{n})}^{\infty} \psi_n^2 e^{-\psi_n x} F_n(x)\, dx \\
&= -\psi_n e^{-\alpha d} F_n\left(\frac{d}{\lambda\sqrt{n}}\right) + \int_{\alpha d}^{\infty} \psi_n e^{-x} F_n\left(\frac{x}{\psi_n}\right) dx \\
&= \int_{\alpha d}^{\infty} \psi_n e^{-x}\left[F_n\left(\frac{x}{\psi_n}\right) - F_n\left(\frac{d}{\lambda\sqrt{n}}\right)\right] dx
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{o}(1)\mathrm{e}^{-\alpha d} + \int_{\alpha d}^{\infty} \psi_n \mathrm{e}^{-x}\left[\Phi\left(\frac{x}{\psi_n}\right) - \Phi\left(\frac{d}{\lambda\sqrt{n}}\right)\right]\mathrm{d}x \\
&\quad + \frac{m_3}{6\sqrt{n}}\int_{\alpha d}^{\infty} \psi_n \mathrm{e}^{-x}\left[\left(1-\left(\frac{x}{\psi_n}\right)^2\right)\phi\left(\frac{x}{\psi_n}\right) - \left(1-\left(\frac{d}{\lambda\sqrt{n}}\right)^2\right)\phi\left(\frac{d}{\lambda\sqrt{n}}\right)\right]\mathrm{d}x.
\end{aligned}
$$

We denote the second term by $I(n)$ and the third one by $II(n)$. Thus,

$$
J(n) = \mathrm{o}(1)\mathrm{e}^{-\alpha d} + I(n) + II(n).
$$

We estimate first I:

$$
\begin{aligned}
\sqrt{2\pi} I(n) &= \int_{\alpha d}^{\infty} \psi_n \mathrm{e}^{-x} \int_{d/(\lambda\sqrt{n})}^{x/(\psi_n)} \mathrm{e}^{-y^2/2}\,\mathrm{d}y\,\mathrm{d}x = \int_{d/(\lambda\sqrt{n})}^{\infty} \psi_n \mathrm{e}^{-y^2/2} \int_{\psi_n y}^{\infty} \mathrm{e}^{-x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{d/(\lambda\sqrt{n})}^{\infty} \psi_n \mathrm{e}^{-\psi_n y} \mathrm{e}^{-y^2/2}\,\mathrm{d}y = -\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\Big|_{d/(\lambda\sqrt{n})}^{\infty} - \int_{d/(\lambda\sqrt{n})}^{\infty} y\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&= \mathrm{e}^{-\alpha d}\mathrm{e}^{-d^2/(2\lambda^2 n)} - \int_{d/(\lambda\sqrt{n})}^{\infty} y\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y.
\end{aligned}
$$

Let $\delta > 0$. We divide the last integral into two parts

$$
\int_{d/(\lambda\sqrt{n})}^{\infty} y\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y = \int_{d/(\lambda\sqrt{n})}^{d/(\lambda\sqrt{n})+\delta/\lambda} y\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y + \int_{d/(\lambda\sqrt{n})+\delta/\lambda}^{\infty} y\mathrm{e}^{-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y
$$

and denote the first one by $I_1(n)$ and the second one by $I_2(n)$. Then

$$
\mathrm{e}^{\alpha d} I_1(n) = \int_{d/(\lambda\sqrt{n})}^{d/(\lambda\sqrt{n})+\delta/\lambda} y\mathrm{e}^{\alpha d-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y \le \frac{\delta}{\lambda}\frac{\theta+\delta}{\lambda}\cdot \mathrm{e}^{-d^2/(2\lambda^2 n)}
$$

and large n we have

$$
\mathrm{e}^{\alpha d} I_2(n) = \int_{d/(\lambda\sqrt{n})+\delta/\lambda}^{\infty} y\mathrm{e}^{\alpha d-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y \le \mathrm{e}^{-\alpha\delta\sqrt{n}}\mathrm{e}^{-d^2/(2\lambda^2 n)} \le \delta\mathrm{e}^{-d^2/(2\lambda^2 n)}.
$$

Thus, we have proved that for large n

$$
\sqrt{2\pi}\mathrm{e}^{\alpha d} I(n) = \mathrm{e}^{-d^2/(2\lambda^2 n)}\big(1 + C(\theta)\delta\big).
$$

We may also write for any $d \ge 0$

$$
\int_{d/(\lambda\sqrt{n})}^{\infty} y\mathrm{e}^{\alpha d-\psi_n y}\mathrm{e}^{-y^2/2}\,\mathrm{d}y \le \int_{d/(\lambda\sqrt{n})}^{\infty} y\mathrm{e}^{-y^2/2}\,\mathrm{d}y \le \mathrm{e}^{-d^2/(2\lambda^2 n)}.
$$

Hence,

$$
\sqrt{2\pi}\mathrm{e}^{\alpha d}\big|I(n)\big| \le 2\mathrm{e}^{-d^2/(2\lambda^2 n)}.
$$

