# Approximation of a stochastic wave equation in dimension three, with application to a support theorem in Hölder norm

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A characterization of the support in Hölder norm of the law of the solution to a stochastic wave equation with three-dimensional space variable is proved. The result is a consequence of an approximation theorem, in the convergence of probability, for a sequence of evolution equations driven by a family of regularizations of the driving noise.

*Keywords:* approximating schemes; stochastic wave equation; support theorem

## 1. Introduction

In this paper, we consider a stochastic wave equation with three-dimensional spatial variable, and we prove a characterization of the topological support of the law of the solution in a space of Hölder continuous functions.

We focus on the stochastic partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t, x) = \sigma\big(u(t, x)\big)\dot{M}(t, x) + b\big(u(t, x)\big),$$

$$u(0, x) = \frac{\partial}{\partial t}u(0, x) = 0, \tag{1.1}$$

where $\Delta$ denotes the Laplacian on $\mathbb{R}^3$, $T > 0$ is fixed, $t \in {]0, T]}$ and $x \in \mathbb{R}^3$. The non-linear terms are defined by functions $\sigma, b : \mathbb{R} \to \mathbb{R}$. The notation $\dot{M}(t, x)$ refers to the formal derivative of a Gaussian random field white in the time variable and with a non-trivial covariance in space. More explicitly, on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ we consider a Gaussian process $M = \{M(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+3})\}$, where $\mathcal{C}_0^\infty(\mathbb{R}^{1+3})$ denotes the space of infinitely differentiable functions with compact support. We assume that $E(M(\varphi)) = 0$ and that the covariance function of $M$ is given by

$$\mathbb{E}\big(M(\varphi)M(\psi)\big) = \int_{\mathbb{R}^+} \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}x)\big(\varphi(s, \cdot) \star \tilde{\psi}(s, \cdot)\big)(x), \tag{1.2}$$

where "$\star$" denotes the convolution operator in the spatial argument and $\tilde{\psi}(t, x) = \psi(t, -x)$. We suppose that $\Gamma$ is a measure on $\mathbb{R}^3$ absolutely continuous with respect to the Lebesgue measure

with density $f$ given by

$$f(x) = |x|^{-\beta}, \qquad x \in \mathbb{R}^3 \setminus \{0\}, \beta \in \, ]0, 2[. \tag{1.3}$$

Let $\mathcal{S}(\mathbb{R}^3)$ be the space of rapidly decreasing functions on $\mathbb{R}^3$. We denote by $\mathcal{F}$ the Fourier transform operator defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^3} \varphi(x) \exp\left(-2\pi i (\xi \cdot x)\right) dx,$$

where the notation "$\cdot$" stands for the Euclidean inner product. The covariance function (1.2) can also be written as

$$\mathbb{E}\left(M(\varphi)M(\psi)\right) = \int_0^\infty ds \int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F}\varphi(s)(\xi) \overline{\mathcal{F}\psi(s)(\xi)},$$

where $\mu = \mathcal{F}^{-1} f$.

We introduce the Hilbert space $\mathcal{H}$ defined by the completion of $\mathcal{S}(\mathbb{R}^3)$ endowed with the semi-inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)}.$$

Assume that $\varphi \in \mathcal{H}$ is a finite measure. Then [14], Lemma 12.12, page 162, gives

$$\|\varphi\|_{\mathcal{H}}^2 = C \int_{\mathbb{R}^3} \left|\mathcal{F}\varphi(\xi)\right|^2 |\xi|^{-(3-\beta)} d\xi = C \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(dx)\varphi(dy)|x-y|^{-\beta} dx \, dy, \tag{1.4}$$

for some finite constant $C$. This identity extends easily to signed finite measures $\varphi \in \mathcal{H}$, by using the decomposition into a difference of positive finite measures. We will apply (1.4) to $\varphi(dx) := G(t, dx)Z(t, x)$, where $G(t, dx)$ is the fundamental solution to the wave equation (the definition is given later) and $Z(t, x)$ is an a.s. finite random variable.

The spaces $\mathcal{H}$ and $\mathcal{H}_t := L^2([0, t]; \mathcal{H})$, $t \in \, ]0, T]$, will play an important role throughout the paper. It is useful to introduce an isometric representation of theses spaces, as follows. Consider a complete orthonormal basis $(e_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3)$ of $\mathcal{H}$. Then the mappings

$$\mathcal{I} : \mathcal{H} \to \ell^2, \qquad \mathcal{I}_T : \mathcal{H}_T \to L^2\left([0, T]; \ell^2\right)$$

defined by

$$\mathcal{I}(g) = \left(\langle g, e_j \rangle_{\mathcal{H}}\right)_{j \in \mathbb{N}}, \qquad \mathcal{I}_T(\varphi)(t) = \left(\langle \varphi(t, *), e_j \rangle_{\mathcal{H}}\right)_{j \in \mathbb{N}}, \qquad t \in [0, T],$$

respectively, are isometries. This provides an identification of $\mathcal{H}$, $\mathcal{H}_T$ with $\ell^2$, $L^2([0, T]; \ell^2)$, respectively.

In a similar vein, the Gaussian process $M$ admits a representation as a sequence $(W_j(t), t \in [0, T])_{j \in \mathbb{N}}$ of independent real-valued standard Brownian motions (see, e.g., [9], Proposi-

tion 2.5). Indeed, this is given by the formula

$$W_j(t) := M(1_{[0,t]}e_j), \qquad j \in \mathbb{N}, t \in [0, T].$$

We refer the reader to [7] for a rigorous derivation of $M(1_{[0,t]}e_j)$ from the process $M$.

Along with the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, we will consider the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ generated by the process $\{W_j(t), j \in \mathbb{N}, t \in [0, T]\}$.

Let $G(t)$ be the fundamental solution to the wave equation in dimension three. It is well-known that $G(t, dx) = \frac{1}{4\pi t}\sigma_t(dx)$, where $\sigma_t(x)$ denotes the uniform surface measure on the sphere of radius $t$ with total mass $4\pi t^2$ (see [11]). We interpret (1.1) as the evolution equation

$$
\begin{aligned}
u(t, x) = & \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)\sigma\big(u(s, y)\big)M(ds, dy) \\
& + \int_0^t \big[G(t - s, \cdot) \star b\big(u(s, \cdot)\big)\big](x)\, ds,
\end{aligned}
\tag{1.5}
$$

where the stochastic integral (also termed stochastic convolution) in (1.5) is defined as

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)\sigma\big(u(s, y)\big)M(ds, dy) \\
& := \sum_{j \in \mathbb{N}} \int_0^t \big\langle G(t - s, x - *)\sigma\big(u(s, *)\big), e_j\big\rangle_{\mathcal{H}} W_j(ds).
\end{aligned}
\tag{1.6}
$$

The notation on the left-hand side of this identity suggests an integration with respect to the martingale measure derived from the Gaussian process $M$, as has been considered in [6], while on the right-hand side, there is an Itô integral with respect to the infinite-dimensional Brownian motion $W = (W_j, j \in \mathbb{N})$. It follows from [9], Propositions 2.6, 2.9, that if $Y(t, x) := \sigma(u(t, x))$, $(t, x) \in [0, T] \times \mathbb{R}^3$, satisfies $\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(|Y(t, x)|^2) < \infty$, then both integrals coincide.

Assume that the functions $\sigma$ and $b$ are Lipschitz continuous. With the definition (1.6), Theorem 4.3 in [9] gives the existence and uniqueness of a random field solution to equation (1.5) satisfying $\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(|u(t, x)|^p) < \infty$, for any $p \in [1, \infty[$. This means a real-valued adapted stochastic process such that (1.5) holds a.s. for all $(t, x) \in [0, T] \times \mathbb{R}^3$. In Theorem A.1, we will give an extension of this result.

In [10], equation (1.5) has been formulated using the stochastic integral introduced by Dalang and Mueller in [8], and a theorem of existence and uniqueness of a random field solution is proved. Moreover, it is also established that the sample paths are almost surely Hölder continuous jointly in $(t, x)$, with degree $\rho$. For the particular covariance density given in (1.3), $\rho \in ]0, \frac{2-\beta}{2}[$. Appealing to [9], Proposition 2.11, this property holds for the solution of (1.5) with the choice of stochastic integral made in (1.6). More precisely, fix $t_0 \in [0, T]$ and a compact set $K \subset \mathbb{R}^3$. For any $\rho \in ]0, 1[$, and every real function $g$, set

$$\|g\|_{\rho,t_0,K} := \sup_{(t,x) \in [t_0,T] \times K} |g(t, x)| + \sup_{\substack{(t,x),(\bar{t},\bar{x}) \in [t_0,T] \times K \\ t \neq \bar{t}, x \neq \bar{x}}} \frac{|g(t, x) - g(\bar{t}, \bar{x})|}{(|t - \bar{t}| + |x - \bar{x}|)^\rho}.$$

We denote by $\mathcal{C}^\rho([t_0, T] \times K)$ the space of real functions $g$ such that $\|g\|_{\rho,t_0,K} < \infty$. Then [10], Theorem 4.11, shows that, for any $\rho \in ]0, \frac{2-\beta}{2}[$, $\|u\|_{\rho,t_0,K} \leq c$, a.s., where $c$ is a finite random variable, a.s. This result tells us that the law of the solution of (1.5), when restricted to $[t_0, T] \times K$, is a probability on $\mathcal{C}^\rho([t_0, T] \times K)$, with $\rho \in ]0, \frac{2-\beta}{2}[$.

The analysis of the topological support under different kinds of norms, like the supremum norm, Hölder norm, weighted Sobolev norms, has been extensively studied for diffusion processes. As a representative sample of references, let us mention [4,12,13,19]. Inspired by [1], Millet and Sanz-Solé have introduced a method for the characterization of the support of a random vector based exclusively on approximations. For solutions to stochastic equations, such approximations entail regularizations of the noise. The paper [16] illustrates the suitability of the method by giving a very simplified proof of an extension of Stroock's support theorem for diffusions. Moreover, the method has also been successfully applied to several examples of stochastic partial differential equations, like a reduced wave equation with $d = 1$, a stochastic heat equation with $d = 1$ and a stochastic wave equation with $d = 2$ (see [3,15] and [17], resp.).

A motivation to study the support of a stochastic evolution equation lies in the analysis of the uniqueness of invariant measures. Recently, R. Cont and D. Fournié have proved results on functional Kolmogorov equations in the framework of a functional Itô calculus (see [5]). Assumptions concerning the support of some functionals play a crucial role in their results. This provides an additional motivation for our work.

In this paper, we apply the approximation method of [15] to obtain a characterization of the topological support of the law of $u$ (the solution to (1.5)) in the Hölder norm $\|\cdot\|_{\rho,t_0,K}$. The core of the work consists of an approximation result for a family of equations more general than equation (1.5) by a sequence of pathwise evolution equations obtained by a smooth approximation of the driving process $M$. In finite dimensions, the celebrated Wong–Zakai approximations for diffusions in the supremum norm could be considered as an analogue. However there are two substantial differences, first the type of equation we consider in this paper is much more complex, and moreover we deal with a stronger topology.

For the sake of completeness, we give a brief description of the procedure of [15] in the particular context of this work, and refer the reader to [15] for further details.

Let $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mu})$ be the canonical space of a standard real-valued Brownian motion on $[0, T]$. In the sequel, the reference probability space will be $(\Omega, \mathcal{G}, \mathbb{P}) := (\bar{\Omega}^{\mathbb{N}}, \bar{\mathcal{G}}^{\otimes \mathbb{N}}, \bar{\mu}^{\otimes \mathbb{N}})$. By the preceding identification of $M$ with $(W_j, j \in \mathbb{N})$, this is the canonical probability space of $M$.

Assume that there exists a measurable mapping $\xi_1 : L^2([0, T]; \ell^2) \to \mathcal{C}^\rho([t_0, T] \times K)$, and a sequence $w^n : \Omega \to L^2([0, T]; \ell^2)$ such that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\{\|u - \xi_1(w^n)\|_{\rho,t_0,K} > \varepsilon\} = 0. \tag{1.7}$$

Then $\mathrm{supp}(u \circ \mathbb{P}^{-1}) \subset \overline{\xi_1(L^2([0, T]; \ell^2))}$, where the closure refers to the Hölder norm $\|\cdot\|_{\rho,t_0,K}$.

Next, we assume that there exists a mapping $\xi_2 : L^2([0, T]; \ell^2) \to \mathcal{C}^\rho([t_0, T] \times K)$ and for any $h \in L^2([0, T]; \ell^2)$, we suppose that there exist a sequence $T_n^h : \Omega \to \Omega$ of measurable transformations such that, for any $n \geq 1$, the probability $\mathbb{P} \circ (T_n^h)^{-1}$ is absolutely continuous with respect

to $\mathbb{P}$ and, for any $h \in L^2([0, T]; \ell^2)$, $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\big\{\big\| u\big(T_n^h\big) - \xi_2(h)\big\|_{\rho, t_0, K} < \varepsilon\big\} > 0. \tag{1.8}$$

Then $\mathrm{supp}(u \circ \mathbb{P}^{-1}) \supset \overline{\xi_2(L^2([0, T]; \ell^2))}$.

For any $h \in L^2([0, T]; \ell^2)$ (or equivalently, $h \in \mathcal{H}_T$), consider the deterministic evolution equation

$$\Phi^h(t, x) = \big\langle G(t - \cdot, x - *)\sigma\big(\Phi^h(\cdot, *)\big), h\big\rangle_{\mathcal{H}_t} + \int_0^t \mathrm{d}s\big[G(t - s, \cdot) \star \big(\Phi^h(s, \cdot)\big)\big](x). \tag{1.9}$$

Similarly as for $u$, the mapping $(t, x) \in [t_0, T] \times K \mapsto \Phi^h(t, x)$ belongs to $\mathcal{C}^\rho([t_0, T] \times K)$.

Let $\xi_1(h) = \xi_2(h) = \Phi^h$, and $(w^n)_{n \geq 1}$ be given by (2.1). From (2.2) and the isometric representation of $\mathcal{H}_T$, we see that $w^n : \Omega \to L^2([0, T]; \ell^2)$. Given $h \in L^2([0, T]; \ell^2)$, we define

$$T_n^h(\omega) = \omega + h - w^n. \tag{1.10}$$

By Girsanov's theorem, the probability $\mathbb{P} \circ (T_n^h)^{-1}$ is absolutely continuous with respect to $\mathbb{P}$.

According to (1.7), (1.8), the final objective is to prove that

$$\lim_{n \to \infty} \Phi^{w^n} = u, \qquad \lim_{n \to \infty} u\big(T_n^h\big) = \Phi^h,$$

in probability and with the Hölder norm $\|\cdot\|_{\rho, t_0, T}$. Then, by the preceding discussion we infer that the support of the law of $u$ in the Hölder norm is the closure of the set of functions $\{\Phi^h, h \in \mathcal{H}_T\}$ (see Theorem 3.1 for the rigorous statement). Notice that the characterization of the support does not depend on the approximating sequence $(w^n)_{n \in \mathbb{N}}$.

The paper is structured as follows. The next Section 2 is devoted to a general approximation result. This is the hard core of the work (see Theorem 2.2). We postpone for a while a more extensive description of its content. Section 3 is devoted to the proof of the characterization of the support of $u$. It is a corollary of Theorem 2.2. Section 4 is of technical character. It is devoted to establish some auxiliary results which are needed in some proofs of Section 2. In the Appendix, a theorem on existence and uniqueness of a random field solution for a quite general evolution equation is proved. It provides the rigorous setting for all the stochastic partial differential equations that appear in this paper. The section also contains two known but fundamental results used at some crucial parts of the proofs of Sections 2 and 3.

We end this introduction with a more detailed description of Section 2 devoted to the proof of the approximation result (see Theorem 2.2). The method we use is similar as in [17], where the case $d = 2$ was studied. Nevertheless, for $d = 3$ its implementation entails substantial differences and new difficulties. The reason for this is that the fundamental solution of the wave equation in dimension three is a measure and not a real-valued function, as in dimension two.

As was formulated in [3], and further developed in [17], there are two main elements in the proof of Theorem 2.2: a control on the $L^p(\Omega)$-increments in time and in space of the processes $X$ and $X_n$, independently of $n$, and $L^p(\Omega)$ convergence of $X_n(t, x)$ to $X(t, x)$, for any $(t, x)$. The precise assertions are given in Theorems 2.3 and 2.4, respectively.

For the sake of illustration, we sketch one of the difficulties encountered in the proof of Theorem 2.3. Consider either stochastic or pathwise integrals with integrands of the form

$$\left[G(\bar{t} - s, x - \mathrm{d}y) - G(t - s, \bar{x} - \mathrm{d}y)\right]Z(s, y), \qquad 0 < t \leq \bar{t} \leq T, x, \bar{x} \in \mathbb{R}^3,$$

where $Z(s, y)$ is a stochastic process. We want estimates of some norms of these expressions in terms of powers of the increments $|\bar{t} - t|$, $|\bar{x} - x|$. In dimension $d = 2$, $G(t, \mathrm{d}x) = G(t, x)\,\mathrm{d}x$ and the problem is solved using direct computations on the function differences $G(\bar{t} - s, x - y) - G(t - s, \bar{x} - y)$. For $d = 3$, this approach fails.

In [10], this problem was tackled by passing increments of the measure $G$ to increments of $Z$, by means of a change of variables. We shall apply repeatedly this idea throughout the paper. However, there are some significant differences between the arguments in [10] and those used here. In [10], the formulation of equation (1.5) is based on Dalang–Mueller stochastic integral – a functional type integral in the spatial variable developed in [8]. Hence, pointwise arguments in the space variable are excluded. Instead they use fractional Sobolev norms and Sobolev's embedding theorem. Moreover, in [10] a regularization of the distribution $G$ is systematically used and final results are obtained by passing to the limit. With the selection of the stochastic integral given in (1.6) it is not necessary to appeal to Sobolev's embedding theorem. Moreover, applying (1.4) we avoid the regularization of $G$. There is yet another difference that deserves to be mentioned. In [10], non-null initial conditions were considered, while here $u_0 = v_0 = 0$. As a consequence, the random fields $X_n$ and $X$ possess the stationary property described in Remark 2.1. This fact is frequently used in the proofs.

For an Itô's stochastic differential equation, smoothing the noise leads to a Stratonovich (or pathwise) type integral, and the correction term between the two kinds of integrals appears naturally in the approximating scheme. In our setting, correction terms explode and therefore they must be avoided. Instead, a control on the growth of the regularized noise is used. This method was introduced in [17] and successfully applied here too. The control is achieved by introducing a localization in $\Omega$ (see (2.10)). With this method, the convergence of the approximating sequence $X_n$ to $X$ takes place in probability.

Let us finally remark that using the method of the proof of Theorem 2.3, a different but simplified proof of [10], Theorem 4.11, in the particular case of null initial conditions can be provided.

Throughout the paper, we shall often call different positive and finite constants by the same notation, even if they differ from one place to another.

