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Local extinction in continuous-state branching processes with immigration

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The purpose of this article is to observe that the zero sets of continuous-state branching processes with immigration (CBI) are infinitely divisible regenerative sets. Indeed, they can be constructed by the procedure of random cutouts introduced by Mandelbrot in 1972. We then show how very precise information about the zero sets of CBI can be obtained in terms of the branching and immigrating mechanism.

Keywords: continuous-state branching process; polarity; random cutout; zero set

1. Introduction

The problem of characterizing the zero set of a real-valued random process is, in general, not straightforward. When the process is a one-dimensional diffusion, several methods have been developed to study their zero sets. When dealing with Markov processes with jumps, the problem is rather involved and remains highly studied. We refer for instance to the recent survey of Xiao [34] where fractal properties for Lévy processes and other Markov processes are discussed. In this paper, we characterize the zero set of continuous-state branching processes with immigration (here called CBI processes).

Fundamental results on CBI processes, including their characterization as the large population limit of Galton–Watson processes with immigration and the complete determination of the generator, are obtained by Kawazu and Watanabe [23]. Since then, this class of processes have been studied extensively in several directions.

The time evolution of the CBI process in general incorporates two kinds of dynamics: reproduction and immigration. Indeed, Kawazu and Watanabe [23] show that the law of a CBI is characterized by the Laplace exponents Ψ and Φ of two independent Lévy processes: a spectrally positive Lévy process (which describes the reproduction) and a subordinator (which describes the immigration).

One of consequences of introducing immigration is that zero is not an absorbing but a reflecting state. We provide necessary and sufficient conditions for zero to be polar, transient or recurrent. These results are obtained thanks to a connection between the zero set of a CBI and the random cutout sets defined by Mandelbrot [28]. Let Y be a CBI process started at zero associated to (Ψ, Φ). Define the random set

$$\mathcal{Z} := \overline{\{t \geq 0; Y_t = 0\}},$$

and denote by v_s the solution to the differential equation

$$\frac{\mathrm{d}v_s}{\mathrm{d}s} = -\Psi(v_s) \quad \text{and} \quad v_{0+} = \infty.$$

Our main result is the following theorem.

Theorem 1.

(i) $\mathcal{Z} = \{0\}$ *if and only if* $\int_0^1 \exp[-\int_1^u \Phi(v_s)\,\mathrm{d}s]\,\mathrm{d}u = \infty$.

(ii) *If* $\int_0^1 \exp[-\int_1^u \Phi(v_s)\,\mathrm{d}s]\,\mathrm{d}u < \infty$, *then*

(a) *The random set* $\mathcal{Z}$ *is the closure of the range of a subordinator with Laplace exponent*

$$\begin{aligned} L(q) &= \left[\int_0^\infty \mathrm{e}^{-qt} \exp\left(\int_t^1 \Phi(v_s)\,\mathrm{d}s\right)\mathrm{d}t\right]^{-1} \\ &= \left[\int_0^\infty \mathrm{e}^{-qt} \exp\left(\int_{v_1}^{v_t} \frac{\Phi(u)}{\Psi(u)}\,\mathrm{d}u\right)\mathrm{d}t\right]^{-1}. \end{aligned}$$

(b) *The random set* $\mathcal{Z}$ *has almost surely a positive Lebesgue measure if and only if* $\int_\theta^\infty \frac{\Phi(s)}{\Psi(s)}\,\mathrm{d}s < \infty$ (*in that case*, *we say that the zero set is heavy*, *otherwise the set is light*).

(c) *The random set* $\mathcal{Z}$ *is almost surely the union of closed nonempty intervals if and only if* Φ *is the Laplace exponent of a compound Poisson process.*

Under an assumption of regular variation on the ratio $R: u \mapsto \Phi(u)/\Psi(u)$, we are able to give more details on the zero set. Loosely speaking, the index of regularity of the map R at $+\infty$ denoted by ρ measures the strength of the immigration over the reproduction. We prove for instance that if $\rho > -1$ then 0 is polar. When $\rho = -1$, some new constants are involved in the nature of the state zero. We also get an upper and a lower bound for the Hausdorff dimension of the zero set.

The connection made between random covering of the real line and the zero set of a CBI can be extended to another celebrated class of processes: the so-called generalized Ornstein–Uhlenbeck processes (OU processes). Besides obtaining results to generalized OU processes which are CBI processes, we provide a characterization of the zero set of Ornstein–Uhlenbeck processes driven by stable Lévy processes.

The paper is organized as follows. Section 2 contains preliminaries on CBI processes and Mandelbrot's random cutout construction. Section 3 contains the first implications of Theorem 1 to properties of the zero set of CBI processes as polarity, transience, recurrence, and Box counting and Hausdorff dimensions. In Section 4, we link the zero set of a CBI process to random cutouts through the spine decomposition and prove Theorem 1. We then particularize to the case when the ratio of the immigration and branching mechanisms of our CBI process is regularly varying in Section 5. Finally, Section 6 is devoted to a characterization of the zero set of an Ornstein–Uhlenbeck process driven by a stable Lévy process.

2. Preliminaries

2.1. Continuous-state branching processes

Let $x \in \mathbb{R}_+$. A Markov process $X(x) = (X_t(x), t \geq 0)$, where $X_0(x) = x$ is called a branching process in continuous time and continuous space (CB for short) if it satisfies the following property: For any $y \in \mathbb{R}_+$

$$X(x+y) \overset{d}{=} X(x) + \tilde{X}(y),$$

where $\tilde{X}(y)$ is an independent copy of $X(y)$.

Let $X(x)$ be a CB issued from x. There exists a unique triplet (d, σ, ν) with $d, \sigma \geq 0$, and ν a measure carried on $\mathbb{R}_+$ satisfying

$$\int_0^\infty (1 \wedge x^2) \nu(\mathrm{d}x) < \infty$$

such that the Laplace transform of the one-dimensional distribution of $X_t(x)$ is given by

$$\mathbb{E}\big[\mathrm{e}^{-\lambda X_t(x)}\big] = \exp\big(-x v_t(\lambda)\big),$$

where the map $t \mapsto v_t(\lambda)$ is the solution to the differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\Psi\big(v_t(\lambda)\big), \qquad v_0(\lambda) = \lambda$$

with

$$\Psi(q) = \frac{\sigma^2}{2} q^2 + dq + \int_0^\infty \big(\mathrm{e}^{-qx} - 1 + qx 1_{x \leq 1}\big) \nu(\mathrm{d}x). \tag{1}$$

The process is said to be critical, subcritical or supercritical according as $\Psi'(0+) = 0$, $\Psi'(0+) > 0$ or $\Psi'(0+) < 0$. For any $x \geq 0$, define the extinction time of the $\mathrm{CB}(X_t(x), t \geq 0)$ by

$$\zeta := \inf\big\{t \geq 0; X_t(x) = 0\big\}.$$

Recall the following result regarding the distribution of the extinction time (see, e.g., Theorems 3.5 and 3.8, pages 59–60 of [27]):

$$\mathbb{P}_x[\zeta \leq t] = \exp(-x v_t),$$

where $v_t = \lim_{\lambda \to \infty} v_t(\lambda)$. Also, recall *Grey's theorem*.

Theorem 2 (Grey [20]). *The* $\mathrm{CB}(X_t, t \geq 0)$ *is absorbed in* 0 *with positive probability if and only if there exists* $\theta > 0$ *such that* $\Psi(z) > 0$ *for* $z \geq \theta$ *and*

$$\int_\theta^\infty \frac{\mathrm{d}q}{\Psi(q)} < \infty. \tag{2}$$

Under that integrability condition (called Grey's condition), the real number $v := \lim \downarrow_{t\to\infty} v_t \in [0,\infty[$ *is the largest root of the equation* $\Psi(x) = 0$ *and*

$$\mathbb{P}_x[\zeta < \infty] = \exp(-xv).$$

In the (sub)critical case, $v = 0$ *and the* $\mathrm{CB}(\Psi)$ *is absorbed at zero almost surely. In the supercritical case,* $v > 0$.

