Bernoulli **19**(5A), 2013, 2000–2009
DOI: 10.3150/12-BEJ438

Integrability properties and limit theorems for the exit time from a cone of planar Brownian motion

STAVROS VAKEROUDIS[1,2,*] and MARC YOR[1,3,**]

[1] *Laboratoire de Probabilités et Modèles Aléatoires (LPMA), CNRS: UMR7599, Université Pierre et Marie Curie – Paris VI, Université Paris-Diderot Paris VII, 4, Place Jussieu, 75252 Paris Cedex 05, France. E-mail:* [*] *stavros.vakeroudis@upmc.fr; url: http://svakeroudis.wordpress.com;* [**] *yormarc@aol.com*
[2] *Probability and Statistics Group, School of Mathematics, University of Manchester, Alan Turing Building, Oxford Road, Manchester M13 9PL, United Kingdom*
[3] *Institut Universitaire de France, Paris, France*

We obtain some integrability properties and some limit theorems for the exit time from a cone of a planar Brownian motion, and we check that our computations are correct via Bougerol's identity.

Keywords: Bougerol's identity; exit time from a cone; planar Brownian motion; skew-product representation

1. Introduction

We consider a standard planar Brownian motion $(Z_t = X_t + \mathrm{i}Y_t, t \geq 0)$, starting from $x_0 + \mathrm{i}0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively, from x_0 and 0 (when we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0; for 2-dimensional Brownian motion, we indicate planar or complex BM).

As is well known Itô and McKean [10], since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{\mathrm{d}Z_s}{Z_s}), t \geq 0$ is well defined. A scaling argument shows that we may assume $x_0 = 1$, without loss of generality, since, with obvious notation:

$$\bigl(Z_t^{(x_0)}, t \geq 0\bigr) \stackrel{(law)}{=} \bigl(x_0 Z^{(1)}_{(t/x_0^2)}, t \geq 0\bigr). \tag{1}$$

Thus, from now on, we shall take $x_0 = 1$.

Furthermore, there is the skew product representation:

$$\log|Z_t| + \mathrm{i}\theta_t \equiv \int_0^t \frac{\mathrm{d}Z_s}{Z_s} = (\beta_u + \mathrm{i}\gamma_u)\bigg|_{u=H_t=\int_0^t \frac{\mathrm{d}s}{|Z_s|^2}}, \tag{2}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$. Thus, the Bessel clock H plays a key role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$ (see, e.g., Yor [21]).

Rewriting (2) as:

$$\log|Z_t| = \beta_{H_t}; \qquad \theta_t = \gamma_{H_t}, \tag{3}$$

we easily obtain that the two σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

We shall also use Bougerol's celebrated identity in law, see, for example, Bougerol [5], Alili, Dufresne and Yor [1] and Yor [24] (page 200), which may be written as:

$$\text{for fixed } t \qquad \sinh(\beta_t) \overset{(law)}{=} \hat{\beta}_{A_t(\beta)}, \tag{4}$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u \mathrm{d}s \exp(2\beta_s)$ and $(\hat{\beta}_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$. For the random times $T_c^{|\theta|} \equiv \inf\{t: |\theta_t| = c\}$, and $T_c^{|\gamma|} \equiv \inf\{t: |\gamma_t| = c\}$, $(c > 0)$ by using the skew-product representation (3) of planar Brownian motion Revuz and Yor [15], we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} \mathrm{d}s \exp(2\beta_s) = H_u^{-1}\Big|_{u=T_c^{|\gamma|}}. \tag{5}$$

Moreover, it has been recently shown that, Bougerol's identity applied with the random time $T_c^{|\theta|}$ instead of t in (4) yields the following Vakeroudis [18].