## 2. Approximations of the wave equation

Consider smooth approximations of $W$ defined as follows. Fix $n \in \mathbb{N}$ and consider the partition of $[0, T]$ determined by $\frac{iT}{2^n}$, $i = 0, 1, \ldots, 2^n$. Denote by $\Delta_i$ the interval $[\frac{iT}{2^n}, \frac{(i+1)T}{2^n}[$ and by $|\Delta_i|$ its length. We write $W_j(\Delta_i)$ for the increment $W_j(\frac{(i+1)T}{2^n}) - W_j(\frac{iT}{2^n})$, $i = 0, \ldots, 2^n - 1$, $j \in \mathbb{N}$. Define differentiable approximations of $(W_j, j \in \mathbb{N})$ as follows:

$$W^n = \left(W_j^n = \int_0^\cdot \dot{W}_j^n(s)\,\mathrm{d}s, j \in \mathbb{N}\right),$$

where for $j > n$, $\dot{W}_j^n = 0$, and for $1 \le j \le n$,

$$
\dot{W}_j^n(t) = \begin{cases} \displaystyle\sum_{i=0}^{2^n-2} 2^n T^{-1} W_j(\Delta_i) 1_{\Delta_{i+1}}(t) & \text{if } t \in \left[2^{-n}T, T\right], \\[4mm] 0 & \text{if } t \in \left[0, 2^{-n}T\right[. \end{cases}
$$

Set

$$
w^n(t, x) = \sum_{j \in \mathbb{N}} \dot{W}_j^n(t) e_j(x). \tag{2.1}
$$

It is easy to check that, for any $p \in [2, \infty[$,

$$
\left\| w^n \right\|_{L^p(\Omega, \mathcal{H}_T)} \le C n^{1/2} 2^{n/2}. \tag{2.2}
$$

In particular, from (2.2) it follows that $w^n$ belongs to $\mathcal{H}_T$ a.s.

In this section, we shall consider the equations

$$
\begin{aligned}
X(t, x) = {}& \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)\big(X(s, y)\big) M(\mathrm{d}s, \mathrm{d}y) \\
&+ \big\langle G(t - \cdot, x - *) D\big(X(\cdot, *)\big), h \big\rangle_{\mathcal{H}_t} \\
&+ \int_0^t \big[ G(t - s, \cdot) \star b\big(X(s, \cdot)\big) \big](x) \, \mathrm{d}s,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
X_n(t, x) = {}& \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y) A\big(X_n(s, y)\big) M(\mathrm{d}s, \mathrm{d}y) \\
&+ \big\langle G(t - \cdot, x - *) B\big(X_n(\cdot, *)\big), w^n \big\rangle_{\mathcal{H}_t} \\
&+ \big\langle G(t - \cdot, x - *) D\big(X_n(\cdot, *)\big), h \big\rangle_{\mathcal{H}_t} \\
&+ \int_0^t \big[ G(t - s, \cdot) \star b\big(X_n(s, \cdot)\big) \big](x) \, \mathrm{d}s,
\end{aligned} \tag{2.4}
$$

where $n \in \mathbb{N}$, $h \in \mathcal{H}_T$, $w^n$ defined as in (2.1) and $A, B, D, b : \mathbb{R} \to \mathbb{R}$.

Moreover, we also need the slight modification of these equations defined by

$$
\begin{aligned}
X_n^-(t, x) = {}& \int_0^{t_n} \int_{\mathbb{R}^3} G(t - s, x - y) A\big(X_n(s, y)\big) M(\mathrm{d}s, \mathrm{d}y) \\
&+ \big\langle G(t - \cdot, x - *) B\big(X_n(\cdot, *)\big) 1_{[0, t_n]}(\cdot), w^n \big\rangle_{\mathcal{H}_t} \\
&+ \big\langle G(t - \cdot, x - *) D\big(X_n(\cdot, *)\big) 1_{[0, t_n]}(\cdot), h \big\rangle_{\mathcal{H}_t} \\
&+ \int_0^{t_n} \big[ G(t - s, \cdot) \star b\big(X_n(s, \cdot)\big) \big](x) \, \mathrm{d}s,
\end{aligned} \tag{2.5}
$$

$$X(t, t_n, x) = \int_0^{t_n} \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)\big(X(s, y)\big)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \big\langle G(t - \cdot, x - *)D\big(X(\cdot, *)\big)1_{[0, t_n]}(\cdot), h\big\rangle_{\mathcal{H}_t} \tag{2.6}$$

$$+ \int_0^{t_n} \big[G(t - s, \cdot) \star b\big(X(s, \cdot)\big)\big](x)\, \mathrm{d}s,$$

where for any $n \in \mathbb{N}$, $t \in [0, T]$, $t_n = \max\{\underline{t}_n - 2^{-n}T, 0\}$, with

$$\underline{t}_n = \max\big\{k2^{-n}T, k = 1, \ldots, 2^n - 1 : k2^{-n}T \le t\big\}. \tag{2.7}$$

We will consider the following assumption.

**Hypothesis (B).** *The coefficients $A$, $B$, $D$, $b : \mathbb{R} \mapsto \mathbb{R}$ are globally Lipschitz continuous.*

Notice that equation (2.4) is more general than (2.3) and (1.5). In Theorem A.1, we prove a result on existence and uniqueness of a random field solution to a class of SPDEs which applies to equation (2.4).

**Remark 2.1.** As a consequence of Remark A.2, we have the following translation invariance of moments:

$$\mathbb{E}\big(\big|X(t, x - y - z) - X(t, y - z)\big|^p\big) = \mathbb{E}\big(\big|X(t, x - y) - X(t, y)\big|^p\big),$$

$$\mathbb{E}\big(\big|X_n(t, x - y - z) - X_n(t, y - z)\big|^p\big) = \mathbb{E}\big(\big|X_n(t, x - y) - X_n(t, y)\big|^p\big), \tag{2.8}$$

for any $x, y, z \in \mathbb{R}^3$ and any $p \in [1, \infty[$. Consequently, a similar property also holds for $X_n^-(t, *)$ and $X_n(t, t_n, *)$ defined in (2.5), (2.6), respectively

The aim of this section is to prove the following theorem.

**Theorem 2.2.** *We assume Hypothesis (B). Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then for any $\rho \in ]0, \frac{2-\beta}{2}[$ and $\lambda > 0$,*

$$\lim_{n \to \infty} \mathbb{P}\big(\|X_n - X\|_{\rho, t_0, K} > \lambda\big) = 0. \tag{2.9}$$

The convergence (2.9) will be proved through several steps. The main ingredients are local $L^p$ estimates of increments of $X_n$ and $X$, in time and in space, and a local $L^p$ convergence of the sequence $X_n(t, x)$ to $X(t, x)$.

Let us describe the *localization* procedure (see [17]). Fix $\alpha > 0$. For any integer $n \ge 1$ and $t \in [0, T]$, define

$$L_n(t) = \Big\{\sup_{1 \le j \le n} \sup_{0 \le i \le [2^n t T^{-1} - 1]^+} \big|W_j(\Delta_i)\big| \le \alpha n^{1/2} 2^{-n/2}\Big\}, \tag{2.10}$$

where $\alpha > (2\ln 2)^{1/2}$. Notice that the sets $L_n(t)$ decrease with $t \geq 0$. Moreover, in [17], Lemma 2.1, it is proved that $\lim_{n\to\infty} \mathbb{P}(L_n(t)^c) = 0$.

It is easy to check that

$$\big\| w^n(t, *) 1_{L_n(t)} \big\|_{\mathcal{H}} \leq C n^{3/2} 2^{n/2}. \tag{2.11}$$

Moreover, for any $0 \leq t \leq t' \leq T$

$$\big\| w^n 1_{L_n(t')} 1_{[t,t']} \big\|_{\mathcal{H}_T} \leq C n^{3/2} 2^{n/2} \big| t' - t \big|^{1/2}.$$

In particular, if $[t, t'] \subset \Delta_i$ for some $i = 0, \ldots, 2^n - 1$, then

$$\big\| w^n 1_{L_n(t')} 1_{[t,t']} \big\|_{\mathcal{H}_T} \leq C n^{3/2}. \tag{2.12}$$

As has been announced in the Introduction, the proof of Theorem 2.2 will follow from Theorems 2.3 and 2.4 below. We denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm.

**Theorem 2.3.** *We assume Hypothesis* (B). *Fix* $t_0 \in \,]0, T[$ *and a compact subset* $K \subset \mathbb{R}^3$. *Let* $t_0 \leq t \leq \bar{t} \leq T$, $x, \bar{x} \in K$. *Then, for any* $p \in [1, \infty)$ *and any* $\rho \in \,]0, \frac{2-\beta}{2}[$, *there exists a positive constant* $C$ *such that*

$$\sup_{n \geq 1} \big\| [X_n(t, x) - X_n(\bar{t}, \bar{x})] 1_{L_n(\bar{t})} \big\|_p \leq C \big( |\bar{t} - t| + |\bar{x} - x| \big)^\rho. \tag{2.13}$$

**Theorem 2.4.** *The assumptions are the same as in Theorem* 2.3. *Fix* $t \in [t_0, T]$, $x \in \mathbb{R}^3$. *Then, for any* $p \in [1, \infty)$

$$\lim_{n\to\infty} \sup_{\substack{t \in [t_0, T] \\ x \in K(t)}} \big\| (X_n(t, x) - X(t, x)) 1_{L_n(t)} \big\|_p = 0, \tag{2.14}$$

*where for* $t \in [0, T]$,

$$K(t) = \big\{ x \in \mathbb{R}^3 : d(x, K) \leq T - t \big\},$$

*and* $d$ *denotes the Euclidean distance.*

The proof of Theorem 2.3 is carried out through two steps. First, we shall consider $t = \bar{t}$ and obtain (2.13), uniformly in $t \in [t_0, T]$. Using this, we will consider $x = \bar{x}$ and establish (2.13), uniformly in $x \in K$. We devote the next two subsections to the proof of these results.

## 2.1. Increments in space

Throughout this section, we fix $t_0 \in \,]0, T[$ and a compact set $K \subset \mathbb{R}^3$. The objective is to prove the following proposition.

**Proposition 2.5.** *Suppose that Hypothesis* (B) *holds. Fix* $t \in [t_0, T]$ *and* $x, \bar{x} \in K$. *Then, for any* $p \in [1, \infty)$ *and* $\rho \in \, ]0, \frac{2-\beta}{2}[$, *there exists a finite constant* $C$ *such that*

$$\sup_{n \geq 0} \sup_{t \in [t_0, T]} \left\| \left( X_n(t, x) - X_n(t, \bar{x}) \right) 1_{L_n(t)} \right\|_p \leq C |x - \bar{x}|^\rho. \tag{2.15}$$

In the next lemma, we give an abstract result that will be used throughout the proofs. We start by introducing some notation.

For a function $f : \mathbb{R}^3 \to \mathbb{R}$, we set

$$Df(u, x) = f(u + x) - f(u),$$

$$\bar{D}^2 f(u, x, y) = f(u + x + y) - f(u + x) - f(u + y) + f(u),$$

$$D^2 f(u, x) = \bar{D}^2 f(u - x, x, x) = f(u - x) - 2f(u) + f(u + x).$$

**Lemma 2.6.** *Consider a sequence of predictable stochastic processes* $\{Z_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$, $n \in \mathbb{N}$, *such that, for any* $p \in [2, \infty[$,

$$\sup_{n} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\left( \left| Z_n(t, x) \right|^p \right) < C, \tag{2.16}$$

*for some finite constant* $C$. *For any* $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^3$, *we define*

$$I_n(t, x, \bar{x}) := \int_0^t ds \left\| Z_n(s, *)\left[ G(t - s, x - *) - G(t - s, \bar{x} - *) \right] \right\|_{\mathcal{H}}^2.$$

*Then, for any* $p \in [2, \infty[$,

$$\mathbb{E}\left( \left| I_n(t, x, \bar{x}) \right|^{p/2} \right)$$

$$\leq C \left\{ |x - \bar{x}|^{\alpha_2 p/2} + \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\left( \left| Z_n(s, x - y) - Z_n(s, \bar{x} - y) \right|^p \right) \right] \right. \tag{2.17}$$

$$\left. + |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\left( \left| Z_n(s, x - y) - Z_n(s, \bar{x} - y) \right|^p \right) \right]^{1/2} \right\},$$

*where* $\alpha_1 \in \, ]0, (2 - \beta) \wedge 1[$ *and* $\alpha_2 \in \, ]0, (2 - \beta)[$.

**Proof.** First, we notice that $I_n(t, x, \bar{x})$ is the second order moment of the stochastic integral

$$\int_0^t \int_{\mathbb{R}^3} Z_n(s, y) \left[ G(t - s, x - y) - G(t - s, \bar{x} - y) \right] M(ds, dy).$$

We write $I_n(t, x, \bar{x})$ using (1.4). This yields

$$I_n(t, x, \bar{x}) = C \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Z_n(s, u) Z_n(s, v) \left[ G(t - s, x - du) - G(t - s, \bar{x} - du) \right]$$

$$\times \left[ G(t - s, x - dv) - G(t - s, \bar{x} - dv) \right] |u - v|^{-\beta}.$$

Then, as in [10], pages 19–20, we see that, by decomposing this expression into the sum of four integrals, by applying a change of variables and rearranging terms, we have

$$I_n(t, x, \bar{x}) = C \sum_{i=1}^{4} J_i^t(x, \bar{x}),$$

where, for $i = 1, \ldots, 4$,

$$J_i^t(x, \bar{x}) = \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) h_i(x, \bar{x}; t, s, u, v)$$

with

$$
\begin{aligned}
h_1(x, \bar{x}; t, s, u, v) &= f(\bar{x} - x + v - u)\big[Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)\big] \\
&\quad \times \big[Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)\big], \\
h_2(x, \bar{x}; t, s, u, v) &= Df(v - u, x - \bar{x}) Z_n(t - s, x - u) \\
&\quad \times \big[Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)\big], \\
h_3(x, \bar{x}; t, s, u, v) &= Df(v - u, \bar{x} - x) Z_n(t - s, \bar{x} - v) \\
&\quad \times \big[Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)\big], \\
h_4(x, \bar{x}; t, s, u, v) &= -D^2 f(v - u, x - \bar{x}) Z_n(t - s, x - u) Z_n(t - s, x - v).
\end{aligned}
$$

Fix $p \in [2, \infty[$. It holds that

$$\mathbb{E}\big(|I_n(t, x, \bar{x})|^{p/2}\big) \leq C \sum_{i=1}^{4} \mathbb{E}\big(|J_i^t(x, \bar{x})|^{p/2}\big). \tag{2.18}$$

The next purpose is to obtain estimates for each term on the right hand-side of (2.18). Let

$$\mu_1(x, \bar{x}) = \sup_{s \in [0,T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) f(\bar{x} - x + v - u).$$

We recall that the inverse Fourier transform of $f(x) = |x|^\beta$ is given by $\mu(d\xi) = |\xi|^{-(3-\beta)} d\xi$, and that $\mathcal{F}G(t, *)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$. Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) f(\bar{x} - x + v - u) &\leq \int_{\mathbb{R}^3} \big|\mathcal{F}G(s, *)(\xi)\big|^2 \mu(d\xi) \\
&= \int_{\mathbb{R}^3} \frac{\sin^2(2\pi s|\xi|)}{4\pi^2 |\xi|^{5-\beta}} d\xi.
\end{aligned}
$$

Consequently, for any $\beta \in ]0, 2[$, $\sup_{x, \bar{x}} \mu_1(x, \bar{x}) < \infty$ (see [10] for a similar result).

Hence using firstly Hölder's inequality and then Cauchy–Schwarz's inequality, we see that

$$
\begin{aligned}
\mathbb{E}\big(\big|J_1^t(x,\bar{x})\big|^{p/2}\big) & \\
&\leq \left(\int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s,du)G(s,dv)f(\bar{x}-x+v-u)\right)^{(p/2)-1} \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s,du)G(s,dv)f(\bar{x}-x+v-u) \\
&\quad \times \mathbb{E}\big(\big|\big[Z_n(t-s,x-u)-Z_n(t-s,\bar{x}-u)\big] \\
&\quad \times \big[Z_n(t-s,x-v)-Z_n(t-s,\bar{x}-v)\big]\big|^{p/2}\big) \\
&\leq C \sup_{x,\bar{x}} \mu_1(x,\bar{x})^{p/2} \\
&\quad \times \int_0^t ds \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z_n(t-s,x-y)-Z_n(t-s,\bar{x}-y)\big|^p\big) \\
&\leq C \int_0^t ds \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z_n(s,x-y)-Z_n(s,\bar{x}-y)\big|^p\big).
\end{aligned}
\tag{2.19}
$$

Set

$$
\mu_2(x,\bar{x}) = \sup_{s\in[0,T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|Df(v-u,x-\bar{x})\big|.
\tag{2.20}
$$

The following property holds: there exists a positive finite constant $C$ such that

$$
\mu_2(x,\bar{x}) \leq C|x-\bar{x}|^{\alpha_1}, \qquad \alpha_1 \in \,]0,(2-\beta)\wedge 1[.
$$

Indeed, this follows from a slight modification of the proof of [10], Lemma 6.1.

Using Hölder's and Cauchy–Schwarz's inequalities, along with (2.16), we have

$$
\begin{aligned}
\mathbb{E}\big(\big|J_2^t(x,\bar{x})\big|^{p/2}\big) &\leq C\left(\int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|Df(v-u,x-\bar{x})\big|\right)^{(p/2)-1} \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|Df(v-u,x-\bar{x})\big| \\
&\quad \times \mathbb{E}\big(\big|Z_n(t-s,x-u)\big[Z_n(t-s,x-v)-Z_n(t-s,\bar{x}-v)\big]\big|^{p/2}\big) \\
&\leq C|x-\bar{x}|^{\alpha_1 p/2} \int_0^t ds \Big[\sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z_n(s,x-y)-Z_n(s,\bar{x}-y)\big|^p\big)\Big]^{1/2},
\end{aligned}
\tag{2.21}
$$

with $\alpha_1 \in \,]0,(2-\beta)\wedge 1[$.

Similarly,

$$\mathbb{E}\big(\big|J_3^t(x,\bar{x})\big|^{p/2}\big) \le C|x-\bar{x}|^{\alpha_1 p/2} \int_0^t ds \Big[\sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z_n(s,x-y)-Z_n(s,\bar{x}-y)\big|^p\big)\Big]^{1/2}, \quad (2.22)$$

with $\alpha_1 \in \,]0, (2-\beta)\wedge 1[$.