2.2. Continuous-state branching processes with immigration

A continuous-state branching processes with immigration (CBI for short) started at x is a Markov process $Y(x) = (Y_t(x), t \geq 0)$ satisfying the following property: for any $y \in \mathbb{R}_+$

$$Y(x+y) \stackrel{d}{=} Y(x) + X(y),$$

where $X(y)$ is an independent CB with mechanism Ψ. Any CBI process is characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = dq + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty \left(\mathrm{e}^{-qu} - 1 + qu 1_{u\in(0,1)}\right)\nu_1(\mathrm{d}u),$$

$$\Phi(q) = \beta q + \int_0^\infty \left(1 - \mathrm{e}^{-qu}\right)\nu_0(\mathrm{d}u),$$

where $\sigma^2, \beta \geq 0$ and ν_0, ν_1 are two Lévy measures such that

$$\int_0^\infty (1\wedge u)\nu_0(\mathrm{d}u) < \infty \quad \text{and} \quad \int_0^\infty \left(1\wedge u^2\right)\nu_1(\mathrm{d}u) < \infty.$$

The measure ν_1 is the Lévy measure of a spectrally positive Lévy process which characterizes the reproduction. The measure ν_0 characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The nonnegative constants σ^2 and β correspond, respectively, to the continuous reproduction and the continuous immigration. Let $\mathbb{P}_x$ be the law of a $\mathrm{CBI}(Y_t(x), t \geq 0)$ started at x, and denote by $\mathbb{E}_x$ the associated expectation. The law of $(Y_t(x), t \geq 0)$ can then be characterized by the Laplace transform of its marginal as follows: for every $q > 0$ and $x \in \mathbb{R}_+$,

$$\mathbb{E}_x\left[\mathrm{e}^{-qY_t}\right] = \exp\left(-xv_t(q) - \int_0^t \Phi\left(v_s(q)\right)\mathrm{d}s\right),$$

where

$$v_t(q) = q - \int_0^t \Psi\left(v_s(q)\right)\mathrm{d}s.$$

The pair (Ψ, Φ) is known as the branching and immigration mechanisms. A CBI process $(Y_t, t \geq 0)$ is said to be conservative if for every $t > 0$ and $x \in [0,\infty[$, $\mathbb{P}_x[Y_t < \infty] = 1$. A result of Kawazu and Watanabe [23] states that $(Y_t, t \geq 0)$ is conservative if and only if for every $\varepsilon > 0$,

$$\int_0^{\varepsilon} \frac{1}{|\Psi(q)|} \mathrm{d}q = \infty.$$

See Theorem 10.3 in Kyprianou [25] for a proof.

Contrary to the simpler setting of continuous-state branching processes without immigration, nondegenerate stationary laws may appear. The following theorem provides a necessary and sufficient condition for the CBI to have a stationary distribution. Properties of the support of its stationary law have been studied by Keller-Ressel and Mijatović in [24].

Theorem 3 (Pinsky [31], Li [27]). *The* CBI(Ψ, Φ) *process has a stationary law if and only if*

$$\Psi'(0+) \geq 0 \quad \textit{and} \quad \int_0^{\theta} \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u < \infty.$$

If the process is subcritical $(\Psi'(0+) > 0)$, *the convergence of this integral is equivalent to the following log-condition*

$$\int_{x \geq 1} \log(x) \nu_1(\mathrm{d}x) < \infty.$$

The main ingredient of the proofs provided in this work relies on a connection between the zero set of the CBI and a particular random set, called random cutout set.

2.3. Random covering of the real half-line

We recall here the definition of a random cutout set studied first by Mandelbrot [28]. For a textbook presentation of the main theorems regarding random cutouts, we refer to the course of Bertoin [3]. The main results we shall use in this paper may be found in. Consider a σ-finite measure μ on $\mathbb{R}_+$ which is finite on compact subsets of $(0, \infty)$. Denote its tail $\mu([x, \infty[)$ by $\bar{\mu}(x)$ for all $x \in \mathbb{R}_+$. Let $\mathcal{N}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\mathrm{d}t \otimes \mu$. Denote by $(t_i, x_i)_{i \in \mathcal{I}}$ its atoms:

$$\mathcal{N} = \sum_{i \in \mathcal{I}} \delta_{(t_i, x_i)}.$$

We recover the half line by the intervals $]t_i, t_i + x_i[$, $i \in \mathcal{I}$. The set of uncovered point is

$$\mathcal{R} = [0, \infty) - \bigcup_{i \in I}]t_i, t_i + x_i[.$$

The measure μ is called the cutting measure. A random set $\mathcal{R}$ is said to be a random cutout set if it is obtained by such a construction. Any random cutout set is a regenerative set (i.e., the closure of the range of a subordinator).

Theorem 4 (Fitzsimmons, Fristedt and Shepp [17]).

$$\text{If } \int_0^1 \exp\left(\int_t^1 \bar{\mu}(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty \text{ then } \mathcal{R} = \{0\} \text{ a.s.}$$

Otherwise $\mathcal{R}$ is the closure of the image of a subordinator with Laplace exponent L given by

$$\frac{1}{L(q)} = \int_0^\infty \mathrm{e}^{-qt} \exp\left(\int_t^1 \bar{\mu}(s)\,\mathrm{d}s\right)\mathrm{d}t.$$

Moreover if $\int_1^\infty \exp(\int_t^1 \bar{\mu}(s)\,\mathrm{d}s)\,\mathrm{d}t = \infty$, then the set $\mathcal{R}$ is unbounded.

These random cutout sets have been intensively studied. As mentioned in the Introduction, we shall prove that the zero set of a CBI process is a random cutout set. Along the article, we shall use results on their geometry and refer the reader to the monography of Bertoin [3]. For instance, we mention that Theorem 4 matches with Theorem 7.2 of [3].

We end this section with the notion of infinitely divisibility for a regenerative set. A regenerative set $\mathcal{R}$ is said to be infinitely divisible if for any $n \geq 1$, there exist n independent identically distributed regenerative sets $(\mathcal{R}^i, 1 \leq i \leq n)$ such that

$$\mathcal{R} \stackrel{\text{law}}{=} \bigcap_{i=1}^n \mathcal{R}^i.$$

Any random-cutout set is infinitely divisible (cf. [17], Theorem 3). To the best of our knowledge, the converse is still an open question (cf. [29], Open problem 2.24, page 334).

3. The zero set of continuous-state branching processes with immigration

In this section, we give some basic consequences of Theorem 1. The latter theorem is proved in Section 4. We shall focus on a CBI(Ψ, Φ) started at 0 such that Ψ satisfies Grey's condition. As we shall see in Section 4, this is of no relevance for the study of the zero set. By a slight abuse of notation, we denote by $(Y_t, t \geq 0)$ the process $(Y_t(0), t \geq 0)$. Define the random set

$$\mathcal{Z} := \overline{\{t \geq 0; Y_t = 0\}}.$$

Proposition 5. *The random set $\mathcal{Z}$ is infinitely divisible.*

Proof. Let $n \geq 1$. Consider n independent CBI processes $(Y^i; 1 \leq i \leq n)$ started at 0 with branching mechanism Ψ and immigrating mechanism $\frac{1}{n}\Phi$. We have clearly the following equalities in law

$$(Y_t, t \geq 0) \stackrel{\text{law}}{=} \left(\sum_{i=1}^n Y_t^i, t \geq 0\right),$$

and therefore

$$\mathcal{Z} \overset{\text{law}}{=} \bigcap_{i=1}^{n} \overline{\{t \geq 0; Y_t^i = 0\}}. \qquad \square$$

If the set $\mathcal{Z}$ is not reduced to $\{0\}$ and bounded (resp., unbounded), the state 0 is transient (resp., recurrent). Recall that the state 0 is said to be polar if

$$\mathbb{P}_x[\exists t \geq 0, Y_t = 0] = 0$$

for any $x \neq 0$.

Remark 3.1. If the process is supercritical, then clearly 0 is polar or transient. This can be easily observed using Theorem 1.

The next result is a corollary of Theorem 1. Since the supercritical case is plain, we focus on (sub)critical reproduction mechanism when studying the recurrence and transience of 0 (statements (ii) and (iii) below).

Corollary 6. *Let $\theta > 0$ such that $\Psi(u) > 0$ for all $u \geq \theta$.*

(i) *The state 0 is polar if and only if*

$$\int_\theta^\infty \exp\left[\int_\theta^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right] \frac{1}{\Psi(z)} \mathrm{d}z = \infty.$$

Assume further that Ψ is (sub)critical, the state 0 is

(ii) *transient if and only if*

$$\int_\theta^\infty \exp\left[\int_\theta^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right] \frac{1}{\Psi(z)} \mathrm{d}z < \infty \quad \textit{and} \quad \int_0^\theta \exp\left[-\int_x^\theta \frac{\Phi(s)}{\Psi(s)} \mathrm{d}s\right] \frac{\mathrm{d}x}{\Psi(x)} < \infty.$$

(iii) *recurrent if and only if*

$$\int_\theta^\infty \exp\left[\int_\theta^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right] \frac{1}{\Psi(z)} \mathrm{d}z < \infty \quad \textit{and} \quad \int_0^\theta \exp\left[-\int_x^\theta \frac{\Phi(s)}{\Psi(s)} \mathrm{d}s\right] \frac{\mathrm{d}x}{\Psi(x)} = \infty.$$

Remark 3.2. We may compare the integral conditions with those for the continuous-state branching process without immigration. The first statement has to be compared with Grey's condition. The probability of blow-up of the CB (and CBI) depends on the behaviour of Ψ at 0. Assume that the process is conservative, which holds if and only if $\int_0^\theta \frac{\mathrm{d}x}{\Psi(x)} = \infty$. If moreover $\int_0^\theta \frac{\Phi(s)}{\Psi(s)} \mathrm{d}s < \infty$, then the second condition in statement (iii) is always satisfied. Note that if the process is not conservative, obviously the state 0 is either polar or transient.