Proposition 1.1. *The distribution of* $T_c^{|\theta|}$ *is characterized by its Gauss–Laplace transform*:

$$E\left[\sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp\left(-\frac{x}{2T_c^{|\theta|}}\right)\right] = \frac{1}{\sqrt{1+x}} \varphi_m(x) \tag{6}$$

for every $x \geq 0$, *with* $m = \frac{\pi}{2c}$, *and*:

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \qquad G_\pm(x) = \sqrt{1+x} \pm \sqrt{x}. \tag{7}$$

The remainder of this article is organized as follows: in Section 2, we study some integrability properties for the exit times from a cone; more precisely, we obtain some new results concerning the negative moments of $T_c^{|\theta|}$ and of $T_c^{\theta} \equiv \inf\{t: \theta_t = c\}$. In Section 3, we state and prove some Limit theorems for these random times for $c \to 0$ and for $c \to \infty$ followed by several generalizations (for extensions of these works to more general planar processes, see, e.g., Doney and Vakeroudis [7]). We use these results in order to obtain (see Remark 3.4) a new simple non-computational proof of Spitzer's celebrated asymptotic theorem Spitzer [16], which states that:

$$\frac{2}{\log t} \theta_t \xrightarrow[t \to \infty]{(law)} C_1, \tag{8}$$

with C_1 denoting a standard Cauchy variable (for other proofs, see, e.g., Williams [20], Durrett [9], Messulam and Yor [13], Bertoin and Werner [2], Yor [23], Vakeroudis [18]). Finally, in Section 4, we use the Gauss–Laplace transform (6) which is equivalent to Bougerol's identity (4) in order to check our results.

2. Integrability properties

Concerning the moments of $T_c^{|\theta|}$, we have the following (a more extended discussion is found in, e.g., Matsumoto and Yor [12]).

Theorem 2.1. *For every* $c > 0$, $T_c^{|\theta|}$ *enjoys the following integrability properties*:

(i) *for* $p > 0$, $E[(T_c^{|\theta|})^p] < \infty$, *if and only if* $p < \frac{\pi}{4c}$,
(ii) *for any* $p < 0$, $E[(T_c^{|\theta|})^p] < \infty$.

Corollary 2.2. *For* $0 < c < d$, *the random times* $T_{-d,c}^{\theta} \equiv \inf\{t: \theta_t \notin (-d, c)\}$, $T_c^{|\theta|}$ *and* T_c^{θ} *satisfy the inequality*:

$$T_c^{\theta} \geq T_{-d,c}^{\theta} \geq T_c^{|\theta|}. \tag{9}$$

Thus, their negative moments satisfy:

$$\text{for } p > 0 \qquad E\left[\frac{1}{(T_c^{\theta})^p}\right] \leq E\left[\frac{1}{(T_{-d,c}^{\theta})^p}\right] \leq E\left[\frac{1}{(T_c^{|\theta|})^p}\right] < \infty. \tag{10}$$

Proofs of Theorem 2.1 and of Corollary 2.2.

(i) The original proof is given by Spitzer [16], followed later by many authors Williams [20], Burkholder [6], Messulam and Yor [13], Durrett [9], Yor [22]. See also Revuz and Yor [15], Ex. 2.21, page 196.

(ii) In order to obtain this result, we might use the representation $T_c^{|\theta|} = A_{T_c^{|\gamma|}}$ together with a recurrence formula for the negative moments of A_t [8], Theorem 4.2, page 417 (in fact, Dufresne also considers $A_t^{(\mu)} = \int_0^t \mathrm{d}s \exp(2\beta_s + 2\mu s)$, but we only need to take $\mu = 0$ for our purpose, and we note $A_t \equiv A_t^{(0)}$), [17]. However, we can also obtain this result by simply remarking that the RHS of the Gauss–Laplace transform (6) in Proposition 1.1 is an infinitely differentiable function in 0 (see also [19]), thus:

$$E\left[\frac{1}{(T_c^{|\theta|})^p}\right] < \infty \qquad \text{for every } p > 0. \tag{11}$$

Now, Corollary 2.2 follows immediately from Theorem 2.1(ii). □

3. Limit theorems for $T_c^{|\theta|}$

3.1. Limit theorems for $T_c^{|\theta|}$, as $c \to 0$ and $c \to \infty$

The skew-product representation of planar Brownian motion allows to prove the three following asymptotic results for $T_c^{|\theta|}$.

Proposition 3.1.