Let

$$\mu_4(x,\bar{x}) = \sup_{s\in[0,T]} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|D^2 f(v-u,x-\bar{x})\big|.$$

Following the arguments of the proof of Lemma 6.2 in [10], we see that, for any $\alpha_2 \in \,]0, (2-\beta)[$,

$$\mu_4(x,\bar{x}) \le C|x-\bar{x}|^{\alpha_2}.$$

Then, Hölder's and Cauchy–Schwarz's inequalities, along with (2.16), imply

$$\begin{aligned}
\mathbb{E}\big(\big|J_4^t(x,\bar{x})\big|^{p/2}\big) &\le C\Big(\int_0^t ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|D^2 f(v-u,x-\bar{x})\big|\Big)^{(p/2)-1}\\
&\quad \times \int_0^t ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,du)G(s,dv)\big|D^2 f(v-u,x-\bar{x})\big|\\
&\quad \times \mathbb{E}\big(\big|Z_n(t-s,x-u)Z_n(t-s,\bar{x}-v)\big|^{p/2}\big)\\
&\le C|x-\bar{x}|^{\alpha_2 p/2}\int_0^t ds \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z_n(t-s,y)\big|^p\big)\\
&\le C|x-\bar{x}|^{\alpha_2 p/2}.
\end{aligned} \qquad (2.23)$$

From (2.18), (2.19), (2.21), (2.22) and (2.23), we obtain (2.17). $\qquad\square$

For any $t \in [t_0, T]$, $x, \bar{x} \in K$, $p \in [1,\infty[$, we set

$$\varphi_{n,p}^0(t,x,\bar{x}) = \mathbb{E}\big(\big|X_n(t,x)-X_n(t,\bar{x})\big|^p 1_{L_n(t)}\big),$$

$$\varphi_{n,p}^-(t,x,\bar{x}) = \mathbb{E}\big(\big|X_n^-(t,x)-X_n^-(t,\bar{x})\big|^p 1_{L_n(t)}\big),$$

$$\varphi_{n,p}(t,x,\bar{x}) = \varphi_n^0(t,x,\bar{x}) + \varphi_n^-(t,x,\bar{x}).$$

Proposition 2.5 is a consequence of the following assertion.

**Proposition 2.7.** *The hypotheses are the same as in Proposition 2.5. Fix $t \in [t_0, T]$, $x, \bar{x} \in K$. Then, for any $p \in [1,\infty[$, $\rho \in \,]0, \frac{2-\beta}{2}[$,*

$$\sup_{n\ge 0} \varphi_{n,p}(t,x,\bar{x}) \le C|x-\bar{x}|^{\rho p}. \qquad (2.24)$$

The proof of this proposition relies on the next lemma and a version of Gronwall's lemma quoted in Lemma A.3.

**Lemma 2.8.** *We assume the same hypotheses as in Proposition 2.5. For any $n \geq 1$, $t \in [t_0, T]$, $x, \bar{x} \in K$, $p \in [2, \infty[$, there exists a finite constant $C$ (not depending on $n$) such that*

$$
\varphi_{n,p}(t, x, \bar{x}) \leq C \left[ f_n + |x - \bar{x}|^{\alpha_2 p/2} + \int_0^t ds \left( \varphi_{n,p}(s, x, \bar{x}) \right) \right.
$$
$$
\left. + |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \left( [\varphi_{n,p}^0(s, x, \bar{x})]^{1/2} + [\varphi_{n,p}^-(s, x, \bar{x})]^{1/2} \right) \right], \tag{2.25}
$$

*where $(f_n, n \geq 1)$ is a sequence of real numbers which converges to zero as $n \to \infty$, $\alpha_1 \in [0, (2 - \beta \wedge 1)[$, $\alpha_2 \in ]0, 2 - \beta[$.*

We postpone the proof of this lemma to the end of this section.

**Proof of Proposition 2.7.** Fix $t \in [t_0, T]$, $x, \bar{x} \in K$, $p \in [2, \infty[$. From Lemma 2.8 along with Jensen's inequality, we have

$$
\varphi_{n,p}(t, x, \bar{x})^2 \leq C \left\{ f_n^2 + |x - \bar{x}|^{\alpha_2 p} + \int_0^t ds \left( \varphi_{n,p}(s, x, \bar{x}) \right)^2 \right.
$$
$$
\left. + |x - \bar{x}|^{\alpha_1 p} \int_0^t ds \left( [\varphi_{n,p}^0(s, x, \bar{x})]^{1/2} + [\varphi_{n,p}^-(s, x, \bar{x})]^{1/2} \right)^2 \right\}
$$
$$
\leq C \left\{ f_n^2 + |x - \bar{x}|^{\alpha_2 p} + \int_0^t ds \left( \varphi_{n,p}(s, x, \bar{x}) \right)^2 \right.
$$
$$
\left. + |x - \bar{x}|^{\alpha_1 p} \int_0^t ds \left( \varphi_{n,p}(s, x, \bar{x}) \right) \right\}.
$$

Since the sequence $(f_n, n \geq 1)$ is bounded, there exists a constant $C_0$ satisfying

$$
\sup_n f_n^2 \leq C_0 t_0 \leq C_0 t \leq C \int_0^t ds \left[ 1 + \left( \varphi_{n,p}(s, x, \bar{x}) \right)^2 \right]
$$

for any $t \in [t_0, T]$. Thus, for some positive constant $C$,

$$
1 + \varphi_{n,p}(t, x, \bar{x})^2 \leq C \left\{ |x - \bar{x}|^{\alpha_2 p} + \int_0^t ds \left[ 1 + \left( \varphi_{n,p}(s, x, \bar{x}) \right)^2 \right] \right.
$$
$$
\left. + |x - \bar{x}|^{\alpha_1 p} \int_0^t ds \left[ 1 + \varphi_{n,p}(s, x, \bar{x})^2 \right]^{1/2} \right\}.
$$

We apply Lemma A.3 in the following particular situation: $u(t) = \varphi_{n,p}(t, x, \bar{x})^2 + 1$, $a = C|x - \bar{x}|^{\alpha_2 p}$, $b(s) \equiv C$, $k(s) \equiv C|x - \bar{x}|^{\alpha_1 p}$, $\bar{p} = \bar{q} = \frac{1}{2}$, $\alpha = 0$, $\beta = T$. This yields

$$
\varphi_{n,p}(t, x, \bar{x})^2 + 1 \leq C \left[ |x - \bar{x}|^{2\alpha_1 p} + |x - \bar{x}|^{\alpha_2 p} \right],
$$

which trivially implies

$$\varphi_{n,p}(t, x, \bar{x}) \leq C\big[|x - \bar{x}|^{\alpha_1 p} + |x - \bar{x}|^{\alpha_2 p/2}\big]. \tag{2.26}$$

We recall that $\alpha_1 \in ]0, (2 - \beta) \wedge 1[$ and $\alpha_2 \in ]0, (2 - \beta)[$. Therefore, (2.26) implies (2.24). This ends the proof of Proposition 2.7. $\qquad\square$

**Proof of Lemma 2.8.** Fix $p \in [2, \infty[$. From (2.4), we have the following:

$$\varphi_{n,p}^0(t, x, \bar{x}) := \mathbb{E}\big(\big|X_n(t, x) - X_n(t, \bar{x})\big|^p 1_{L_n(t)}\big) \leq C \sum_{i=1}^{6} R_n^i(t, x, \bar{x}),$$

with

$$R_n^1(t, x, \bar{x}) = \mathbb{E}\bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} \big[G(t - s, x - y) - G(t - s, \bar{x} - y)\big]A\big(X_n(s, y)\big)M(\mathrm{d}s, \mathrm{d}y)\bigg|^p 1_{L_n(t)}\bigg),$$

$$R_n^2(t, x, \bar{x}) = \mathbb{E}\big(\big|\langle[G(t - \cdot, x - *) - G(t - \cdot, \bar{x} - *)]B\big(X_n^-(\cdot, *)\big), w^n\rangle_{\mathcal{H}_t}\big|^p 1_{L_n(t)}\big),$$

$$R_n^3(t, x, \bar{x}) = \mathbb{E}\big(\big|\langle[G(t - \cdot, x - *) - G(t - \cdot, \bar{x} - *)]\big[B(X_n) - B\big(X_n^-\big)\big](\cdot, *), w^n\rangle_{\mathcal{H}_t}\big|^p 1_{L_n(t)}\big),$$

$$R_n^4(t, x, \bar{x}) = \mathbb{E}\big(\big|\langle[G(t - \cdot, x - *) - G(t - \cdot, \bar{x} - *)]D\big(X_n(\cdot, *)\big), h\rangle_{\mathcal{H}_t}\big|^p 1_{L_n(t)}\big),$$

$$R_n^5(t, x, \bar{x}) = \mathbb{E}\bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} \big[G(t - s, x - \mathrm{d}y) - G(t - s, \bar{x} - \mathrm{d}y)\big]b\big(X_n(s, y)\big)\mathrm{d}s\bigg|^p 1_{L_n(t)}\bigg).$$

Using Burkholder's inequality and then Plancherel's identity, we have

$$R_n^1(t, x, \bar{x}) = \mathbb{E}\bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} \big[G(t - s, x - y) - G(t - s, \bar{x} - y)\big]A\big(X_n(s, y)\big)M(\mathrm{d}s, \mathrm{d}y)\bigg|^p 1_{L_n(t)}\bigg)$$

$$= \mathbb{E}\bigg(\bigg|\sum_{j \in \mathbb{N}} \int_0^t \langle[G(t - s, x - *)$$

$$- G(t - s, \bar{x} - *)]A\big(X_n(s, *)\big), e_k(*)\rangle_{\mathcal{H}} \mathrm{d}W_j(s)\bigg|^p 1_{L_n(t)}\bigg)$$

$$\leq C\mathbb{E}\bigg(\bigg[\int_0^t \mathrm{d}s \sum_{j \in \mathbb{N}} |\langle[G(t - s, x - *)$$

$$- G(t - s, \bar{x} - *)]A\big(X_n(s, *)\big), e_k(*)\rangle_{\mathcal{H}}|^2 1_{L_n(s)}\bigg]\bigg)^{p/2} \tag{2.27}$$

$$= C\mathbb{E}\bigg(\bigg|\int_0^t \mathrm{d}s \big\|[G(t - s, x - *) - G(t - s, \bar{x} - *)]A\big(X_n(s, *)\big)\big\|_{\mathcal{H}}^2 1_{L_n(s)}\bigg)^{p/2}.$$

The process $\{Z_n(t, x) := A(X_n(t, x))1_{L_n(t)}, (t, x) \in [0, T] \times \mathbb{R}^3\}$ satisfies the assumption (2.16). Indeed, this is a consequence of the linear growth of $A$ and (4.9). Then, by applying Lemma 2.6

and using the Lipschitz continuity of $A$, we obtain

$$R_n^1(t, x, \bar{x}) \leq C \left\{ |x - \bar{x}|^{\alpha_2 p/2} \right.$$

$$+ \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E} \left( |X_n(s, x - y) - X_n(s, \bar{x} - y)|^p 1_{L_n(s)} \right) \right] \quad (2.28)$$

$$\left. + |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E} \left( |X_n(s, x - y) - X_n(s, \bar{x} - y)|^p 1_{L_n(s)} \right) \right]^{1/2} \right\},$$

with $\alpha_1 \in ]0, (2 - \beta) \wedge 1[$ and $\alpha_2 \in ]0, (2 - \beta)[$.

For a given function $\rho : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $t \in [0, T]$, let $\tau_n$ be the operator defined by

$$\tau_n(\rho) = \rho \left( (s + 2^{-n}) \wedge t, x \right). \quad (2.29)$$

Let $\mathcal{E}_n$ be the closed subspace of $\mathcal{H}_T$ generated by the orthonormal system of functions

$$2^n T^{-1} 1_{\Delta_i}(\cdot) \otimes e_j(*), \qquad i = 0, \ldots, 2^n - 1, \ j = 1, \ldots, n,$$

and denote by $\pi_n$ the orthogonal projection on $\mathcal{E}_n$. Notice that $\pi_n \circ \tau_n$ is a bounded operator on $\mathcal{H}_T$, uniformly in $n$.

Since $X_n^-(s, *)$ is $\mathcal{F}_{s_n}$-measurable, by using the definition of $w^n$ we easily see that

$$R_n^2(t, x, \bar{x}) = \mathbb{E} \left( \left| \int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n) \left( [G(t - \cdot, x - *) - G(t - \cdot, \bar{x} - *)] \right. \right. \right.$$

$$\left. \left. \left. \times B \left( X_n^-(\cdot, *) \right) \right) (s, y) M(ds, dy) \right|^p 1_{L_n(s)} \right).$$

By Burkholder's inequality and the properties of the operator $\pi_n \circ \tau_n$, this last expression is bounded up to a constant by

$$\mathbb{E} \left( \int_0^t ds \left\| \left[ G(t - s, x - *) - G(t - s, \bar{x} - *) \right] B \left( X_n^-(s, *) \right) \right\|_{\mathcal{H}}^2 1_{L_n(s)} \right)^{p/2}.$$

The properties of the function $B$ along with (4.9) imply that the process $\{ Z_n(t, x) := B(X_n^-(t, x)) 1_{L_n(t)}, (t, x) \in [0, T] \times \mathbb{R}^3 \}$ satisfies the hypotheses of Lemma 2.6. This yields

$$R_n^2(t, x, \bar{x}) \leq C \left\{ |x - \bar{x}|^{\alpha_2 p/2} \right.$$

$$+ \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E} \left( |X_n^-(s, x - y) - X_n^-(s, \bar{x} - y)|^p 1_{L_n(s)} \right) \right] \quad (2.30)$$

$$\left. + |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E} \left( |X_n^-(s, x - y) - X_n^-(s, \bar{x} - y)|^p 1_{L_n(s)} \right) \right]^{1/2} \right\},$$

where as before, $\alpha_1 \in ]0, (2 - \beta) \wedge 1[$ and $\alpha_2 \in ]0, (2 - \beta)[$.

Cauchy–Schwarz's inequality along with (2.11) yield

$$R_n^3(t, x, \bar{x}) \leq Cn^{3p/2}2^{np/2}$$
$$\times \mathbb{E}\bigg(\int_0^t ds \big\|\big[G(t-s, x-*) - G(t-s, \bar{x}-*)\big]$$
$$\times \big[B(X_n) - B(X_n^-)\big](s, *)1_{L_n(s)}\big\|_{\mathcal{H}}^2\bigg)^{p/2}.$$

Notice that an upper bound for the second factor on the right-hand side of the preceding inequality could be obtained using Lemma 2.6 with $Z_n(t, x) := [B(X_n(t, x)) - B(X_n^-(t, x))]1_{L_n}(t)$. However, this would not be a good strategy to compensate the first factor (which explodes when $n \to \infty$). Instead, we will try to quantify the discrepancy between $B(X_n(t, x))$ and $B(X_n^-(t, x))$. This can be achieved by transferring again the increments of the Green function to increments of the process

$$\hat{B}(X_n(t, x)) = \big[B(X_n(t, x)) - B(X_n^-(t, x))\big], \tag{2.31}$$

in the same manner as we did in the proof of Lemma 2.6 (see [10], pages 19–20).

Indeed, similarly as in (2.18), we obtain

$$R_n^3(t, x, \bar{x}) \leq Cn^{3p/2}2^{np/2}\sum_{i=1}^4 \mathbb{E}\big(\big|K_i^t(x, \bar{x})\big|^{p/2}1_{L_n(t)}\big), \tag{2.32}$$

where for any $i = 1, \ldots, 4$, $K_i^t(x, \bar{x})$ is given by $J_i^t(x, \bar{x})$ of Lemma 2.6 with $Z_n$ replaced by $\hat{B}(X_n)$.

Using Remark 2.1, we have

$$\mathbb{E}\big(\big|X_n(t, x-y) - X_n(t, \bar{x}-y)\big|^p 1_{L_n(t)}\big) = \mathbb{E}\big(\big|X_n(t, x) - X_n(t, \bar{x})\big|^p 1_{L_n(t)}\big). \tag{2.33}$$

With this property and the definition of $\hat{B}(X_n)$ given in (2.31), we easily get

$$\mathbb{E}\big(\big|\hat{B}(X_n(s, x-y)) - \hat{B}(X_n(s, \bar{x}-y))\big|^p 1_{L_n(s)}\big)$$
$$\leq C\big[\mathbb{E}\big(\big|X_n(s, x-y) - X_n^-(s, x-y)\big|^p 1_{L_n(s)}\big)$$
$$+ \mathbb{E}\big(\big|X_n(s, \bar{x}-y) - X_n^-(s, \bar{x}-y)\big|^p 1_{L_n(s)}\big)\big] \tag{2.34}$$
$$\leq Cn^{3p/2}2^{-np(3-\beta)/2},$$

uniformly in $(s, x, y) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$, where the last bound is obtained by using (4.10). This estimate will be applied to the study of the right-hand side of (2.32).

For $i = 1$, (2.19) with $Z_n(s, y) := \hat{B}(X_n(s, y))1_{L_n(s)}$, along with (2.34) yields

$$\mathbb{E}\big(\big|K_1^t(x, \bar{x})\big|^{p/2}1_{L_n(t)}\big) \leq Cn^{3p/2}2^{-np(3-\beta)/2}. \tag{2.35}$$

Let $\mu_2(x, \bar{x})$ be as in (2.20). Since $x, \bar{x} \in K$, and $K$ is bounded,

$$\sup_{x,\bar{x}\in K} \mu_2(x, \bar{x}) \le C,$$

for some finite constant $C > 0$. Hence, (2.21), (2.22) (with the same choice of $Z_n$ as before) together with (2.34) gives

$$\mathbb{E}\big(\big|K_2^t(x, \bar{x})\big|^{p/2} 1_{L_n(t)}\big) + \mathbb{E}\big(\big|K_3^t(x, \bar{x})\big|^{p/2} 1_{L_n(t)}\big) \le C n^{3p/2} 2^{-np(3-\beta)/2}. \tag{2.36}$$

Proceeding as in (2.23), but replacing $Z_n(s, y)$ by $\hat{B}(X_n(s, y))1_{L_n(s)}$, we obtain

$$\mathbb{E}\big(\big|K_4^t(x, \bar{x})\big|^{p/2} 1_{L_n(t)}\big) \le C|x - \bar{x}|^{\alpha_2 p/2} \int_0^t ds \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|\hat{B}\big(X_n(s, y)\big)\big|^p 1_{L_n(s)}\big).$$

By the definition of $\hat{B}(X_n)$, and applying (4.10), we have

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|\hat{B}\big(X_n(s, y)\big)\big|^p 1_{L_n(s)}\big) \le C n^{3p/2} 2^{-np(3-\beta)/2}.$$

Thus,

$$\mathbb{E}\big(\big|K_4^t(x, \bar{x})\big|^{p/2} 1_{L_n(t)}\big) \le C n^{3p/2} 2^{-np(3-\beta)/2}. \tag{2.37}$$

Putting together (2.32) and (2.35)–(2.37) yields

$$R_n^3(t, x, \bar{x}) \le C f_n, \tag{2.38}$$

where $f_n = n^{3p} 2^{-np[(3-\beta)/2-1/2]}$. Since $\beta \in ]0, 2[$, $\lim_{n\to\infty} f_n = 0$.