Proof of Corollary 6. Statements of the corollary are easily obtained by substitution in the integrals appearing in Theorem 1. Recall that $v_{0+} = \infty$, the first statement is equivalent to the statement (i) of Theorem 1 by considering $u = v_s$. The transience or the recurrence of zero hold, respectively, if the set $\mathcal{Z}$ is bounded or unbounded. As already explained, this is equivalent for the underlying subordinator to be killed or not. If the state 0 is not polar, then the zero set is bounded if and only if $L(0) > 0$. By the same substitution in the integral condition of Theorem 1, we get $\int_1^\infty \Phi(v_t)\,\mathrm{d}t = \int_v^{v_1} \frac{\Phi(u)}{\Psi(u)}\,\mathrm{d}u$. The constant v is the limit of $(v_t, t \geq 0)$ when t goes to ∞ and equals to 0 since we focus on the (sub)critical case. The constant v_1 does not play any role here and may be replaced by θ. □

Recall the definition of the lower and upper box-counting dimension. See, for instance, Section 3 of [34] or Chapter 5 of [3]. For every nonempty bounded subset E of $\mathbb{R}_+$, let $N_\varepsilon(E)$ be the smallest number of intervals of length ε needed to cover E. The upper and lower box-counting dimension of E are defined as

$$\overline{\dim}(E) := \limsup_{\varepsilon \to 0} \frac{\log(N_\varepsilon(E))}{\log(1/\varepsilon)}$$

and

$$\underline{\dim}(E) := \liminf_{\varepsilon \to 0} \frac{\log(N_\varepsilon(E))}{\log(1/\varepsilon)}.$$

We provide now a last general result on the zero set. The proof is postponed at the end of Section 4.

Lemma 7. *The random set $\mathcal{Z}$ has the following upper and lower box-counting dimensions. For every $t > 0$,*

$$\overline{\dim}\big(\mathcal{Z} \cap [0,t]\big) = 1 - \liminf_{u \to 0} \frac{1}{\log(1/u)} \int_{v_1}^{v_u} \frac{\Phi(s)}{\Psi(s)}\,\mathrm{d}s \qquad a.s.$$

and

$$\underline{\dim}\big(\mathcal{Z} \cap [0,t]\big) = 1 - \limsup_{u \to 0} \frac{1}{\log(1/u)} \int_{v_1}^{v_u} \frac{\Phi(s)}{\Psi(s)}\,\mathrm{d}s \qquad a.s.$$

Moreover if $\mathcal{Z}$ is bounded almost surely, then the law of the last zero of $(Y_t, t \geq 0)$,

$$g_\infty := \sup\{s \geq 0; s \in \mathcal{Z}\}$$

is given by

$$\mathbb{P}[g_\infty \in \mathrm{d}t] = k^{-1} \exp\left(\int_t^1 \Phi(v_s)\,\mathrm{d}s\right)\mathrm{d}t = k^{-1} \exp\left(\int_{v_1}^{v_t} \frac{\Phi(u)}{\Psi(u)}\,\mathrm{d}u\right)\mathrm{d}t,$$

with k the renormalization constant.

Remark 3.3. The lower box-counting dimension and the Hausdorff dimension (denoted by $\dim_H$) of a regenerative set coincides almost surely (see Corollary 5.3 of [3]).

4. Analysis of the zero set: Spine decomposition and random covering

The objective of this section is to recall a construction of CBI processes which will then be used to prove Theorem 1.

A $\mathrm{CB}(\Psi)$ process reaches zero with positive probability if and only if the branching mechanism Ψ satisfies Grey's condition. Intuitively, a $\mathrm{CBI}(\Psi, \Phi)$ cannot touch zero, unless the corresponding $\mathrm{CB}(\Psi)$ can reach zero. We begin by establishing rigorously this idea. Let $x > 0$; the branching property stated in Section 2.2 provides that the $\mathrm{CBI}(\Psi, \Phi)$ process $Y(x)$ satisfies

$$Y(x) \stackrel{\text{law}}{=} Y(0) + X(x)$$

for a $\mathrm{CB}(\Psi)$ $X(x)$ that starts at x. If $x > 0$ and $Y(x)$ reaches zero with positive probability, then plainly the $\mathrm{CB}(\Psi)$ $X(x)$ reaches also zero with positive probability. Theorem 2 ensures then that Ψ verifies Grey's condition. In order to handle the case $x = 0$, we note that $\Phi \neq 0$ implies that

$$\mathbb{E}\bigl(\mathrm{e}^{-\lambda Y_t(0)}\bigr) = \mathrm{e}^{-\int_0^t \Phi(v_s(\lambda))\,\mathrm{d}s} < 1$$

and so $Y_t(0) > 0$ with positive probability for any $t > 0$. If $Y(0)$ reaches zero with positive probability, then it does so after time t for some $t > 0$. Applying the Markov property at time t on the set $Y_t(0) > 0$, we are reduced to the previous case and conclude that Grey's condition holds. Thus, we see that imposing Grey's condition on the branching mechanism merely rules out a trivial case in which the zero-set of $Y(0)$ is almost surely $\{0\}$.

In order to prove Theorem 1, we now recall a Poissonian decomposition of CBI processes, also called the spine decomposition, which is obvious in the simpler setting of Galton–Watson processes in immigration and indeed has been considered by a number of authors in the continuous setting. For example, [32] use it to obtain a decomposition of Bessel bridges or in [15] it allows representations of superprocesses conditioned on nonextinction, a work which has continued in [14,18].

The spine decomposition of CBI processes is found in Section 2.1 of [9] or as a particular case of the construction of Section 2.2 of [1]. In the case of CB processes with stable reproduction mechanism conditioned on nonextinction, which are self-similar CBI processes, the spine decomposition is found in [16]. The spine decomposition is based on the $\mathbb{N}$-measures constructed for superprocesses in [13] and specialized to the case of CBI processes in Theorem 1.1 of [9]. We now give a streamlined exposition of the construction of this specialized $\mathbb{N}$-measure, based on [32], by assuming Grey's condition.

Starting in [32], a Poisson process representation for (continuous) CBI processes has been achieved by means of a σ-finite measure which can be understood as the excursion law of a $\mathrm{CB}(\Psi)$ although, in general, one is not able to concatenate excursions to obtain a recurrent extension of a $\mathrm{CB}(\Psi)$ (cf. [32], page 440).

For a $\mathrm{CB}(\Psi)$, we have

$$\mathbb{E}_x^{\Psi}\bigl(\mathrm{e}^{-\lambda X_t}\bigr) = \mathrm{e}^{-x v_t(\lambda)}$$

for any $x \geq 0$. Hence, $\lambda \mapsto v_t(\lambda)$ is the Laplace exponent of a subordinator, under (2), it is a driftless subordinator since

$$\lim_{\lambda \to \infty} \frac{v_t(\lambda)}{\lambda} = 0.$$

See, for instance, Corollary 3.11 page 61 of [27]. (This is the fundamental simplification in comparison with [9].) Let $(P_t, t \geq 0)$ be the semigroup of the $\mathrm{CB}(\Psi)$ and η_t stand for the Lévy measure of the Laplace exponent v_t so that

$$v_t(\lambda) = \int_0^\infty \left(1 - \mathrm{e}^{-\lambda x}\right) \eta_t(\mathrm{d}x).$$

By the composition property $v_t \circ v_s(\lambda) = v_{t+s}(\lambda)$ and so

$$\begin{aligned}
\lambda \int \mathrm{e}^{-\lambda y} \eta_{t+s}(y, \infty)\,\mathrm{d}y &= \int 1 - \mathrm{e}^{-\lambda x} \eta_{t+s}(\mathrm{d}x) = v_{t+s}(\lambda) \\
&= \int 1 - \mathrm{e}^{-x v_s(\lambda)} \eta_t(\mathrm{d}x) \\
&= \lambda \int \mathrm{e}^{-\lambda y} \int \mathbb{P}_x(X_s > y) \eta_t(\mathrm{d}x)\,\mathrm{d}y.
\end{aligned}$$

We then see that η_{t+s} and $\eta_t P_s$ have the same tails and that therefore, $(\eta_t, t > 0)$ is an entrance law for the semigroup P_t. Hence, we may consider the σ-finite measure Q on the space of càdlàg excursions starting and ending at zero characterized by its finite-dimensional marginals: for $0 < t_1 < \cdots < t_n$

$$Q(X_{t_1} \in \mathrm{d}x_1, \ldots, X_{t_n} \in \mathrm{d}x_n) = \mathbf{1}_{x_1, \ldots, x_n > 0} \eta_{t_1}(\mathrm{d}x_1) P^{\Psi}_{t_2 - t_1}(x_1, \mathrm{d}x_2) \cdots P^{\Psi}_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n).$$