(a) *For* $c \to 0$, *we have*:

$$\frac{1}{c^2} T_c^{|\theta|} \underset{c\to 0}{\overset{(law)}{\longrightarrow}} T_1^{|\gamma|}. \tag{12}$$

(b) *For* $c \to \infty$, *we have*:

$$\frac{1}{c} \log\bigl(T_c^{|\theta|}\bigr) \underset{c\to\infty}{\overset{(law)}{\longrightarrow}} 2|\beta|_{T_1^{|\gamma|}}. \tag{13}$$

(c) *For* $\varepsilon \to 0$, *we have*:

$$\frac{1}{\varepsilon^2}\bigl(T_{c+\varepsilon}^{|\theta|} - T_c^{|\theta|}\bigr) \underset{\varepsilon\to 0}{\overset{(law)}{\longrightarrow}} \exp(2\beta_{T_c^{|\gamma|}}) T_1^{\gamma'}, \tag{14}$$

where γ' *stands for a real Brownian motion*, *independent from* γ, *and* $T_1^{\gamma'} = \inf\{t\colon \gamma_t' = 1\}$.

Proof. We rely upon (5) for the three proofs. By using the scaling property of BM, we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \overset{(law)}{=} A_u(\beta)|_{u=c^2 T_1^{|\gamma|}}$$

thus:

$$\frac{1}{c^2} T_c^{|\theta|} \overset{(law)}{=} \int_0^{T_1^{|\gamma|}} \mathrm{d}v \exp(2c\beta_v). \tag{15}$$

(a) For $c \to 0$, the RHS of (15) converges to $T_1^{|\gamma|}$, thus we obtain part (a) of the proposition.

(b) For $c \to \infty$, taking logarithms on both sides of (15) and dividing by c, on the LHS we obtain $\frac{1}{c}\log(T_c^{|\theta|}) - \frac{2}{c}\log c$ and on the RHS:

$$\frac{1}{c}\log\biggl(\int_0^{T_1^{|\gamma|}} \mathrm{d}v \exp(2c\beta_v)\biggr) = \log\biggl(\int_0^{T_1^{|\gamma|}} \mathrm{d}v \exp(2c\beta_v)\biggr)^{1/c},$$

which, from the classical Laplace argument: $\|f\|_p \overset{p\to\infty}{\longrightarrow} \|f\|_\infty$, converges for $c \to \infty$, towards:

$$2 \sup_{v \le T_1^{|\gamma|}} (\beta_v) \overset{(law)}{=} 2|\beta|_{T_1^{|\gamma|}}.$$

This proves part (b) of the proposition.

(c)

$$\begin{aligned} T^{|\theta|}_{c+\varepsilon} - T^{|\theta|}_{c} &= \int_{T^{|\gamma|}_{c}}^{T^{|\gamma|}_{c+\varepsilon}} \mathrm{d}u \exp(2\beta_u) \\ &= \int_0^{T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}} \mathrm{d}v \exp(2\beta_{T^{|\gamma|}_{c}}) \exp\bigl(2(\beta_{v+T^{|\gamma|}_{c}} - \beta_{T^{|\gamma|}_{c}})\bigr) \\ &= \exp(2\beta_{T^{|\gamma|}_{c}}) \int_0^{T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}} \mathrm{d}v \exp(2B_v), \end{aligned} \tag{16}$$

where $(B_s \equiv \beta_{s+T^{|\gamma|}_{c}} - \beta_{T^{|\gamma|}_{c}}, s \geq 0)$ is a BM independent of $T^{|\gamma|}_{c}$.

We study now $\tilde{T}^{|\gamma|}_{c,c+\varepsilon} \equiv T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}$, the first hitting time of the level $c+\varepsilon$ from $|\gamma|$, starting from c. Thus, we define: $\rho_u \equiv |\gamma_u|$, starting also from c. Thus, $\rho_u = c + \delta_u + L_u$, where $(\delta_s, s \geq 0)$ is a BM and $(L_s, s \geq 0)$ is the local time of ρ at 0. Thus,

$$\begin{aligned} \tilde{T}^{|\gamma|}_{c,c+\varepsilon} &= \inf\{u \geq 0\colon\ \rho_u = c+\varepsilon\} \equiv \inf\{u \geq 0\colon\ \delta_u + L_u = \varepsilon\} \\ &\overset{u=\varepsilon^2 v}{=} \varepsilon^2 \inf\left\{v \geq 0\colon\ \frac{1}{\varepsilon}\delta_{v\varepsilon^2} + \frac{1}{\varepsilon} L_{v\varepsilon^2} = 1\right\}. \end{aligned} \tag{17}$$