Now we compute the second term $II(n)$. Denote $g(x) = \sqrt{2\pi}(1 - x^2)\phi(x)$. Then

$$\begin{aligned}
\sqrt{2\pi}II(n) &= \frac{m_3\alpha\lambda}{6}\int_{\alpha d}^{\infty} \mathrm{e}^{-x}\left[g\left(\frac{x}{\psi_n}\right) - g\left(\frac{d}{\lambda\sqrt{n}}\right)\right]\mathrm{d}x \\
&= C\int_{\alpha d}^{\infty} \mathrm{e}^{-x}\int_{d/(\lambda\sqrt{n})}^{x/\psi_n} g'(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= C\int_{d/(\lambda\sqrt{n})}^{\infty} g'(y)\int_{\psi_n y}^{\infty} \mathrm{e}^{-x}\,\mathrm{d}x\,\mathrm{d}y \\
&= C\int_{d/(\lambda\sqrt{n})}^{\infty} \mathrm{e}^{-\psi_n y} g'(y)\,\mathrm{d}y.
\end{aligned}$$

Hence,

$$\sqrt{2\pi}\left|II(n)\right| \le C\int_{d/(\lambda\sqrt{n})}^{\infty} \mathrm{e}^{-\psi_n y}\,\mathrm{d}y = -\frac{C}{\psi_n}\mathrm{e}^{-\psi_n y}\bigg|_{d/(\lambda\sqrt{n})}^{\infty} = \frac{C}{\lambda\alpha\sqrt{n}}\mathrm{e}^{-\alpha d},$$

and so

$$\mathrm{e}^{\alpha d}\left|II(n)\right| = \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$

Finally,

$$\sqrt{2\pi}\mathrm{e}^{\alpha d} J \le \mathrm{o}(1) + 2\mathrm{e}^{-d^2/(2\lambda^2 n)} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right),$$

which shows (2.3) and

$$\sqrt{2\pi}\mathrm{e}^{\alpha d} J = \mathrm{o}(1) + \mathrm{e}^{-d^2/(2\lambda^2 n)}\big(1 + C(\theta)\delta\big) + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$

We may always take $\delta = \delta(n) = \mathrm{o}(1)$. Hence (2.5) follows. □

Acknowledgements

The authors were supported in part by NCN Grant DEC-2012/05/B/ST1/00692.

References

[1] Addario-Berry, L. and Reed, B. (2009). Minima in branching random walks. *Ann. Probab.* **37** 1044–1079. MR2537549

[2] Alsmeyer, G., Biggins, J.D. and Meiners, M. (2012). The functional equation of the smoothing transform. *Ann. Probab.* **40** 2069–2105. MR3025711

[3] Alsmeyer, G., Damek, E. and Mentemeier, S. Precise tail index of fixed points of the two-sided smoothing transform. arXiv:1206.3970v1.
[4] Alsmeyer, G. and Meiners, M. (2012). Fixed points of inhomogeneous smoothing transforms. *J. Difference Equ. Appl.* **18** 1287–1304. MR2956046
[5] Alsmeyer, G. and Meiners, M. (2013). Fixed points of the smoothing transform: Two-sided solutions. *Probab. Theory Related Fields* **155** 165–199. MR3010396
[6] Bassetti, F. and Ladelli, L. (2012). Self-similar solutions in one-dimensional kinetic models: A probabilistic view. *Ann. Appl. Probab.* **22** 1928–1961. MR3025685
[7] Buraczewski, D., Damek, E., Mentemeier, S. and Mirek, M. (2013). Heavy tailed solutions of multivariate smoothing transforms. *Stochastic Process. Appl.* **123** 1947–1986. MR3038495
[8] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. New York: Springer. MR1619036
[9] Durrett, R. and Liggett, T.M. (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete* **64** 275–301. MR0716487
[10] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II. Second Edition*. New York: Wiley. MR0270403
[11] Goldie, C.M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1** 126–166. MR1097468
[12] Hu, Y. and Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.* **37** 742–789. MR2510023
[13] Jelenković, P.R. and Olvera-Cravioto, M. (2010). Information ranking and power laws on trees. *Adv. in Appl. Probab.* **42** 1057–1093. MR2796677
[14] Jelenković, P.R. and Olvera-Cravioto, M. (2012). Implicit renewal theory and power tails on trees. *Adv. in Appl. Probab.* **44** 528–561. MR2977407
[15] Jelenković, P.R. and Olvera-Cravioto, M. (2012). Implicit renewal theorem for trees with general weights. *Stochastic Process. Appl.* **122** 3209–3238. MR2946440
[16] Neininger, R. and Rüschendorf, L. (2004). A general limit theorem for recursive algorithms and combinatorial structures. *Ann. Appl. Probab.* **14** 378–418. MR2023025
[17] Rösler, U. (2001). On the analysis of stochastic divide and conquer algorithms. *Algorithmica* **29** 238–261. Average-case analysis of algorithms (Princeton, NJ, 1998). MR1887306
[18] Volkovich, Y. and Litvak, N. (2010). Asymptotic analysis for personalized web search. *Adv. in Appl. Probab.* **42** 577–604. MR2675117

Received October 2012 and revised October 2013