The last part of the proof consists of getting estimates for the term $R_n^4(t, x, \bar{x})$. This is done using first Cauchy–Schwarz's inequality and then, applying Lemma 2.6 with $Z_n$ replaced by $D(X_n)1_{L_n}$. The Lipschitz continuity of $D$ along with the estimate (4.9) ensure that assumption (2.16) is satisfied. We obtain

$$R_n^4(t, x, \bar{x}) \le \|h\|_{\mathcal{H}_t}^p \mathbb{E}\big(\big\|\big[G(t - \cdot, x - *) - G(t - \cdot, \bar{x} - *)\big]D\big(X_n(\cdot, *)\big)1_{L_n(t)}\big\|_{\mathcal{H}_t}^2 \big|^{p/2}\big)$$

$$\le C\bigg\{|x - \bar{x}|^{\alpha_2 p/2} + \int_0^t ds \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, x - y) - X_n(s, \bar{x} - y)\big|^p 1_{L_n(s)}\big) \tag{2.39}$$

$$+ |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \Big[\sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, x - y) - X_n(s, \bar{x} - y)\big|^p 1_{L_n(s)}\big)\Big]^{1/2}\bigg\},$$

where $\alpha_1 \in ]0, (2 - \beta) \wedge 1[$ and $\alpha_2 \in ]0, (2 - \beta)[$.

After having applied the change of variable $u \mapsto x - \bar{x} + y$, we have

$$R_n^5(t, x, \bar{x}) = \mathbb{E}\bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} G(t - s, x - \mathrm{d}y)\big[b\big(X_n(s, y)\big) - b\big(X_n(s, y - x + \bar{x})\big)\big] ds\bigg|^p 1_{L_n(t)}\bigg).$$

Applying Hölder's inequality, we obtain

$$
\begin{aligned}
R_n^5(t, x, \bar{x}) \\
\leq \left( \int_0^t \int_{\mathbb{R}^3} G(t - s, x - dy)\, ds \right)^{p-1} \\
\times \int_0^t \int_{\mathbb{R}^3} G(t - s, x - dy) \mathbb{E}\big( \big| b\big(X_n(s, y)\big) - b\big(X_n(s, y - x + \bar{x})\big) \big|^p 1_{L_n(s)} \big)\, ds \\
\leq C \int_0^t ds \sup_{y \in \mathbb{R}^3} \mathbb{E}\big( \big| X_n(s, x - y) - X_n(s, \bar{x} - y) \big|^p 1_{L_n(s)} \big).
\end{aligned}
\tag{2.40}
$$

Bringing together the inequalities (2.28), (2.30), (2.38), (2.39) and (2.40), yields

$$
\begin{aligned}
\mathbb{E}\big( \big| X_n(t, x) - X_n(t, \bar{x}) \big|^p 1_{L_n(t)} \big) \\
\leq C \bigg\{ f_n + |x - \bar{x}|^{\alpha_2 p/2} + \int_0^t ds \Big[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\big( \big| X_n(s, x - y) - X_n(s, \bar{x} - y) \big|^p 1_{L_n(s)} \big) \Big] \\
+ |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \Big[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\big( \big| X_n(s, x - y) - X_n(s, \bar{x} - y) \big|^p 1_{L_n(s)} \big) \Big]^{1/2} \\
+ \int_0^t ds \Big[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\big( \big| X_n^-(s, x - y) - X_n^-(s, \bar{x} - y) \big|^p 1_{L_n(s)} \big) \Big] \\
+ |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \Big[ \sup_{y \in \mathbb{R}^3} \mathbb{E}\big( \big| X_n^-(s, x - y) - X_n^-(s, \bar{x} - y) \big|^p 1_{L_n(s)} \big) \Big]^{1/2} \bigg\}.
\end{aligned}
$$

By Remark 2.1, the right-hand side of this inequality is equal (up to a constant) to

$$
\begin{aligned}
f_n + |x - \bar{x}|^{\alpha_2 p/2} + \int_0^t \mathbb{E}\big( \big| X_n(s, x) - X_n(s, \bar{x}) \big|^p 1_{L_n(s)} \big)\, ds \\
+ \int_0^t \mathbb{E}\big( \big| X_n^-(s, x) - X_n^-(s, \bar{x}) \big|^p 1_{L_n(s)} \big)\, ds \\
+ |x - \bar{x}|^{\alpha_1 p/2} \int_0^t \big[ \mathbb{E}\big( \big| X_n(s, x) - X_n(s, \bar{x}) \big|^p 1_{L_n(s)} \big) \big]^{1/2}\, ds \\
+ |x - \bar{x}|^{\alpha_1 p/2} \int_0^t \big[ \mathbb{E}\big( \big| X_n^-(s, x) - X_n^-(s, \bar{x}) \big|^p 1_{L_n(s)} \big) \big]^{1/2}\, ds.
\end{aligned}
$$

With this, we see that $\varphi_{n,p}^0(t, x, \bar{x})$ is bounded by the right-hand side of (2.25).

Finally, we prove that the same bound holds for $\varphi_{n,p}^-(t, x, \bar{x})$ too. Indeed, For every $i = 1, \ldots, 5$, we consider the terms $R_n^i(t, x, \bar{x})$ defined in the first part of the proof, and we replace the domain of integration of the time variable $s$ ($[0, t]$) by $[0, t_n]$. We denote the corresponding

new expressions by $S_n^i(t, x, \bar{x})$. From (2.5), we obtain the following

$$\varphi_{n,p}^-(t, x, \bar{x}) \le C \sum_{i=1}^{5} S_n^i(t, x, \bar{x}).$$

Since $t_n \le t$, it can be checked that, similarly as for $R_n^i(t, x, \bar{x})$, $S_n^i(t, x, \bar{x})$, $i = 1, \dots, 5$, are bounded by (2.28), (2.30), (2.38), (2.39), (2.40), respectively. This ends the proof of the lemma. $\qquad\square$

## 2.2. Increments in time

Throughout this section, we fix $t_0 \in ]0, T]$, and a compact set $K \subset \mathbb{R}^3$. We shall prove the following proposition.

**Proposition 2.9.** *Assume that Hypothesis* (B) *holds. Fix $t, \bar{t} \in [t_0, T]$. Then for any $p \in [1, \infty)$ and $\rho \in ]0, \frac{2-\beta}{2}[$, there exists a finite constant $C$ such that*

$$\sup_{n \ge 1} \sup_{x \in K} \left\| (X_n(t, x) - X_n(\bar{t}, x)) 1_{L_n(\bar{t})} \right\|_p \le C|t - \bar{t}|^\rho. \tag{2.41}$$

The next lemma is meant to play a similar rôle than Lemma 2.6 but in this case, for integrals containing increments in time of the Green function $G(t)$.

**Lemma 2.10.** *Consider a sequence of stochastic processes $\{D_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$, $n \ge 1$, satisfying the following conditions:*
*For any $p \in [2, \infty[$,*

$$\sup_n \sup_{(t,x) \in [t_0,T] \times \mathbb{R}^3} \mathbb{E}\left(|D_n(t, x)|^p\right) \le C. \tag{2.42}$$

*There exists $\rho_1 > 0$ and for any $x, y \in K$,*

$$\sup_n \sup_{t \in [t_0,T]} \mathbb{E}\left(|D_n(t, x) - D_n(t, y)|^p\right) \le C|x - y|^{\rho_1 p}, \tag{2.43}$$

*where $C$ is a finite constant and $\rho_1 > 0$.*
*For $0 \le t_0 \le t \le \bar{t} \le T$ and $x \in K$, set*

$$J_n(t, \bar{t}, x) = \int_0^t ds \left\| D_n(x, *)[G(\bar{t} - s, x - *) - G(t - s, x - *)] \right\|_{\mathcal{H}}^2.$$

*Then, for any $p \in [2, \infty[$ there exists a finite constant $C > 0$ such that*

$$\mathbb{E}\left(J_n(t, \bar{t}, x)^{p/2}\right) \le C\left(|\bar{t} - t|^{\rho_1 p} + |\bar{t} - t|^{(\rho_1 + \alpha_1)p/2} + |\bar{t} - t|^{\alpha_2 p/2}\right), \tag{2.44}$$

*with $\alpha_1 \in ]0, 1 \wedge (2 - \beta)[$ and $\alpha_2 \in ]0, (2 - \beta)[$.*

**Proof.** First of all we notice that, as a consequence of Burkholder's inequality, the $L^p$-moment of the stochastic integral

$$\int_0^t \int_{\mathbb{R}^3} D_n(x, y)\big[G(\bar{t} - s, x - y) - G(t - s, x - y)\big]M(ds, dy),$$

is bounded up to a positive constant, by $\mathbb{E}(J_n(t, \bar{t}, x)^{p/2})$.

We write $J_n(t, \bar{t}, x)$ using (1.4). This gives

$$J_n(t, \bar{t}, x) = C \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} D_n(x, y)\big[G(\bar{t} - s, x - y) - G(t - s, x - y)\big]$$
$$\times D_n(x, z)\big[G(\bar{t} - s, x - z) - G(t - s, x - z)\big]|y - z|^{-\beta}.$$

Then, as in [10] page 28 (see the study of the term $T_2^n(t, \bar{t}, x)$ in this reference), we have

$$\mathbb{E}\big(J_n(t, \bar{t}, x)^{p/2}\big) \leq C \sum_{k=1}^4 \mathbb{E}\big(|Q^i(t, \bar{t}, x)|^{p/2}\big), \tag{2.45}$$

where for $i = 1, \ldots, 4$,

$$Q^i(t, \bar{t}, x) := \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - s, du)G(t - s, dv)r_i(t, \bar{t}, s, x, u, v) \tag{2.46}$$

and

$$r_1(t, \bar{t}, s, x, u, v) := \frac{\bar{t} - s}{t - s}f\left(v\frac{\bar{t} - s}{t - s} - u\right)\left[D_n\left(s, x - \frac{\bar{t} - s}{t - s}u\right) - D_n(s, x - u)\right]$$
$$\times \left[D_n\left(s, x - \frac{\bar{t} - s}{t - s}v\right) - D_n(s, x - v)\right],$$

$$r_2(t, \bar{t}, s, x, u, v) := \left\{\left(\frac{\bar{t} - s}{t - s}\right)^2 f\left(\frac{\bar{t} - s}{t - s}(v - u)\right) - \frac{\bar{t} - s}{t - s}f\left(v\frac{\bar{t} - s}{t - s} - u\right)\right\}$$
$$\times D_n\left(s, x - \frac{\bar{t} - s}{t - s}u\right)\left[D_n\left(s, x - \frac{\bar{t} - s}{t - s}v\right) - D_n(s, x - v)\right],$$

$$r_3(t, \bar{t}, s, x, u, v) := \left\{\left(\frac{\bar{t} - s}{t - s}\right)^2 f\left(\frac{\bar{t} - s}{t - s}(v - u)\right) - \frac{\bar{t} - s}{t - s}f\left(v - u\frac{\bar{t} - s}{t - s}\right)\right\}$$
$$\times \left[D_n\left(s, x - \frac{\bar{t} - s}{t - s}u\right) - D_n(s, x - u)\right]D_n(s, x - v),$$

$$r_4(t, \bar{t}, s, x, u, v) := \left\{\left(\frac{\bar{t} - s}{t - s}\right)^2 f\left(\frac{\bar{t} - s}{t - s}(v - u)\right) - \frac{\bar{t} - s}{t - s}f\left(v\frac{\bar{t} - s}{t - s} - u\right)\right.$$
$$\left. - \frac{\bar{t} - s}{t - s}f\left(v - u\frac{\bar{t} - s}{t - s}\right) + f(v - u)\right\}D_n(s, x - u)D_n(s, x - v).$$

Let

$$v_1(s, t, \bar{t}) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-s, du) G(t-s, dv) \frac{\bar{t}-s}{t-s} f\left(v\frac{\bar{t}-s}{t-s} - u\right).$$

Following the arguments of the proof of Lemma 6.3 in [10] (with $G_n$ replaced by $G$), we see that

$$\sup_{0 \le s \le t \le \bar{t} \le T} v_1(s, t, \bar{t}) < \infty. \tag{2.47}$$

Applying Hölder's and then Cauchy–Schwarz' inequalities, along with (2.43) yield

$$
\begin{aligned}
\mathbb{E}\left(\left|Q^1(t, \bar{t}, x)\right|^{p/2}\right) \le & \left(\sup_{0 \le s \le t \le \bar{t} \le T} v_1(s, t, \bar{t})\right)^{p/2-1} \\
& \times \int_0^t ds \int_{R^3} \int_{R^3} G(t-s, du) G(t-s, dv) \frac{\bar{t}-s}{t-s} f\left(v\frac{\bar{t}-s}{t-s} - u\right) \\
& \times \left[\mathbb{E}\left(\left|D_n\left(s, x-\frac{\bar{t}-s}{t-s}u\right) - D_n(s, x-u)\right|^p\right)\right]^{1/2} \\
& \times \left[\mathbb{E}\left(\left|D_n\left(s, x-\frac{\bar{t}-s}{t-s}v\right) - D_n(s, x-v)\right|^p\right)\right]^{1/2} \\
\le & C \int_0^t ds \int_{R^3} \int_{R^3} G(t-s, du) G(t-s, dv) \frac{\bar{t}-s}{t-s} f\left(v\frac{\bar{t}-s}{t-s} - u\right) \\
& \times \left|\frac{\bar{t}-t}{t-s}u\right|^{\rho_1 p/2} \left|\frac{\bar{t}-t}{t-s}v\right|^{\rho_1 p/2}.
\end{aligned}
\tag{2.48}
$$

The support of the measure $G(t)$ is $\{x \in \mathbb{R}^3 : |x| = t\}$. Using this property and (2.47), we obtain

$$\mathbb{E}\left(\left|Q^1(t, \bar{t}, x)\right|^{p/2}\right) \le C|t-\bar{t}|^{\rho_1 p}. \tag{2.49}$$

Let

$$
\begin{aligned}
v_2(s, t, \bar{t}) := & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-s, du) G(t-s, dv) \\
& \times \left|\left(\frac{\bar{t}-s}{t-s}\right)^2 f\left(\frac{\bar{t}-s}{t-s}(v-u)\right) - \frac{\bar{t}-s}{t-s} f\left(\frac{\bar{t}-s}{t-s}v - u\right)\right|.
\end{aligned}
$$

A slight modification of Lemma 6.4 in [10] (where $G_n$ is replaced by $G$), yields

$$\sup_{s \le t \le \bar{t} \le T} v_2(s, t, \bar{t}) \le C|t-\bar{t}|^{\alpha_1}, \tag{2.50}$$

with $\alpha_1 \in ]0, (2 - \beta) \wedge 1[$. Then, Hölder's and Cauchy–Schwarz's inequalities along with (2.42), (2.43) and (2.50) imply

$$
\mathbb{E}\big(\big|Q^2(t, \bar{t}, x)\big|^{p/2}\big) \leq \Big(\sup_{0 \leq s \leq t \leq \bar{t} \leq T} v_2(s, t, \bar{t})\Big)^{p/2-1} \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - s, \mathrm{d}u) G(t - s, \mathrm{d}v)
$$

$$
\times \left| \left(\frac{\bar{t} - s}{t - s}\right)^2 f\left(\frac{\bar{t} - s}{t - s}(v - u)\right) - \frac{\bar{t} - s}{t - s} f\left(\frac{\bar{t} - s}{t - s} v - u\right) \right|
$$

$$
\times \left[ \left( \mathbb{E}\left| D_n\left(s, x - \frac{\bar{t} - s}{t - s} u\right) \right|^p \right) \right]^{1/2}
$$

$$
\times \left[ \left( \mathbb{E}\left| D_n\left(s, x - \frac{\bar{t} - s}{t - s} v\right) - D_n(s, x - v) \right|^p \right) \right]^{1/2} \tag{2.51}
$$

$$
\leq C|\bar{t} - t|^{\rho_1 p/2} \Big(\sup_{0 \leq s \leq t \leq \bar{t} \leq T} v_2(s, t, \bar{t})\Big)^{p/2}
$$

$$
\leq C|\bar{t} - t|^{(\rho_1 + \alpha_1)p/2}, \tag{2.52}
$$

with $\alpha_1 \in ]0, 1 \wedge (2 - \beta)[$.

Similarly,

$$
\mathbb{E}\big(\big|Q^3(t, \bar{t}, x)\big|^{p/2}\big) \leq C|\bar{t} - t|^{(\rho_1 + \alpha_1)p/2}, \tag{2.53}
$$

$\alpha_1 \in ]0, 1 \wedge (2 - \beta)[$.

Define

$$
v_4(s, t, \bar{t}) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - s, \mathrm{d}u) G(t - s, \mathrm{d}v)
$$

$$
\times \left\{ \left(\frac{\bar{t} - s}{t - s}\right)^2 f\left(\frac{\bar{t} - s}{t - s}(v - u)\right) \right.
$$

$$
\left. - \frac{\bar{t} - s}{t - s} f\left(v\frac{\bar{t} - s}{t - s} - u\right) - \frac{\bar{t} - s}{t - s} f\left(v - u\frac{\bar{t} - s}{t - s}\right) + f(v - u) \right\}.
$$

Replacing $G_n$ by $G$ in [10], Lemma 6.5 yields

$$
\sup_{s \leq t \leq \bar{r} \leq T} v_4(s, t, \bar{t}) \leq C|\bar{t} - t|^{\alpha_2}, \tag{2.54}
$$

where $\alpha_2 \in ]0, (2 - \beta)[$.

By applying Hölder's and Cauchy–Schwarz's inequalities along with (2.42), we get

$$
\mathbb{E}\big(\big|Q^4(t, \bar{t}, x)\big|^{p/2}\big)
$$

$$
\leq \Big(\sup_{0 \leq s \leq t \leq \bar{t} \leq T} v_4(s, t, \bar{t})\Big)^{p/2-1} \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - s, \mathrm{d}u) G(t - s, \mathrm{d}v)
$$

$$\times \left\{ \left(\frac{\bar{t}-s}{t-s}\right)^2 f\left(\frac{\bar{t}-s}{t-s}(v-u)\right) - \frac{\bar{t}-s}{t-s} f\left(v\frac{\bar{t}-s}{t-s} - u\right) \right.$$

$$\left. - \frac{\bar{t}-s}{t-s} f\left(v - u\frac{\bar{t}-s}{t-s}\right) + f(v-u) \right\}$$

$$\times \left[\mathbb{E}\big(|D_n(s, x-u)|^p 1_{L_n(s)}\big)\right]^{1/2} \left[\mathbb{E}\big(|D_n(s, x-v)|^p\big)\right]^{1/2} \tag{2.55}$$

$$\leq C\left(\sup_{0\leq s\leq t\leq \bar{t}\leq T} v_4(s, t, \bar{t})\right)^{p/2}$$

$$\leq C|\bar{t}-t|^{\alpha_2 p/2}, \tag{2.56}$$

with $\alpha_2 \in {]0, (2-\beta)[}$.