From Skorokhod's lemma Revuz and Yor [15]:

$$L_u = \sup_{y \leq u}\bigl((-c - \delta_y) \vee 0\bigr)$$

we deduce:

$$\frac{1}{\varepsilon} L_{v\varepsilon^2} = \sup_{y \leq v\varepsilon^2}\bigl((-c - \delta_y) \vee 0\bigr) \overset{y=\varepsilon^2\sigma}{=} \sup_{\sigma \leq v}\left(\left(-c - \varepsilon\frac{1}{\varepsilon}\delta_{\sigma\varepsilon^2}\right) \vee 0\right) = 0. \tag{18}$$

Hence, with γ' denoting a new BM independent from γ, (16) writes:

$$T^{|\theta|}_{c+\varepsilon} - T^{|\theta|}_{c} = \exp(2\beta_{T^{|\gamma|}_{c}}) \int_0^{\varepsilon^2 T^{\gamma'}_1} \mathrm{d}v \exp(2B_v). \tag{19}$$

Thus, dividing both sides of (19) by ε^2 and making $\varepsilon \to 0$, we obtain part (c) of the proposition. □

Remark 3.2. The asymptotic result (c) in Proposition 3.1 may also be obtained in a straightforward manner from (16) by analytic computations. Indeed, using the Laplace transform of the first hitting time of a fixed level by the absolute value of a linear Brownian motion

$E[\mathrm{e}^{-(\lambda^2/2)T_b^{|\gamma|}}] = \frac{1}{\cosh(\lambda b)}$ (see, e.g., Proposition 3.7, page 71 in Revuz and Yor [15]), we have that for $0 < c < b$, and $\lambda \geq 0$:

$$E\left[\mathrm{e}^{-(\lambda^2/2)(T_b^{|\gamma|} - T_c^{|\gamma|})}\right] = \frac{\cosh(\lambda c)}{\cosh(\lambda b)}. \tag{20}$$

Using now $b = c + \varepsilon$, for every $\varepsilon > 0$, the latter equals:

$$\frac{\cosh(\lambda c/\varepsilon)}{\cosh((\lambda/\varepsilon)(c+\varepsilon))} \xrightarrow{\varepsilon \to 0} \mathrm{e}^{-\lambda}.$$

The result follows now by remarking that $\mathrm{e}^{-\lambda}$ is the Laplace transform (for the argument $\lambda^2/2$) of the first hitting time of 1 by a linear Brownian motion γ', independent from γ.

3.2. Generalizations

Obviously, we can obtain several variants of Proposition 3.1, by studying $T^{\theta}_{-bc,ac}$, $0 < a, b \leq \infty$, for $c \to 0$ or $c \to \infty$, and a, b fixed. We define $T^{\gamma}_{-d,c} \equiv \inf\{t\colon \gamma_t \notin (-d, c)\}$ and we have:

- $\frac{1}{c^2} T^{\theta}_{-bc,ac} \underset{c \to 0}{\overset{(law)}{\longrightarrow}} T^{\gamma}_{-b,a}$.
- $\frac{1}{c} \log(T^{\theta}_{-bc,ac}) \underset{c \to \infty}{\overset{(law)}{\longrightarrow}} 2|\beta|_{T^{\gamma}_{-b,a}}$.

In particular, we can take $b = \infty$, hence the following corollary.

Corollary 3.3.

(a) *For $c \to 0$, we have*

$$\frac{1}{c^2} T^{\theta}_{ac} \underset{c \to 0}{\overset{(law)}{\longrightarrow}} T^{\gamma}_{a}. \tag{21}$$

(b) *For $c \to \infty$, we have*

$$\frac{1}{c} \log\left(T^{\theta}_{ac}\right) \underset{c \to \infty}{\overset{(law)}{\longrightarrow}} 2|\beta|_{T^{\gamma}_{a}} \overset{(law)}{=} 2|C_a|, \tag{22}$$

where $(C_a, a \geq 0)$ is a standard Cauchy process.