This can be thought of as an excursion law for the $\mathrm{CB}(\Psi)$. It is not a trivial point that $X_t \to 0$ as $t \to 0+$ Q-almost everywhere. Pitman and Yor argue, in the case of diffusions, by stating a William's type decomposition of the measure Q: on the set where X reaches a height $> x$, the measure Q is proportional to the probability measure which concatenates the law of a $\mathrm{CB}(\Psi)$ conditioned to stay positive (started at zero) until the process reaches a height $> x$ with a trajectory of a $\mathrm{CB}(\Psi)$ until it reaches zero. Although not explicitly stated, this point of view is the basis for the proof given in [26], Section 2.4, that $X_{0+} = 0$ under Q. Also, under Q, X is Markovian and with the semigroup of the $\mathrm{CB}(\Psi)$. To characterize the image of the length of the excursions under Q, say ζ, notice that

$$\begin{aligned}
Q(\zeta > t) &= \lim_{h \to 0+} Q(\zeta > t + h) = \lim_{h \to 0+} \int \mathbb{P}^{\Psi}_x(\zeta > t) \eta_h(\mathrm{d}x) \\
&= \lim_{h \to 0+} \int \left(1 - \mathrm{e}^{-x v_t}\right) \eta_h(\mathrm{d}x) = \lim_{h \to 0+} v_h(v_t) = v_t.
\end{aligned}$$

Now use Φ and Q to construct the σ-finite measure $\mathbb{N}$ on excursion space by means of

$$\mathbb{N} = \beta Q + \int \nu_0(\mathrm{d}x) \mathbb{P}^{\Psi}_x,$$

which will be the intensity of the Poisson point process

$$\Theta = \sum_{i \in \mathcal{I}} \delta_{(t_i, X^i)}.$$

Finally, let

$$Y_t = \sum_{t_i \le t} X^i_{t-t_i}.$$

Theorem 8. *Y is a* $\mathrm{CBI}(\Psi, \Phi)$ *which starts at* 0.

This is what we refer to as a spine decomposition of a CBI process.

Proof of Theorem 8. We start with a preliminary computation which uses the exponential formula for Poisson point processes:

$$\begin{aligned}
\mathbb{E}\bigl(\mathrm{e}^{-\lambda Y_t}\bigr) &= \mathbb{E}\biggl(\exp -\lambda \sum_{i \in \mathcal{I}; t_i \le t} X^i_{t-t_i}\biggr) = \exp\biggl(-\int_0^t \mathbb{N}\bigl(1 - \mathrm{e}^{-\lambda X_{t-s}}\bigr)\,\mathrm{d}s\biggr) \\
&= \exp\biggl(-\int_0^t \int_0^\infty \bigl(1 - \mathrm{e}^{-y v_{t-s}(\lambda)}\bigr)\nu(\mathrm{d}y)\,\mathrm{d}s - \beta \int_0^t \int \bigl(1 - \mathrm{e}^{-\lambda x}\bigr)\eta_{t-s}(\mathrm{d}x)\,\mathrm{d}s\biggr) \\
&= \exp\biggl(-\int_0^t \Phi\bigl(v_{t-s}(\lambda)\bigr)\,\mathrm{d}s\biggr) = \exp\biggl(-\int_0^t \Phi\bigl(v_s(\lambda)\bigr)\,\mathrm{d}s\biggr).
\end{aligned}$$

Hence, at least Y has the correct one-dimensional distributions. To prove that Y is a CBI, we need to compute conditional expectations. Let

$$\mathcal{F}_t = \sigma\biggl\{\sum_{t_i \le t} \delta_{(t_i, X^i_{t-t_i})}\biggr\}.$$

Then

$$\begin{aligned}
\mathbb{E}\biggl[\exp\biggl(-\lambda \sum_{t \le t_i \le t+s} X^i_{t+s-t_i}\biggr)\biggm|\mathcal{F}_t\biggr] &= \mathbb{E}\biggl[\exp\biggl(-\lambda \sum_{t \le t_i \le t+s} X^i_{t+s-t_i}\biggr)\biggr] \\
&= \exp\biggl(-\int_t^{t+s} \Phi\bigl(v_{t+s-r}(\lambda)\bigr)\,\mathrm{d}r\biggr) \\
&= \exp\biggl(-\int_0^s \Phi\bigl(v_u(\lambda)\bigr)\,\mathrm{d}u\biggr).
\end{aligned}$$

On the other hand:

$$\mathbb{E}\biggl[\exp\biggl(-\lambda \sum_{t_i \le t} X^i_{t+s-t_i}\biggr)\biggm|\mathcal{F}_t\biggr] = \exp\biggl(-\sum_{t_i \le t} X^i_{t-t_i} v_s(\lambda)\biggr) = \exp\bigl(-v_s(\lambda) Y_t\bigr).$$

Hence,

$$\mathbb{E}\big(e^{-\lambda Y_{t+s}}|\mathcal{F}_t\big) = \exp\bigg(-Y_t v_s(\lambda) - \int_0^s \Phi\big(v_u(\lambda)\big)\,du\bigg).$$

Because Y is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ and since the conditional expectations of functionals of Y_{t+s} given $\mathcal{F}_t$ depend only on Y_t we see that Y is a homogeneous Markov process. Since the Laplace transform of the semigroup of Y coincides with that of a CBI(Ψ, Φ), we conclude that Y is a CBI(Ψ, Φ) process started at 0. □

Remark 4.1. We mention that for the excursion measure Q to exist (and then for this spine decomposition to hold), one only needs that $\sigma > 0$ or $\int_0^\infty (1 \wedge u)\nu_1(du) = \infty$. We refer the reader to Duquesne and Labbé [11]. Moreover, this spine decomposition has also been used in the framework of stationary CBIs by Bi [4] to study the time of the most recent common ancestor.

Proof of Theorem 1. From this spine decomposition, we establish a useful connection between the random covering procedure and the set of the zeros of the CBI. Results due to Fitzsimmons *et al.* [17], recalled in Section 2.3, will allow us to study the set $\mathcal{Z}$. Denote for each $i \in \mathcal{I}$, $\zeta_i = \inf\{t \geq 0; X_t^i = 0\}$. By standard properties of random Poisson measures, the random measure $\sum_{i\in I} \delta_{(t_i,\zeta_i)}$ is a Poisson random measure with intensity $dt \otimes \mathbb{N}(\zeta \in dt)$. Plainly, we have

$$\{t \geq 0, Y_t = 0\} = \mathbb{R}_+ \Big\backslash \Big[\bigcup_{i\in\mathcal{I}; X_0^i>0} [t_i, t_i + \zeta_i[\cup \bigcup_{i\in\mathcal{I}; X_0^i=0}]t_i, t_i + \zeta_i[\Big].$$

The following crucial equality holds

$$\mathcal{Z} = \mathbb{R}_+ \Big\backslash \bigcup_{i\in I}]t_i, t_i + \zeta_i[. \tag{3}$$

Notice that some t_i may belong to $\mathbb{R}_+ \setminus \bigcup_{i\in I}]t_i, t_i + \zeta_i[$ and not to $\{t \geq 0, Y_t = 0\}$. However such point t_i is a left accumulation point of the set $\{t \geq 0, Y_t = 0\}$. Namely, consider $t_i \in \mathcal{Z}$, assume by contradiction that there exists $\varepsilon > 0$ such that $[t_i - \varepsilon, t_i[\cap \{t \geq 0; Y_t = 0\} = \varnothing$, then taking the closure we have $[t_i - \varepsilon, t_i] \cap \mathcal{Z} = \varnothing$ which is impossible since $t_i \in \mathcal{Z}$. Hence, equality (3) holds almost surely. Moreover, we have

$$\begin{aligned}
\mathbb{N}(\zeta > t) &= \beta Q[\zeta > t] + \int_0^\infty \mathbb{P}_x(\zeta > t)\nu_0(dx) \\
&= \beta v_t + \int_0^\infty \big(1 - \exp(-x v_t)\big)\nu_0(dx) \\
&= \Phi(v_t).
\end{aligned}$$

In order to establish the statement, we can directly use the results of Section 2.3. Part (i) and statement (a) of (ii) readily follow from Theorem 4, taking

$$\bar{\mu}(t) = \mathbb{N}(\zeta > t) = \Phi(v_t)$$

for all $t \geq 0$. We get $\mathcal{Z} = \{0\}$ if and only if

$$\int_0^1 \exp\left[-\int_1^u \Phi(v_s)\,\mathrm{d}s\right]\mathrm{d}u = \infty.$$

Using the differential equation satisfied by v, the substitution $z = v_t$ and the condition $v_{0+} = \infty$ we get

$$\int_0^1 \exp\left[-\int_1^u \Phi(v_s)\,\mathrm{d}s\right]\mathrm{d}u = \int_{v_1}^{\infty} \exp\left[\int_{v_1}^{z} \frac{\Phi(u)}{\Psi(u)}\,\mathrm{d}u\right]\frac{1}{\Psi(z)}\,\mathrm{d}z = \infty.$$

Statement (b) follows from Proposition 1 of [17] (see also Corollary 7.3 of [3]). Namely the set $\mathcal{Z}$ has a positive Lebesgue measure if and only if

$$\int_0^{\infty} (s \wedge 1)\mu(\mathrm{d}s) = \int_0^1 \Phi(v_s)\,\mathrm{d}s < \infty.$$

By the same substitution, this yields

$$\int_{v_1}^{\infty} \frac{\Phi(s)}{\Psi(s)}\,\mathrm{d}s < \infty.$$

The last statement (d) follows from Corollary 2 of [17]. Namely, the necessary and sufficient condition for the uncovered set to be a union of intervals is that the intensity measure $\mathbb{N}(\zeta \in \mathrm{d}t)$ has a finite mass. Therefore, assume that $\Phi(v_t) \longrightarrow_{t\to 0} c < \infty$. We have $v_t \longrightarrow_{t\to 0} \infty$ and

$$\Phi(v_t) = \beta v_t + \int_0^{\infty} \left(1 - \mathrm{e}^{-v_t x}\right)\nu_0(\mathrm{d}x).$$

By monotone convergence

$$\int_0^{\infty} \left(1 - \mathrm{e}^{-v_t x}\right)\nu_0(\mathrm{d}x) \underset{t\to 0}{\longrightarrow} \nu_0\big([0, \infty[\big).$$

Thus we get the conditions $\beta = 0$ and $\nu_0([0, \infty[) < \infty$. $\square$

We now provide the proof of Lemma 7. The arguments are directly those used for the fractal dimensions of random cutout set.