The inequalities (2.49), (2.52), (2.53), (2.56), together with (2.45) imply (2.44). $\qquad\square$

**Proof of Proposition 2.9.** Fix $0 \leq t \leq \bar{t} \leq T$, $x \in K$, $p \in [2, \infty[$, and according to (2.4) consider the decomposition

$$\mathbb{E}\big(|X_n(\bar{t}, x) - X_n(t, x)|^p 1_{L_n(\bar{t})}\big) \leq C \sum_{i=1}^{6} R_n^i(t, \bar{t}, x),$$

where

$$R_n^1(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^{\bar{t}} \int_{\mathbb{R}^3} [G(\bar{t}-s, x-y) - G(t-s, x-y)] \right.\right.$$

$$\left.\left. \times A(X_n(s, y)) M(\mathrm{d}s, \mathrm{d}y)\right|^p 1_{L_n(\bar{t})}\right),$$

$$R_n^2(t, \bar{t}, x) = \mathbb{E}\big(\big|\langle [G(\bar{t}-\cdot, x-*) - G(t-\cdot, x-*)] B(X_n^-(\cdot, *)), w^n\rangle_{\mathcal{H}_{\bar{t}}}\big|^p 1_{L_n(\bar{t})}\big),$$

$$R_n^3(t, \bar{t}, x) = \mathbb{E}\big(\big|\langle [G(\bar{t}-\cdot, x-*) - G(t-\cdot, x-*)]$$

$$\times [B(X_n) - B(X_n^-)](\cdot, *), w^n\rangle_{\mathcal{H}_{\bar{t}}}\big|^p 1_{L_n(\bar{t})}\big),$$

$$R_n^4(t, \bar{t}, x) = \mathbb{E}\big(\big|\langle [G(\bar{t}-\cdot, x-*) - G(t-\cdot, x-*)] D(X_n(\cdot, *)), h\rangle_{\mathcal{H}_{\bar{t}}}\big|^p 1_{L_n(\bar{t})}\big),$$

$$R_n^5(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^{\bar{t}} \int_{\mathbb{R}^3} [G(\bar{t}-s, x-\mathrm{d}y) - G(t-s, x-\mathrm{d}y)] b(X_n(s, y)) \mathrm{d}s\right|^p 1_{L_n(\bar{t})}\right).$$

Similarly as for the term $R_n^1(t, x, \bar{x})$ in the proof of Lemma 2.8 (see (2.27)), we have

$$R_n^1(t, \bar{t}, x) \leq C\mathbb{E}\left(\int_0^{\bar{t}} \mathrm{d}s \big\| [G(\bar{t}-s, x-*) \right.$$

$$\left. - G(t-s, x-*)] A(X_n(s, *))\big\|_{\mathcal{H}}^2 1_{L_n(s)}\right)^{p/2}. \tag{2.57}$$

This is bounded up to a positive constant by $R_n^{1,1}(t,\bar{t},x) + R_n^{1,2}(t,\bar{t},x)$, where

$$R_n^{1,1}(t,\bar{t},x) = \mathbb{E}\left(\left|\int_t^{\bar{t}} \left\|G(\bar{t}-s,x-*)A\big(X_n(s,*)\big)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\, ds\right|\right)^{p/2}$$

$$= \mathbb{E}\left(\left|\int_0^{\bar{t}-t} \left\|G(s,x-*)A\big(X_n(\bar{t}-s,*)\big)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\, ds\right|\right)^{p/2} \tag{2.58}$$

and

$$R_n^{1,2}(t,\bar{t},x) = \mathbb{E}\left(\left|\int_0^t ds \left\|\big[G(\bar{t}-s,x-*)\right.\right.\right.$$

$$\left.\left.\left. - G(t-s,x-*)\big]A\big(X_n(s,*)\big)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right|\right)^{p/2}. \tag{2.59}$$

Set

$$\mu_1(t,\bar{t},x) := \int_0^{\bar{t}-t} ds \int_{\mathbb{R}^3} d\xi \left|\mathcal{F}G(s)(\xi)\right|^2 \mu(d\xi).$$

Lemma 2.2 in [10] shows that

$$\mu_1(t,\bar{t},x) \le C|\bar{t}-t|^{3-\beta}. \tag{2.60}$$

Then, using Hölder's inequality, the linear growth of $A$ and (4.9), we obtain

$$R_n^{1,1}(t,\bar{t},x) \le C\big(\mu_1(t,\bar{t},x)\big)^{p/2}\left(1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(|X_n(t,x)|^p 1_{L_n(t)}\big)\right) \tag{2.61}$$

$$\le C|\bar{t}-t|^{p(3-\beta)/2}. \tag{2.62}$$

Set $D_n(t,x) = A(X_n(t,x))1_{L_n(t)}$. Owing to Hypothesis (B), (4.9) and Proposition 2.5, the conditions (2.42), (2.43) of Lemma 2.10 are satisfied with $\rho_1 \in ]0, \frac{2-\beta}{2}[$. Thus,

$$R_n^{1,2}(t,\bar{t},x) \le C\big(|\bar{t}-t|^{\rho_1 p} + |\bar{t}-t|^{(\rho_1+\alpha_1)p/2} + |\bar{t}-t|^{\alpha_2 p/2}\big), \tag{2.63}$$

with $\rho_1 \in ]0, \frac{2-\beta}{2}[$, $\alpha_1 \in ]0, 1 \wedge (2-\beta)[$ and $\alpha_2 \in ]0, (2-\beta)[$.

It is easy to check that $\frac{2-\beta}{2} + (1 \wedge (2-\beta)) \ge (2-\beta)$. Hence, from (2.63) we obtain

$$R_n^{1,2}(t,\bar{t},x) \le C|\bar{t}-t|^{\rho p}, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.64}$$

Since $\frac{3-\beta}{2} \ge \frac{2-\beta}{2}$, (2.62) and (2.64) imply

$$R_n^1(t,\bar{t},x) \le C|\bar{t}-t|^{\rho p}, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.65}$$

With the same arguments as those applied in the study of the term $R_n^2(t, x, \bar{x})$ in the proof of Lemma 2.8, we have

$$R_n^2(t, \bar{t}, x) \leq C \mathbb{E}\left(\int_0^{\bar{t}} ds \left\|\left[G(\bar{t} - s, x + *) - G(t - s, x - *)\right]B\left(X_n^-(s, *)\right)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right)^{p/2}.$$

This yields $R_n^2(t, \bar{t}, x) \leq C(R_n^{2,1}(t, \bar{t}, x) + R_n^{2,2}(t, \bar{t}, x))$, where

$$R_n^{2,1}(t, \bar{t}, x) = \mathbb{E}\left(\int_0^t ds \left\|\left[G(\bar{t} - s, x + *) - G(t - s, x - *)\right]B\left(X_n^-(s, *)\right)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right)^{p/2},$$

$$R_n^{2,2}(t, \bar{t}, x) = \mathbb{E}\left(\int_0^{\bar{t}-t} \left\|G(s, x - *)B\left(X_n^-(s, *)\right)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right)^{p/2}.$$

The term $R_n^{2,1}(t, \bar{t}, x)$ is similar as $R_n^{1,2}(t, \bar{t}, x)$, with $A(X_n)$ replaced by $B(X_n^-)$. Hence both can be studied using the same approach. First, we see that the process $D_n(t, x) := B(X_n^-(t, x))1_{L_n(t)}$ satisfies the hypothesis of Lemma 2.10 with $\rho_1 \in ]0, \frac{2-\beta}{2}[$. In fact, this is a consequence of (4.9) and Proposition 2.7. Therefore, as for $R_n^{1,2}(t, \bar{t}, x)$, we have

$$R_n^{2,1}(t, \bar{t}, x) \leq C|\bar{t} - t|^{\rho p}, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.66}$$

As for $R_n^{2,2}(t, \bar{t}, x)$, it is analogous to $R_n^{1,1}$ with $A(X_n)$ replaced by $B(X_n^-)$. As in (2.62), we have

$$R_n^{2,2}(t, \bar{t}, x) \leq C|\bar{t} - t|^{p(3-\beta)/2}. \tag{2.67}$$

Consequently, from (2.66), (2.67), we obtain

$$R_n^2(t, \bar{t}, x) \leq C|\bar{t} - t|^{\rho p}, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.68}$$

Let $\hat{B}(X_n(\cdot, *))$ be defined by (2.31). Using Cauchy–Schwarz's inequality and (2.11) we have

$$R_n^3(t, \bar{t}, x) \leq Cn^{3p/2}2^{np/2}\left[R_n^{3,1}(t, \bar{t}, x) + R_n^{3,2}(t, \bar{t}, x)\right], \tag{2.69}$$

where

$$R_n^{3,1}(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t ds \left\|\left[G(\bar{t} - s, x - *) - G(t - s, x - *)\right]\hat{B}\left(X_n(s, *)\right)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right|\right)^{p/2},$$

$$R_n^{3,2}(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^{\bar{t}-t} ds \left\|G(s, x - *)\hat{B}\left(X_n(\bar{t} - s, *)\right)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right|\right)^{p/2}.$$

From (4.10), it follows that

$$\sup_{(t,x)\in[0,T\times\mathbb{R}^3]} \mathbb{E}\left(\left|\hat{B}\left(X_n(t, x)\right)\right|^p 1_{L_n(t)}\right) \leq Cn^{3p/2}2^{-np(3-\beta)/2}. \tag{2.70}$$

Let us study $R_n^{3,2}(t, \bar{t}, x)$. This term is similar to $R_n^{1,1}(t, \bar{t}, x)$ with $A(X_n)$ replaced here by $\hat{B}(X_n)$. Hence, as in (2.61) we have

$$
\begin{aligned}
R_n^{3,2}(t, \bar{t}, x) &\leq \left(\mu_1(t, \bar{t}, x)\right)^{p/2} \left( \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\left(\left|\hat{B}\left(X_n(t, x)\right)\right|^p 1_{L_n(t)}\right) \right) \\
&\leq C|\bar{t} - t|^{p(3-\beta)/2} n^{3p/2} 2^{-np(3-\beta)/2},
\end{aligned}
\tag{2.71}
$$

where in the last inequality we have applied (2.70).

The analysis of $R_n^{3,1}$ relies on a variant of Lemma 2.10 where the process $D_n$ is replaced by $\hat{B}(X_n)$. By (2.70), this process satisfies a stronger assumption than (2.42). This fact is expected to compensate the factor $n^{3p/2} 2^{np/2}$ in (2.69).

As in the proof of Lemma 2.10 (see also [10], page 28), we consider the decomposition

$$
R_n^{3,1}(t, \bar{t}, x) \leq \sum_{k=1}^{4} \mathbb{E}\left(\left|Q^i(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right),
$$

where $Q^i(t, \bar{t}, x)$, $i = 1, \ldots, 4$, are defined in (2.46) with $D_n := \hat{B}(X_n) 1_{L_n}$.

From (2.70) and the triangular inequality, we obtain

$$
\mathbb{E}\left(\left|\hat{B}\left(X_n\left(s, x - \frac{\bar{t} - s}{t - s} u\right)\right) - \hat{B}\left(X_n(s, x - u)\right)\right|^p 1_{L_n(s)}\right) \leq C n^{3p/2} 2^{-np(3-\beta)/2}.
\tag{2.72}
$$

Consider the expression (2.48) with $D_n = \hat{B}(X_n) 1_{L_n}$. The above estimate (2.72) yields

$$
\begin{aligned}
&\mathbb{E}\left(\left|Q^1(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right) \\
&\leq C n^{3p/2} 2^{-np(3-\beta)/2} \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - s, \mathrm{d}u) G(t - s, \mathrm{d}v) \frac{\bar{t} - s}{t - s} f\left(v \frac{\bar{t} - s}{t - s} - u\right).
\end{aligned}
$$

Along with (2.47), this implies

$$
\mathbb{E}\left(\left|Q^1(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right) \leq C n^{3p/2} 2^{-np(3-\beta)/2}.
\tag{2.73}
$$

Consider the expression (2.51) with $D_n = \hat{B}(X_n) 1_{L_n}$. Using (2.31), the Lipschitz property of $B$ and (4.10), we obtain

$$
\mathbb{E}\left(\left|Q^2(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right) \leq C n^{3p/2} 2^{-np(3-\beta)/2}.
\tag{2.74}
$$

Similarly,

$$
\mathbb{E}\left(\left|Q^3(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right) \leq C n^{3p/2} 2^{-np(3-\beta)/2}.
\tag{2.75}
$$

Let us now consider the expression (2.55) with $D_n = \hat{B}(X_n) 1_{L_n}$. Appealing to (2.70), we obtain

$$
\mathbb{E}\left(\left|Q^4(t, \bar{t}, x)\right|^{p/2} 1_{L_n(\bar{t})}\right) \leq C n^{3p/2} 2^{-np(3-\beta)/2}.
\tag{2.76}
$$

From (2.73)–(2.76) it follows that

$$R_n^{3,1}(t,\bar{t},x) \leq C n^{3p/2} 2^{-np(3-\beta)/2}, \tag{2.77}$$

where $C$ is a finite constant.

Set $f_n := n^{3p} 2^{-np((3-\beta)/2 - 1/2)}$. From (2.69), (2.71), (2.77), it follows that

$$R_n^3(t,\bar{t},x) \leq C|\bar{t} - t|^{\rho p} + C f_n, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.78}$$

By applying Cauchy–Schwarz's inequality, we see that

$$R_n^4(t,x,\bar{x}) \leq C\mathbb{E}\left(\int_0^{\bar{t}} ds \left\|\left[G(\bar{t}-s,x-*) - G(t-s,x-*)\right] D\big(X_n(s,*)\big)\right\|_{\mathcal{H}}^2 1_{L_n(s)}\right)^{p/2}.$$

The last expression is similar as (2.57) with the function $A$ replaced by $D$. Therefore, as in (2.65) we obtain

$$R_n^4(t,\bar{t},x) \leq C|\bar{t} - t|^{\rho p}, \qquad \rho \in \left]0, \frac{2-\beta}{2}\right[. \tag{2.79}$$

Finally, we consider $R_n^5(t,\bar{t},x)$. Clearly,

$$R_n^5(t,\bar{t},x) \leq C\big[R_n^{5,1}(t,\bar{t},x) + R_n^{5,2}(t,\bar{t},x)\big],$$

where

$$R_n^{5,1}(t,\bar{t},x) := \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(\bar{t}-s,x-dy) - G(t-s,x-dy)\right] b\big(X_n(s,y)\big) ds\right|^p 1_{L_n(\bar{t})}\right),$$

$$R_n^{5,2}(t,\bar{t},x) := \mathbb{E}\left(\left|\int_t^{\bar{t}} \int_{\mathbb{R}^3} G(\bar{t}-s,x-dy) b\big(X_n(s,y)\big) ds\right|^p 1_{L_n(\bar{t})}\right).$$

Applying the change of variable, $y \mapsto \frac{y-x}{\bar{t}-s} + x$ and $y \mapsto \frac{y-x}{t-s} + x$, we see that

$$R_n^{5,1}(t,\bar{t},x) = \mathbb{E}\big(\big|T_1(t,\bar{t},x) - T_2(t,\bar{t},x)\big|^p 1_{L_n(\bar{t})}\big),$$

where

$$T_1(t,\bar{t},x) = \int_0^t (\bar{t}-s) \int_{\mathbb{R}^3} G(1,x-dy) b\big(X_n\big(s,(\bar{t}-s)(y-x)+x\big)\big) ds,$$

$$T_2(t,\bar{t},x) = \int_0^t (t-s) \int_{\mathbb{R}^3} G(1,x-dy) b\big(X_n\big(s,(t-s)(y-x)+x\big)\big) ds.$$

By adding and subtracting $t$ in $T_1$ we get

$$T_1(t,\bar{t},x) = \int_0^t (\bar{t}-t) \int_{\mathbb{R}^3} G(1,x-dy) b\big(X_n\big(s,(\bar{t}-s)(y-x)+x\big)\big) ds$$

$$+ \int_0^t (t-s) \int_{\mathbb{R}^3} G(1,x-dy) b\big(X_n\big(s,(\bar{t}-s)(y-x)+x\big)\big) ds.$$

Then, Hölder's inequality yields

$$R_n^{5,1}(t, \bar{t}, x)$$

$$\leq C|\bar{t} - t|^p \int_0^t ds \int_{\mathbb{R}^3} G(1, x - dy) \mathbb{E}\big(\big|b\big(X_n\big(s, (\bar{t} - s)(y - x) + x\big)\big)\big|^p 1_{L_n(s)}\big)$$

$$+ C \int_0^t |t - s|^p ds \int_{\mathbb{R}^3} G(1, x - dy) \mathbb{E}\big(\big|b\big(X_n\big(s, (\bar{t} - s)(y - x) + x\big)\big)$$

$$- b\big(X_n\big(s, (t - s)(y - x) + x\big)\big)\big|^p 1_{L_n(s)}\big).$$

Owing to (4.9), the first term on the right hand-side of the last inequality is bounded up to a constant by $|\bar{t} - t|^p$. For the second one, we use the Hypothesis (B) along with (2.15) to obtain

$$\int_0^t |t - s|^p ds \int_{\mathbb{R}^3} G(1, x - dy)$$

$$\times \mathbb{E}\big(\big|b\big(X_n\big(s, (\bar{t} - s)(y - x) + x\big)\big) - b\big(X_n\big(s, (t - s)(y - x) + x\big)\big)\big|^p 1_{L_n(s)}\big)$$

$$\leq C \int_0^t ds \int_{\mathbb{R}^3} G(1, x - dy)$$

$$\times \mathbb{E}\big(\big|X_n\big(s, (\bar{t} - s)(y - x) + x\big) - X_n\big(s, (t - s)(y - x) + x\big)\big|^p 1_{L_n(s)}\big)$$

$$\leq C|t - \bar{t}|^{\rho p},$$

with $\rho \in \,]0, \frac{2-\beta}{2}[$.

Hölder inequality along with (4.9) clearly yields

$$R_n^{5,2}(t, \bar{t}, x) \leq C|\bar{t} - t|^{p-1} \int_t^{\bar{t}} \int_{\mathbb{R}^3} G(\bar{t} - s, x - dy) \mathbb{E}\big(\big|b\big(X_n(s, y)\big)\big|^p 1_{L_n(s)}\big) ds$$

$$\leq C|\bar{t} - t|^p. \tag{2.80}$$

Hence, we have proved that

$$R_n^5(t, \bar{t}, x) \leq C|\bar{t} - t|^{\rho p}, \qquad \rho \in \,\left]0, \frac{2-\beta}{2}\right[. \tag{2.81}$$

With the inequalities (2.65), (2.68), (2.78), (2.79) and (2.81), we have

$$\mathbb{E}\big(\big|X_n(\bar{t}, x) - X_n(t, x)\big|^p 1_{L_n(\bar{t})}\big) \leq C\big[|\bar{t} - t|^{\rho p} + f_n\big],$$

with $\rho \in \,]0, \frac{2-\beta}{2}[$.