Remark 3.4 (Yet another proof of Spitzer's theorem). Taking $a = 1$, from Corollary 3.3(b), we can obtain yet another proof of Spitzer's celebrated asymptotic theorem stated in (8). Indeed, (22) can be equivalently stated as:

$$P\left(\log T^{\theta}_{c} < cx\right) \underset{c \to \infty}{\overset{(law)}{\longrightarrow}} P\left(2|C_1| < x\right). \tag{23}$$

Now, the LHS of (23) equals:

$$\begin{aligned} P\big(\log T_c^\theta < cx\big) &\equiv P\big(T_c^\theta < \exp(cx)\big) \equiv P\Big(\sup_{u \le \exp(cx)} \theta_u > c\Big) \\ &= P\big(|\theta_{\exp(cx)}| > c\big) = P\left(|\theta_t| > \frac{\log t}{x}\right), \end{aligned} \tag{24}$$

with $t = \exp(cx)$. Thus, because $|C_1| \overset{(law)}{=} |C_1|^{-1}$, (23) now writes:

$$\text{for every } x > 0 \text{ given} \qquad P\left(|\theta_t| > \frac{\log t}{x}\right) \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} P\left(|C_1| > \frac{2}{x}\right), \tag{25}$$

which yields precisely Spitzer's theorem (8).

3.3. Speed of convergence

We can easily improve upon Proposition 3.1 by studying the speed of convergence of the distribution of $\frac{1}{c^2} T_c^{|\theta|}$ towards that of $T_1^{|\gamma|}$, that is, the following proposition.

Proposition 3.5. *For any function $\varphi \in \mathcal{C}^2$, with compact support,*

$$\begin{aligned} &\frac{1}{c^2}\left(E\left[\varphi\left(\frac{1}{c^2} T_c^{|\theta|}\right)\right] - E\big[\varphi\big(T_1^{|\gamma|}\big)\big]\right) \\ &\quad \underset{c\to 0}{\longrightarrow} E\left[\varphi'\big(T_1^{|\gamma|}\big)\big(T_1^{|\gamma|}\big)^2 + \frac{2}{3}\varphi''\big(T_1^{|\gamma|}\big)\big(T_1^{|\gamma|}\big)^3\right]. \end{aligned} \tag{26}$$

Proof. We develop $\exp(2c\beta_v)$, for $c \to 0$, up to the second order term, that is,

$$\mathrm{e}^{2c\beta_v} = 1 + 2c\beta_v + 2c^2\beta_v^2 + \cdots.$$

More precisely, we develop up to the second order term, and we obtain

$$\begin{aligned} E\left[\varphi\left(\frac{1}{c^2} T_c^{|\theta|}\right)\right] &= E\left[\varphi\left(\int_0^{T_1^{|\gamma|}} \mathrm{d}v \exp(2c\beta_v)\right)\right] \\ &= E\left[\varphi\big(T_1^{|\gamma|}\big) + \varphi'\big(T_1^{|\gamma|}\big) \int_0^{T_1^{|\gamma|}} \big(2c\beta_v + 2c^2\beta_v^2\big)\,\mathrm{d}v\right] \\ &\quad + \frac{1}{2} E\left[\varphi''\big(T_1^{|\gamma|}\big) 4c^2 \left(\int_0^{T_1^{|\gamma|}} \beta_v\,\mathrm{d}v\right)^2\right] + c^2 \mathrm{o}(c). \end{aligned}$$

We then remark that $E[\int_0^t \beta_v\,\mathrm{d}v] = 0$, $E[\int_0^t \beta_v^2\,\mathrm{d}v] = t^2/2$ and $E[(\int_0^t \beta_v\,\mathrm{d}v)^2] = t^3/3$, thus we obtain (26). □

4. Checks via Bougerol's identity

So far, we have not made use of Bougerol's identity (4), which helps us to characterize the distribution of $T_c^{|\theta|}$ [18]. In this subsection, we verify that writing the Gauss–Laplace transform in (6) as:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(1/c^2)T_c^{|\theta|}}}\exp\left(-\frac{xc^2}{2T_c^{|\theta|}}\right)\right] = \frac{1}{\sqrt{1+xc^2}}\varphi_m\left(xc^2\right), \tag{27}$$

with $m = \pi/(2c)$, we find asymptotically for $c \to 0$ the Gauss–Laplace transform of $T_1^{|\gamma|}$. Indeed, from (27), for $c \to 0$, we obtain:

$$\begin{aligned} &E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right] \\ &\quad = \lim_{c\to 0}\frac{2}{(\sqrt{1+xc^2}+\sqrt{xc^2})^{\pi/(2c)}+(\sqrt{1+xc^2}-\sqrt{xc^2})^{\pi/(2c)}}. \end{aligned} \tag{28}$$