Proof of Lemma 7. Combining Theorem 5.1 and Corollary 7.6 of [3], we get that for every $t > 0$,

$$\overline{\dim}\big(\mathcal{Z} \cap [0, t]\big) = 1 - \liminf_{u\to 0} \frac{\int_u^1 \Phi(v_s)\,\mathrm{d}s}{\log(1/u)} \qquad \text{a.s.}$$

and

$$\underline{\dim}\big(\mathcal{Z} \cap [0, t]\big) = 1 - \limsup_{u\to 0} \frac{\int_u^1 \Phi(v_s)\,\mathrm{d}s}{\log(1/u)} \qquad \text{a.s.}$$

We get the statement by substitution $u = v_s$. The second statement concerning the largest zero is a direct application of Corollary 7.4 of [3]. □

5. Regularly varying branching-immigrating mechanisms

In this section, we shall focus on specific mechanisms Ψ and Φ such that the function *ratio* $R(x) = \frac{\Phi(x)}{\Psi(x)}$ is regularly varying at ∞. The index of R, called in the sequel, ρ may be interpreted as representing the strength of the immigration over the reproduction. For sake of conciseness, we only work with a critical branching mechanism Ψ. A remarkable phenomenon occurs when $\rho = -1$. In such case, other quantities are involved. Surprisingly the quantities $\underline{r} := \liminf_{s\to\infty} sR(s)$ and $\overline{r} := \limsup_{s\to\infty} sR(s)$ play a crucial role.

We recall that a positive measurable function f defined on some neighbourhood of ∞ (resp., 0) is said to be regularly varying at ∞ (resp., 0) with index ρ if for all $c > 0$,

$$f(cx)/f(x) \longrightarrow c^{\rho} \qquad \text{when } x \to \infty \text{ (resp., 0)}.$$

5.1. Polarity of zero and Hausdorff dimension of the zero set

We first recall the definition of the upper and lower indices at infinity of a Laplace exponent (we refer the reader to the seminal work of Blumenthal and Getoor [6] and to Duquesne and Le Gall's work [12], page 557).

$$\underline{\mathrm{Ind}}(\Psi) = \sup\left\{\alpha \ge 0; \lim_{\lambda\to\infty} \Psi(\lambda)\lambda^{-\alpha} = \infty\right\} = \liminf_{\lambda\to\infty} \frac{\log(\Psi(\lambda))}{\log(\lambda)},$$
$$\overline{\mathrm{Ind}}(\Psi) = \inf\left\{\alpha \ge 0; \lim_{\lambda\to\infty} \Psi(\lambda)\lambda^{-\alpha} = 0\right\} = \limsup_{\lambda\to\infty} \frac{\log(\Psi(\lambda))}{\log(\lambda)}.$$

This definition holds also for the Laplace exponent of a subordinator Φ, replace only Ψ by Φ above.

Theorem 9. *Assume that R is regularly varying at $+\infty$ with index ρ.*

(i) *If $\rho > -1$, then 0 is polar.*
(ii) *If $\rho < -1$, then 0 is not polar and the zero set is heavy.*
(iii) *If $\rho = -1$. Define the quantities*

$$\overline{r} := \limsup_{s\to\infty} s\frac{\Phi(s)}{\Psi(s)}$$

and

$$\underline{r} := \liminf_{s\to\infty} s\frac{\Phi(s)}{\Psi(s)}.$$

Both quantities belong to $[0, \infty]$.

(a) *If* $\overline{r} < \underline{\mathrm{Ind}}(\Psi) - 1$, *then* 0 *is not polar. Moreover if* $\underline{r} > 0$, *then the zero set is light*.
(b) *If* $\underline{r} \geq \overline{\mathrm{Ind}}(\Psi) - 1$, *then* 0 *is polar.*

Proof. Assume that $\rho > -1$, therefore $z \mapsto \int_\theta^z R(u)\,\mathrm{d}u$ is regularly varying with index $\rho + 1$ and by Karamata's theorem (see, e.g., Proposition 1.5.8 of [5]) we have

$$\text{For all } z \geq \theta, \qquad \int_\theta^z R(u)\,\mathrm{d}u = C + z^{\rho+1} l(z) \tag{4}$$

with C a constant and l a slowly varying function. By Corollary 6, we have to study the following integral

$$\mathcal{I} := \int_\theta^\infty \exp\biggl(\int_\theta^z R(u)\,\mathrm{d}u\biggr) \frac{1}{\Psi(z)}\,\mathrm{d}z.$$

(i) If $\rho > -1$, since l is slowly varying, we have $z^{(\rho+1)/2} l(z) \longrightarrow_{z\to\infty} +\infty$, moreover observing that $\Psi(z) = \mathrm{O}(z^2)$, we have clearly that $\mathcal{I} = \infty$.
(ii) If $\rho < -1$, by Proposition 1.5.10 of [5], we have $\int_\theta^\infty R(u)\,\mathrm{d}u < \infty$, then $\mathcal{I} < \infty$ and furthermore by statement (ii)(b) of Theorem 1, the zero set has a positive Lebesgue measure.
(iii) In the case $\rho = -1$, the map R is regularly varying with index -1. We mention that integrals of such regularly varying functions yields to de Haan functions (see Chapter 3 of [5]).
(a) Let $\varepsilon > 0$. For s large enough, $R(s) \leq \frac{\overline{r}}{s} + \frac{\varepsilon}{s}$. Therefore for large enough z,

$$\exp\biggl(\int_\theta^z R(s)\,\mathrm{d}s\biggr) \frac{1}{\Psi(z)} \leq C \frac{z^{\overline{r}+\varepsilon}}{\Psi(z)}.$$

Moreover, $\Psi(z) \geq C' z^{\underline{\mathrm{Ind}}(\Psi)-\varepsilon}$ and thus

$$\exp\biggl(\int_\theta^z R(s)\,\mathrm{d}s\biggr) \frac{1}{\Psi(z)} \leq C'' z^{\overline{r}-\underline{\mathrm{Ind}}(\Psi)+2\varepsilon}.$$

As ε is arbitrarily small, if $\overline{r} - \underline{\mathrm{Ind}}(\Psi) < -1$ then

$$\int_\theta^\infty z^{\overline{r}-\underline{\mathrm{Ind}}(\Psi)+2\varepsilon}\,\mathrm{d}z < \infty.$$

This implies that 0 is not polar. Assume $\underline{r} > 0$ for s large enough, $R(s) \geq \underline{r}/2s$, we have clearly $\int_\theta^\infty R(s)\,\mathrm{d}s = \infty$.
(b) For s large enough, $sR(s) \geq \underline{r} - \varepsilon$ and then $\exp(\int_\theta^z R(s)\,\mathrm{d}s)\frac{1}{\Psi(z)} \geq C\frac{z^{\underline{r}-\varepsilon}}{\Psi(z)}$ By the same reasoning, we have $\Psi(z) \leq C' z^{\overline{\mathrm{Ind}}(\Psi)+\varepsilon}$ and we get

$$\exp\biggl(\int_\theta^z R(s)\,\mathrm{d}s\biggr) \frac{1}{\Psi(z)} \geq C'' z^{\underline{r}-\overline{\mathrm{Ind}}(\Psi)-2\varepsilon}.$$

If $\underline{r} - \overline{\mathrm{Ind}}(\Psi) > -1$, since ε is arbitrarily small, then $\int_\theta^\infty z^{\underline{r}-\overline{\mathrm{Ind}}(\Psi)-2\varepsilon}\,\mathrm{d}z = \infty$ and $\mathcal{I} = \infty$.