For a given fixed $\bar{t} \in [t_0, T]$, we introduce the function

$$\Psi_{n,x,p}^{\bar{t}}(t) := \mathbb{E}\big(\big|X_n(\bar{t}, x) - X_n(t, x)\big|^p 1_{L_n(\bar{t})}\big),$$

for $t_0 \leq t \leq \bar{t}$.

Notice that $\lim_{n \to \infty} f_n = 0$ and thus, $\sup_n f_n \leq C$. Thus, there exists a constant $0 < C_0 < \infty$, such that

$$\sup_n f_n \leq C_0 t_0 \leq C_0 \bar{t} \leq C_0 \int_0^{\bar{t}} ds \left[ 1 + \Psi_{n,x,p}^{\bar{t}}(s) \right].$$

With a similar argument, there exists $0 < C_1 < \infty$ such that

$$1 \leq C_1 t_0 \leq C_1 \bar{t} \leq C_1 \int_0^{\bar{t}} ds \left[ 1 + \Psi_{n,x,p}^{\bar{t}}(s) \right].$$

Therefore,

$$1 + \Psi_{n,x,p}^{\bar{t}}(t) \leq C \left\{ |\bar{t} - t|^{\rho p} + \int_0^{\bar{t}} ds \left[ 1 + \Psi_{n,x,p}^{\bar{t}}(s) \right] \right\}.$$

Then, by Gronwall's lemma,

$$1 + \Psi_{n,x,p}^{\bar{t}}(t) \leq C \left( |\bar{t} - t|^{\rho p} \right),$$

where $\rho \in \left]0, \frac{2-\beta}{2}\right[$. This finish the proof of the proposition.     □

## 2.3. Pointwise convergence

This section is exclusively devoted to the proof of Theorem 2.4. Using equations (2.3), (2.4), we write the difference $X_n(t, x) - X(t, x)$ grouped into comparable terms in order to prove their convergence to zero. The main difficulty lies in the proof of the convergence of $\langle G(t - \cdot, x - *) B(X_n(\cdot, *)), w^n \rangle_{\mathcal{H}_T}$ to $\int_0^t \int_{\mathbb{R}^3} B(X(s, y)) M(ds, dy)$. We write

$$X_n(t, x) - X(t, x) = \sum_{i=1}^{8} U_n^i(t, x),$$

where

$$U_n^1(t, x) = \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y) \left[ (A + B)(X_n(s, y)) - (A + B)(X(s, y)) \right] M(ds, dy),$$

$$U_n^2(t, x) = \langle G(t - \cdot, x - *) \left[ D(X_n(\cdot, *)) - D(X(\cdot, *)) \right], h \rangle_{\mathcal{H}_t},$$

$$U_n^3(t, x) = \int_0^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \left[ b(X_n(s, y)) - b(X(s, y)) \right],$$

$$U_n^4(t, x) = \langle G(t - \cdot, x - *) \left[ B(X_n(\cdot, *)) - B(X_n^-(\cdot, *)) \right], w^n \rangle_{\mathcal{H}_t},$$

$$U_n^5(t, x) = \langle G(t - \cdot, x - *) \left[ B(X_n^-(\cdot, *)) - B(X^-(\cdot, *)) \right], w^n \rangle_{\mathcal{H}_t},$$

$$U_n^6(t, x) = \langle G(t - \cdot, x - *) B(X^-(\cdot, *)), w^n \rangle_{\mathcal{H}_t}$$

$$-\int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) B(X^-(s,y)) M(\mathrm{d}s, \mathrm{d}y),$$

$$U_n^7(t,x) = \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \big[ B(X^-(s,y)) - B(X_n^-(s,y)) \big] M(\mathrm{d}s, \mathrm{d}y),$$

$$U_n^8(t,x) = \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \big[ B(X_n^-(s,y)) - B(X_n(s,y)) \big] M(\mathrm{d}s, \mathrm{d}y).$$

Here, we have used the abridged notation $X^-(\cdot, *)$ for the stochastic process $X^-(t,x) := X(t, t_n, x)$ defined in (2.6). Notice that, although this is not apparent in the notation $X^-(\cdot, *)$ does depend on $n$.

Fix $p \in [2, \infty[$. Clearly,

$$\mathbb{E}\big( |X_n(t,x) - X(t,x)|^p 1_{L_n(t)} \big) \le C \sum_{i=1}^8 \mathbb{E}\big( |U_n^i(t,x)|^p 1_{L_n(t)} \big).$$

Next, we analyze the contribution of each term $U_n^i(t,x)$, $i = 1, \ldots, 8$.

Burkholder's and Hölder's inequalities yield

$$\mathbb{E}\big( |U_n^1(t,x)|^p 1_{L_n(t)} \big) \le C \int_0^t \mathrm{d}s \Big[ \sup_{y \in K(s)} \mathbb{E}\big( |X_n(s,y) - X(s,y)|^p 1_{L_n(s)} \big) \Big]. \tag{2.82}$$

Cauchy–Schwarz's inequality implies

$$\mathbb{E}\big( |U_n^2(t,x)|^p 1_{L_n(t)} \big) \le \|h\|_{\mathcal{H}_t}^p \mathbb{E}\big( \big\| G(t-\cdot, x-*) \big[ D(X_n(\cdot, *)) - D(X(\cdot, *)) \big] 1_{L_n(t)} \big\|_{\mathcal{H}_t}^2 \big)^{p/2}.$$

Then, by using Hölder's inequality we obtain

$$\mathbb{E}\big( |U_n^2(t,x)|^p 1_{L_n(t)} \big) \le C \int_0^t \mathrm{d}s \Big[ \sup_{y \in K(s)} \mathbb{E}\big( |X_n(s,y) - X(s,y)|^p 1_{L_n(s)} \big) \Big]. \tag{2.83}$$

For $U_n^3(t,x)$, we apply Hölder's inequality. This yields

$$\mathbb{E}\big( |U_n^3(t,x)|^p 1_{L_n(t)} \big) \le C \int_0^t \mathrm{d}s \Big[ \sup_{y \in K(s)} \mathbb{E}\big( |X_n(s,y) - X(s,y)|^p 1_{L_n(s)} \big) \Big]. \tag{2.84}$$

Let $\tau_n$ and $\pi_n$ be the operators defined in the proof of Lemma 2.8 (see (2.29) and lines thereafter). Let $I_{\mathcal{H}_t}$ be the identity operator on $\mathcal{H}_t$. $\Upsilon_t := (\pi_n \circ \tau_n) - I_{\mathcal{H}_t}$ is a contraction operator on $\mathcal{H}_t$.

After having applied Burkholder's inequality, we obtain

$$\mathbb{E}\big[ \big( |U_n^5(t,x) + U_n^7(t,x)|^p \big) 1_{L_n(t)} \big]$$

$$\le C\mathbb{E}\big( \big\| \Upsilon_t \big[ G(t-\cdot, x-*) \{ B(X_n^-) - B(X^-) \} \big](\cdot, *) 1_{L_n(\cdot)} \big\|_{\mathcal{H}_t}^p \big)$$

$$\le C\mathbb{E}\bigg( \int_0^t \mathrm{d}s \big\| \big[ G(t-s, x-*) \{ B(X_n^-) - B(X^-) \} \big](s, *) 1_{L_n(s)} \big\|_{\mathcal{H}}^2 \bigg)^{p/2}.$$

Similarly as for $U_n^2(t, x)$, we have

$$
\mathbb{E}\big[\big(\big|U_n^5(t, x) + U_n^7(t, x)\big|^p\big)1_{L_n(t)}\big]
$$
$$
\leq C \int_0^t ds\Big[\sup_{y \in K(s)} \mathbb{E}\big(\big|X_n^-(s, y) - X^-(s, y)\big|^p 1_{L_n(s)}\big)\Big]. \tag{2.85}
$$

This clearly implies

$$
\mathbb{E}\big[\big(\big|U_n^5(t, x) + U_n^7(t, x)\big|^p\big)1_{L_n(t)}\big] \leq C\Big\{\int_0^t ds\Big[\sup_{y \in K(s)} \mathbb{E}\big(\big|X_n^-(s, y) - X_n(s, y)\big|^p 1_{L_n(s)}\big)\Big]
$$
$$
+ \int_0^t ds\Big[\sup_{y \in K(s)} \mathbb{E}\big(\big|X_n(s, y) - X(s, y)\big|^p 1_{L_n(s)}\big)\Big]
$$
$$
+ \int_0^t ds\Big[\sup_{y \in K(s)} \mathbb{E}\big(\big|X(s, y) - X^-(s, y)\big|^p 1_{L_n(s)}\big)\Big]\Big\}.
$$

Recall that $X^-(s, y) = X(s, s_n, y)$. By applying (4.1) and (4.10), we obtain

$$
\mathbb{E}\big[\big(\big|U_n^5(t, x) + U_n^7(t, x)\big|^p\big)1_{L_n(t)}\big] \leq C \int_0^T ds\Big[\sup_{y \in K(s)} \mathbb{E}\big(\big|X_n(s, y) - X(s, y)\big|^p 1_{L_n(s)}\big)\Big]
$$
$$
+ Cn^{3p/2}2^{-np(3-\beta)/2}. \tag{2.86}
$$

Next, we will prove that

$$
\lim_{n \to \infty}\Big(\sup_{t \in [0, T]}\sup_{x \in K(t)} \mathbb{E}\big(\big|U_n^i(t, x)\big|^p 1_{L_n(t)}\big)\Big) = 0, \qquad i = 4, 6, 8. \tag{2.87}
$$

Consider $i = 4$. Cauchy–Schwarz' inequality along with (2.11) implies

$$
\mathbb{E}\big(\big|U_n^4(t, x)\big|^p 1_{L_n(t)}\big)
$$
$$
\leq Cn^{3p/2}2^{np/2}\mathbb{E}\Big(\int_0^t ds\,\big\|G(t-s, x-*)[B(X_n) - B(X_n^-)](s, *)1_{L_n(s)}\big\|_{\mathcal{H}}^2\Big)^{p/2}.
$$

Then, the Lipschitz continuity of $B$ and (4.10) yield

$$
\mathbb{E}\big(\big|U_n^4(t, x)\big|^p 1_{L_n(t)}\big) \leq Cn^{3p/2}2^{np/2}\int_0^t ds\Big[\sup_{y \in \mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, y) - X_n^-(s, y)\big|^p 1_{L_n(s)}\big)\Big]
$$
$$
\leq Cn^{3p}2^{-np[(3-\beta)/2-1/2]}.
$$

Since $\beta \in ]0, 2[$, this implies (2.87) for $i = 4$.

The arguments based on Burkholder's and Hölder's inequalities, already applied many times, give

$$\mathbb{E}\big(|U_n^8(t,x)|^p 1_{L_n(t)}\big) \le C \int_0^t ds \sup_{y \in \mathbb{R}^3} \mathbb{E}\big(|X_n^-(s,y) - X_n(s,y)|^p 1_{L_n(s)}\big)$$

$$\le C n^{3p/2} 2^{-np(3-\beta)/2},$$

where, in the last inequality we have used (4.10). Thus, (2.87) holds for $i = 8$.

Let us now consider the case $i = 6$. Define

$$U_n^{6,1}(t,x) = \int_0^t \int_{\mathbb{R}^3} \big\{\pi_n\big[\tau_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]\big]$$

$$- G(t-\cdot,x-*)\tau_n\big[B\big(X^-(\cdot,*)\big)\big]\big]\big](s,y)\big\} M(ds,dy),$$

$$U_n^{6,2}(t,x) = \int_0^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot,x-*)\tau_n\big[B\big(X^-(\cdot,*)\big)\big]$$

$$- G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big](s,y) M(ds,dy),$$

$$U_n^{6,3}(t,x) = \int_0^t \int_{\mathbb{R}^3} \big\{\pi_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]$$

$$- G(t-s,x-y)B\big(X^-(s,y)\big)\big\} M(ds,dy).$$

Clearly,

$$U_n^6(t,x) = U_n^{6,1}(t,x) + U_n^{6,2}(t,x) + U_n^{6,3}(t,x).$$

To facilitate the analysis, we write $U_n^{6,1}(t,x)$ more explicitly, as follows

$$U_n^{6,1}(t,x) = \int_0^t \int_{\mathbb{R}^3} \big\{\pi_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]\big((s+2^{-n}) \wedge t, y\big)$$

$$\tag{2.88}$$

$$- \pi_n\big[G(t-\cdot,x-*)B\big(X^-((\cdot+2^{-n}) \wedge t, *)\big)\big](s,y)\big\} M(ds,dy).$$

We are assuming that $t \ge t_0 > 0$. Hence, for $n$ big enough, $t - 2^{-n} > 0$. Consider the first integral on the right-hand side of (2.88). We have

$$\int_0^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]\big((s+2^{-n}) \wedge t, y\big) M(ds,dy)$$

$$= \int_0^{t-2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big](s+2^{-n}, y) M(ds,dy)$$

$$+ \int_{t-2^{-n}}^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big](t,y) M(ds,dy) \tag{2.89}$$

$$= \int_0^{t-2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(\cdot, *)\big)\big](s+2^{-n}, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$= \int_{2^{-n}}^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(s, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y).$$

Indeed, the integral on the domain $[t-2^{-n}, t]$ vanishes, and we have applied the change of variable $s \mapsto s+2^{-n}$.

For the second integral on the right-hand side of (2.88), we split the domain of integration of the $s$-variable into three disjoint sets, as follows:

$$\int_0^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-((\cdot+2^{-n})\wedge t, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$= \int_0^{2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(\cdot+2^{-n}, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y) \qquad (2.90)$$

$$+ \int_{2^{-n}}^{t-2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(\cdot+2^{-n}, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \int_{t-2^{-n}}^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(t, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y).$$

Then (2.89), (2.90) yield

$$U_n^{6,1}(t, x) = -\int_0^{2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(\cdot+2^{-n}, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \int_{2^{-n}}^{t-2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)\big(B\big(X^-(\cdot, *)\big)$$

$$- B\big(X^-(\cdot+2^{-n}, *)\big)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$- \int_{t-2^{-n}}^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(t, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \int_{t-2^{-n}}^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)B\big(X^-(\cdot, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y).$$

From this, we see that $\mathbb{E}(|U_n^{6,1}(t, x)|^p 1_{L_n(t)}) \leq C \sum_{i=1}^4 V_n^{6,i}(t, x)$, where

$$V_n^{6,1}(t, x) = \mathbb{E}\bigg(\bigg|\int_0^{t-2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot, x-*)$$

$$\times \big\{B\big(X^-(\cdot, *)\big) - B\big(X^-(\cdot+2^{-n}, *)\big)\big\}\big](s, y)M(\mathrm{d}s, \mathrm{d}y)\bigg|1_{L_n(t)}\bigg)^p,$$

$$V_n^{6,2}(t, x) = \mathbb{E}\left(\left|\int_0^{2^{-n}} \int_{\mathbb{R}^3} \pi_n\big[G(t - \cdot, x - *)B\big(X^-(\cdot, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)\right|\right)^p,$$

$$V_n^{6,3}(t, x) = \mathbb{E}\left(\left|\int_{t-2^{-n}}^t \pi_n\big[G(t - \cdot, x - *)B\big(X^-(t, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)\right|\right)^p,$$

$$V_n^{6,4}(t, x) = \mathbb{E}\left(\left|\int_{t-2^{-n}}^t \pi_n\big[G(t - \cdot, x - *)B\big(X^-(\cdot, *)\big)\big](s, y)M(\mathrm{d}s, \mathrm{d}y)\right|\right)^p.$$

By Burkholder's and Hölder's inequalities, we have

$$V_n^{6,1}(t, x) \le C \int_0^t \mathrm{d}s \sup_{y \in \mathbb{R}^3} \mathbb{E}\big(\big|X^-(s, y) - X^-\big(s + 2^{-n}, y\big)\big|^p 1_{L_n(t)}\big)$$

$$= C \int_0^t \mathrm{d}s \sup_{y \in \mathbb{R}^3} \mathbb{E}\big(\big|X(s, s_n, y) - X^-\big(s + 2^{-n}, s_n + 2^{-n}, y\big)\big|^p 1_{L_n(t)}\big) \le C 2^{-np\rho},$$

with $\rho \in ]0, \frac{2-\beta}{2}[$. Indeed, the last inequality is obtained by using the triangular inequality along with (4.1) and (2.41).

For $s \in [0, 2^{-n}]$, $X^-(s, y) = X(s, s_n, y) = 0$. Therefore,

$$V_n^{6,2}(t, x) \le C \left(\int_0^{2^{-n}} \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)\big|\mathcal{F}G(t - s, *)(\xi)\big|^2\right)^{p/2} \le C 2^{-np/2},$$

where in the last inequality we have used the property

$$\int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)\big|\mathcal{F}\big(G(r, *)\big)(\xi)\big|^2 = C r^{2-\beta}.$$

In a rather similar way,

$$V_n^{6,3}(t, x) + V_n^{6,4}(t, x)$$

$$\le C\left(1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big|X(t, t_n, x)\big|^p\right)\left(\int_0^{2^{-n}} \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)\big|\mathcal{F}\big(G(s, *)\big)(\xi)\big|^2\right)^{p/2}$$

$$\le C 2^{-np(3-\beta)/2}.$$

Thus, we have established the convergence

$$\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times K(t)} \mathbb{E}\big(\big|U_n^{6,1}(t, x)\big|^p 1_{L_n(t)}\big) = 0. \tag{2.91}$$

Next, we consider the term $U_n^{6,2}(t, x)$. As usually for these type of terms, we apply Burkholder's and then Hölder's inequalities, along with the contraction property of the projection $\pi_n$. This

yields,

$$
\mathbb{E}\big(|U_n^{6,2}(t,x)|^p 1_{L_n(t)}\big)
$$

$$
= \mathbb{E}\Bigg(\bigg|\int_0^t \int_{\mathbb{R}^3} \pi_n\big[G(t-\cdot,x-*)\{B\big(X^-((\cdot+2^{-n})\wedge t,*)\big) - B\big(X^-(\cdot,*)\big)\}\big](s,y)
$$

$$
\times M(\mathrm{d}s,\mathrm{d}y)\bigg|^p 1_{L_n(t)}\Bigg)
$$

$$
\leq C \int_0^t \sup_{x\in\mathbb{R}^3} \mathbb{E}\big(|X\big((s+2^{-n})\wedge t,(s_n+2^{-n})\wedge t,x\big) - X(s,s_n,x)|\big)^p.
$$

Equation (2.3) is a particular case of equation (2.4). Therefore, Proposition 2.9 also holds with $X_n$ replaced by $X$. Then, by virtue of (4.1) and (2.41), this is bounded up to a constant by $2^{-np(3-\beta)/2} + 2^{-np\rho}$, with $\rho \in \,]0,\frac{2-\beta}{2}[$. Consequently,

$$
\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(|U_n^{6,2}(t,x)|^p 1_{L_n(t)}\big) = 0. \tag{2.92}
$$

For $U_n^{6,3}(t,x)$, after having applied Burkholder's inequatily we have

$$
\mathbb{E}\big(|U_n^{6,3}(t,x)|^p 1_{L_n(t)}\big) \leq C\mathbb{E}\big(\big\|(\pi_n - I_{\mathcal{H}_t})\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]1_{L_n(\cdot)}\big\|_{\mathcal{H}_t}^p\big).
$$

We want to prove that the right-hand side of this inequality tends to zero as $n\to\infty$, uniformly in $(t,x)\in[t_0,T]\times K(t)$. For this, we will use a similar approach as in [17], pages 906–909.