Let us now study:

$$\begin{aligned} \left(\sqrt{1+xc^2}+\sqrt{xc^2}\right)^{\pi/(2c)} &= \exp\left(\frac{\pi}{(2c)}\log\left[1+\left(\sqrt{1+xc^2}-1\right)+\sqrt{xc^2}\right]\right) \\ &\sim \exp\left(\frac{\pi}{2c}\left[c\sqrt{x}+\frac{xc^2}{2}\right]\right)\underset{c\to 0}{\longrightarrow}\exp\left(\frac{\pi\sqrt{x}}{2}\right). \end{aligned}$$

A similar calculation finally gives

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right] = \frac{1}{\cosh((\pi/2)\sqrt{x})}, \tag{29}$$

a result which is in agreement with the law of $\beta_{T_1^{|\gamma|}}$, whose density is

$$E\left[\frac{1}{\sqrt{2\pi T_1^{|\gamma|}}}\exp\left(-\frac{y^2}{2T_1^{|\gamma|}}\right)\right] = \frac{1}{2\cosh((\pi/2)y)}. \tag{30}$$

Indeed, the law of $\beta_{T_c^{|\gamma|}}$ may be obtained from its characteristic function which is given by Revuz and Yor [15], page 73:

$$E\left[\exp(\mathrm{i}\lambda\beta_{T_c^{|\gamma|}})\right] = \frac{1}{\cosh(\lambda c)}.$$

It is well known that Lévy [11], Biane and Yor [4]:

$$\begin{aligned} E\big[\exp(\mathrm{i}\lambda\beta_{T_c^{|\gamma|}})\big] &= \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda c/\pi)} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\lambda c/\pi)y}\frac{1}{2\pi}\frac{1}{\cosh(y/2)}\,\mathrm{d}y \\ &\overset{x=cy/\pi}{=} \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda x}\frac{1}{2\pi}\frac{\pi/c}{\cosh(x\pi/(2c))}\,\mathrm{d}x = \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda x}\frac{1}{2c}\frac{1}{\cosh(x\pi/(2c))}\,\mathrm{d}x. \end{aligned} \tag{31}$$

So, the density $h_{-c,c}$ of $\beta_{T_c^{|\gamma|}}$ is:

$$h_{-c,c}(y) = \left(\frac{1}{2c}\right)\frac{1}{\cosh(y\pi/(2c))} = \left(\frac{1}{c}\right)\frac{1}{\mathrm{e}^{y\pi/(2c)} + \mathrm{e}^{-y\pi/(2c)}}$$

and for $c = 1$, we obtain (30).

We recall from Remark 3.2 that (see also Pitman and Yor [14], where further results concerning the infinitely divisible distributions generated by some Lévy processes associated with the hyperbolic functions cosh, sinh and tanh can also be found):

$$E\left[\exp\left(-\frac{\lambda^2}{2}T_c^{|\gamma|}\right)\right] = \frac{1}{\cosh(\lambda c)}, \tag{32}$$

thus, for $c = 1$ and $\lambda = \frac{\pi}{2}\sqrt{x}$, (29) now writes:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right] = E\left[\exp\left(-\frac{x\pi^2}{8}T_1^{|\gamma|}\right)\right], \tag{33}$$

a result which gives a probabilistic proof of the reciprocal relation that was obtained in Biane, Pitman and Yor [3] (using the notation of this article, Table 1, page 442):

$$f_{C_1}(x) = \left(\frac{2}{\pi x}\right)^{3/2} f_{C_1}\left(\frac{4}{\pi^2 x}\right).$$

References

[1] Alili, L., Dufresne, D. and Yor, M. (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement Brownien avec drift. In *Exponential Functionals and Principal Values Related to Brownian Motion* (M. Yor, ed.). *A collection of research papers*. *Biblioteca de la Revista Matematica*, *Ibero-Americana* 3–14. Madrid: Revista Matemática Iberoamericana.