If $\underline{r} - \overline{\mathrm{Ind}}(\Psi) = -1$, the direct computation

$$\int_\theta^\infty z^{-1-2\varepsilon}\,\mathrm{d}z = \frac{\theta^{-2\varepsilon}}{2\varepsilon}$$

yields the lower bound $\mathcal{I} \geq C'' \frac{\theta^{-2\varepsilon}}{2\varepsilon}$. Letting ε going to 0, we get $\mathcal{I} = \infty$. □

Proposition 10. *Assume that $\overline{r} < \underline{\mathrm{Ind}}(\Psi) - 1$ then for all $t > 0$,*

$$1 - \frac{\overline{r}}{\overline{\mathrm{Ind}}(\Psi) - 1} \leq \overline{\dim}\big(\mathcal{Z} \cap [0,t]\big) \leq 1 - \frac{\underline{r}}{\overline{\mathrm{Ind}}(\Psi) - 1}$$

and

$$1 - \frac{\overline{r}}{\underline{\mathrm{Ind}}(\Psi) - 1} \leq \underline{\dim}\big(\mathcal{Z} \cap [0,t]\big) \leq 1 - \frac{\underline{r}}{\underline{\mathrm{Ind}}(\Psi) - 1}.$$

Before tackling the proof, we need a lemma.

Lemma 11 (Duquesne Le Gall, Lemma 5.6 of [12]). *The following equalities hold*

$$\liminf_{t\to 0} \frac{\log(v_t)}{\log(1/t)} = \frac{1}{\overline{\mathrm{Ind}}(\Psi) - 1}$$

and

$$\limsup_{t\to 0} \frac{\log(v_t)}{\log(1/t)} = \frac{1}{\underline{\mathrm{Ind}}(\Psi) - 1}.$$

Proof. Duquesne and Le Gall established

$$\liminf_{t\to 0} \frac{\log(v_t)}{\log(1/t)} \leq \frac{1}{\overline{\mathrm{Ind}}(\Psi) - 1}$$

and

$$\limsup_{t\to 0} \frac{\log(v_t)}{\log(1/t)} \leq \frac{1}{\underline{\mathrm{Ind}}(\Psi) - 1}.$$

The other inequalities are also true. We provide here a proof. Denote $\eta := \overline{\mathrm{Ind}}(\Psi)$, there exists $\varepsilon > 0$ arbitrarily small such that

$$q^{-(\eta+\varepsilon)}\Psi(q) \underset{q\to\infty}{\longrightarrow} 0.$$

For all $C > 0$, for large enough q we have $\frac{1}{\Psi(q)} \geq \frac{C}{q^{\eta+\varepsilon}}$. Recall that $\eta > 1$, therefore for small enough t, we have

$$\int_{v_t}^{\infty} \frac{\mathrm{d}q}{\Psi(q)} = t \geq C \int_{v_t}^{\infty} \frac{\mathrm{d}q}{q^{\eta+\varepsilon}} = \frac{C}{\eta+\varepsilon-1} v_t^{1-\eta-\varepsilon}.$$

This yields

$$\log\left(\frac{1}{t}\right) \leq \log\left(\frac{\eta+\varepsilon-1}{C}\right) + (\eta-1+\varepsilon)\log(v_t).$$

We then have

$$\liminf_{t\to 0} \frac{\log(v_t)}{\log(1/t)} \geq \frac{1}{\eta-1+\varepsilon}.$$

As ε is arbitrarily small, the result follows and the first equality in the statement is established.

We study now the second equality. Denote $\gamma := \underline{\mathrm{Ind}}(\Psi)$. Let $\gamma' > \gamma$. There exists a sequence $(u_n, n \geq 1)$ such that $u_n \longrightarrow_{n\to\infty} \infty$ and $\Psi(2u_n) \leq 2^{\gamma'} u_n^{\gamma'}$. For $u \in]0, 2u_n]$, we have by convexity $\frac{\Psi(u)}{u} \leq \frac{\Psi(2u_n)}{2u_n}$ and therefore $\Psi(u) \leq 2^{\gamma'-1} u u_n^{\gamma'-1}$ when $u \in]0, 2u_n]$. Define the function $F(a) = \int_a^{\infty} \frac{\mathrm{d}u}{\Psi(u)}$, we have

$$F(u_n) = \int_{u_n}^{\infty} \frac{\mathrm{d}u}{\Psi(u)} \geq \int_{u_n}^{2u_n} \frac{\mathrm{d}u}{\Psi(u)} \geq 2^{1-\gamma'} u_n^{1-\gamma'} \log(2).$$

Then

$$\liminf_{a\to\infty} \frac{\log(1/F(a))}{\log(a)} \leq \lim_{n\to\infty} \frac{\log(1/F(u_n))}{\log(u_n)} \leq \gamma' - 1.$$

As on page 592 of [12], observe that by definition of v_t:

$$\left(\limsup_{t\to 0} \frac{\log(v_t)}{\log(1/t)}\right)^{-1} = \liminf_{a\to\infty} \frac{\log(1/F(a))}{\log(a)}.$$

We deduce that

$$\limsup_{t\to 0} \frac{\log(v_t)}{\log(1/t)} \geq \frac{1}{\gamma'-1},$$

by letting γ' go to γ, we obtain the wished inequality. □

Proof of Proposition 10. Let $\varepsilon > 0$, by assumption, we have for C large enough

$$\sup_{s\in[C,\infty[} sR(s) \leq \overline{r} + \varepsilon \quad \text{and} \quad \inf_{s\in[C,\infty[} sR(s) \geq \underline{r} - \varepsilon.$$

Therefore,

$$(\underline{r} - \varepsilon)\big[\log(v_t) - \log(C)\big] \leq \int_C^{v_t} \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u \leq (\overline{r} + \varepsilon)\big[\log(v_t) - \log(C)\big].$$

By using the previous lemma and Lemma 7, we plainly get

$$1 - \frac{\overline{r} + \varepsilon}{\overline{\overline{\mathrm{Ind}}}(\Psi) - 1} \leq \overline{\dim}\big(\mathcal{Z} \cap [0, t]\big) \leq 1 - \frac{\underline{r} - \varepsilon}{\overline{\overline{\mathrm{Ind}}}(\Psi) - 1}.$$

As ε is arbitrarily small, we get the statement. Same arguments hold for the lower box-counting dimension. $\square$

5.2. Recurrence and regular variation at 0+

To study the recurrence of zero, we need information on the behaviour of the map R in the neighbourhood of $0+$ (see statement (iii) in Corollary 6). In the same vein as our previous result on polarity, a natural assumption is to consider the map R with regular variation at $0+$.

In order to state the following result, we need to introduce the lower and upper indices of a Laplace exponent at $0+$:

$$\underline{\mathrm{ind}}(\Psi) = \sup\Big\{\alpha \geq 0; \lim_{\lambda \to 0} \Psi(\lambda)\lambda^{-\alpha} = 0\Big\} = \liminf_{\lambda \to 0} \frac{\log(\Psi(\lambda))}{\log(\lambda)},$$

$$\overline{\mathrm{ind}}(\Psi) = \inf\Big\{\alpha \geq 0; \lim_{\lambda \to 0} \Psi(\lambda)\lambda^{-\alpha} = \infty\Big\} = \limsup_{\lambda \to 0} \frac{\log(\Psi(\lambda))}{\log(\lambda)}.$$

Theorem 12. *Assume that the map R is regularly varying at $+\infty$ with index ρ and at $0+$ with index κ. If $\rho < -1$ or $\rho = -1$ and $\overline{r} < \underline{\mathrm{Ind}}(\Psi) - 1$ (then 0 is not polar) and*

(i) *if $\kappa < -1$ then 0 is transient,*
(ii) *if $\kappa > -1$ then 0 is recurrent,*
(iii) *if $\kappa = -1$. Define the quantities*

$$\overline{\kappa} := \limsup_{x \to 0} \, x R(x)$$

and

$$\underline{\kappa} := \liminf_{x \to 0} \, x R(x).$$

Both quantities belong to $[0, \infty]$.
(a) *If $\overline{\kappa} - \underline{\underline{\mathrm{ind}}}(\Psi) \leq -1$, then 0 is recurrent.*
(b) *If $\underline{\kappa} - \overline{\overline{\mathrm{ind}}}(\Psi) > -1$, then 0 is transient.*

Proof. We assume that 0 is not polar, then we have to study the following integral

$$\mathcal{J} := \int_0^\theta \exp\bigg[-\int_x^\theta R(u)\,\mathrm{d}u\bigg] \frac{\mathrm{d}x}{\Psi(x)}.$$

Assume that R is regularly varying at 0 with index $\kappa \in \mathbb{R}$, then by an easy adaptation of Propositions 1.5.8 and 1.5.10 of [5] to the setting of regular variation at 0+, we get

$$\lim_{x\to 0}\int_x^\theta R(u)\,\mathrm{d}u = \begin{cases} = \infty, & \text{if } \kappa < -1, \\ < \infty, & \text{if } \kappa > -1. \end{cases}$$

More precisely, if $\kappa < -1$, we have $\int_x^\theta R(u)\,\mathrm{d}u = x^{\kappa+1}l(x)$ with l a slowly varying function at 0+.