Set

$$
\tilde{Z}_n(t,x) = \big\|(\pi_n - I_{\mathcal{H}_t})\big[G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)\big]1_{L_n(\cdot)}\big\|_{\mathcal{H}_t}.
$$

Since $\pi_n$ is a projection on the Hilbert space $\mathcal{H}_t$, the sequence $\{\tilde{Z}_n(t,x), n\geq 1\}$ decreases to zero as $n\to\infty$. Assume that

$$
\mathbb{E}\Big(\sup_n \big\|G(t-\cdot,x-*)B\big(X^-(\cdot,*)\big)1_{L_n(\cdot)}\big\|_{\mathcal{H}_t}^p\Big) < \infty. \tag{2.93}
$$

Remember that $X^-(s,y)$ stands for $X(s,s_n,y)$, defined in (2.6), and therefore it depends on $n$. Then, by bounded convergence, this would imply $\lim_{n\to\infty}\mathbb{E}(\tilde{Z}_n(t,x))^p = 0$. Set $Z_n(t,x) = \mathbb{E}(\tilde{Z}_n(t,x))^p$. Proceeding as in the proof of Lemmas 2.6, 2.10, we can check that

$$
\big|\big(Z_n(t,x)\big)^{1/p} - \big(Z_n(\bar{t},\bar{x})\big)^{1/p}\big| \leq C\big(|t-\bar{t}| + |x-\bar{x}|\big)^\rho,
$$

with $\rho \in \,]0,\frac{2-\beta}{2}[$.

Hence, $(Z_n)_n$ is a sequence of monotonically decreasing continuous functions defined on $[0,T]\times\mathbb{R}^3$ which converges pointwise to zero. Appealing to Dini's theorem, we obtain

$$
\lim_{n\to\infty} \sup_{(t,x)\in[t_0,T]\times K(t)} \mathbb{E}\big(\tilde{Z}_n(t,x)\big)^p = 0. \tag{2.94}
$$

This yields the expected result on $U_n^{6,3}$.

It remains to prove (2.93). We will sketch the arguments, leaving the details to the reader. As usually, we write $\|G(t-\cdot, x-*)B(X^-(\cdot,*))1_{L_n(\cdot)}\|_{\mathcal{H}_t}$ using the identity (1.4). By applying Hölder's inequality with respect to the measure on $[0,t]\times\mathbb{R}^3\times\mathbb{R}^3$ with density $G(t-s, x-y)G(t-s, x-z)|y-z|^{-\beta}\,ds\,dy\,dz$, and using the linear growth of the function $B$, we obtain as upper bound for the left-hand side of (2.93)

$$C\Big[1+\sup_{t,x\in[0,T]\times\mathbb{R}^3} E\Big(\sup_n\big|X(t,t_n,x)\big|^p\Big)\Big]. \tag{2.95}$$

Looking back to the definition of $X(t, t_n, x)$, we see that for the second and third terms in (2.6), the supremum in $n$ can be easily handled, since they are defined pathwise. For the stochastic integral term, we consider the discrete martingale

$$\Big\{\int_0^{t_n}\int_{\mathbb{R}^3} G(s_0-s, x-y)(A+B)\big(X(s,y)\big)M(ds,dy), \mathcal{F}_{t_n}, n\in\mathbb{N}\Big\},$$

where $s_0\in]0,T]$ is fixed. By applying first Doob's maximal inequality and then Burkholder's inequality, we have

$$E\Big(\sup_n\Big|\int_0^{t_n}\int_{\mathbb{R}^3} G(s_0-s, x-y)(A+B)\big(X(s,y)\big)M(ds,dy)\Big|^p\Big)$$

$$\leq CE\Big(\Big|\int_0^t\int_{\mathbb{R}^3} G(s_0-s, x-y)(A+B)\big(X(s,y)\big)M(ds,dy)\Big|^p\Big)$$

$$\leq CE\big(\big\|G(s_0-\cdot, x-*)(A+B)\big(X(\cdot,*)\big)\big\|_{\mathcal{H}_t}^{p/2}\big).$$

Finally, we take $s_0:=s$. Using the property $\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E(|X(t,x)|^p)$, we obtain that the expression (2.95) is finite.

Owing to (2.91), (2.92) and (2.94), we have

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times K(t)} \mathbb{E}\big(\big|U_n^6(t,x)\big|^p 1_{L_n(t)}\big)=0. \tag{2.96}$$

In order to conclude the proof, let us consider the estimates (2.82), (2.83), (2.84), (2.86), along with (2.87). We see that

$$\mathbb{E}\big(\big|X_n(t,x)-X(t,x)\big|^p 1_{L_n(t)}\big)\leq C_1\theta_n+C_2\int_0^t ds\Big[\sup_{x\in K(s)}\mathbb{E}\big(\big|X_n(s,x)-X(s,x)\big|^p 1_{L_n(s)}\big)\Big],$$

where $(\theta_n, n\geq 1)$ is a sequence of real numbers which converges to zero as $n\to\infty$. Applying Gronwall's lemma, we finish the proof of the theorem. $\qquad\square$

## 2.4. Proof of Theorem 2.2

Fix $t_0>0$ and a compact set $K\subset\mathbb{R}^3$. Let $Y_n(t,x):=X_n(t,x)-X(t,x)$ and $B_n(t):=L_n(t)$, $n\geq 1$, $(t,x)\in[t_0,T]\times K$, $p\in[1,\infty[$. From Theorems 2.3 and 2.4, we see that the conditions

(P1) and (P2) of Lemma A.4 are satisfied with $\delta = p\rho - 4$, for any $\rho \in ]0, \frac{2-\beta}{2}[$. We infer that

$$\lim_{n \to \infty} \mathbb{E}\big(\|X_n - X\|_{\rho,t_0,K}^p 1_{L_n(t)}\big) = 0, \tag{2.97}$$

for any $p \in [1, \infty[$ and $\rho \in ]0, \frac{2-\beta}{2}[$.

Fix $\varepsilon > 0$. Since $\lim_{n \to \infty} \mathbb{P}(L_n(t)^c) = 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\mathbb{P}(L_n(t)^c) < \varepsilon$. Then, for any $\lambda > 0$ and $n \geq N_0$,

$$\mathbb{P}\big(\|X_n - X\|_{\rho,t_0,K} > \lambda\big) \leq \varepsilon + \mathbb{P}\big(\big(\|X_n - X\|_{\rho,t_0K} > \lambda\big) \cap L_n(t)\big)$$

$$\leq \varepsilon + \lambda^{-p}\mathbb{E}\big(\|X_n - X\|_{\rho,t_0K}^p 1_{L_n(t)}\big).$$

Since $\varepsilon > 0$ is arbitrary, this finishes the proof of the theorem. $\square$

## 3. Support theorem

This section is devoted to the characterization of the topological support of the law of the random field solution to the stochastic wave equation (1.5). As has been explained in the Introduction, this is a corollary of Theorem 2.2.

**Theorem 3.1.** *Assume that the functions $\sigma$ and $b$ are Lipschitz continuous. Fix $t_0 \in ]0, T[$ and a compact set $K \subset \mathbb{R}^3$. Let $u = \{u(t, x), (t, x) \in [t_0, T] \times K\}$ be the random field solution to (1.5). Fix $\rho \in ]0, \frac{2-\beta}{2}[$. Then the topological support of the law of $u$ in the space $\mathcal{C}^\rho([t_0, T] \times K)$ is the closure in $\mathcal{C}^\rho([t_0, T] \times K)$ of the set of functions $\{\Phi^h, h \in \mathcal{H}_T\}$, where $\{\Phi^h(t, x), (t, x) \in [t_0, T] \times K\}$ is the solution of (1.9).*

Let $\{w^n, n \geq 1\}$ be the sequence of $\mathcal{H}_T$-valued random variables defined in (2.1). For any $h \in \mathcal{H}_T$, we consider the sequence of transformations of $\Omega$ defined in (1.10). As has been pointed out in Section 1, $P \circ (T_n^h)^{-1} \ll P$.

Notice also that the process $v_n(t, x) := (u \circ T_n^h)(t, x), (t, x) \in [t_0, T] \times \mathbb{R}^3$, satisfies the equation

$$v_n(t, x) = \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)\sigma\big(v_n(s, y)\big)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \big\langle G(t - \cdot, x - *)\sigma\big(v_n(\cdot, *)\big), h - w^n\big\rangle_{\mathcal{H}_t} + \int_0^t \mathrm{d}s\big[G(t - s, \cdot) \star b\big(v_n(s, \cdot)\big)\big](x). \tag{3.1}$$

**Proof of Theorem 3.1.** According to the method developed in [15] (see also [3] and Section 1 for a summary), the theorem will be a consequence of the following convergences:

$$\lim_{n \to \infty} \mathbb{P}\big\{\big\|u - \Phi^{w^n}\big\|_{\rho,t_0,K} > \eta\big\} = 0, \tag{3.2}$$

$$\lim_{n \to \infty} \mathbb{P}\big\{\big\|u \circ T_n^h - \Phi^h\big\|_{\rho,t_0,K} > \eta\big\} = 0, \tag{3.3}$$

where $\eta$ is an arbitrary positive real number.

This follows from the general approximation result developed in Section 2. Indeed, consider equations (2.3) and (2.4) with the choice of coefficients $A = D = 0$, $B = \sigma$. Then the processes $X$ and $X_n$ coincide with $u$ and $\Phi^{w^n}$, respectively. Hence, the convergence (3.2) follows from Theorem 2.2. Next, we consider again equations (2.3) and (2.4) with a new choice of coefficients: $A = D = \sigma$, $B = -\sigma$. In this case, the processes $X$ and $X_n$ are equal to $\Phi^h$ and $v_n := u \circ T_n^h$, respectively. Thus, Theorem 2.2 yields (3.3). □

## 4. Auxiliary results

The most difficult part in the proof of Theorem 2.2 consists of establishing (2.13). In particular, handling the contribution of the pathwise integral (with respect to $w^n$) requires a careful analysis of the discrepancy between this integral and the stochastic integral with respect to $M$. This section gathers several technical results that have been applied in the analysis of such questions in the preceding Section 2.

The first statement in the next lemma provides a measure of the discrepancy between the processes $X(t, x)$ and $X(t, t_n, x)$ defined in (2.3), (2.6), respectively.

**Lemma 4.1.** *Suppose that Hypothesis* (B) *is satisfied. Then for any* $p \in [1, \infty)$ *and every integer* $n \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \left\| X(t, x) - X(t, t_n, x) \right\|_p \leq C 2^{-n(3-\beta)/2} \tag{4.1}$$

*and*

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \left\| X(t, t_n, x) \right\|_p \leq C < \infty, \tag{4.2}$$

*where $C$ is a positive constant not depending on $n$.*

**Proof.** Fix $p \in [2, \infty[$. From equations (2.3), (2.6), we obtain

$$\left\| X(t, x) - X(t, t_n, x) \right\|_p^p \leq C \big( V_1(t, x) + V_2(t, x) + V_3(t, x) \big),$$

where

$$V_1(t, x) := \left\| \int_{t_n}^t \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)\big( X(s, y) \big) M(\mathrm{d}s, \mathrm{d}y) \right\|_p^p,$$

$$V_2(t, x) := \left\| G(t - \cdot, x - *) D\big( X(\cdot, *) \big) 1_{[t_n, t]}(\cdot), h \rangle_{\mathcal{H}_t} \right\|_p^p,$$

$$V_3(t, x) := \left\| \int_{t_n}^t G(t - s, \cdot) \star b\big( X(s, \cdot) \big)(x) \, \mathrm{d}s \right\|_p^p.$$

Applying first Burholder's and then Hölder's inequalities, we obtain

$$V_1(t, x) \leq C \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(t - s)(\xi)|^2 \right)^{p/2} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big( |(A + B)(X(t, x))|^p \big)$$

$$\leq C \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(t - s)(\xi)|^2 \right)^{p/2} \left( 1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big( |X((t, x)|^p) \big) \right).$$

Applying the inequality (2.60) along with (A.3), imply

$$V_1(t, x) \leq C 2^{-np(3-\beta)/2}.$$

For the study of $V_2$, we apply first Cauchy–Schwarz inequality and then Hölder's inequality. We obtain

$$V_2(t, x) \leq \| h 1_{[t_n,t]}(\cdot) \|_{\mathcal{H}_t}^{p/2} \mathbb{E} \left( \int_0^t ds \| G(t - s, x - *) D(X(s, *)) 1_{[t_n,t]}(s) \|_{\mathcal{H}}^2 \right)^{p/2}.$$

Hence, similarly as for $V_1$ we have

$$V_2(t, x) \leq C 2^{-np(3-\beta)/2}.$$

By applying Hölder's inequality, we get

$$V_3(t, x) \leq \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \right)^{p-1} \int_{t_n}^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \mathbb{E}\big( |b(X(s, y))|^p \big)$$

$$\leq C \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \right)^p \left( 1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big( |X(s, y)|^p \big) \right)$$

$$\leq C 2^{-2np}.$$

The condition $\beta \in ]0, 2[$ implies $2^{-2np} < 2^{-np(3-\beta)/2}$. Thus from the estimates on $V_i(t, x)$, $i = 1, 2, 3$ (which hold uniformly on $(t, x) \in [0, T] \times \mathbb{R}^3$) we obtain (4.1).

Finally, (4.2) is a consequence of the triangular inequality, (4.1) and (A.3).  □

The next result states an analogue of Lemma 4.1 for the stochastic processes $X_n$, $X_n^-$ defined in (2.4), (2.5), respectively, this time including a localization by $L_n$.

**Lemma 4.2.** *We assume Hypothesis* (B). *Then for any* $p \in [2, \infty)$ *and* $t \in [0, T]$,

$$\sup_{(s,y) \in [0,t] \times \mathbb{R}^3} \mathbb{E}\big( |X_n(s, y) - X_n^-(s, y)|^p 1_{L_n(s)} \big)$$

$$\leq C n^{3p/2} 2^{-np(3-\beta)/2} \left[ 1 + \sup_{(s,y) \in [0,t] \times \mathbb{R}^3} \mathbb{E}\big( |X_n(s, y)|^p 1_{L_n(s)} \big) \right]. \tag{4.3}$$

**Proof.** Fix $p \in [2, \infty[$ and consider the decomposition

$$\mathbb{E}\big(\big|X_n(t, x) - X_n^-(t, x)\big|^p 1_{L_n(t)}\big) \leq C \sum_{i=1}^{4} T_{n,i}^k(t, x), \tag{4.4}$$

where

$$T_{n,1}(t, x) = \mathbb{E}\bigg(\bigg|\int_{t_n}^{t} \int_{\mathbb{R}^3} G(t - s, x - y) A\big(X_n(s, y)\big) M(\mathrm{d}s, \mathrm{d}y)\bigg|^p 1_{L_n(t)}\bigg),$$

$$T_{n,2}(t, x) = \mathbb{E}\big(\big|\langle G(t - \cdot, x - *) B\big(X_n(\cdot, *)\big) 1_{[t_n, t]}(\cdot), w^n\rangle_{\mathcal{H}_t}\big|^p 1_{L_n(t)}\big),$$

$$T_{n,3}(t, x) = \mathbb{E}\big(\big|\langle G(t - \cdot, x - *) D\big(X_n(\cdot, *)\big) 1_{[t_n, t]}(\cdot), h\rangle_{\mathcal{H}_t}\big|^p 1_{L_n(t)}\big),$$

$$T_{n,4}(t, x) = \mathbb{E}\bigg(\bigg|\int_{t_n}^{t} G(t - s, \cdot) \star b\big(X_n(s, \cdot)\big)(x) \, \mathrm{d}s\bigg|^p 1_{L_n(t)}\bigg).$$

By the same arguments used for the analysis of $V_1(t, x)$ in the preceding lemma, we obtain

$$T_{n,1}(t, x) \leq C 2^{-np(3-\beta)/2} \times \Big[1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, y)\big|^p 1_{L_n(s)}\big)\Big]. \tag{4.5}$$

For $T_{n,2}(t, x)$, we first use Cauchy–Schwarz' inequality to obtain

$$T_{n,2}(t, x)$$
$$\leq \mathbb{E}\big(\big|\big\| w^n 1_{[t_n, t]} 1_{L_n(t)}\big\|_{\mathcal{H}_t} \big\| G(t - \cdot, x - *) B\big(X_n(\cdot, *)\big) 1_{[t_n, t]}(\cdot) 1_{L_n(t)}\big\|_{\mathcal{H}_t}\big|^p\big).$$

Appealing to (2.12), this yields

$$T_{n,2}(t, x) \leq C n^{3p/2} \mathbb{E}\bigg(\bigg|\int_{t_n}^{t} \mathrm{d}s \big\| G(t - s, x - *) B\big(X_n(s, *)\big)(s) 1_{L_n(s)}\big\|_{\mathcal{H}}^2\bigg|^{p/2}\bigg).$$

We can now proceed as for the term $V_2((t, x)$ in the proof of Lemma 4.1. We obtain

$$T_{n,2}(t, x) \leq C n^{3p/2} 2^{-np(3-\beta)/2}\Big[1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, y)\big|^p 1_{L_n(s)}\big)\Big]. \tag{4.6}$$

The difference between the terms $T_{n,3}(t, x)$ and $T_{n,2}(t, x)$ is that $w^n$ in the latter is replaced by $h$ in the former. Hence, following similar arguments as for the study of $T_{n,2}(t, x)$, and using that $\|h 1_{[t_n, t]} 1_{L_n(t)}\|_{\mathcal{H}_T} < \infty$, we prove

$$T_{n,3}(t, x) \leq C 2^{-np(3-\beta)/2} \times \Big[1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\big(\big|X_n(s, y)\big|^p 1_{L_n(s)}\big)\Big]. \tag{4.7}$$

Finally, we notice the similitude between $T_{n,4}(t, x)$ and $V_3(t, x)$ in Lemma 4.1. Proceeding as for the study of this term, we obtain

$$
\begin{aligned}
T_{n,4}(t, x) &\le C \left( \int_{t_n}^t \mathrm{d}s \int_{\mathbb{R}^3} G(t - s, x - \mathrm{d}y) \right)^p \left[ 1 + \sup_{(s,y) \in [0,t] \times \mathbb{R}^3} \mathbb{E}\big( |X_n(s, y)|^p 1_{L_n(s)} \big) \right] \\
&\le C 2^{-np(3-\beta)/2} \left[ 1 + \sup_{(s,y) \in [0,t] \times \mathbb{R}^3} \mathbb{E}\big( |X_n(s, y)|^p 1_{L_n(s)} \big) \right].
\end{aligned}
\tag{4.8}
$$