[2] Bertoin, J. and Werner, W. (1994). Asymptotic windings of planar Brownian motion revisited via the Ornstein–Uhlenbeck process. In *Séminaire de Probabilités*, *XXVIII*. *Lecture Notes in Math*. **1583** 138–152. Berlin: Springer. MR1329109

[3] Biane, P., Pitman, J. and Yor, M. (2001). Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull*. *Amer. Math*. *Soc*. (*N.S.*) **38** 435–465 (electronic). MR1848256

[4] Biane, P. and Yor, M. (1987). Valeurs principales associées aux temps locaux browniens. *Bull*. *Sci*. *Math*. **111** 23–101. MR0886959

[5] Bougerol, P. (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré Sect. B* (*N.S.*) **19** 369–391. MR0730116
[6] Burkholder, D.L. (1977). Exit times of Brownian motion, harmonic majorization, and Hardy spaces. *Advances in Math.* **26** 182–205. MR0474525
[7] Doney, R.A. and Vakeroudis, S. (2013). Windings of planar stable processes. In *Sém. Prob. XLV, Lect. Notes in Mathematics*. Berlin/Heidelberg/New York: Springer. To appear.
[8] Dufresne, D. (2000). Laguerre series for Asian and other options. *Math. Finance* **10** 407–428. MR1785163
[9] Durrett, R. (1982). A new proof of Spitzer's result on the winding of two-dimensional Brownian motion. *Ann. Probab.* **10** 244–246. MR0637391
[10] Itô, K. and McKean, H.P. Jr. (1965). *Diffusion Processes and Their Sample Paths*. *Die Grundlehren der Mathematischen Wissenschaften* **125**. Berlin/Heidelberg/New York: Springer. MR0199891
[11] Lévy, P. (1980). *Œuvres de Paul Lévy. Processus stochastiques* **4** Paris: Gauthier-Villars. Published under the direction of Daniel Dugué with the collaboration of Paul Deheuvels and Michel Ibéro. MR0586765
[12] Matsumoto, H. and Yor, M. (2005). Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.* **2** 312–347. MR2203675
[13] Messulam, P. and Yor, M. (1982). On D. Williams' "pinching method" and some applications. *J. London Math. Soc.* (2) **26** 348–364. MR0675178
[14] Pitman, J. and Yor, M. (2003). Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.* **55** 292–330. MR1969794
[15] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Berlin: Springer. MR1725357
[16] Spitzer, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87** 187–197. MR0104296
[17] Vakeroudis, S. (2011). Nombres de tours de certains processus stochastiques plans et applications à la rotation d'un polymère. (Windings of some planar stochastic processes and applications to the rotation of a polymer.) PhD thesis, Université Pierre et Marie Curie (Paris VI), Paris.
[18] Vakeroudis, S. (2011). On hitting times of the winding processes of planar Brownian motion and of Ornstein–Uhlenbeck processes, via Bougerol's identity. *Teor. Veroyatnost. i Primenen.–SIAM Theory Probab. Appl.* **56** 566–591 (in TVP).
[19] Vakeroudis, S. and Yor, M. (2012). Some infinite divisibility properties of the reciprocal of planar Brownian motion exit time from a cone. *Electron. Commun. Probab.* **17** 9. MR2950189
[20] Williams, D. (1974). A simple geometric proof of Spitzer's winding number formula for 2-dimensional Brownian motion. Preprint. Swansea: Univ. College.
[21] Yor, M. (1980). Loi de l'indice du lacet brownien, et distribution de Hartman–Watson. *Z. Wahrsch. Verw. Gebiete* **53** 71–95. MR0576898
[22] Yor, M. (1985). Une décomposition asymptotique du nombre de tours du mouvement brownien complexe. *Astérisque* **132** 103–126. Colloquium in honor of Laurent Schwartz, Vol. 2 (Palaiseau, 1983). MR0816763
[23] Yor, M. (1997). Generalized meanders as limits of weighted Bessel processes, and an elementary proof of Spitzer's asymptotic result on Brownian windings. *Studia Sci. Math. Hungar.* **33** 339–343. MR1454119
[24] Yor, M. (2001). *Exponential Functionals of Brownian Motion and Related Processes*. *Springer Finance*. Berlin: Springer. With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson. MR1854494

Received January 2012