- Assume $\kappa < -1$. Let $\varepsilon > 0$, we have for small enough x, $\Psi(x) \geq Cx^{\overline{\mathrm{ind}}(\Psi)+\varepsilon}$. We shall prove that the map

  $$x \mapsto \frac{\exp(-x^{\kappa+1}l(x))}{x^{\overline{\mathrm{ind}}(\Psi)+\varepsilon}}$$

  is bounded in the neighbourhood of 0. This will imply that $\mathcal{J}$ is finite. Taking the logarithm, we have

  $$\log\left(\frac{\exp(-x^{\kappa+1}l(x))}{x^{\overline{\mathrm{ind}}(\Psi)+\varepsilon}}\right) = -x^{\kappa+1}l(x)\left(1+\left(\overline{\mathrm{ind}}(\Psi)+\varepsilon\right)\frac{1}{l(x)}\frac{\log(x)}{x^{\kappa+1}}\right).$$

  On the one hand, $x^{\kappa+1}l(x) \longrightarrow_{x\to 0} +\infty$, on the other hand the map $x \mapsto \frac{\log(x)}{l(x)}$ is slowly varying at 0+, and thus using Potter's bound (see, e.g., Theorem 1.5.6(iii) in [5]), we have $x^{-(\kappa+1)}\frac{\log(x)}{l(x)} \longrightarrow_{x\to 0} 0$. Finally,

  $$-x^{\kappa+1}l(x)\left(1+\left(\overline{\mathrm{ind}}(\Psi)+\varepsilon\right)\frac{1}{l(x)}\frac{\log(x)}{x^{\kappa+1}}\right) \underset{x\to 0}{\longrightarrow} -\infty$$

  and the map is bounded. We deduce that $\mathcal{J} < \infty$ and then 0 is transient.
- If $\kappa > -1$, provided that $\int_{0+}\frac{\mathrm{d}x}{\Psi(x)} = \infty$, we have $\mathcal{J} = \infty$ and 0 is recurrent.
- We deal now with the case $\kappa = -1$. We prove first the recurrence criterion (a). Let $\varepsilon > 0$, for u small enough, we have

  $$R(u) \leq \frac{\overline{\kappa}}{u} + \frac{\varepsilon}{u} \quad \text{and} \quad \Psi(u) \leq Cu^{\underline{\mathrm{ind}}(\Psi)-\varepsilon}.$$

  We deduce that

  $$\frac{1}{\Psi(x)}\exp\left(-\int_x^\theta R(u)\,\mathrm{d}u\right) \geq Cx^{\overline{\kappa}-\underline{\mathrm{ind}}(\Psi)+2\varepsilon}.$$

  If $\overline{\kappa} - \underline{\mathrm{ind}}(\Psi) < -1$, since ε is arbitrarily small, we have $\mathcal{J} = \infty$.

  If $\overline{\kappa} - \underline{\mathrm{ind}}(\Psi) = -1$, then

  $$\int_0^\theta x^{-1+2\varepsilon}\,\mathrm{d}x = \frac{\theta^{2\varepsilon}}{2\varepsilon},$$

and $\mathcal{J} \geq C\frac{\theta^{2\varepsilon}}{2\varepsilon}$ and letting ε going to 0, we obtain $\mathcal{J} = \infty$. Therefore, 0 is recurrent. We prove now statement (b). Assume $\underline{\kappa} - \overline{\mathrm{ind}}(\Psi) > -1$. We have for u small enough $uR(u) \geq \underline{\kappa} - \varepsilon$ and $\Psi(u) \geq u^{\overline{\mathrm{ind}}(\Psi)+\varepsilon}$. Therefore,

$$\frac{1}{\Psi(x)} \exp\left(-\int_x^\theta R(u)\,\mathrm{d}u\right) \leq Cx^{\underline{\kappa}-\overline{\mathrm{ind}}(\Psi)-2\varepsilon}.$$

Since ε is arbitrarily small, we can choose one such that $\underline{\kappa} - \overline{\mathrm{ind}}(\Psi) - 2\varepsilon > -1$. This implies $\mathcal{J} < \infty$ and the transience follows. $\square$

Remark 5.1. If $\kappa > -1$, then $\int_{0+} R(u)\,\mathrm{d}u < \infty$ and by Theorem 3, the CBI has a stationary law.

We study in the sequel the specific case of stable and gamma mechanisms.

5.2.1. *Stable and gamma mechanisms*

Consider (Ψ, Φ) of the form $\Psi(q) = dq^\alpha$ and $\Phi(q) = d'q^\beta$ with $\alpha \in (1, 2]$, $\beta \in (0, 1]$ and $d, d' \in (0, \infty)$. The CBI process $(Y_t, t \geq 0)$ associated is said to be *stable*. Obviously, we have $\overline{\mathrm{Ind}}(\Psi) = \alpha$, $\underline{\mathrm{Ind}}(\Phi) = \beta$. The map R is regularly varying at $0+$ and at $+\infty$ with index $\rho = \kappa = \beta - \alpha$. We can thus apply the previous results in Theorem 9 and Theorem 12.

- If $\beta > \alpha - 1$, then 0 is polar,
- if $\beta < \alpha - 1$, then 0 is transient and the zero set is heavy,
- if $\beta = \alpha - 1$, we have $r := \lim_{x\to\infty} xR(x) = \frac{d'}{d}$ and $\kappa = -1$. Two cases may occur.
  - If $\frac{d'}{d} \geq \alpha - 1$, then 0 is polar,
  - if $\frac{d'}{d} < \alpha - 1$, then 0 is recurrent and by Proposition 10

$$\dim_H\big(\mathcal{Z} \cap [0, t]\big) = 1 - \frac{1}{\alpha - 1}\frac{d'}{d}.$$

Notice that in this stable framework, we cannot have $\rho < -1$ and $\kappa \geq -1$.

Proposition 13. *Let $(Y_t, t \geq 0)$ denote a stable critical CBI started at 0 with parameters α, β satisfying $\beta = \alpha - 1$.*

$$\mathcal{Z} = \overline{\{t \geq 0; Y_t = 0\}} = \overline{\{\sigma_t, t \geq 0\}}$$

with $(\sigma_t, t \geq 0)$ a stable subordinator with index $\gamma = 1 - \frac{1}{\alpha-1}\frac{d'}{d}$.

Proof. In order to get that the subordinator $(\sigma_t, t \geq 0)$ is a γ-stable one, we shall use self-similarity result. The self-similarity of $(Y_t, t \geq 0)$ follows by inspection (see also [30]). Namely, we have

$$\mathbb{E}_0\big[\mathrm{e}^{-qY_t}\big] = \exp\left(-\int_0^t \Phi\big(v_s(q)\big)\,\mathrm{d}s\right)$$

with $v_s(q) = q[1 + d(\alpha - 1)q^{\alpha-1}s]^{-1/(\alpha-1)}$. An easy computation yields that the processes $(kY_t, t \geq 0)$ and $(Y_{k^{\alpha-1}t}, t \geq 0)$ have the same law. We deduce that the regenerative set $\mathcal{Z}$ is self-similar meaning that for any $k > 0$, $k\mathcal{Z} \overset{\text{law}}{=} \mathcal{Z}$. The only regenerative sets satisfying this property are the closure of the range of stable subordinator (see, e.g., Section 3.1.1 of [3]). Proposition 10 provides the Hausdorff dimension of the set and therefore the index of stability (see Theorem 5.1 of [3]). □

Consider the immigration mechanism $\Phi(q) = \frac{\Gamma(\beta+q)}{\Gamma(\beta)\Gamma(q)} \sim_{q\to\infty} \frac{q^\beta}{\Gamma(\beta)}$. The subordinator with Laplace exponent Φ is called Lamperti stable subordinator (see, e.g., [8]). The map Φ is regularly varying at $+\infty$ with index β and at $0+$ with index 1. Assume that the reproduction mechanism Ψ is α-stable, as previously we can apply Theorems 9 and 12. We easily get

- if $\beta > \alpha - 1$, then 0 is polar,
- if $\beta < \alpha - 1$, then 0 is recurrent and the zero set is heavy,
- if $\beta = \alpha - 1$, we have $r := \lim_{s\to\infty} sR(s) = \frac{1}{d\Gamma(\alpha-1)}$ and $\kappa = 1 - \alpha \geq -1$. Two cases may occur.
  - if $d \leq \frac{1}{\Gamma(\alpha)}$ then 0 is polar,
  - if $d > \frac{1}{\Gamma(\alpha)}$ then 0 is not polar, and we easily verify that 0 is recurrent:

$$\text{if } \alpha = 2 \text{ then } \underline{\kappa} = \overline{\kappa} = 1, \text{ and 0 is recurrent,}$$
$$\text{if } \alpha \in (1,2) \text{ then } \underline{\kappa} = \overline{\kappa} = 0, \text{ and 0 is recurrent.}$$

Finally, by Proposition 10 we get

$$\dim_H\big(\mathcal{Z} \cap [0,t]\big) = 1 - \frac{1}{\Gamma(\alpha)d}.$$

Last, consider now the case of a Gamma immigration mechanism and a stable branching one. Let $a > 0, b > 0$ and $\alpha \in (1,2], d > 0$.

$$\Phi(x) = a\log(1 + x/b) = \int_0^\infty \big(1 - \mathrm{e}^{-xu}\big) a u^{-1} \mathrm{e}^{-bu}\,\mathrm{d}u,$$
$$\Psi(x) = dx^\alpha.$$

We can observe that Φ is slowly varying at ∞ and regularly varying at $0+$ with index 1. Thus $R: x \mapsto \frac{\Phi(x)}{\Psi(x)}$ is regularly varying at 0 with index $\kappa = 1 - \alpha$ and at $+\infty$ with index $\rho = -\alpha$. Applying Theorems 9 and 12, we obtain that the zero set is heavy and further:

- if $\alpha \in (1,2)$, then $\kappa > -1$ and 0 is recurrent,
- if $\alpha = 2$, then $\kappa = -1$ and $\overline{\kappa} = \underline{\kappa} = \frac{a}{bd}$. Therefore, 0 is recurrent if $\frac{a}{b} \leq d$, otherwise 0 is transient.