From (4.4)–(4.8) we obtain (4.3). $\qquad\square$

**Lemma 4.3.** *We assume Hypothesis* (B). *Then, for any $p \in [1, \infty)$, there exists a finite constant $C$ such that*

$$
\sup_{n \ge 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\big[ \big( |X_n(t, x)|^p + |X_n^-(t, x)|^p \big) 1_{L_n(t)} \big] \le C.
\tag{4.9}
$$

*Moreover,*

$$
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \big\| \big( X_n(t, x) - X_n^-(t, x) \big) 1_{L_n(t)} \big\|_p \le C n^{3/2} 2^{-n(3-\beta)/2}.
\tag{4.10}
$$

**Proof.** For $0 \le r \le t$, define

$$
\begin{aligned}
X_n(t, r; x) &= \int_0^r \int_{\mathbb{R}^3} G(t - s, x - y) A\big( X_n(s, y) \big) M(\mathrm{d}s, \mathrm{d}y) \\
&\quad + \big\langle G(t - \cdot, x - *) B\big( X_n(\cdot, *) \big) 1_{[0,r]}(\cdot), w^n \big\rangle_{\mathcal{H}_t} \\
&\quad + \big\langle G(t - \cdot, x - *) D\big( X_n(\cdot, *) \big) 1_{[0,r]}(\cdot), h \big\rangle_{\mathcal{H}_t} + \int_0^r G(t - s, \cdot) \star b\big( X_n(s, \cdot) \big)(x) \, \mathrm{d}s.
\end{aligned}
$$

Fix $p \in [2, \infty[$ and consider the decomposition

$$
\mathbb{E}\big( |X_n(t, r; x)|^p 1_{L_n(t)} \big) \le C \sum_{i=1}^5 T_{n,i}(t, r; x),
$$

where

$$
T_{n,1}(t, r; x) = \mathbb{E}\left( \left| \int_0^r \int_{\mathbb{R}^3} G(t - s, x - y) A\big( X_n(s, y) \big) M(\mathrm{d}s, \mathrm{d}y) \right|^p 1_{L_n(t)} \right),
$$

$$
T_{n,2}(t, r; x) = \mathbb{E}\big( \big| \big\langle G(t - \cdot, x - *) B\big( X_n^-(\cdot, *) \big) 1_{[0,r]}(\cdot), w^n \big\rangle_{\mathcal{H}_t} \big|^p 1_{L_n(t)} \big),
$$

$$
T_{n,3}(t, r; x) = \mathbb{E}\big( \big| \big\langle G(t - \cdot, x - *) \big[ B\big( X_n(\cdot, *) \big) - B\big( X_n^-(\cdot, *) \big) \big] 1_{[0,r]}(\cdot), w^n \big\rangle_{\mathcal{H}_t} \big|^p 1_{L_n(t)} \big),
$$

$$
T_{n,4}(t, r; x) = \mathbb{E}\big( \big| \big\langle G(t - \cdot, x - *) D\big( X_n(\cdot, *) \big) 1_{[0,r]}(\cdot), h \big\rangle_{\mathcal{H}_t} \big|^p 1_{L_n(t)} \big),
$$

$$
T_{n,5}^k(t, r; x) = \mathbb{E}\left( \left| \int_0^r G(t - s, \cdot) \star b\big( X_n(s, \cdot) \big)(x) \, \mathrm{d}s \right|^p 1_{L_n(t)} \right).
$$

Similarly as for the term $V_1(t, x)$ in Lemma 4.1, we have

$$
\begin{aligned}
T_{n,1}(t, r; x) \leq C &\left( \int_0^r ds \int_{\mathbb{R}^3} \mu(d\xi) \left| \mathcal{F}G(t-s)(\xi) \right|^2 \right)^{p/2-1} \\
&\times \int_0^r ds \left[ 1 + \sup_{(\hat{s}, y) \in [0,s] \times \mathbb{R}^3} \mathbb{E}\left( \left| X_n(\hat{s}, y) \right|^p 1_{L_n(\hat{s})} \right) \right] \\
&\times \left( \int_{\mathbb{R}^3} \mu(d\xi) \left| \mathcal{F}G(t-s)(\xi) \right|^2 \right) \\
\leq C &\int_0^r ds \left[ 1 + \sup_{(\hat{s}, y) \in [0,s] \times \mathbb{R}^3} \mathbb{E}\left( \left| X_n(\hat{s}, y) \right|^p 1_{L_n(\hat{s})} \right) \right].
\end{aligned}
\tag{4.11}
$$

Let $\tau_n$ and $\pi_n$ be as in the proof of Lemma 2.8 (see (2.29) and the successive lines). Since $X_n^-(s, y)$ is $\mathcal{F}_{s_n}$-measurable, the definition of $w^n$ implies

$$
\begin{aligned}
&T_{n,2}(t, r; x) \\
&= \mathbb{E}\left( \left| \int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n) \left[ G(t - \cdot, x - *) B\left( X_n^-(\cdot, *) \right) \right](s, y) 1_{L_n(t)} 1_{[0,r]}(\cdot) M(ds, dy) \right|^p \right).
\end{aligned}
$$

Then, applying Burkholder's inequality, using the boundedness of the operator $\pi_n \circ \tau_n$, and similar arguments as for the term $T_{n,1}(t, r; x)$ we obtain

$$
T_{n,2}(t, r; x) \leq C \int_0^r ds \left[ 1 + \sup_{(\hat{s}, y) \in [0,s] \times \mathbb{R}^3} \mathbb{E}\left( \left| X_n^-(\hat{s}, y) \right|^p 1_{L_n(\hat{s})} \right) \right].
\tag{4.12}
$$

To study $T_{n,3}(t, r; x)$, we apply Cauchy–Schwarz and then Hölder's inequality. This yields

$$
\begin{aligned}
&T_{n,3}(t, r; x) \\
&\leq \mathbb{E}\left( \left\| w^n 1_{[0,r]} 1_{L_n(t)} \right\|_{\mathcal{H}_t}^2 \left\| G(t - \cdot, x - *) \left[ B(X_n) - B\left( X_n^- \right) \right](\cdot, *) 1_{[0,r]}(\cdot) 1_{L_n(t)} \right\|_{\mathcal{H}_t}^2 \right|^{p/2} \right) \\
&\leq C n^{3p/2} 2^{np/2} \mathbb{E}\left( \int_0^t ds \left\| G(t - s, x - *) \left[ B(X_n) - B\left( X_n^- \right) \right](s, *) 1_{[0,r]}(s) 1_{L_n(s)} \right\|_{\mathcal{H}}^2 \right)^{p/2} \\
&\leq C n^{3p/2} 2^{np/2} \left( \int_0^r ds \int_{\mathbb{R}^3} \mu(d\xi) \left| \mathcal{F}G(t-s) \right|^2(\xi) \right)^{p/2-1} \\
&\quad \times \int_0^r ds \sup_{(\hat{s}, y) \in [0,s] \times \mathbb{R}^3} \mathbb{E}\left( \left| X_n(\hat{s}, y) - X_n^-(\hat{s}, u) \right|^p 1_{L_n(\hat{s})} \right) \\
&\quad \times \left( \int_{\mathbb{R}^3} \mu(d\xi) \left| \mathcal{F}G(t-s) \right|^2(\xi) \right),
\end{aligned}
$$

where we have used (2.11) and the Lipschitz continuity of the function $B$. By applying (4.3), we obtain

$$T_{n,3}(t,r;x) \leq Cn^{3p}2^{-np[(3-\beta)/2-1/2]} \int_0^r ds \Big[1 + \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(|X_n(\hat{s},y)|^p 1_{L_n(\hat{s})}\big)\Big]$$

$$\leq C \int_0^r ds \Big[1 + \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(|X_n(\hat{s},y)|^p 1_{L_n(\hat{s})}\big)\Big],$$

where in the last inequality we have used that $\sup_n \{n^{3p}2^{-np[(3-\beta)/2-1/2]}\} < \infty$.

We now consider $T_{n,4}(t,r;x)$. With similar arguments as those used in the analysis of $T_{n,3}(t,x)$ in Lemma 4.2, we prove

$$T_{n,4}(t,r;x) \leq C \int_0^r ds \Big[1 + \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(|X_n(\hat{s},y)|^p 1_{L_n(\hat{s})}\big)\Big]. \tag{4.13}$$

Finally, we notice that $T_{n,5}(t,r;x)$ is very similar to $T_{n,4}(t,x)$ in Lemma 4.2. With similar arguments as those used in the analysis of this term, we have

$$T_{n,5}(t,r;x) \leq C \int_0^r ds \Big[1 + \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(|X_n(\hat{s},y)|^p 1_{L_n(\hat{s})}\big)\Big]. \tag{4.14}$$

Bringing together (4.11), (4.12)–(4.14) yields

$$\mathbb{E}\big(|X_n(t,r;x)|^p 1_{L_n(t)}\big)$$
$$\leq C\Big\{1 + \int_0^r \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(\{|X_n(\hat{s},y)|^p + |X_n^-(\hat{s},y)|^p\}1_{L_n(\hat{s})}\big) ds\Big\}. \tag{4.15}$$

Notice that $X_n(t,t;x) = X_n(t,x)$. Hence, for $r := t$, (4.15) tells us

$$\mathbb{E}\big(|X_n(t,x)|^p 1_{L_n(t)}\big)$$
$$\leq C\Big\{1 + \int_0^t \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(\{|X_n(\hat{s},y)|^p + |X_n^-(\hat{s},y)|^p\}1_{L_n(\hat{s})}\big) ds\Big\}. \tag{4.16}$$

Next, take $r := t_n$ and remember that $X_n(t,t_n;x) = X_n^-(t,x)$. From (4.15), and since $t_n \leq t$, we obtain

$$\mathbb{E}\big(|X_n^-(t,x)|^p 1_{L_n(t)}\big)$$
$$\leq C\Big\{1 + \int_0^t \sup_{(\hat{s},y)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\big(\{|X_n(\hat{s},y)|^p + |X_n^-(\hat{s},y)|^p\}1_{L_n(\hat{s})}\big) ds\Big\}. \tag{4.17}$$

For $t \in [0,T]$, set

$$\varphi_n(t) = \sup_{(s,y)\in[0,t]\times\mathbb{R}^3} \mathbb{E}\big[\big(|X_n(s,y)|^p + |X_n^-(s,y)|^p\big)1_{L_n(s)}\big].$$

The inequalities (4.16), (4.17) imply $\varphi_n(t) \leq C\{1 + \int_0^t \varphi_n(s)\,ds\}$. By Gronwall's lemma, this implies (4.9). Finally, the inequality (4.10) is a consequence of (4.3) and (4.9) $\qquad\square$

# Appendix

We start this section with a theorem on existence and uniqueness of solution to a class of equations which in particular applies to (2.3), and therefore also to (1.5), and to (2.4). For related results, we refer the reader to [6], Theorem 13, [9], Theorem 4.3 and [18], Proposition 4.0.4. In comparison with these references, here we state the theorem in spatial dimension $d = 3$, and we assume that $G$ is the fundamental solution of the wave equation in dimension three.

**Theorem A.1.** *Let $G$ denote the fundamental solution to the wave equation in dimension three and $M$ a Gaussian process as given in the Introduction. Consider the stochastic evolution equation defined by*

$$
\begin{aligned}
Z(t, x) = & \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)\sigma\big(Z(s, y)\big)M(ds, dy) \\
& + \big\langle G(t - \cdot, x - *)g\big(Z(\cdot, *)\big), H\big\rangle_{\mathcal{H}_T} \\
& + \int_0^t \big[G(t - s, \cdot) \star b\big(Z(s, \cdot)\big)\big](x),
\end{aligned}
\tag{A.1}
$$

*where the functions $\sigma, g, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous.*

  (i) *Assume that $H = \{H_t, t \in [0, T]\}$ is an $\mathcal{H}$-valued predictable stochastic process such that $C_0 := \sup_\omega \|H(\omega)\|_{\mathcal{H}_T} < \infty$.*
      *Then, there exists a unique real-valued adapted stochastic process $Z = \{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ satisfying (A.1), a.s., for all $(t, x) \in [0, T] \times \mathbb{R}^3$. Moreover, the process $Z$ is continuous in $L^2$ and satisfies*

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|Z(t, x)\big|^p\big) \leq C < \infty,
$$

  *for any $p \in [1, \infty[$, where the constant $C$ depends among others on $C_0$.*
 (ii) *Assume that there exist an increasing sequence of events $\{\Omega_n, n \geq 1\}$ such that $\lim_{n\to\infty} \mathbb{P}(\Omega_n) = 1$, and that $H_n = \{H_n(t), t \in [0, T]\}$ is a sequence of $\mathcal{H}$-valued predictable stochastic processes such that $C_n := \sup_\omega \|H(\omega)1_{\Omega_n}(\omega)\|_{\mathcal{H}_T} < \infty$. Then, the conclusion on existence and uniqueness of solution to (A.1) stated in part (i) also holds.*

*The process $Z$ is termed a random field solution to (A.1).*

**Sketch of the proof.** We start with part (i). Consider the Picard iteration scheme

$$
Z^0(t, x) = 0,
$$

$$Z^{(k+1)}(t, x) = \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)\sigma\big(Z^{(k)}(s, y)\big)M(\mathrm{d}s, \mathrm{d}y)$$

$$+ \big\langle G(t - \cdot, x - *)g\big(Z^{(k)}(\cdot, *)\big), H\big\rangle_{\mathcal{H}_T} + \int_0^t \big[G(t - s, \cdot) \star b\big(Z^{(k)}(s, \cdot)\big)\big](x),$$

$k \geq 0$.

Fix $p \in [2, \infty[$. First, we prove by induction on $k \geq 0$ that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|Z^k(t, x)\big|^p\big) \leq C < \infty, \tag{A.2}$$

with a constant $C$ independent of $k$. Second, we prove that

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}\big(\big|Z^{(k+1)}(t, x) - Z^{(k)}(t, x)\big|^p\big)$$

$$\leq C(1 + C_0)\left[\int_0^t \mathrm{d}s \sup_{y\in\mathbb{R}^3} \mathbb{E}\big(\big|Z^{(k)}(s, y) - Z^{(k-1)}(s, y)\big|^p\big)\right].$$

With this, we conclude that the sequence of processes $\{Z^{(k)}(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$, $k \geq 0$ converges in $L^p(\Omega)$ as $k \to \infty$, uniformly in $(t, x) \in [0, T] \times \mathbb{R}^3$. The limit is a random field that satisfies the properties of the statement. We refer the reader to [6,9,18], for more details on the proof.

The proof of part (ii) is done by localizing the preceding Picard scheme using the sequence $\{\Omega_n, n \geq 1\}$. $\qquad\square$

In comparison with the equation considered in [9], Theorem 4.3, (A.1) has null initial conditions, and the extra term $\langle G(t - \cdot, x - *)g(Z(\cdot, *)), H\rangle_{\mathcal{H}_T}$.

Part (i) of Theorem A.1 can be applied to (1.5), (2.3). Therefore, we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\big(\big|X(t, x)\big|^p\big) < \infty. \tag{A.3}$$

Let $\Omega_n = L_n(t)$ as given in (2.10). The sequence $H_n := w^n$ defined in (2.1) satisfies the assumptions of part (ii) of Theorem A.1 (see (2.11)). Therefore the conclusion applies to the stochastic process solution of (2.4).

**Remark A.2.** Set $Z^{(z)}(s, x) = Z(s, x + z)$, $z \in \mathbb{R}^3$. Similarly as in [6], we can argue that the finite dimensional distributions of the process $\{Z^{(z)}(s, x), (s, x) \in [0, T] \times \mathbb{R}^3\}$ do not depend on $z$. This is a consequence from the fact that the martingale measure $M$ has a spatial stationary covariance, and that the initial condition of the SPDE vanishes.

At several points, we have applied the following version of Gronwall's lemma whose proof can be found in [2], Theorem 4.9.

**Lemma A.3.** *Let $u$, $b$ and $k$ be nonnegative continuous functions defined on the interval $J = [\alpha, \beta]$. Let $\bar{p} \geq 0$, $\bar{p} \neq 1$ and $a > 0$ be constants. Suppose that*

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)\,\mathrm{d}s + \int_{\alpha}^{t} k(s)u^{\bar{p}}(s)\,\mathrm{d}s, \quad t \in J.$$

*Then*

$$u(t) \leq \exp\left(\int_{\alpha}^{\beta} b(s)\,\mathrm{d}s\right)\left[a^{\bar{q}} + \bar{q}\int_{\alpha}^{\beta} k(s)\exp\left(-\bar{q}\int_{\alpha}^{s} b(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{1/\bar{q}}, \quad \text{(A.4)}$$

*for every $t \in [\alpha, \beta_1)$, where $\bar{q} = 1 - \bar{p}$ and $\beta_1$ is choosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha, \beta_1)$ ($\beta_1 = \beta$ if $\bar{q} > 0$).*

In the proof of Theorem 2.2, we have used the lemma below. For its proof, we refer the reader to Lemma A.2 in [17], with a trivial change on the spacial dimension ($d = 3$ in [17], while $d = 4$ in Lemma A.4).

**Lemma A.4.** *Fix $[t_0, T]$ with $t_0 \geq 0$ and a compact set $K \subset \mathbb{R}^3$. Let $\{Y_n(t, x), (t, x) \in [t_0, T] \times K, n \geq 1\}$ be a sequence of processes and $\{B_n(t), t \in [t_0, T]\} \subset \mathcal{F}$ be a sequence of adapted events which, for every $n$, decreases in $t$. Assume that for every $p \in ]1, \infty[$ the following conditions hold:*

(P1) *There exists $\delta > 0$ and $C > 0$ such that, for any $t_0 \leq t \leq \bar{t} \leq T$, $x, \bar{x} \in K$,*

$$\sup_n \mathbb{E}\left(\left|Y_n(t, x) - Y_n(\bar{t}, \bar{x})\right|^p 1_{B_n(\bar{t})}\right) \leq C\left(|t - \bar{t}| + |x - \bar{x}|\right)^{4+\delta}.$$

(P2) *For every $(t, x) \in [t_0, T] \times K$,*

$$\lim_{n \to \infty} \mathbb{E}\left(\left|Y_n(t, x)\right|^p 1_{B_n(t)}\right) = 0.$$

*Then, for any $\eta \in ]0, \delta/p[$ and any $r \in [1, p[$,*

$$\lim_{n \to \infty} \mathbb{E}\left(\|Y_n\|_{\eta, t_0, K}^r 1_{B_n(T)}\right) = 0.$$

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