6. Ornstein–Uhlenbeck processes

We recall here basics on Markov processes of Ornstein–Uhlenbeck type. Contrary to CBI processes, these processes in general take values in $\mathbb{R}$. Consider $x \in \mathbb{R}$, $\gamma \in \mathbb{R}_+$, and $(A_t, t \geq 0)$ a one-dimensional Lévy process with characteristic function given by η such that

$$\mathbb{E}\big[e^{\mathrm{i}zA_t}\big] = \exp\big(t\eta(z)\big),$$
$$\eta(z) = -\frac{\sigma^2}{2}z^2 + \mathrm{i}bz + \int_{\mathbb{R}} \big(e^{\mathrm{i}zx} - 1 - \mathrm{i}zx1_{\{|x|\leq 1\}}\big)\nu(\mathrm{d}x),$$

where $\sigma \geq 0$, $b \in \mathbb{R}$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. A process $(X_t, t \geq 0)$ is said to be an OU type process if it satisfies the following equation

$$X_t = x - \gamma \int_0^t X_s \,\mathrm{d}s + A_t.$$

Generalized Ornstein–Uhlenbeck processes valued in $\mathbb{R}_+$ belong actually to the class of CBI processes. When A is subordinator with Lévy measure ν and drift d, the Ornstein–Uhlenbeck is a CBI with $\Psi(z) = \gamma z$ and $\Phi(z) = \mathrm{d}z + \int_0^\infty (1 - e^{-zx})\nu(\mathrm{d}x)$. In this case, Grey's condition is not fulfilled and clearly the state 0 is polar. Most of the Ornstein–Uhlenbeck processes are not CBI processes (we refer, e.g., to the discussion about their stationary laws in Proposition 4.7 in Keller-Ressel and Mijatović [24]). However, there is an interesting class of OU processes whose zero sets are random cutout sets: Ornstein–Uhlenbeck processes whose driving Lévy process is self-similar of index α. A Lévy process A is self-similar of index α if for any $c, t > 0$ there is equality in law between A_{ct} and $c^{1/\alpha}A_t$. Note that it excludes the asymmetric Cauchy process ($\alpha = 1$) which is not strictly stable. The corresponding characteristic function η is rather involved and we refer the reader to page 11 of Kyprianou [25].

Theorem 14. *Let $(X_t, t \geq 0)$ be an Ornstein–Uhlenbeck process started at 0 driven by a self-similar Lévy process of index α. If $\alpha \in (0, 1]$, then the zero set of X equals $\{0\}$ almost surely. If $\alpha \in (1, 2]$, then the zero set of X is a random cutout set whose cutting measure has density with respect to Lebesgue measure given by*

$$z \mapsto (1 - \beta)e^z / \big(e^z - 1\big)^2,$$

where $\beta = 1 - 1/\alpha$.

Proof. Consider the process

$$\tilde{X}_t = Ce^{-\gamma t} A_{e^{\gamma t \alpha}}, \qquad \text{where } C = \frac{1}{(\alpha\gamma)^{1/\alpha}}.$$

Because A is self-similar, it follows that $\tilde{X}$ is stationary. Using integration by parts, it follows that

$$\mathrm{d}\tilde{X}_t = Ce^{-\gamma t}\,\mathrm{d}A_{e^{\gamma t \alpha}} - \gamma \tilde{X}_t \,\mathrm{d}t.$$

Then, we can use Kallenberg's results on time-changes of stable stochastic integrals found in [22] (cf. equation (1.4)) to infer first the existence of a Lévy process $\tilde{A}$ with the same law as A such that for $t \geq 0$:

$$A_{e^{\alpha\gamma t}} - A_1 \stackrel{d}{=} A_{e^{\alpha\gamma t}-1} = A_{\int_0^t (e^{\gamma s}/C)^\alpha \, ds} = \frac{1}{C}\int_0^t e^{\gamma s} \, d\tilde{A}_s.$$

Then, by associativity of the stochastic integral, we see that

$$\int_0^t Ce^{-\gamma s} \, dA_{e^{\gamma s\alpha}} = \tilde{A}_t.$$

(A similar argument extends when integrating from s to t.) Hence, $\tilde{X}$ is a stationary version of the Ornstein–Uhlenbeck process driven by A.

On the other hand, the zero set of $\tilde{X}$ is then the logarithm of the zero set of A. The latter is a self-similar regenerative set and therefore a random cutout set. Actually, the zero set of A is empty if $\alpha \in (0, 1]$ (cf. [2], page 63) while if $\alpha \in (1, 2]$ then it has the law of the (closure) of the image of a β-stable subordinator with $\beta = 1 - 1/\alpha$ (see [7] and [19]).

Let

$$\Xi = \sum_{i\in\mathcal{I}} \delta_{(t_i, x_i)}$$

be a Poisson point process with intensity $dt \otimes \mu$ where $\mu(dx) = (1-\beta)x^{-2}\,dx$. Then the zero set of A has the law of a random cutout set based on Ξ and so the zero set of $\tilde{X}$ is the random cutout set (on $\mathbb{R}$) obtained by removing the intervals

$$(s_i, s_i + z_i) = \bigl(\log(t_i), \log(t_i + x_i)\bigr).$$

Namely, we have

$$\mathcal{Z} := \{t \in \mathbb{R}; \tilde{X}_t = 0\} = \mathbb{R} - \bigcup_{i\in I}]s_i, s_i + z_i[.$$

The intensity of the point process

$$\tilde{\Xi} = \sum_{i\in\mathcal{I}} \delta_{(s_i, z_i)}$$

is the image of the measure $dt \otimes \mu(dx)$ by $(t, x) \mapsto (s, z) = (\log(t), \log(t + x))$. A notable cancellation occurs and it is found to be equal to $(1-\beta)e^z/(e^z - 1)^2\,ds\,dz$. Using the identity established between the cutting measures of random cutouts on $(-\infty, \infty)$ and random cutouts on $(0, \infty)$ (cf. Theorem 2 of [17]) we see that the zero set of the α-stable Ornstein–Uhlenbeck process (started at zero) is a random cutout set whose cutting measure has density

$$(1-\beta)e^z/\bigl(e^z - 1\bigr)^2\,dz. \qquad \square$$

The cutting measure with density $(1-\beta)\mathrm{e}^z/(\mathrm{e}^z-1)^2$ was studied in detail in Example 8 of [17]. There, the authors show that the associated subordinators have zero drift and Lévy measure ν given by

$$\nu(x,\infty) = \frac{C}{(\mathrm{e}^x-1)^{1-\beta}}.$$

Note that the density of ν can also be written as

$$\nu(\mathrm{d}x) = \frac{C\mathrm{e}^x}{(\mathrm{e}^x-1)^{1-\beta}}\,\mathrm{d}x = \frac{C'\mathrm{e}^{(\beta/2)x}}{(\sinh(x/2))^{2-\beta}}\,\mathrm{d}x.$$

Hence, in the special case of the Ornstein–Uhlenbeck process associated to Brownian motion, we recover the results of [33] (which go back to [21]). However, we also deduce that in the general stable case, the zero set is the image of the Lamperti stable subordinators introduced in [10] and studied in general in [8].

Corollary 15. *If $\alpha \in (1,2]$, the random set $\mathcal{Z} = \overline{\{t \geq 0, X_t = 0\}}$ is infinitely divisible. Moreover, $\mathcal{Z}$ is almost surely not bounded* (0 *is recurrent*) *and we have for all $t > 0$*

$$\dim_H\bigl(\mathcal{Z}\cap[0,t]\bigr) = 1/\alpha.$$

Proof. We only have to give a proof for the Hausdorff dimension. The Laplace exponent of the Lamperti stable subordinator involved in Theorem 14 is $\kappa(\gamma) = \frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)\Gamma(\gamma)}$ (see equation (27) in [17]). Therefore $\dim_H(\mathcal{Z}\cap[0,t]) = \underline{\mathrm{Ind}}(\kappa) = 1-\beta = 1/\alpha$. $\square$

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